Appendix 3
Homological Algebra

A3.1 Introduction

A **complex** of modules over a ring R is a sequence of R-modules and homomorphisms

$$\mathcal{F}: \ \cdots \to F_{i+1} \xrightarrow{\varphi_{i+1}} F_i \xrightarrow{\varphi_i} F_{i-1} \to \cdots$$

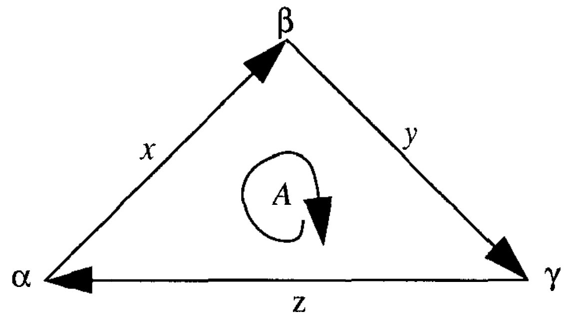
such that $\varphi_i \varphi_{i+1} = 0$ for each i. The **homology** of F at F_i is defined to be

$$H_i\mathcal{F} := \ker \varphi_i / \operatorname{im} \varphi_{i+1}.$$

Homological algebra is, roughly speaking, the study of complexes of modules and their homology.

Some basic terminology: The module F_i is called the **term of degree *i*** of F. For reasons we shall explain the maps φ_i are often called the "boundary operators," or "differentials," of $\mathcal{F}$. The elements of the image of φ_{i+1} are accordingly called boundaries, and the elements of the kernel of φ_i are called cycles. We think of $\mathcal{F}$ as having infinitely many terms, but we shall almost always be concerned only with complexes where $F_i = 0$ either for all $i < 0$ or for all $i > 0$. It is often convenient not to indicate explicitly the terms that are zero. The complex $\mathcal{F}$ is said to be **exact** at F_i if $H_iF = 0$; we say that $\mathcal{F}$ is exact if it is exact at every F_i.

Complexes appear in the work of Cayley fairly explicitly as early as [1858]. They were used by Hilbert in his famous work [1890] to compare a factor ring of a polynomial ring to the polynomial ring itself (just as we shall use free resolutions to compare an arbitrary module to free modules); the context of his application was explained in Chapter 1. Our current terminology was introduced much later. The name "complex," for example, arose from the simplicial complexes of topology: To an oriented simplicial complex Poincaré [1899] associated a "chain complex," with geometrically defined "boundary operator." The case of a triangle is illustrated in the figure. The formulation in terms of groups and maps came later, apparently suggested by Emmy Noether to several people in the mid-1920s.



$$
\begin{aligned}
\mathbf{Z}A &\to \mathbf{Z}x \oplus \mathbf{Z}y \oplus \mathbf{Z}z \to \mathbf{Z}\alpha \oplus \mathbf{Z}\beta \oplus \mathbf{Z}\gamma \\
\partial A &= x + y + z, \\
\partial x &= \beta - \alpha,\ \partial y = \gamma - \beta,\ \partial z = \alpha - \gamma
\end{aligned}
$$

Another part of the prehistory of homological algebra is Poincaré's study of the complex of differential forms on a manifold that we now call the de Rham complex: Poincaré's lemma asserts that the de Rham complex of $\mathbf{R}^n$ is exact. (de Rham's name attaches to the complex because he was the first to prove—in the 1940s—that the cohomology of the de Rham complex is a topological invariant.) The maps in the de Rham complex are derived from differentiation, and it was natural to call them differentials:

$$
\mathcal{C}^{\infty}_{\mathbf{R}} \xrightarrow{\partial} T^*_{\mathbf{R}} \to \wedge^2 T^*_{\mathbf{R}} \to \cdots \qquad \partial(f(x)) = f'(x)\, dx.
$$

If M is an R-module, we may consider M as a complex

$$
\cdots \to 0 \to M \to 0 \to \cdots
$$

with only one nonzero term. Thus homological algebra includes the study of modules. In commutative algebra, homological algebra is usually pursued in order to study modules more closely. Complexes give us a way of comparing an arbitrary module with nicer ones—with free, or projective, or injective modules. Perhaps the most complete expression of this idea is in the construction of the derived category, which we describe briefly in the last section of this appendix.

Complexes arise naturally from the study of systems of linear equations: A system of n_0 linear equations in n_1 unknowns over a ring R corresponds to an $n_0 \times n_1$ matrix φ over R (the matrix of coefficients of the equations), or alternately as a map of free R-modules, $\varphi : F_1 = R^{n_1} \to R^{n_0} = F_0$. A family of solutions to the (homogeneous) equations

$$\varphi X = 0$$

may be described by a map $F_2 \to F_1$ making the sequence

$$F_2 \to F_1 \to F_0$$

a complex. Solving the equations means giving a "complete" set of solutions; that is, a map as above making the complex exact at F_1.

If R is a field, then of course there is a finite linearly independent set of solutions in terms of which all others can be expressed. For more general rings, this is no longer the case: It may be impossible to choose a generating set for the kernel consisting of linearly independent elements.

An example will make the situation clear. Let $R = k[a, b, c]$ be a polynomial ring in three variables, and consider the linear equation in three unknowns

$$aX_1 + bX_2 + cX_3 = 0$$

corresponding to the map $\varphi : R^3 \to R$ with matrix (a, b, c). By analogy with our experience of linear equations over a field, we should say that the rank of this system is 1, so we should expect $3 - 1 = 2$ independent solutions. However, the three columns of the matrix

$$\begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}$$

are all solutions (elements of ker φ). It is easy to see that they actually generate ker φ, but that no two elements generate it. Furthermore, these three generators are linearly dependent in the sense that if we multiply the first column by a, the second by b, and the third by c, and add, we get 0. It is not hard to show (see Chapter 20) that every complete set of solutions must be linearly dependent. Thus we have a situation that could not have arisen over a field: a system of linear equations such that any complete set of solutions is linearly dependent.

If we wish to describe the solutions to our original system of equations as linear combinations of the solutions in a complete set of solutions, then we must describe the linear dependencies (otherwise, we won't be able to tell which linear combinations give the trivial solution). If we have n_2 solutions, and we define a new map φ_2 of free R-modules $F_2 = R^{n_2}$ to F_1 by sending the basis elements of F_2 to our solutions, then the dependencies are the elements of the kernel of φ_2. We may regard φ_2 as being a new system of linear equations, and the process of solving begins again. With hindsight we rename φ as φ_1, and continuing the process above we finally arrive at a complex:

$$\cdots \to R^{n_i} \xrightarrow{\varphi_i} \cdots \xrightarrow{\varphi_2} R^{n_1} \xrightarrow{\varphi_1} R^{n_0};$$

in fact, this is an especially interesting sort of complex, called a **free resolution**. In the example above, the resolution actually ends at the next step beyond the one we have given, in the sense that the kernel of φ_2 is itself free (that is, φ_2 has a complete system of solutions with no dependencies). It may be exhibited as in Figure A3.1.

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} a \\ b \\ c \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}} R^3 \xrightarrow{(abc)} R$$

FIGURE A3.1.

This phenomenon is typical of rings called regular rings; see Chapter 19. The complex given here is called a **Koszul complex** (the name, though universal, is misleading: Such complexes appeared in the works of Cayley and Hilbert before Koszul was born).

We shall now take up these notions systematically, if somewhat sketchily. The proofs that we have omitted are all easy, and we leave them as exercises for the reader.[1] The goal of the first half of this appendix is the theory of derived functors; Ext, Tor, and local cohomology are the most important ones here. One of our less traditional topics in this part is the theory of injective modules over a Noetherian ring. The second half of the appendix is an introduction to spectral sequences.

As everywhere in this book, we shall work with modules over a commutative ring, but the reader should know that nearly everything here can be generalized with just a little effort to modules over an arbitrary ring, or even to objects in a nice Abelian category. Jans [1964], Rotman [1979], and Maclane [1963] are readable sources for more information, roughly in order of increasing difficulty and comprehensiveness. The book of Gelfand and Manin [1989] should soon be available in English.

Part I: Resolutions and Derived Functors

Let R be a commutative ring; the modules in this chapter will all be R-modules.

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[1]This is not so bad. A famous exercise from Serge Lang's influential textbook *Algebra* [Addison-Wesley, Reading, MA, 1965, p. 105] reads: "Take any book on homological algebra, and prove all the theorems without looking at the proofs given in that book."

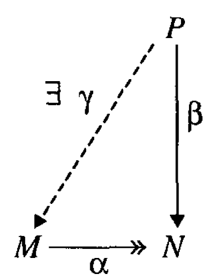
A3.2 Free and Projective Modules

The easiest modules to understand are the free modules: direct sums of copies of the ring. From our point of view free modules are useful because it is easy to define a map from a free module: Namely, suppose F is free on a set of generators p_i (that is, $F \cong \oplus_i R$, and we denote the generator of the ith summand by p_i). To define a map from F to any module M it is enough to tell where to send the generators p_i, and any choice of images for these elements will do. That is,

$$\mathrm{Hom}_{R\text{-modules}}(F, M) = \mathrm{Hom}_{\mathrm{Sets}}(\{p_i\}, M).$$

(In the language of category theory (Appendix 5), the "free module functor is left-adjoint to the forgetful functor"; but we shall not use this formulation here.) This property makes them **projective** in the following sense:

Definition. *A module P is projective if for every epimorphism of modules $\alpha : M \twoheadrightarrow N$ and every map $\beta : P \to N$, there exists a map $\gamma : P \to M$ such that $\beta = \alpha\gamma$, as in the following figure.*



Free modules are projective because if P is free on a set of generators p_i, then we may choose elements q_i of M that map to the elements $\beta(p_i) \in N$, and take γ to be the map sending p_i to q_i.

The definition of projectivity has several useful reformulations:

Proposition A3.1. *Let P be an R-module. The following are equivalent:*

a. *P is projective.*

b. *For every epimorphism of modules $\alpha : M \twoheadrightarrow N$, the induced map $\mathrm{Hom}(P, M) \to \mathrm{Hom}(P, N)$ is an epimorphism.*

c. *For some epimorphism $F \twoheadrightarrow P$, where F is free, the induced map $\mathrm{Hom}(P, F) \to \mathrm{Hom}(P, P)$ is an epimorphism.*

d. *P is a direct summand of a free module.*

e. *Every epimorphism $\alpha : M \twoheadrightarrow P$ "splits": That is, there is a map $\beta : P \to M$ such that $\alpha\beta = 1_P$.*

Proof.

a $\Leftrightarrow$ b: This is a restatement of the definition.

b $\Rightarrow$ c: Obvious.

c $\Rightarrow$ d: Any map $\varphi \in \text{Hom}(P, F)$ in the preimage of the identity map $1 \in \text{Hom}(P, P)$ is a splitting of the epimorphism $F \twoheadrightarrow P$, so P is a summand of F.

d $\Rightarrow$ b: This follows because for any modules P and Q we have

$$\text{Hom}(P \oplus Q, -) = \text{Hom}(P, -) \oplus \text{Hom}(Q, -).$$

We have now shown that a, b, c, and d are equivalent.

a $\Rightarrow$ e: Apply the definition in the case where β is the identity map of P.

e $\Rightarrow$ d: Obvious. □

As a first example, the reader may check that a finitely generated **Z**-module is projective iff it is torsion-free iff it is free.

Not all projective modules are free; perhaps the simplest example is the ideal $(2, 1+\sqrt{-5}) \subset \mathbf{Z}[\sqrt{-5}]$; see Chapter 11. In general, whether or not a projective module is free is quite a hard question. (We have already discussed the connection of this question to number theory.)

Geometrically, projective modules correspond to algebraic vector bundles: The set of sections of a vector bundle on a variety X is a module over the ring of regular functions on X. This connection is sketched in Corollary A3.3. The relation of algebraic vector bundles to the structure of X is more subtle than in the topological case. For example, topological vector bundles on contractible spaces are easily shown to be trivial, but it was only recently shown (by Quillen and Suslin, in answer to a celebrated problem of Serre; see Lam [1978]) that algebraic vector bundles on $\mathbf{A}^n_k$ are trivial—that is, that projective modules over $k[x_1, \ldots, x_r]$, where k is a field, are free.

Projective modules behave well under localization. Moreover, there is a useful "local criterion" for projectivity, established in Exercises 4.11 and 4.12 and their hints, and summarized as follows:

Theorem A3.2 (Characterization of projectives). *Let M be a finitely presented module over a Noetherian ring R. The following are equivalent:*

a. M is a projective module.

b. M_P is a free module for every maximal ideal (and thus for every prime ideal) P of R.

c. *There is a finite set of elements* $x_1, \ldots, x_r \in R$ *that generate the unit ideal of* R*, such that* $M[x_i^{-1}]$ *is free over* $R[x_i^{-1}]$ *for each* i.

In particular, every projective module over a local ring is free. Every graded projective module over a positively graded ring R *with* R_0 *a field is a graded free module.* □

Corollary A3.3. *Finitely generated projective modules over the affine ring* A *of a variety* X *correspond to vector bundles on* X*: Given a vector bundle* E*, its sections* $\Gamma(E)$ *form a finitely generated projective* A*-module, and any finitely generated projective module arises from a unique vector bundle in this way.*

Proof (*sketch, for those who know about sheaves*). If E is a vector bundle, then there is a covering of X by affine open sets $X_i = \{p \in X | x_i(p) \neq 0\}$ such that $E|_{X_i}$ is trivial. Thus $\Gamma(E|_{X_i}) = \Gamma(E)[x_i^{-1}]$ is free, and $\Gamma(E)$ is projective by Theorem A3.2. Conversely, suppose M is a finitely generated projective module. By Theorem A3.2 we may find elements $x_1, \ldots, x_r$ that generate the unit ideal and such that $M[x_i^{-1}]$ is free (of some rank r_i) for each i. Let E_i be the trivial bundle on X_i of rank r_i. Choose an isomorphism $\alpha_i : \Gamma(E_i) \to M[x_i^{-1}]$. On $X_i \cap X_j$ we may form the composite

$$\Gamma(E_{i|X_j}) \xrightarrow{\alpha_i} M[(x_i x_j)^{-1}] \xrightarrow{\alpha_j^{-1}} \Gamma(E_{j|X_i}),$$

and this determines an isomorphism of bundles $a_{ij} : E_{i|X_i \cap X_j} \to E_{j|X_i \cap X_j}$. Using the maps a_{ij} as gluing maps, we reconstruct a vector bundle E on X. An easy computation using Exercise 2.19 shows that $M = \Gamma(E)$. Further, if M is the module of sections of a vector bundle E' to begin with, then the identification of modules of sections

$$\Gamma(E_i) \to M[x_i^{-1}] = \Gamma(E'_{|X_i})$$

comes from an isomorphism $E_{|X_i} = E_i \to E'_{|X_i}$. Since these isomorphisms are compatible with the gluings, we get $E' \cong E$. □

A3.3 Free and Projective Resolutions

As we have already noted, every module M is an epimorphic image of a free (and thus projective) module—just choose a set of generators $\{g_i\}$ for M, and map a free module on a corresponding set of generators $\{e_i\}$ to M by sending e_i to g_i. This makes it easy to compare any module to free modules: if $\alpha : F_0 \twoheadrightarrow M$ is an epimorphism, then we may say that F_0 differs from M by the module $\ker \alpha$. We may thus express M in terms of free modules "better" by mapping a free module F_1 onto $\ker \alpha$. Taking φ_1 to be the composite

$$F_1 \twoheadrightarrow \ker \alpha \rightarrowtail F_0,$$

we may say instead that $M = \operatorname{coker} \varphi_1 : F_1 \to F_0$. Unfortunately, there is still a (possibly) nonfree module lurking in this description: the kernel of φ_1. We can repair this defect to some extent by taking a free module F_2 that maps onto $\ker \varphi_1$. Writing $\varphi_2 : F_2 \to F_1$ for the composite

$$F_2 \twoheadrightarrow \ker \varphi_1 \hookrightarrow F_1,$$

we may think of M as given by the sequence of free modules

$$F_2 \to F_1 \to F_0.$$

There is still the problem that $\ker \varphi_2$ might not be free. Repeating the process above indefinitely if necessary, we may at last obtain a sequence of free modules

$$F : \cdots \to F_{i+1} \xrightarrow{\varphi_{i+1}} F_i \xrightarrow{\varphi_i} F_{i-1} \to \cdots \xrightarrow{\varphi_1} F_0$$

with the properties that φ_{i+1} maps F_{i+1} onto the kernel of φ_i for each $i \geq 1$, and that M is the cokernel of φ_1. A sequence of free modules F_i and maps φ_i with these properties is called a **free resolution** of M. If the F_i are merely projective, it is called a **projective resolution**. Note that F is a complex in the sense above (regarding all the F_i with $i < 0$ as 0).

Example. Perhaps the simplest nontrivial, finite free resolutions are the Koszul complexes; see Chapter 17. The simplest nontrivial, infinite resolution might be the following:

Let $S = k[x]$ be a polynomial ring in one variable (k could be any ring, but might as well be taken to be a field). Let $R = S/(x^n)$, and let M be R/Rx^m, with $0 < m < n$, regarded as an R-module. Here is a free resolution of M as an R-module:

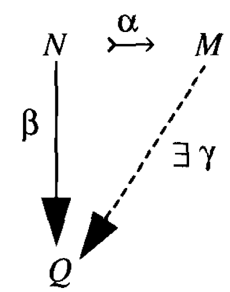
$$\cdots \xrightarrow{x^{n-m}} R \xrightarrow{x^m} R \xrightarrow{x^{n-m}} R \xrightarrow{x^m} R,$$

where we have written x^a for the map that is multiplication by x^a. We leave the easy verification to the reader.

A3.4 Injective Modules and Resolutions

The notion of an **injective** module is dual to that of a projective module, but perhaps because injective modules are almost never finitely generated, they are not so familiar.

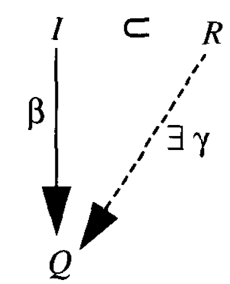
Definition. *An R-module Q is* ***injective*** *if for every monomorphism of R-modules $\alpha : N \rightarrowtail M$ and every homomorphism of R-modules $\beta : N \to Q$,* *there exists a homomorphism of R-modules* $\gamma : M \to Q$ *such that* $\beta = \gamma\alpha$, *as in the figure.*



Although the definition of injective modules is precisely dual to that of projective modules, the theory is not dual at all (the category of modules is quite different from its dual category, so this should not be a surprise). The subject is quite beautiful, and we shall explain its beginning.

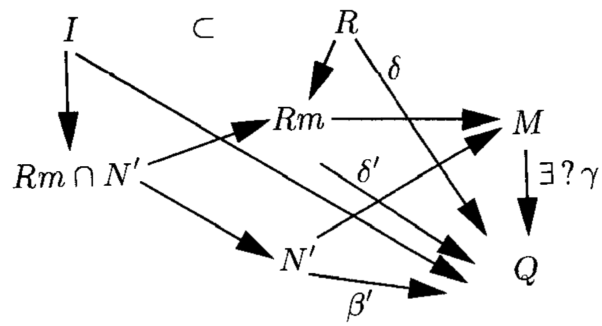
We begin with a result of Reinhold Baer (who defined injective modules in [1940]), showing that in the definition of injectives it is enough to check the case where α is the inclusion of an ideal in the ring.

Lemma A3.4 (Baer). *Let* Q *be an* R*-module. If for every ideal* $I \subset R$, *every homomorphism* $\beta : I \to Q$ *extends to* R *as in the diagram, then* Q *is injective.*



Proof. Suppose M and N are arbitrary R-modules. Let $\beta : N \to Q$ be a homomorphism, and let $\alpha : N \rightarrowtail M$ be a monomorphism. If N' is a submodule of M containing N, then we shall say that $\beta' : N' \to Q$ is an extension of β if β' restricts to β on N. We wish to show that there is an extension of β to M. By Zorn's lemma, there is a submodule N' and an extension β' of β to N' that is maximal in the sense that β' can be extended no further. If $N' = M$, we are done.

Supposing that $N' \neq M$, we shall derive a contradiction. Let $m \in M$ be outside of N', and consider the submodule $N'' = N' + Rm$. Let $I = \{r \in R | rm \in N'\}$. By hypothesis the map $I \to Q$ sending $r \in I$ to $\beta'(rm) \in Q$ extends to a map $\delta : R \to Q$. The map δ induces a map $\delta' : Rm \to Q$ because the kernel of the map $R \to Rm$ is contained in $\ker \delta$, and δ' agrees with β' on $Rm \cap N'$ by definition. We may thus define an extension β'' of β' to N'' by letting β'' be β' on N' and δ' on Rm. This contradicts the maximality of N' and β'. All the necessary maps are shown in the figure. □



Injective modules over **Z**—that is, injective Abelian groups—are easy to describe:

Proposition A3.5. *An Abelian group Q is injective iff it is divisible in the sense that for every $q \in Q$ and every $o \neq n \in \mathbf{Z}$ there exists $q' \in Q$ such that $nq' = q$.*

Proof. Let Q be injective, and let $q \in Q, o \neq n\ n \in \mathbf{Z}$. Let $\beta : \mathbf{Z} \to Q$ be the map sending 1 to q and let $\alpha : \mathbf{Z} \to \mathbf{Z}$ be multiplication by n. Since Q is injective there is a map $\gamma : \mathbf{Z} \to Q$ with $\beta = \gamma\alpha$. It follows that $n\gamma(1) = q$, so Q is divisible.

Conversely, suppose that Q is divisible. We apply Baer's lemma, A3.4: Let $(n) \subset \mathbf{Z}$ be the inclusion of an ideal. Suppose a map $\beta : (n) \to Q$ takes n to q. Since Q is divisible we may choose $q' \in Q$ with $nq' = q$. The map $\gamma : \mathbf{Z} \to Q$ sending 1 to q' obviously extends β.

From Proposition A3.5 we easily derive a result that is dual to the statement that subgroups of free groups are free.

Corollary A3.6. *If Q is an injective Abelian group, and K is any subgroup, then Q/K is an injective Abelian group.*

Proof. If Q is divisible, then Q/K is divisible too.

We can now show that every Abelian group may be embedded in an injective Abelian group:

Corollary A3.7 (Baer [1940]). *There are "enough" injective Abelian groups, in the sense that for every module M there is a monomorphism $i : M \to Q$ with Q injective.*

Proof. Write $M = F/K$, with F a free module. F is contained in the **Q**-vector space $F \otimes_{\mathbf{Z}} \mathbf{Q}$, which is obviously divisible. Thus M is contained in the divisible group $(F \otimes_{\mathbf{Z}} \mathbf{Q})/K$. □

Remarkably enough, the corresponding statement for modules over any ring, which is the main goal of our development, is an immediate consequence. (In fact the same argument works still more generally, for example in categories of sheaves of modules over a "ringed space"—a fact that is exploited in the cohomology theory of sheaves. See, for example, Hartshorne [1977, p. 207].) The key observation is this:

Lemma A3.8. *If R is an S-algebra, and Q' is an injective S-module, then $Q := \mathrm{Hom}_S(R, Q')$ is an injective R-module (the R-module structure comes via the action of R on the first factor of* $\mathrm{Hom}_S(R, Q')$*).*

For a partial converse, see Exercise A3.7a.

Proof. Let $N \subset M$ be a submodule, and let $\beta : N \to Q$ be a homomorphism; we must show that β extends to M. There is a natural map of S-modules $Q \to Q'$, sending a homomorphism φ to $\varphi(1)$. Let β' be the composite of β and this map $Q \to Q'$, and let γ' be an extension of β' to M, regarded as an S-module. We may define the desired map $\gamma : M \to Q$ of R-modules by sending m to the map φ defined by $\varphi(r) = \gamma'(rm)$.

Corollary A3.9. *For any ring R, the category of R-modules has* **enough injective objects**, *in the sense that for every module M there is a monomorphism $i : M \to Q$ with Q injective.*

Proof (Eckmann and Schöpf [1953]). There is a monomorphism $\alpha : M \to \mathrm{Hom}_{\mathbf{Z}}(R, M)$ sending m to the map φ given by $\varphi(r) = rm$. Temporarily viewing M as an Abelian group, we know that there is a monomorphism of Abelian groups, $\beta : M \to Q'$ of M into an injective Abelian group Q'; applying the functor $\mathrm{Hom}_{\mathbf{Z}}(R, -)$ we get a monomorphism $\beta' : \mathrm{Hom}_{\mathbf{Z}}(R, M) \to \mathrm{Hom}_{\mathbf{Z}}(R, Q')$. By Lemma A3.8 the module $\mathrm{Hom}_{\mathbf{Z}}(R, Q')$ is an injective R-module. Thus $\beta'\alpha$ is a monomorphism of M to an injective module, as desired. □

If M is an R-module, then by Corollary A3.9 we may embed M in an injective module Q_0. We may then embed the cokernel, Q_0/M, in an injective module Q_1. Continuing in this way, we get an **injective resolution**

$$0 \to M \to Q_0 \to Q_1 \to Q_2 \to \cdots$$

of M; that is, an exact sequence of the given form in which all the Q_i are injectives. We shall see how such resolutions are used in the upcoming section on derived functors.

Example. The most familiar injective modules are the divisible Abelian groups. Perhaps the simplest interesting injective resolution is that of $\mathbf{Z}$ as a $\mathbf{Z}$-module:

$$0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0.$$

In general, injective modules have an interesting and simple structure; see the exercises for more information.

In Chapter 20 it is shown that if R is a local ring then every finitely generated R-module has a unique minimal projective (actually free) resolution. The situation for injective modules is much better: Any module over any ring has a unique minimal injective resolution! The key idea is that of **injective envelope** (or **injective hull**). First a preliminary definition:

Let R be a ring and let $M \subset E$ be R-modules. We say that M is an **essential submodule** of E, or that E is an **essential extension** of M if every nonzero submodule of E intersects M nontrivially.

Proposition–Definition A3.10. *Let R be a ring.*

a. *Given any R-modules $M \subset F$, there is a maximal submodule E of F containing M such that $M \subset E$ is essential.*

b. *If F is injective, then so is E.*

c. *There is, up to isomorphism, a unique essential extension E of M that is an injective R-module; this E is called the **injective envelope** of M, written $E(M)$.*

Proof.

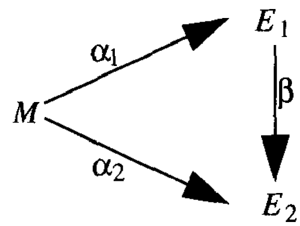
a. If $M \subset E_1 \subset E_2 \subset \cdots \subset F$ with $M \subset E_i$ essential, then any submodule N of $\cup_i E_i$ meets some E_i nontrivially, and thus meets M nontrivially. Thus M is essential in $\cup E_i$. Since M is essential in M, it follows by Zorn's lemma that there exist maximal essential extensions of M contained in F.

b. Suppose now that F is injective, and $M \subset E \subset F$ with $M \subset E$ a maximal essential extension of M by a submodule of F. If E' were an essential extension of E in F, then any nontrivial submodule of E' would meet E, and thus M, nontrivially, so E' would be an essential extension of M and so $E' = E$ by hypothesis. It thus suffices to treat the case where $M = E$. Let N be a submodule of F maximal among those not meeting E; such submodules exist by Zorn's lemma. Since E and N do not meet, we see that $E \oplus N \cong E + N \subset F$. We shall show that $F = E + N$, from which it follows that $E \oplus N \cong F$, so E is injective.

Consider the composite map $\alpha : E \subset F \twoheadrightarrow F/N$. Because N does not meet E, α is an inclusion. It is essential, for if a submodule N' of F/N failed to meet E, then its preimage in F would be a submodule larger than N and not meeting E, contradicting our hypothesis. Since F is injective, we may find a map $\beta : F/N \to F$ extending α. Since $(\ker \beta) \cap E = \ker \alpha = 0$, and E is essential in F/N, we see that $\ker \beta = 0$. In particular, $\beta(F/N)$ is an essential extension of E. It follows from the maximality of E that $\beta(F/N) = E$, so $F/N = E$, and $E + N = F$ as desired.

c. By **Corollary A3.9** there exist monomorphisms from M to an injective R-module F. From parts a and b we see that a maximal submodule $E \subset F$, such that $M \subset E$ is essential, is injective too.

For uniqueness, suppose that $\alpha_1 : M \to E_1$ and $\alpha_2 : M \to E_2$ are both essential inclusions, with E_1 and E_2 injective, then by the injectivity of E_2 there exists a map $\beta : E_1 \to E_2$ extending α_2 in the sense that the following diagram



commutes. Since $\ker \beta|_M = \ker \alpha_2 = 0$, and $\alpha_1(M)$ is essential in E_1, we see that $\ker \beta = 0$. Thus $\beta(E_1)$ is an injective submodule of E_2. It follows that $E_2 = \beta(E_1) \oplus E_2'$ for some submodule E_2' of E_2. Since $\alpha_2(M)$ is essential in E_2, and $\alpha_2(M) \subset \beta(E_1)$, we must have $E_2' = 0$, and β is the required isomorphism. □

We now say that an injective resolution

$$(*) \qquad 0 \to M \to Q_0 \to Q_1 \to Q_2 \to \cdots$$

is a **minimal injective resolution** if, setting $M_i = \operatorname{coker}(Q_{i-1} \to Q_i)$, we have $Q_{i+1} = E(M_i)$, and the map $Q_i \to Q_{i+1}$ to be the composite of the natural maps

$$Q_i \to M_i \to E(M_i) = Q_{i+1}.$$

As an immediate consequence of Proposition A3.10 we have:

Corollary A3.11. *If R is any ring and M is any R-module, then M has a unique minimal injective resolution.* □

A3.4.1 Exercises

Injective Envelopes

Exercise A3.1: The following principle was used several times in the text: Show that if $N \subset M$ is an essential submodule then any map $M \to E$ of modules that restricts to a monomorphism on N is a monomorphism.

Injective Modules over Noetherian Rings

Exercise A3.2 a. (Bass' Characterization of Noetherian rings):* Show that arbitrary direct products of injective modules are injective.

Show, however, that a ring R is Noetherian iff every direct sum of injective R-modules is injective. (This observation from Bass' graduate student days appears, with reference to Bass, in Chase [1960].)

b. Again, assume that R is a Noetherian ring. Use part a to show that any injective module is a direct sum of indecomposable injective modules.

Exercise A3.3 (Injectives and primes): We shall say that an injective module E is indecomposable if it cannot be written as a direct sum $E = E' \oplus E''$ with both E and E'' nonzero. Suppose that R is a Noetherian ring. Use primary decomposition to show that if E is an indecomposable injective R-module, then $E \cong E(R/P)$ for some prime ideal P of R. Show that if P and Q are primes, then $E(R/P) \cong E(R/Q)$ iff $P = Q$. Thus there is a one-to-one correspondence between indecomposable injectives and prime ideals.

Exercise A3.4: We can compute injective envelopes in some simple cases:

a.* Let R be a Noetherian ring and let P be any ideal of R. Set $E = E(R/P)$. For any ideal $I \subset R$ and map $\varphi : I \to R/P$, use the Artin-Rees lemma (Lemma 5.1) to show that there is a number d such that φ factors through $I/(P^d \cap I) \cong (P^d + I)/P^d$. Deduce that $E' = \cup_d (0 :_E P^d) \subset E$, the set of elements annihilated by some power of P, is injective, and thus that $E = E'$.

b.* With notation as in part a, suppose that P is a maximal ideal. Show that $(0 :_E P^d)$ is the injective hull of R/P over the Artinian ring R/P^d. By Corollary 21.3 it is a module of the same finite length as R/P^d.

c. Let $R = k[x_1, \ldots, x_r]$, and let $P = (x_1, \ldots, x_r)$. Let $E = \oplus_d \operatorname{Hom}_k(R_d, k)$ be the graded dual of R. We have $E \subset E_1 := \operatorname{Hom}_k(R, k) = \Pi_d \operatorname{Hom}_k(R_d, k)$, which is an injective R-module by Lemma A3.8. Show that E is an essential extension of $k = \operatorname{Hom}_k(k, k) \subset \operatorname{Hom}_k(R, k)$. Show that $E = \cup_d (0 :_{E_1} P^d)$. Conclude from part a that E is the injective envelope of k.

d. Show that the indecomposable injective Abelian groups are $\mathbf{Q}$ and, for each prime p, the group

$$\mathbf{Z}/p^\infty := \varinjlim (\mathbf{Z}/p \subset \mathbf{Z}/p^2 \subset \mathbf{Z}/p^3 \subset \cdots) = \mathbf{Z}[p^{-1}]/\mathbf{Z}.$$

Show that $\mathbf{Q}/\mathbf{Z} \cong \oplus_p \mathbf{Z}/p^\infty$.

What is the injective resolution of $\mathbf{Z}/p$ as a $\mathbf{Z}$-module?

Exercise A3.5 (Graded injective modules and injective graded modules): Let $R = \oplus_d R_d$ be a $\mathbf{Z}$-graded ring. If $M = \oplus M_d$ is a graded R-module, and E is an R_0-module, we write

$$\mathrm{Hom}_{\mathrm{gr}}(M, E) := \oplus_d \mathrm{Hom}_{R_0}(M_d, E).$$

This is a graded R-module, and is generally much smaller than $\mathrm{Hom}_{R_0}(M, E) = \Pi_d \mathrm{Hom}_{R_0}(M_d, E)$.

a. Show that if E is an injective R_0-module then $Q = \mathrm{Hom}_{\mathrm{gr}}(R, E)$ is an injective in the category of graded R-modules in the sense that Q satisfies the definition given in the text whenever N and M are graded modules and α is a homomorphism of graded modules. (One way to do this is first to prove an analogue of Lemma A3.4). Conclude that every graded module has a graded-injective resolution.

b. Let $R = k[x]$, where k is a field. Show that $k[x, x^{-1}]$ is injective in the category of graded R-modules. Show that in the category of all R-modules, $k(x)$ is the injective hull of $k[x]$. Conclude that $k[x, x^{-1}]$ is not injective in the category of all R-modules, and that in fact there is no degree-preserving inclusion of R into a graded module that is injective in the category of all modules.

c. Suppose $R = \oplus_{d\geq 0} R_d$ is a positively graded Noetherian ring, and that R_0 is a field. Extend the method of Exercise 3.4c to show that $\mathrm{Hom}_{\mathrm{gr}}(R, k)$ is the injective hull of $k = \mathrm{Hom}_k((R/\oplus_{d>0} R_d), k)$ in the category of all R-modules, not just the category of graded R-modules.

Exercise A3.6 (Injective envelopes and primary decomposition): Still assuming that R is Noetherian, let M be any finitely generated R-module.

a. Let P be a prime. Show that if $\alpha : M \to E(R/P)$ is any map, then $\ker \alpha$ is a P-primary submodule of M.

b.* Show that the injective envelope $E(M)$ is a finite direct sum of indecomposable injectives. Let $M \to E(M) = \oplus E(R/P_i)$ be the injective envelope of M. Show that if P is a prime ideal and if $M(P)$ is the kernel of the composite map $M \to E(M) = \oplus E(R/P_i) \to \oplus_{P_i = P} E(R/P_i)$, then $M(P)$ is P-primary. Show that $0 = \cap M(P)$ is a primary decomposition of 0, and that the set of P that occur among the P_i above is precisely the set $\mathrm{Ass}(M)$.

Exercise A3.7 (More on the Noetherian property): Let $R \subset S$ be rings, and suppose that S is finitely generated as an R-module.

a.* Let F be an R-module. Show that F is injective as an R-module iff $\mathrm{Hom}_R(S, F)$ is injective as an S-module.

b. (Eakin's Theorem) Use part a and the criterion of Exercise A3.2 to show that R is Noetherian iff S is Noetherian. (This result is due to Eakin [1968]; the argument is from Eisenbud [1970]. A direct and more general proof was given by Formanek [1973] and is reproduced in Matsumura [1986, Theorem 3.6].)

A3.5 Basic Constructions with Complexes

A3.5.1 Notation and Definitions

To simplify the notation in what follows, we think of R as a trivially graded ring—that is, the degree $= 0$ part is R and all the other homogeneous components are 0. If

$$F: \cdots \to F_{i+1} \xrightarrow{\varphi_{i+1}} F_i \xrightarrow{\varphi_i} F_{i-1} \to \cdots$$

is a complex, we think of F as a graded R-module (the degree-i component of F is F_i) together with an endomorphism φ of degree -1. As usual we shall make the convention that maps of graded modules have degree 0 unless otherwise specified: Writing $F[i]$ for the graded module obtained from F by the rule $F[i]_j = F_{i+j}$, we could also have said that φ is a map from F to $F[-1]$. Often the grading does not matter, and we define a **differential module** (F, φ) to be an R-module F with an endomorphism φ such that $\varphi^2 = 0$. As for complexes, we define a **cycle** of F to be an element of $\ker \varphi$ and a **boundary** of F to be an element of $\operatorname{im} \varphi$.

Definitions. *Let F be a complex as above. The ith homology module of F is defined to be*

$$H_i(F) = \ker \varphi_i / \operatorname{im} \varphi_{i+1}.$$

We sometimes write $H(F)$ for the direct sum $\oplus_i H_i(F)$ of all the homology modules. If F is simply a differential module, with differential φ, then we set $H(F) = \ker \varphi / \operatorname{im} \varphi$; in case F is a complex, this is again $\oplus_i H_i(F)$.

We say that the complex (or differential module) F is **exact** if $H(F) = 0$. A complex

$$F: \cdots \to F_{i+1} \xrightarrow{\varphi_{i+1}} F_i \xrightarrow{\varphi_i} F_{i-1} \to \cdots \xrightarrow{\varphi_1} F_0$$

is called a **(left) resolution** (of $H_0(F) = \operatorname{coker} \varphi_1$) if $H_i(F) = 0$ for all $i > 0$. (It is sometimes convenient to regard F as continuing to the right forever with 0 maps and modules, thus:

$$F: \cdots \to F_{i+1} \xrightarrow{\varphi_{i+1}} F_i \xrightarrow{\varphi_i} F_{i-1} \to \cdots \xrightarrow{\varphi_1} F_0 \to 0 \to 0 \to \cdots .)$$

If the F_i are projective (respectively, free), then such a resolution is called a **projective (respectively, free) resolution.** Dually, a complex

$$I: I_0 \to I_{-1} \to \cdots I_{-i+1} \to I_{-i} \to I_{-i-1}$$

is called a **(right) resolution** if its only nonzero homology is $H_0(I) = \ker \varphi_0$. A right resolution is called an **injective resolution** if all the I_j are injective modules.

A3.6 Maps and Homotopies of Complexes

Projective (or free) resolutions of modules are in general far from unique (though over a local ring minimal resolutions of finitely generated modules are unique up to noncanonical isomorphisms—see Chapter 20). Thus, if we are to examine modules by studying their resolutions, it is necessary to ask what connects two different resolutions of the same module. This question turns out to have a simple answer. The necessary idea is useful in a more general form.

Definition. *If (F, φ) and (G, ψ) are differential modules, then a* ***map of differential modules*** *is a map of modules $\alpha : F \to G$ such that $\alpha\varphi = \psi\alpha$. If F and G are complexes, then we insist that α preserve the grading as well. Explicitly, if*

$$F: \cdots \to F_{i+1} \xrightarrow{\varphi_{i+1}} F_i \xrightarrow{\varphi_i} F_{i-1} \to \cdots$$

and

$$G: \cdots \to G_{i+1} \xrightarrow{\psi_{i+1}} G_i \xrightarrow{\psi_i} G_{i-1} \to \cdots$$

are complexes of modules, then a map of complexes $\alpha : F \to G$ is a collection of maps

$$\alpha_i : F_i \to G_i$$

of modules making the diagrams

$$\begin{array}{ccccccc} \cdots & \to & F_i & \xrightarrow{\varphi_i} & F_{i-1} & \to & \cdots \\ & & \downarrow{\scriptstyle \alpha_i} & & \downarrow{\scriptstyle \alpha_{i-1}} & & \\ \cdots & \to & G_i & \xrightarrow[\psi_i]{} & G_{i-1} & \to & \cdots \end{array}$$

commutative.

If $\alpha : (F, \varphi) \to (G, \psi)$ is a map of differential modules, then α carries $\ker \varphi$ *to* $\ker \psi$ *and* $\operatorname{im} \varphi$ *to* $\operatorname{im} \psi$. *Thus α gives rise to an* ***induced map on homology****, which we also call α:*

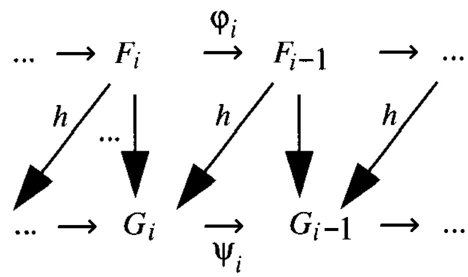
$$\alpha : HF = \frac{\ker \varphi}{\operatorname{im} \varphi} \to \frac{\ker \psi}{\operatorname{im} \psi} = HG.$$

If $\alpha : F \to G$ is a map of complexes, then the grading is preserved, and we get

$$\alpha_i : H_i F = \frac{\ker \varphi_i}{\operatorname{im} \varphi_{i+1}} \to \frac{\ker \psi_i}{\operatorname{im} \psi_{i+1}} = H_i G.$$

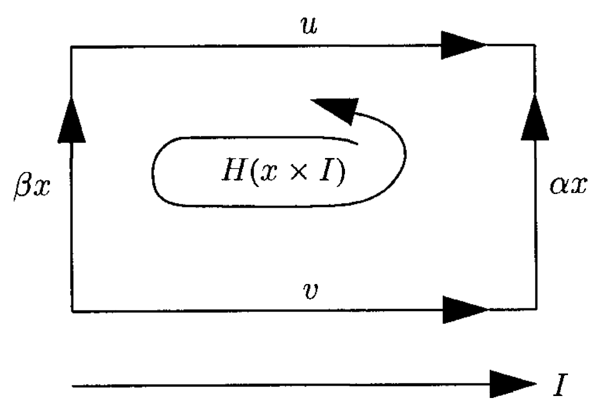
When do two maps of a complex F to a complex G induce the same map on homology? This is a subtle question in general, but there is a very important sufficient condition that may be given in terms of equations. This sufficient condition is called homotopy equivalence.

Definition. *If $\alpha, \beta : (F, \varphi) \to (G, \psi)$ are two maps of differential modules, then α is* **homotopy equivalent** *to β (or simply* **homotopic** *to β) if there is a map of modules $h : F \to G$ such that $\alpha - \beta = \psi h + h\varphi$. If F and G are complexes (so that F and G are graded modules and φ and ψ have degree -1), then we insist that h have degree 1:*



Note that α is homotopy equivalent to β iff $\alpha - \beta$ is equivalent to 0.

The homotopy terminology comes from topology: If α and β are continuous maps from a space X to a space Y, then they induce maps of complexes from the (say, singular) chain complex of X to that of Y. A homotopy $H : X \times I \to Y$ from α to β determines a chain map $h(x) := H(x \times I)$ that raises dimensions by 1. If we orient everything appropriately, we get $\alpha(x) - \beta(x) = \partial(h(x)) - h\partial(x))$ as in Figure A3.2:



$h(x) = H(x \times I);$

$\partial h(x) = \alpha x - u - \beta x + v;$

$h(\delta x) = H(\delta x \times I) = u - v;$

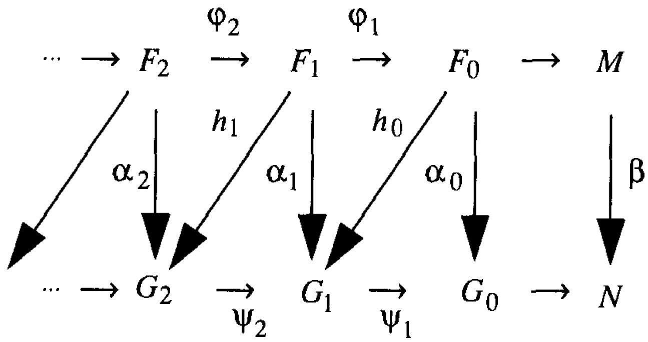
$x\alpha - \beta x = \partial h(x) + h(\partial x).$

FIGURE A3.2.

One of the fundamental properties of homotopic maps in topology is that they induce the same map on homology. The topological proof works by considering the map h on the level of chain complexes. It generalizes immediately to the following algebraic form.

Proposition A3.12. *If $\alpha, \beta : (F, \varphi) \to (G, \psi)$ are two maps of differential modules, and α is homotopy equivalent to β, then α and β induce the same map on homology.*

Proof. It suffices to show that $\alpha - \beta$ induces the map 0 on homology. Thus we may simplify the notation by replacing α by $\alpha - \beta$, and assume from the outset that $\beta = 0$. Let h be the homotopy, so that $\alpha = \psi h + h\varphi$.



Let $x \in \ker\varphi$ be a cycle of F; we must show that $\alpha(x)$ is a boundary of G. From the formula for the homotopy h we get

$$\alpha(x) = \psi(h(x)) + h(\varphi(x)) = \psi(h(x)) + h(0) = \psi(h(x)),$$

as desired. □

An important idea in homological algebra is that one can usefully replace a module with a projective (or dually an injective) resolution. Suppose that F and G are projective resolutions of modules M and N. It turns out that maps from M to N are the same thing as homotopy classes of maps from F to G. An equally useful dual statement, with injective resolutions, can be proved by "dualizing" the following argument; we leave the formulation and proof to the reader.

Proposition A3.13. *Let*

$$F:\ \cdots \to F_i \xrightarrow{\varphi_i} F_{i-1} \cdots \to F_1 \xrightarrow{\varphi_1} F_0$$

and

$$G:\ \cdots \to G_i \xrightarrow{\psi_i} G_{i-1} \cdots \to G_1 \xrightarrow{\psi_1} G_0$$

be complexes of modules, and set $M = \operatorname{coker}\varphi_1 = H_0F$, $N = \operatorname{coker}\psi_1 = H_0G$. If the modules F_i are projective and the homology of G vanishes except for $H_0G = N$, then every map of modules $\beta : M \to N$ is the map induced on H_0 by a map of complexes $\alpha : F \to G$, and α is determined by β up to homotopy.

Proof. Both the existence and the homotopy uniqueness of α are proved by induction; we give the first step and leave the (easy) continuation to the reader.

Existence: Since G_0 maps onto N, the composite map $F_0 \to M \to N$ may be lifted to a map $\alpha_0 : F_0 \to G_0$. It is immediate that $\alpha_0\varphi_1$ maps F_1 to $\ker(G_0 \to N) = \operatorname{im}(G_1 \to G_0)$, so $\alpha_0\varphi_1$ has a lifting $\alpha_1 : F_1 \to G_1$; continuing in this way we get the map of complexes α:

Homotopy uniqueness: If we are given two maps α and α' of complexes lifting the same map $\beta : M \to N$, then subtracting we see that $\alpha - \alpha'$

is a lifting of the zero map. Thus, changing notation, it suffices to show that if α is a lifting of the zero map, then α is homotopic to zero, that is, $\alpha_i = h_{i-1}\varphi_i + \psi_{i+1}h_i$ for some maps $h_i : F_i \to G_{i+1}$. First, since α_0 induces zero : $\operatorname{coker}\varphi_1 \to \operatorname{coker}\psi_1$, it takes F_0 into $\operatorname{im}\psi_1$. Thus there is a lifting $h_0 : F_0 \to G_1$ such that $\psi_1 h_0 = \alpha_0$. Now

$$\psi_1(h_0\varphi_1 - \alpha_1) = \alpha_0\varphi_1 - \psi_1\alpha_1 = 0,$$

so $h_0\varphi_1 - \alpha_1$ maps into $\ker\psi_1 = \operatorname{im}\psi_2$. Since F_1 is projective, we may lift this to a map $h_1 : F_1 \to G_2$. Continuing in this way we get the desired homotopy.

We can at last give the answer to the question with which we began, of what connects different projective resolutions of a module. For later use, we give a version with a functor in it. Recall that a functor F from a category of modules to another category of modules is called **additive** if it preserves the addition of homomorphisms: That is, if a, $b : M \to N$ are homomorphisms, then $F(a + b) = F(a) + F(b) : FM \to FN$. This is the property that we need in order that F preserve homotopy equivalences.

Corollary A3.14.

a. *Any two projective resolutions P and P' of the same module are homotopy equivalent in the sense that there are maps $\alpha : P \to P'$ and $\beta : P' \to P$ such that $\alpha\beta$ is homotopic to the identity map of P' and $\beta\alpha$ is homotopic to the identity map of P.*

b. *If F is any additive functor and we write FP, FP' for the results of applying F to the complexes P and P', then for each i the homology modules $H_i(FP)$ and $H_i(FP')$ are canonically isomorphic.*

Proof.

a. Suppose that P and P' are projective resolutions of a module M. By Proposition A3., there are maps $\alpha : P \to P'$ and $\beta : P' \to P$ of complexes inducing the identity map on M. The composites $\alpha\beta : P' \to P'$ also induces the identity map on M. But the identity map $P' \to P'$ induces the same map on M, so $\alpha\beta$ is homotopic to the identity by the other part of Proposition A3.13. Of course the same argument holds for $\beta\alpha$.

b. Suppose α is as above, and fix an index i. We claim that the map $H_i(F\alpha) : H_i(FP) \to H_i(FP')$ is a canonical isomorphism—that is, an isomorphism independent of the choice of α.

First, if $\alpha' : P \to P'$ were another choice of a map of complexes inducing the identity on M, then by Proposition A3.13 α is homotopic to α', say by a homotopy s with $\alpha - \alpha' = ds + sd$, where d denotes the differential both in P and in P'. Applying F, we get $F\alpha - F\alpha' = FdFs + FsFd$, so $F\alpha$ and $F\alpha'$ induce homotopic maps $FP \to FP'$. By Proposition A3.12, the induced maps $H_i(F\alpha)$ and $H_i(F\alpha')$ are the same.

Next, to see that $H_i(F\alpha)$ is an isomorphism, note simply that $H_i(F\alpha)H_i(F\beta) = H_iF(\alpha\beta) = 1$ because $\alpha\beta$ is homotopic to the identity, and the same argument works for $H_i(F\beta)H_i(F\alpha)$. □

A3.7 Exact Sequences of Complexes

If $\alpha : F' \to F$ and $\beta : F \to F''$ are maps of complexes, with $\beta\alpha = 0$, then we say that

$$0 \to F' \xrightarrow{\alpha} F \xrightarrow{\beta} F'' \to 0$$

is a **short exact sequence of complexes** if for each i the sequence

$$0 \to F_i' \xrightarrow{\alpha_i} F_i \xrightarrow{\beta_i} F_i'' \to 0$$

is exact. Given such a short exact sequence, we get induced maps $\alpha_i : H_iF' \to H_iF$ and $\beta_i : H_iF \to H_iF''$. Somewhat more surprisingly, we get a natural map

$$\delta_i : H_iF'' \to H_{i-1}F'$$

called the **connecting homomorphism**, defined as follows: Write φ', φ, and φ'' for the boundary maps of F', F, and F'', respectively. If $h \in H_iF''$ we choose a cycle $x \in \ker \varphi_i''$ whose homology class is x. Let $y \in F_i$ be an element such that $\beta_i(y) = x$; such a y exists because β_i is surjective. Since $\beta_{i-1}\varphi_i(y) = \varphi_i''\beta_i(y) = \varphi_i''(x) = 0$, there is an element $z \in F_{i-1}'$ such that $\alpha_{i-1}(z) = \varphi_i(y)$. Since α_{i-2} is a monomorphism and $\alpha_{i-2}\varphi_{i-1}'(z) = \varphi_{i-1}\alpha_{i-1}(z) = \varphi_{i-1}\varphi_i(y) = 0$, we see that z is a cycle of F'. We define $\delta_i(h)$ to be the image of z in $H_{i-1}F'$ (see Figure A3.3).

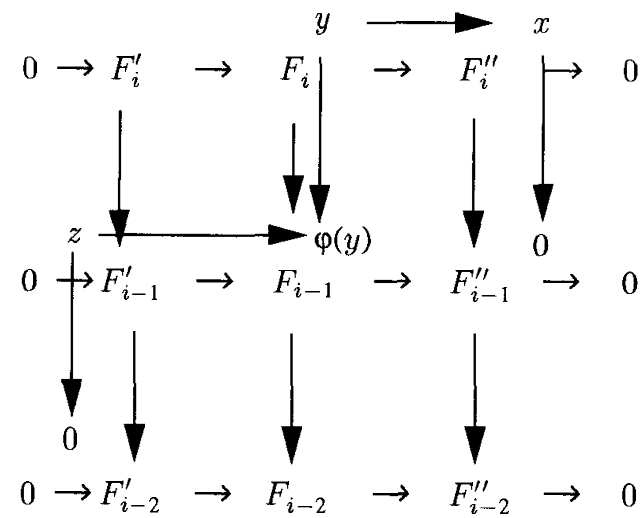


FIGURE A3.3.

A3.7.1 Exercises

Exercise A3.8: Show that $\delta_i(h)$ is independent of the choices made in the definition, and δ_i is a map of modules.

Exercise A3.9: A parallel construction works for exact sequences of differential modules; give it explicitly. The case of complexes becomes a special case if we remark that in the case of complexes the connecting homomorphism can be taken homogeneous and of degree -1.

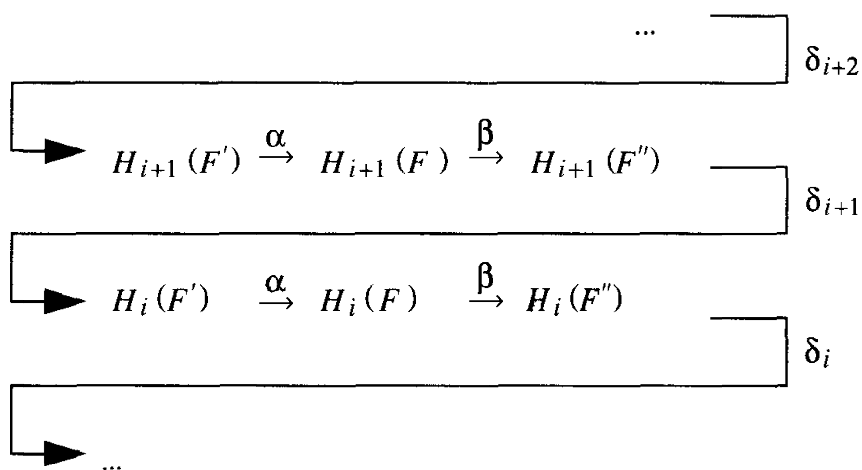


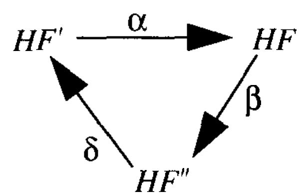
FIGURE A3.4.

A3.8 The Long Exact Sequence in Homology

Proposition A3.15. *If*

$$(*) \qquad 0 \to F' \xrightarrow{\alpha} F \xrightarrow{\beta} F''$$

is a short exact sequence of complexes, then the sequence shown in Figure A3.4, called the **long exact sequence in homology** *of* $(*)$, *is exact. More generally, if* $(*)$ *is a short exact sequence of differential modules, then the connecting homomorphism makes the following triangle exact (in the sense that the image of each map is the kernel of the next map).*



Proof. We leave the easy verification to the reader. □

Extending the principle embodied in Proposition A3.13, that phenomena regarding modules are well reflected in projective resolutions, we now show that a short exact sequence of modules corresponds to a short exact sequence of projective resolutions in a certain natural sense.

Proposition A3.16. *Let*

$$0 \to M' \xrightarrow{\beta'} M \xrightarrow{\beta} M'' \to 0$$

be a short exact sequence of modules. If

$$F' : \cdots \to F_i \xrightarrow{\varphi'_i} F_{i-1} \cdots \to F_1 \xrightarrow{\varphi'_1} F_0,$$

and

$$F'' : \cdots \to F''_i \xrightarrow{\varphi''_i} F''_{i-1} \cdots \to F''_1 \xrightarrow{\varphi''_1} F''_0$$

are projective resolutions of M' *and* M'', *respectively, then there is a projective resolution* F *of* M *and a short exact sequence of complexes*

$$0 \to F' \xrightarrow{\alpha'} F \xrightarrow{\alpha} F'' \to 0$$

such that α' *and* α *induce the maps* β' *and* β, *respectively.*

Note that because the F''_i are projective, it follows that $F_i = F'_i \oplus F''_i$ for each i. However, the differentials $\varphi_i : F_i \to F_{i-1}$ of F will generally not be the direct sums of φ'_i and φ''_i.

Proof. Again, we only describe the beginning of the induction, leaving the rest to the reader. Because F''_0 is projective the map from it to M'' can be lifted to a map $F''_0 \to M$. Of course we also have a composite map $F'_0 \to M' \to M$. Taking the sum of these maps we get a map $F_0 := F'_0 \oplus F''_0 \to M$, and it is easy to check that this is an epimorphism. Replacing M', M, and M'' by the kernels of the maps $F'_0 \to M'$, $F_0 \to M$, and $F''_0 \to M''$, respectively, we may repeat this argument. □

A3.8.1 Exercises

Diagrams and Syzygies

Exercises A3.10–A3.12 are three arguments with diagrams that come up so frequently that they have acquired names.

Exercise A3.10 (Snake Lemma):* If

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

is a commutative diagram of modules with exact rows, show that there is an exact sequence

$$0 \to \ker \alpha \to \ker \beta \to \ker \gamma \to \operatorname{coker} \alpha \to \operatorname{coker} \beta \to \operatorname{coker} \gamma \to 0.$$

Show that if we drop the assumptions that $A \to B$ is a monomorphism and that $B' \to C'$ is an epimorphism, then the six-term sequence is still exact except at the ends.

Where is the "snake"? Look at Figure A3.5.

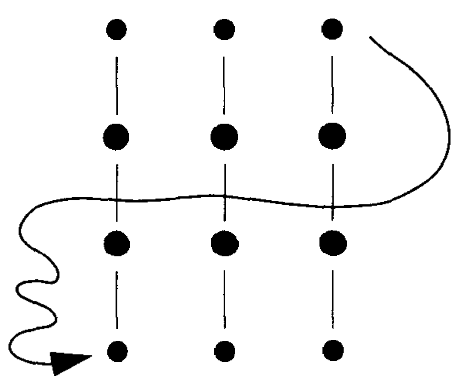


FIGURE A3.5.

Exercise A3.11 (5-Lemma): If

$$\begin{array}{ccccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\
\alpha_1 \big\downarrow & & \beta_1 \big\downarrow & & \gamma \big\downarrow & & \beta_2 \big\downarrow & & \alpha_2 \big\downarrow \\
B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5
\end{array}$$

is a commutative diagram of modules with exact rows, show that if β_1 and β_2 are isomorphisms, α_1 is an epimorphism, and α_2 is a monomorphism, then γ is an isomorphism. This is often applied when A_1, B_1, A_5, and B_5 are 0.

Exercise A3.12 (9-Lemma): Suppose that the diagram in Figure A3.6

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & A'' & \longrightarrow & B'' & \longrightarrow & C'' & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

FIGURE A3.6.

is a commutative diagram of modules with exact columns, and exact middle row. Show that if either $0 \to A \to B \to C \to 0$ or $0 \to A'' \to B'' \to C'' \to 0$ is exact, then both are.

Exercise A3.13 (Schanuel's Lemma):* Show that if

$$0 \to N_F \to F \to M \to 0$$

and

$$0 \to N_G \to G \to M \to 0$$

are exact sequences with F and G projective, then

$$N_F \oplus G \cong \ker(F \oplus G \to M) \cong N_G \oplus F,$$

where the map in the middle expression is the sum of the two given maps $F \to M$ and $G \to M$.

The module N_F is usually called a **first syzygy module** of M, and its uniqueness "up to projective summand" is another way of saying in what sense the projective resolution of M is unique. (The nth syzygy module is defined inductively as the first syzygy module of the $(n-1)$st syzygy module. Since the first syzygy module of a direct sum may be taken to be the direct sum of the first syzygy modules, all the syzygy modules of M are uniquely defined up to projective summands.)

Exercise A3.14: Let R be a ring and let

$$\mathcal{F}: \ \cdots \to F_d \to \cdots \to F_0 \to M \to 0$$

be a projective resolution of M. Let d be the smallest number such that $\mathrm{im}(F_d \to F_{d-1})$ is projective. Use Schanuel's lemma (Exercise A3.13) to show that d is independent of the resolution chosen, so that $d = \mathrm{pd}\ M$.

A3.9 Derived Functors

One of the main applications of projective and injective resolutions is defining **derived functors**. The idea is this: Often one has a functor F (say, for simplicity, from R-modules to R-modules) that is additive and that takes short exact sequences

$$0 \to A \to B \to C \to 0$$

of modules into sequences that are exact only at one end, say at the right:

$$FA \to FB \to FC \to 0.$$

Such a functor is said to be **right-exact**; an example is the functor $M \otimes_R -$, which takes an R-module to its tensor product with a fixed R-module M. (If the sequence is only exact on the left, we speak of a left-exact functor; an example is $\mathrm{Hom}_R(M, -)$. We shall stick with right-exact functors in the description that follows, and remark on the dualization to the left-exact case at the end. The reader should be warned that we shall apply both notions.)

If F is an interesting right-exact functor, then it is generally interesting to have a description of when a zero can be added on the left end of the right-exact sequence

$$FA \to FB \to FC \to 0$$

and still have an exact sequence; or more generally, to have a good description of the kernel of the left-hand map. Derived functors provide this. In the situation above, for example, there is a "first left-derived functor L_1F" and a map $\delta : L_1F(C) \to FA$ such that

$$L_1F(C) \xrightarrow{\delta} FA \to FB \to FC \to 0$$

is exact. (Here δ must depend on the short exact sequence given, but the module $L_1F(C)$ does not!) Of course, one should then ask about the kernel of δ. In fact, the theory provides a whole sequence of left-derived functors, which answer the sequence of questions beginning in this way:

Definition. *Suppose F is a right-exact functor on the category of R-modules. If A is an R-module, let*

$$P : \ \cdots \to P_i \xrightarrow{\varphi_i} P_{i-1} \cdots \to P_1 \xrightarrow{\varphi_1} P_0$$

be a projective resolution of A, and define the ***ith left-derived functor*** *of F to be $L_iF(A) = H_iFP$, where FP is the complex*

$$FP : \ \cdots \to FP_i \xrightarrow{F\varphi_i} FP_{i-1} \cdots \to FP_1 \xrightarrow{F\varphi_1} FP_0,$$

the result of applying F to P.

We have:

Proposition A3.17. *The left-derived functors of F are independent of the choice of resolution and have the following properties:*

a. *$L_0F = F$.*

b. *If A is a projective module, then $L_iF(A) = 0$ for all $i > 0$.*

c. *For every short exact sequence*

$$0 \to A \xrightarrow{u} B \xrightarrow{v} C \to 0,$$

there is a long exact sequence as shown in Figure A3.7.

d. *The "connecting homomorphisms" δ_i in the long exact sequence are* **natural***: That is, if*

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \to & A' & \to & B' & \to & C' & \to & 0 \end{array}$$

is a commutative diagram with exact rows (a "map of short exact sequences") then the diagrams

$$\begin{array}{ccc} L_{i+1}FC & \xrightarrow{\delta_{i+1}} & L_iFA \\ {\scriptstyle L_{i+1}F\gamma}\downarrow & & \downarrow{\scriptstyle L_iF\alpha} \\ L_{i+1}FC' & \xrightarrow{\delta_i} & L_iFA' \end{array}$$

commute.

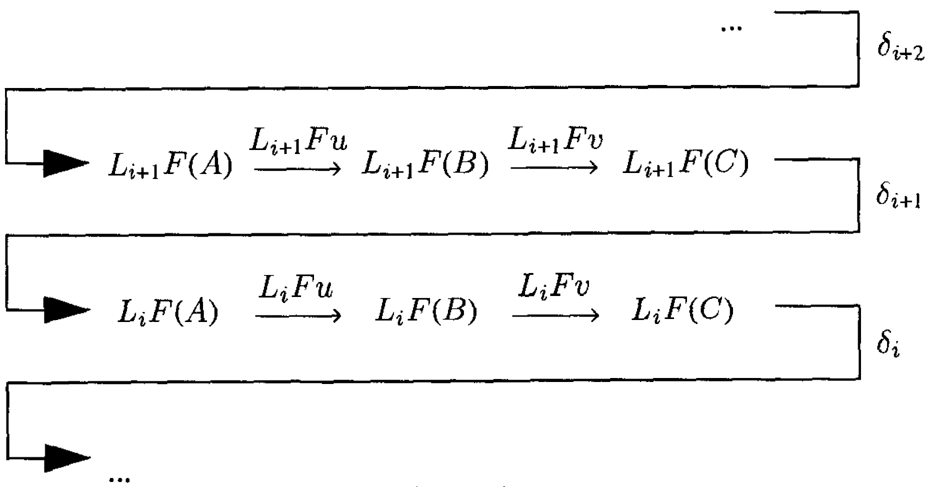


FIGURE A3.7.

Proof. The independence of resolution is the content of Corollary A3.14b.

a. To show that $L_0F(A) = F(A)$, just use the right-exactness of F: From the definition

$$L_0F(A) = H_0(\cdots \to FP_1 \to FP_0),$$

we get $L_0F(A) = \operatorname{coker} FP_1 \to FP_0 = FA$.

b. This is immediate from the independence of resolution, since if A is projective then we may take as projective resolution the complex

$$\cdots \to 0 \to 0 \to A.$$

c. This is immediate from Propositions A3.15 and A3.16.

d. Form the projective resolutions of each of the two short exact sequences as in Proposition A3.16. The maps α, β, and γ lift to comparison maps between these resolutions. If we use these maps of resolutions to define the maps $L_iF(\alpha)$ and $L_iF(\beta)$, then the verification of the commutativity of the diagram in part d is easy. We leave the details to the reader. □

Dually, if F is a left-exact functor, then we define the right-derived functors R^iF of F: If A is a module, we let

$$Q : 0 \to Q_0 \to Q_{-1} \to \cdots$$

be an injective resolution of A, and we set

$$R^iF(A) = H_{-i}(FQ),$$

where FQ is the complex

$$FQ : 0 \to FQ_0 \to FQ_{-1} \to \cdots .$$

Proposition A3.17 dualizes to this setting.

A3.9.1 Exercise on Derived Functors

Exercise A3.15:* Show that the conditions of Proposition A3.17 characterize the functors L_iF.

A3.10 Tor

Let A be an R-module. The left-derived functors of the functor $M \otimes_R -$ are called $\mathrm{Tor}_i^R(M, -)$. The tensor product itself is commutative in the sense that $M \otimes_R N \cong N \otimes_R M$, and this property carries over to the Tor_i, as we shall prove in the section on spectral sequences. Thus Tor_i^R may be regarded as a functor of two variables, $\mathrm{Tor}_i^R(-, -)$, and we get long exact sequences from short exact sequences in either variable. When the ring R is understood, we suppress it from the notation. We give a few very useful computations as exercises; the reader is urged to do at least the first three.

A3.10.1 Exercises: Tor

The name "Tor" comes from the following computation, which connects Tor with torsion.

Exercise A3.16:* Let $x \in R$ be a nonzerodivisor. Show that

$$\mathrm{Tor}_1(R/x, M) = \{m \in M | xm = 0.\}$$

Exercise A3.17: If I and J are any ideals of R, then $IJ \subset I \cap J$. Show that $\mathrm{Tor}_1(R/I, R/J) = (I \cap J)/(IJ)$. This usefully encapsulates several often-used cases (of course these can also be proven directly). For example, use it to show that $I \cap J = IJ$ in the following cases:

a. $I + J = R$.

b. I is generated by a sequence of elements that form a regular sequence mod J.

Exercise A3.18 ("Betti" numbers): Let $(R, \mathfrak{m})$ be a local ring. We say that a free resolution

$$F : \cdots \to F_{i+1} \xrightarrow{\varphi_{i+1}} F_i \xrightarrow{\varphi_i} F_{i-1} \to \cdots \xrightarrow{\varphi_1} F_0$$

of a module M is **minimal** if each φ_i has an image contained in $\mathfrak{m}F_{i-1}$. (If the F_i are finitely generated modules, then Nakayama's lemma shows that this is equivalent to a more obviously natural formulation. See Chapter 20.) If F as above is a minimal free resolution of M and $\operatorname{rank} F_i = b_i$, then show that $\operatorname{Tor}_i(R/\mathfrak{m}, M) = (R/\mathfrak{m})^{b_i}$. The b_i are called Betti numbers of M, in loose analogy with the situation in topology, where F is a chain complex.

Exercise A3.19 (Serre's Intersection Formula): Let X and Y be subvarieties of $\mathbf{A}_k^r$, of dimensions d and $n-d$, defined by ideals I and $J \subset k[x_1, \ldots, x_r] = S$, and suppose that $X \cap Y$ has the origin 0 as an isolated point. A crucial part of algebraic geometry is devoted to the question, in this and similar cases, of defining an "intersection multiplicity $i(X, Y; 0)$ of X and Y at 0" that will have desirable properties. If X and Y are themselves nice (for example, nonsingular at 0), then this is not too hard; writing R for the localization of S at $(x_1, \ldots, x_r)$, the right answer turns out to be the vector space dimension of $R/I \otimes_R R/J = R/(I+J)$. Such a formula is correct also in the case of plane curves, but in general the dimension of $R/(I+J)$ turns out only to be the first term of an alternating sum. The following definition is due to Serre [1957]:

$$i(X, Y; 0) := \sum_j (-1)^j \dim_k \operatorname{Tor}_j^R(R/I, R/J).$$

Show that $\operatorname{Tor}_j^R(R/I, R/J)$ is annihilated by both I and J, and therefore has finite length. Let $r = 4$, and take $I = (x_1, x_2) \cap (x_3, x_4)$, the ideal corresponding to the union X of two two-planes, meeting in the point 0, and $J = (x_1 - x_3, x_2 - x_4)$, the ideal corresponding to another two-plane Y, transverse to each of the first two and meeting them at the origin. Compute the $\operatorname{Tor}_j^R(R/I, R/J)$ and show that $i(X, Y; 0) = 2$. Note that Y meets each of the two-planes in X transversely in a single point (multiplicity 1) so Y "should" meet X with multiplicity 2; however, the length of $R/I \otimes_R R/J = R/(I+J)$ is not 2.

Exercise A3.20 (Tor as an algebra): For any R-modules A, A', B, B', define a natural "external multiplication" map

$$e : \operatorname{Tor}_m^R(A, B) \otimes_R \operatorname{Tor}_n^R(A', B') \to \operatorname{Tor}_{m+n}^R(A \otimes_R A', B \otimes_R B')$$

as follows. Let P and P' be projective resolutions of A and A'. Represent elements α, β of $\operatorname{Tor}_m^R(A, B)$ and $\operatorname{Tor}_n^R(A', B')$ as cycles of the complexes $P \otimes B$ and $P' \otimes B'$ (where for simplicity we write $\otimes$ for $\otimes_R$). Show that $\alpha \otimes \beta$ is then naturally a cycle in the tensor product complex $(P \otimes B) \otimes (P' \otimes B') \cong P \otimes P' \otimes B \otimes B'$. (Here the tensor product of two complexes may be defined as the total complex of the double complex with terms $P_i \otimes B \otimes P'_j \otimes B'$—see the section on double complexes below if this is unfamiliar.) If P'' is a free resolution of $A \otimes A'$, there is a map of complexes $P \otimes P' \to P''$ inducing the identity on $H_0 = A \otimes A'$. Use this to define e.

If A and B are R-algebras, take $A' = A$ and $B' = B$ and combine the map above with the multiplication maps of A and B to get a multiplication

$$\mu : \mathrm{Tor}_m^R(A,B) \otimes_R \mathrm{Tor}_n^R(A,B) \to \mathrm{Tor}_{m+n}^R(A,B).$$

Show that this makes $\mathrm{Tor}_*^R(A,B)$ into a graded associative R-algebra that is "graded-commutative" in the sense that for elements α, β of degrees a and b we have

$$\beta\alpha = (-1)^{ab}\alpha\beta.$$

Remarks: A good deal of work has been done on the structure of this algebra in the case where $A = B = k$, the residue class field of a local ring R. In that case Tate and Gulliksen showed, for example, that $\mathrm{Tor}_*^R(k,k)$ is a free graded-commutative divided power algebra (that is, the tensor product of a divided power algebra on even degree generators and an exterior algebra on odd degree generators). It was hoped for a long time that the "Poincaré series" of R, namely the power series

$$P_R(t) = \sum_n \dim_k(\mathrm{Tor}_n^R(k,k))t^n,$$

would be a rational function of t, but Anick [1982] showed that this is false in general. The hope behind this hope was perhaps that the ranks of the free modules in a minimal free resolution of k are "finitely determined." It remains an open problem to give a description simpler than the one obtained by computing the minimal free resolution.

One important point in this development is that the algebra structure on Tor can be computed from a resolution that is an algebra in a nice way:

Exercise A3.21: Let R be a ring with augmentation onto a factor ring $R \to k$. Suppose that

$$P : \cdots \xrightarrow{d} P_1 \xrightarrow{d} P_0$$

is a projective resolution of k over R, with $P_0 = R$. Suppose that the complex P has an algebra structure such that d is a derivation, $d(pq) = d(p)q + (-1)^q pd(q)$. Show that this algebra structure induces the natural algebra structure on the homology $\mathrm{Tor}(k,k) = H_*(P \otimes k)$.

Exercise A3.22 (Auslander's Transpose Functor): The long exact sequence in Tor is not the only answer to the question of how to measure the inexactness of the functor $\otimes$. Suppose that M is a finitely presented R-module. Following ideas of Auslander [1966], we define the **transpose** of M as follows:

Let $\varphi : F \to G$ be a **projective presentation of** M—that is, a map of projective modules with $\operatorname{coker}\varphi = M$. Write $-^*$ for $\mathrm{Hom}_R(-,R)$, so that $\varphi^* : G^* \to F^*$ is the "transpose" of φ. Define $T(\varphi)$, the transpose of M, to be $T(\varphi) = \operatorname{coker}\varphi^*$.

a.* Show that like the first syzygy of M, $T(\varphi)$ depends, up to a projective summand, only on M in the sense that if φ' is another projective presentation of M, then there are projective modules P and P' such that $T(\varphi) \oplus P' \cong T(\varphi') \oplus P$.

Notation: We shall write $T(M)$ for any (fixed) choice $T(\varphi)$. We may choose things so that $T(T(M)) = M$.

b. Show that if

$$\alpha : 0 \to A \to B \to C \to 0$$

is a short exact sequence of R-modules, and M is a finitely presented R-module, then there is an exact sequence

$$0 \to \operatorname{Hom}(T(M), A) \to \operatorname{Hom}(T(M), B) \to \operatorname{Hom}(T(M), C) \to \\ M \otimes A \to M \otimes B \to M \otimes C \to 0.$$

This sequence gives another way of "measuring" the inexactness of the functor $M \otimes -$. If N is any module, and we choose $M = T(N)$, then since $T(T(N)) = N$, we may also think of it as a measure for the inexactness of $\operatorname{Hom}(N, -)$.

c. Here is an application: We say that $A \subset B$ is a **pure** R-submodule if for every module M the induced map $M \otimes_R A \to M \otimes_R B$ is a monomorphism. Show that if $\alpha : 0 \to A \to B \to C \to 0$ is a short exact sequence with $A \to B$ pure, and if N is a finitely presented R-module, then

$$\operatorname{Hom}(N, \alpha) : 0 \to \operatorname{Hom}(N, A) \to \operatorname{Hom}(N, B) \to \operatorname{Hom}(N, C) \to 0$$

is a short exact sequence. Deduce that if C is finitely presented, then α splits. Note that it is even enough to know that $N \otimes A \to N \otimes B$ is a monomorphism for every finitely presented module N. (This is actually the same as purity, since every module is the filtered direct limit of finitely presented modules—see Exercise A6.5.)

A3.11 Ext

We now turn from $\otimes$ to Hom. The functor $\operatorname{Hom}_R(M, -)$ is left-exact, so we may apply the dual theory, the theory of right-derived functors, as follows: For any R-module N, let

$$I : I_0 \to I_1 \to \cdots$$

be an injective resolution of N, and define the right-derived functor $R^i \operatorname{Hom}(M, -)(N)$, which we shall write more compactly as $\operatorname{Ext}^i_R(M, N)$, to be $H_{-i}(\operatorname{Hom}_R(M, I))$, where $\operatorname{Hom}_R(M, I)$ is the complex

$$\operatorname{Hom}_R(M, I) : 0 \to \operatorname{Hom}_R(M, I_0) \to \operatorname{Hom}_R(M, I_1) \to \cdots .$$

As we shall prove by spectral sequences later (another proof, done by identifying both results with the "Yoneda Ext," is given in the exercises), we could also compute this from a projective resolution

$$F : \cdots \to F_1 \to F_0$$

of M as $\operatorname{Ext}^i_R(M, N) = H_{-i}(\operatorname{Hom}_R(F, N))$, where $\operatorname{Hom}_R(F, N)$ is the complex

$$\operatorname{Hom}_R(F, N) : 0 \to \operatorname{Hom}_R(F_0, N) \to \operatorname{Hom}_R(F_1, N) \to \cdots .$$

Here is a classic application of Ext, due to Auslander, showing that the global dimension of a ring can be computed from finitely generated modules—even from cyclic modules. The original proof used a direct limit argument; the proof given here, using injective modules, is due to Serre. The result is very general: It holds for non-Noetherian rings too, and even for noncommutative rings if we specify left or right modules and ideals throughout.

Theorem A3.18 (Auslander [1955]). *The following conditions on a ring R are equivalent:*

a. $\operatorname{gl\,dim} R \le n$*—that is,* $\operatorname{pd} M \le n$ *for every R-module M.*

b. $\operatorname{pd} R/I \le n$ *for every ideal I.*

c. injective dimension $N \le n$ for every R-module N.

d. $\operatorname{Ext}^i_R(M, N) = 0$ *for all $i > n$ and all R-modules M and N.*

Proof.

a $\Rightarrow$ b is trivial.

b $\Rightarrow$ c: Suppose that condition b holds and let

$$0 \to N \to E_0 \to \cdots \to E_{n-1} \to X \to 0$$

be an exact sequence with the E_i injective; we shall show that X is injective, proving c. Breaking the long exact sequence above into short exact sequences, and considering the long exact sequences obtained from these by applying $\mathrm{Ext}_R^*(R/I,-)$, we see that

$$\mathrm{Ext}_R^1(R/I,X)\cong\mathrm{Ext}_R^{n+1}(R/I,M)=0,$$

the last equality coming from the hypothesis b. Thus it suffices to show that a module X is injective if $\mathrm{Ext}_R^1(R/I,X)=0$ for all ideals I. Computing $\mathrm{Ext}_R^1(R/I,X)$ from a projective resolution of R/I, we see that this hypothesis is equivalent to saying that if $\psi: I\to X$ is any map, then there is a map $R\to X$ such that the composition $I\to R\to X$ is ψ. By Lemma A3.4, X is injective.

c $\Rightarrow$ d: Compute $\mathrm{Ext}_R(M,N)$ from an injective resolution of N.

d $\Rightarrow$ a: Assume that condition d holds, and let

$$0\to X\to F_n\to\cdots\to F_0\to M\to 0$$

be an exact sequence with the F_i projective. It will suffice to show that X is projective. Applying the long exact sequences in Ext to the short exact sequences

$$0\to\ker(F_{i+1}\to F_i)\to F_i\to\ker(F_i\to F_{i-1})\to 0$$

obtained from this resolution, we see that

$$\mathrm{Ext}_R^1(X,N)\cong\mathrm{Ext}_R^{n+1}(M,N)=0,$$

for every module N. We shall show that this condition implies that X is projective. (Note the duality of this with the preceding argument —but here there is no restriction on N, and the proof is easier.)

To this end we must show that if

$$\mathcal{P}\cdots\to P_1\to P_0\to X\to 0$$

is a projective resolution, then the map $P_0\to X$ splits. Let

$$N=\ker(P_0\to X).$$

The natural map $\varphi: P_1\to N$ is a *cycle* of $\mathrm{Hom}(\mathcal{P},N)$ and thus defines an element of $\mathrm{Ext}_R^1(X,N)$; since this group vanishes, the element is a boundary so there exists a map $P_0\to N$ extending φ. This map is a splitting of the inclusion $N\to P_0$, and thus coker $P_0\to X$ splits too. This concludes the proof of the equivalence of conditions a–d. □

As with Tor, we offer the reader some simple exercises to become comfortable with Ext.

A3.11.1 Exercises: Ext

Exercise A3.23: If x is a nonzerodivisor in a ring R, compute $\mathrm{Ext}^i_R\,(R/x, M)$. In particular, compute $\mathrm{Ext}^i_{\mathbf{Z}}(\mathbf{Z}/n, \mathbf{Z}/m)$ for any integers n, m.

Exercise A3.24: Show that a finitely generated Abelian group A is free iff $\mathrm{Ext}^1_{\mathbf{Z}}(A, \mathbf{Z}) = 0$. It was conjectured by Whitehead that this would hold for all groups, but the truth turns out to depend on your set theory (Shelah [1974]).

Exercise A3.25:* For any ring R and ideal $I \subset R$, show from the definitions and Exercise A3.17 that

$$\mathrm{Ext}^1_R(R/I, R/I) = \mathrm{Hom}_R(I/I^2, R/I) = \mathrm{Hom}(\mathrm{Tor}_1(R/I, R/I), R/I).$$

In a geometric context, supposing that R is the affine coordinate ring of a variety X and that I is the ideal of a subvariety Y, this module $\mathrm{Hom}_R(I/I^2, R/I)$ plays the role of the "normal bundle" of Y in X; see Exercise 16.8 for more information.

Exercise A3.26 (Yoneda's description of Ext^1): The ideas in this and the next exercise give a useful and appealing interpretation of the elements of Ext. See, for example, MacLane [1963, Chapter III] for more details.

a. If

$$\begin{array}{ll} \alpha: & 0 \to B \to X \to A \to 0 \\ \alpha': & 0 \to B \to X' \to A \to 0 \end{array}$$

are short exact sequences, we say that α is **Yoneda equivalent** to α' if there exists a map $f : X \to X'$ making the diagram

$$\begin{array}{ccccccccc} 0 & \to & B & \to & X & \to & A & \to & 0 \\ & & \| & & {\scriptstyle f}\downarrow & & \| & & \\ 0 & \to & B & \to & X' & \to & A & \to & 0 \end{array}$$

commute. Show that Yoneda equivalence is an equivalence relation (reflexive, symmetric, and transitive). Show that α is Yoneda equivalent to the "split" sequence

$$0 : 0 \to A \to A \oplus B \to B \to 0$$

iff α is itself split.

We shall write $[\alpha]$ for the Yoneda equivalence class of a short exact sequence α.

We now define $E^1(A, B)$ to be the set of equivalence classes of short exact sequences as above. We shall see that $E^1_R(A, B)$ is naturally isomorphic to $\mathrm{Ext}^1_R(A, B)$.

b. Functoriality in A: Show that $E_R^1(A, B)$ is a contravariant functor of A as follows: If

$$\alpha : 0 \to B \to X \xrightarrow{a} A \to 0$$

is a short exact sequence and $v : A' \to A$ is a map, define

$$X' = \ker(-a, v) : X \oplus A' \to A,$$

X' is called the **pull-back (or fibered product)** of X and A' over A. Show that there is a short exact sequence

$$\alpha' : 0 \to B \to X' \to A' \to 0$$

and a commutative diagram

$$\begin{array}{ccccccccccc} \alpha' : & 0 & \to & B & \to & X' & \to & A' & \to & 0 \\ & & & \| & & \downarrow & & \downarrow v & & \\ \alpha : & 0 & \to & B & \to & X & \to & A & \to & 0. \end{array}$$

We define $v([\alpha])$ to be α'. Show that this makes $E_R^1(A, B)$ into a contravariant functor of A as claimed.

c. Functoriality in B: Given a map $b : B \to X$ and another map $u : B \to B'$, the **push-out (or fibered coproduct)** of X and B' under B is by definition $\operatorname{coker}(-b, u) : B \to X \oplus B'$. Dualize the argument of part b, using the push-out construction, to show that $E_R^1(A, B)$ is a covariant functor of A.

d. Prove that $E_R^1(A, B) \cong \operatorname{Ext}_R^1(A, B)$ as follows: Let

$$Q : Q_0 \xrightarrow{\psi_0} Q_{-1} \xrightarrow{\psi_{-1}} \cdots$$

be an injective resolution of B, and let $b : B \to Q_0$ be the injection of B to Q_0 that is the kernel of ψ_0. An element ν of $\operatorname{Ext}_R^1(A, B)$ is represented by a cycle of $\operatorname{Hom}_R(A, Q)$, which is a map $v : A \to Q_{-1}$ such that $\psi_{-1} v = 0$; that is, a map $v : A \to \ker \psi_{-1} = Q_0/B$. Let α be the short exact sequence

$$\alpha : 0 \to B \to Q_0 \to Q_0/B \to 0,$$

and let $\nu' \in E_R^1(A, B)$ be the element $v([\alpha])$. Show that

$$\varepsilon : \operatorname{Ext}_R^1(A, B) \to E_R^1(A, B); \nu \mapsto \varepsilon(\nu) := \nu'$$

is a bijection of sets, natural in the sense that if $A' \to A$ or $B \to B'$ are homomorphisms, then the induced maps on $\operatorname{Ext}_R^1(A, B)$ and $E_R^1(A, B)$ correspond. If $P : \cdots \to P_1 \to P_0$ is a projective resolution of A, show dually that $E_R^1(A, B)$ may be identified with $H_{-1}(\operatorname{Hom}_R(P, B))$. This proves that $\operatorname{Ext}_R^1(A, B)$ could be computed from a projective resolution of A as well as from an injective resolution of B.

e. The module structure on E^1_R : If $r \in R$, the underlying ring, then multiplication by r is an endomorphism of any module B, and thus induces a map on $E^1_R(A, B)$ by functoriality in B. Of course, it also induces a map by functoriality in A. Show that these two maps are the same; we use them to define the action of R on $E^1_R(A, B)$. To define an addition on $E^1_R(A, B)$, let α, α' be short exact sequences as in part a. Let $d : A \to A \oplus A$ be the diagonal map $d(a) = (a, a)$, and let $s : B \oplus B \to B$ be the sum map $s(b, b') = b + b'$. Let $\alpha \oplus \alpha'$ be the direct sum of α and α',

$$\alpha \oplus \alpha' : 0 \to B \oplus B \to X \oplus X' \to A \oplus A \to 0$$

and set

$$[\alpha] + [\alpha'] = sd[\alpha \oplus \alpha'] = ds[\alpha \oplus \alpha'].$$

Show that these definitions make $E^1_R(A, B)$ a module and ε an isomorphism of modules.

f. If

$$\beta : 0 \to B' \to B \to B'' \to 0$$

is a short exact sequence of modules, define a "connecting homomorphism" $\delta : \mathrm{Hom}_R(A, B'') \to E^1_R(A, B')$ for $b \in \mathrm{Hom}_R(A, B'')$ by $\delta(b) = b[\beta] \in E^1_R(A, B')$. Show that there is an exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_R(A, B') &\to \mathrm{Hom}_R(A, B) \to \mathrm{Hom}_R(A, B'') \\ &\xrightarrow{\delta} E^1_R(A, B') \to E^1_R(A, B) \to E^1_R(A, B''), \end{aligned}$$

and that if we identify E^1_R with Ext^1_R, then δ is the usual connecting homomorphism.

Exercise A3.27 (Ext as an algebra; the Yoneda Ext in general):

a. Higher Exts: Two exact sequences from A to B "of length n"

$$\begin{aligned} \alpha :\ & 0 \to A \to X_1 \to X_2 \to \cdots \to X_n \to B \to 0 \\ \alpha' :\ & 0 \to A \to X'_1 \to X'_2 \to \cdots \to X'_n \to B \to 0 \end{aligned}$$

are **primitively equivalent** if there is a commutative diagram

$$\begin{array}{cccccccccccccccc} \alpha: & 0 & \to & A & \to & X_1 & \to & X_2 & \to & \cdots & \to & X_n & \to & B & \to & 0 \\ & & & \| & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \| & & \\ \alpha': & 0 & \to & A & \to & X'_1 & \to & X'_2 & \to & \cdots & \to & X'_n & \to & B & \to & 0. \end{array}$$

This is not an equivalence relation (it is not symmetric), but we may define Yoneda equivalence to be the equivalence relation it generates. Define $E^n_R(A, B)$ to be the set of Yoneda equivalence classes of exact sequences of length n from A to B. Analogously with the case done in the previous exercise, show that $E^n_R(A, B)$ is naturally isomorphic to $\mathrm{Ext}^n_R(A, B)$ (computed from either an injective resolution of B or a projective resolution of A).

b. The Yoneda product: The functoriality of $\mathrm{Ext}^1(A, B)$ may be thought of as giving rise to "multiplication" maps

$$\begin{aligned}\mu : \quad &\mathrm{Hom}_R(B, C) \otimes_R \mathrm{Ext}^n_R(A, B) \to \mathrm{Ext}^n_R(A, C);\\ \mu : \quad &\mathrm{Ext}^m_R(B, C) \otimes_R \mathrm{Hom}_R(A, B) \to \mathrm{Ext}^m_R(A, C).\end{aligned}$$

Thinking of Hom as Ext^0 is the first step in defining an "algebra structure," which is a natural pairing called the Yoneda product

$$\mu : \ \mathrm{Ext}^n_R(B, C) \otimes_R \mathrm{Ext}^m_R(A, B) \to \mathrm{Ext}^{n+m}_R(A, C)$$

defined for all m and n. Namely, if

$$\alpha : \ 0 \to A \to X_1 \to X_2 \to \cdots \to X_m \xrightarrow{b} B \to 0$$

and

$$\beta : \ 0 \to B \xrightarrow{b'} Y_1 \to Y_2 \to \cdots \to Y_n \to C \to 0$$

are exact sequences, then we define $\mu([\beta] \otimes [\alpha])$ to be the class of the exact sequence

$$\beta\alpha : \ 0 \to A \to X_1 \to \cdots \to X_m \xrightarrow{b'b} Y_1 \to \cdots \to Y_n \to C \to 0.$$

Prove that this multiplication is well defined on Yoneda equivalence classes and that it is associative. (The only case that needs work is where one of the factors is in $\mathrm{Ext}^0 = \mathrm{Hom}$.)

Note that the Yoneda algebra defined above is graded by the positive integers and pairs of modules! However, if we fix a module A and take $A = B$, we get a more reasonable object, a (noncommutative) algebra $\mathrm{Ext}_R(A, A) := \oplus_{n \geq 0} \mathrm{Ext}^n_R(A, A)$ that is graded by the positive integers. Very little is known in general about the properties of this algebra, although extensive work has been done on the case where R is local and $A = k$ is its residue class field. The natural commutative algebra structure on $\mathrm{Tor}^R(k, k) := \oplus_n \mathrm{Tor}^R_n(k, k) = \oplus_n \mathrm{Hom}_k(\mathrm{Ext}^n_R(k, k), k)$, described in the exercises on Tor, makes $\mathrm{Ext}^n_R(k, k)$ into a cocommutative Hopf algebra. Good references are Gulliksen-Levin [1969] for the early work and Anick [1988] for more recent developments. One important point (used in the exercises of Chapter 17) is that the product on $\mathrm{Ext}(k, k)$ may be computed from an appropriate coalgebra structure of the resolution of k.

Exercise A3.28:* Let R be a ring with augmentation onto a factor ring $R \to k$. Suppose that

$$P : \ \cdots \xrightarrow{d} P_1 \xrightarrow{d} P_0$$

is a projective resolution of k over R, with $P_0 = R$. Suppose that the complex $P^* = \mathrm{Hom}_R(P, R)$ has the structure of a graded algebra over R

and that the differential d^* of P^* is defined by multiplication by some fixed element $x \in P_1$, that is $d^*(p) = xp$. Show that the algebra structure on P^* induces the Yoneda algebra structure on $\mathrm{Ext}_R(k,k)$, in the sense that the cycles of P^* form a subalgebra of P^* and the map from the cycles onto $\mathrm{Ext}_R(k,k)$ is an algebra homomorphism. Show that if R is a regular local ring then the hypothesis on d^* is satisfied.

Exercise A3.29 (Miyata [1967]):

a.* (Apparently split implies split) If $\alpha : 0 \to A \to B \to C \to 0$ is a short exact sequence of finitely generated modules over a Noetherian ring and $B \cong A \oplus C$, then α splits. If you find the general case difficult, try the case where A, B, and C are finite Abelian groups.

b. One could try to classify the R-modules X that are extensions of one given module B by another, A, by classifying elements of $\mathrm{Ext}^1_R(A,B)$. One problem with this approach is that one can have two short exact sequences

$$\alpha : 0 \to B \to X \to A \to 0$$

and

$$\alpha' : 0 \to B \to X' \to A \to 0$$

with $X \cong X'$ without α being Yoneda-equivalent to α'. Give an example of sequences of finite Abelian groups where this happens. In general, it is hard to say even what relationship $[\alpha]$ and $[\alpha'] \in \mathrm{Ext}^1_R(A,B)$ have. However, part a shows that $[\alpha] = 0$ iff $[\alpha'] = 0$. Extend this by proving, with notation as above, that if $X \cong X'$, then

$$\mathrm{rad}(\mathrm{ann}[\alpha]) = \mathrm{rad}(\mathrm{ann}[\alpha']).$$

A3.11.2 Local Cohomology

The third derived functor of great use in commutative algebra is **local cohomology**. (The coherent sheaf cohomology of the algebraic geometers can also be expressed in terms of it, at least for projective varieties, and local cohomology with I the ideal of a subvariety plays in a certain sense the role of "relative" cohomology; see Grothendieck [1967].) For any ideal I of R, let $\Gamma_I(M) = \{m \in M | I^p m = 0 \text{ for sufficiently large } p\}$. It is easy to see that Γ_I is a left-exact functor, and we define

$$H^i_I(M) = R^i\Gamma_I(M),$$

again as the homology of the complex obtained by applying Γ_I to an injective resolution of M. We explain something of the properties of this derived functor in the central case where R is a local ring and I is the maximal ideal in Appendix 4.

Part II: From Mapping Cones to Spectral Sequences

A3.12 The Mapping Cone and Double Complexes

If $\alpha : F \to G$ is a map of complexes, then in many contexts we would like to know about the kernel and cokernel of the map induced by α on homology. If α were part of a short exact sequence of complexes—that is, if either all the α_i were monomorphisms or all were epimorphisms, then we could study this problem by looking at the corresponding long exact sequence in homology. Of more general usefulness is the following simple way of producing an exact sequence of complexes

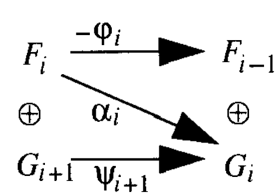
$$0 \to G \to M \to F[-1] \to 0$$

whose connecting homomorphisms are the maps on homology

$$\alpha_i : H_{i-1}F = H_i(F[-1]) \to H_{i-1}G$$

induced by α. Here we make the convention that if F is a complex with differential φ, then $F[i]$ is the complex where $F[i]_j = F_{i+j}$ and with differential $(-1)^i\varphi$. Of course the change of sign of the differential has no effect on the homology module (indeed, the complexes with signs changed or not are isomorphic—the map is -1 in every degree), but turns out to be convenient.

Definition. *If $\alpha : F \to G$ is a map of complexes, and we write φ and ψ, respectively, for the differentials of F and G, then the* **mapping cone** *$M(\alpha)$ of α is the complex such that $M(\alpha)_i = F_{i-1} \oplus G_i$, with differential*



That is, on G_{i+1} the map is the differential of G, but on F_i the map is the sum of the differential of F and the given map α of complexes.

Again, the motivation for this construction is topological: If $\alpha : X \to Y$ is a continuous map between topological spaces, then we may form the union $X \times I \cup Y$. Let M be the space obtained by identifying $X \times \{0\}$ to a point, and $X \times \{1\}$ to $\alpha(X)$ in Y, as in Figure A3.8. The d-dimensional chains of M are generated by the d-dimensional chains of Y and in addition for every (d-1)-chain x of X, a d-dimensional chain that we may describe as

$$\frac{\tilde{x} := (x \times I) \cup \alpha(x)}{x \times \{0\} = \text{point}, x \times \{1\} = \alpha(x) \subset Y}.$$

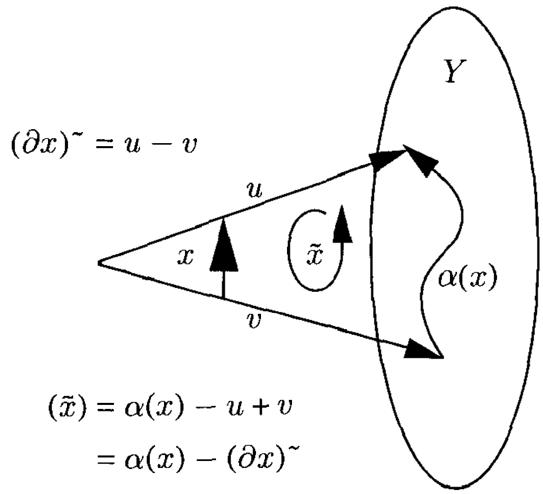


FIGURE A3.8.

With orientations as in Figure A3.8, we have $\partial(\tilde{x}) = -(\partial(x))^{\sim} + \alpha(x)$.

Proposition A3.19. *With notation as in the preceding definition, the natural inclusion makes G into a subcomplex of $M(\alpha)$, and $M(\alpha)/G \cong F[-1]$, so that there is a short exact sequence*

$$0 \to G \to M(\alpha) \to F[-1] \to 0$$

of complexes. In the corresponding long exact sequence in homology,

$$\cdots \to H_i(G) \to H_i M(\alpha) \to H_i(F[-1]) \xrightarrow{\delta} H_{i-1}(G) \to \cdots,$$

the connecting homomorphism δ is the map $H_i(F[-1]) = H_{i-1}F \to H_{i-1}G$ induced on homology by $\alpha : F \to G$.

Proof. The fact that G is a subcomplex of $M(\alpha)$ with quotient $F[-1]$ (that is, F shifted by 1 in degree) follows at once from the definition. To compute the effect of the connecting homomorphism, recall that if $[z]$ is the homology class in $H_i(F[-1])$ of a cycle z of degree i, then $\delta([z])$ is by definition the homology class of $d\tilde{z}$, where $\tilde{z}$ is a preimage of z in $M(\alpha)$ and d is the differential of $M(\alpha)$, and we regard $d\tilde{z}$ as an element of the subcomplex G. But we may take $\tilde{z}$ to be $(z, 0) \in M(\alpha)_i = F_{i-1} \oplus G_i$, and then $d\tilde{z} = (0, \alpha_{i-1}(z))$, whence the assertion. □

Applications of the mapping cone to the proof of exactness of the Koszul complex and the Taylor complex are given in Chapter 17. A natural generalization of the mapping cone is the total complex of a double complex; we give the construction here, though we shall not use it seriously until we develop the language of spectral sequences.

For agreement with what we do later, we make a small change in notation. Up to this point we have usually dealt with complexes whose differentials d have degree -1:

$$\cdots \to F_n \xrightarrow{d} F_{n-1} \to \cdots .$$

In the interest of agreeing with most of the standard treatments of spectral sequences and double complexes, we shall now switch to complexes with differential of degree +1, and we shall write them with *upper* indices

$$\cdots \to F^m \to F^{m+1} \to \cdots .$$

If we take $m = -n$ and identify F_n with F^{-n}, we recover our previous notation. We shall generally adopt this convention for dealing with upper and lower indices. It has the advantage of avoiding negative indices. Thus we shall write an injective resolution of a module M as

$$0 \to M \to Q^0 \to Q^1 \to Q^2 \to \cdots,$$

and we can write a free resolution M either in the form

$$\cdots \to F^{-2} \to F^{-1} \to F^0 \to M \to 0$$

or in the old form

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0.$$

I believe that the price of making such translations is more than repaid by the convenience, in dealing with spectral sequences, of always having the arrows point the same way.

Definition. ***A double complex*** *is a commutative diagram as in Figure A3.9 (extending infinitely in all four directions) where each row and each column is an ordinary complex; that is, a commutative diagram F as shown, with $d_{\text{hor}}^2 = 0 = d_{\text{vert}}^2$.*

Of course, any ordinary complex may be considered a double complex in which only one row is nonzero, and a map of ordinary complexes may be thought of as a double complex in which only two rows are nonzero. From the latter example, we have seen how to make an ordinary complex, the mapping cone. The natural generalization of this construction is a way of making an ordinary complex, called the associated **total complex**, from any double complex.

Definition. ***The total complex*** *of F is a complex whose kth term is*

$$\oplus_{i+j=k} F^{i,j},$$

with differential as in Figure A3.10. Somewhat more directly, one may think of a term of the total complex as the sum of the terms of the double complex along a diagonal, as shown by the line through the summands of $(\text{tot}\, F)^{i+j+1}$

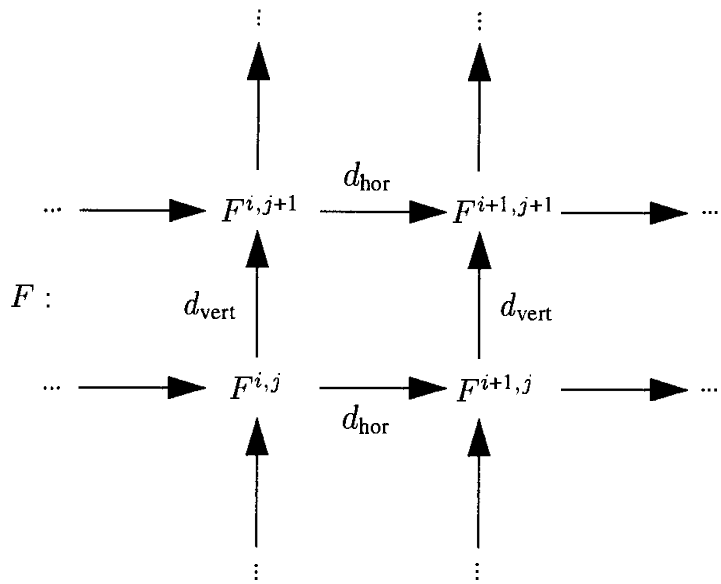


FIGURE A3.9.

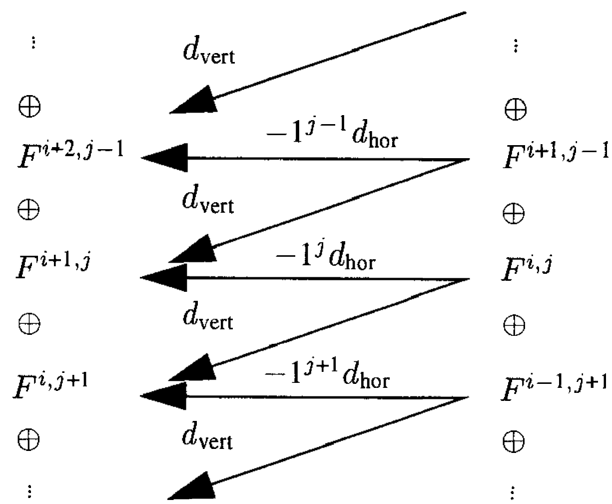


FIGURE A3.10.

in Figure A3.11. The differential is equal to the sum of all the maps shown, the maps in the jth row being multiplied by $(-1)^j$.

If F and G are ordinary complexes, with differentials φ and ψ, then the tensor product of F and G (as graded modules) becomes a double complex with terms $F_{i,j} = F^{-i,-j} := F_i \otimes G_j$ and differentials $d_{\text{hor}} = \varphi \otimes 1, d_{\text{vert}} = 1 \otimes \psi$, as in Figure A3.12.

Similarly, $\text{Hom}(F, G)$ is a double complex with terms $F^{i,-j} := \text{Hom}(F_i, G_j)$ and differentials $d_{\text{hor}} = \text{Hom}(\varphi, 1)$ and $d_{\text{vert}} = \text{Hom}(1, \psi)$. The homology of the total complex of $\text{Hom}(F, G)$ has a nice interpretation, given in the following exercises.

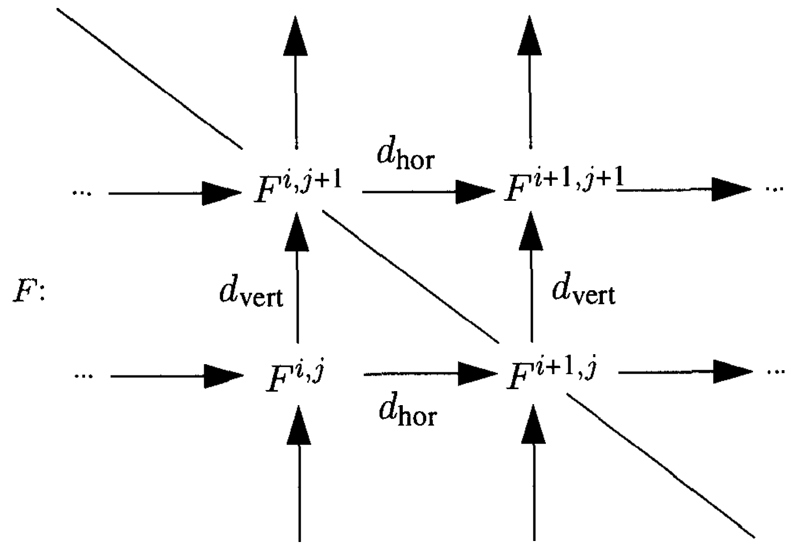


FIGURE A3.11.

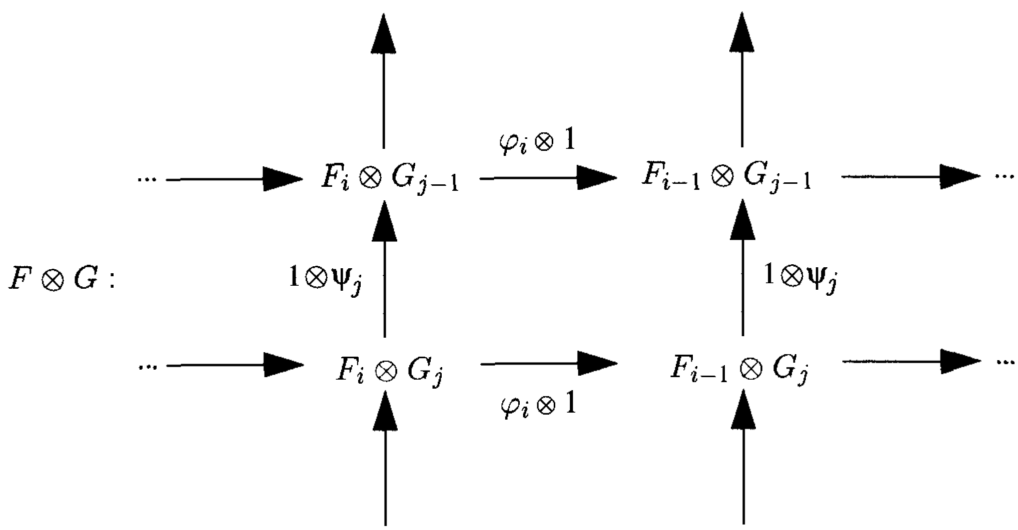


FIGURE A3.12.

A3.12.1 Exercises: Mapping Cones and Double Complexes

Exercise A3.30 (Resolution of an ideal from a factor ring): Suppose that R is a graded ring such that R_0 is a field, $I \subset R$ is an ideal, and J is an R-module. Suppose that

$$F: \cdots \to F_s \to \cdots \to F_1 \to R \to R/I \to 0$$

and

$$G: \cdots \to G_s \to \cdots \to G_1 \to G_0 \to J \to 0$$

are free resolutions of R/I and J. Given a monomorphism $a: J \to R/I$, identifying J with an ideal in R/I, let J' be the preimage of $a(J)$ in R. Given also maps $\alpha_i : G_i \to F_i$ forming a map of complexes $\alpha : F \to G$ lifting the map a, show that the mapping cone of α is a free resolution

$$M: \cdots \to F_s \oplus G_{s-1} \to \cdots \to F_1 \oplus G_0 \to R \to R/J' \to 0$$

of R/J'.

If R is local or graded, we would sometimes like to have a minimal free resolution of R/J'. Unfortunately, M need not be minimal even if F and G are, but there is one moderately common case in which we can prove that M is minimal. Suppose that $R = R_0 \oplus R_1 \oplus \cdots$ is a graded ring, and write each F_i and G_j as a sum of twists of $R: F_i = \oplus_j R(f_{ij})$ and $G_i = \oplus_j R(g_{ij})$. Show that if $f_{ij} > g_{ik}$ for all i, j, k, then M is minimal.

Exercise A3.31: If α is an isomorphism of complexes, show that the complex $M(\alpha)$ is "homotopically trivial" in the sense that the identity map from $M(\alpha)$ to itself is homotopic to the zero map.

Exercise A3.32: A **quasi-isomorphism of complexes** is a map of complexes that induces an isomorphism on homology; two complexes are quasi-isomorphic if there is a quasi-isomorphism between them (in either direction). A homotopy equivalence of complexes F and G is a map $\alpha : F \to G$ such that there is a map $\beta : G \to F$ with the property that $\alpha\beta$ and $\beta\alpha$ are each homotopic to the identity. Show that a homotopy equivalence is a quasi-isomorphism. Show by example that not every quasi-isomorphism is a homotopy equivalence. Show by example that two complexes may have the same homology without being quasi-isomorphic.

Exercise A3.33: Suppose that

$$0 \to F' \xrightarrow{\alpha} F \xrightarrow{\beta} F'' \to 0$$

is a short exact sequence of complexes. Show that F'' is quasi-isomorphic to $M(\alpha)$ by showing that there is a short exact sequence of complexes

$$0 \to M(\alpha') \to M(\alpha) \to F'' \to 0,$$

where α' is the isomorphism of F' onto $\alpha(F') \subset F$, and using Exercise A3.32. Similarly, show that F' is quasi-isomorphic to $M(\beta)$ (up to a shift of degree).

Exercise A3.34: Show that if

$$F: \cdots \to F^{n-2} \to F^{n-1} \to F^n \to 0 \to 0 \to \cdots$$

is a complex "bounded above" and

$$G: \cdots \to 0 \to 0 \to G^m \to G^{m+1} \to G^{m+2} \to \cdots$$

is a complex "bounded below," then the cycles of degree i in $\mathrm{tot}(\mathrm{Hom}(F, G))$ are the degree-i maps of complexes from F to G (that is, collections of maps $F_{-j} = F^j \to G^{j+1}$ that commute with the differentials), and the boundaries are the maps homotopic to 0; thus $H_i(\mathrm{tot}(\mathrm{Hom}(F, G)))$ is the group of homotopy classes of maps of degree i from F to G. The same thing is true if F is bounded below and G is bounded above.

A3.13 Spectral Sequences

General references for spectral sequences:[2] Serre [1957] does the case of a filtered complex quite directly (I learned the subject from this source). Other good treatments may be found in Maclane [1963], Cartan and Eilenberg [1956], Godement [1958], Grothendieck [1957] and Hilton and Stammbach [1971]. For a particularly gentle exposition of the subject with topological intentions, see Bott and Tu [1982] (but watch out for misprints).

Spectral sequences first arose in the work of Leray [1946, 1950] on topology and independently in the work of Lyndon [1946, 1948] on group cohomology. The topologists are the primary consumers of the theory, but there are plenty of applications in commutative algebra, in various algebraic cohomology theories, and in other areas as well.

It is easy to describe a spectral sequence.

Definition. *A **spectral sequence** is a sequence of modules* ${}^rE$ *for* $r \geq 1$, *each with a "differential"* $d_r : {}^rE \to {}^rE$ *satisfying* $d_r d_r = 0$, *such that* ${}^{r+1}E \cong \ker d_r / \operatorname{im} d_r$ *(or, as we shall prefer to write it,* ${}^{r+1}E = H({}^rE)$, *the homology of* ${}^rE$*).*

From these data one can define a "limit" term ${}^\infty E$. A spectral sequence may be interesting because ${}^\infty E$ may be identified with some inherently interesting object, to which the ${}^rE$ become "successive approximations"; or, on occasion, because the ${}^rE$ are interesting and ${}^\infty E$ is somehow trivial, which shows that some of the maps d_r must be very nontrivial.

To define ${}^\infty E$, we first define submodules

$$0 = {}^1B \subset {}^2B \subset \cdots \subset {}^rB \subset \cdots \subset \cdots \subset {}^rZ \subset \cdots \subset {}^2Z \subset {}^1Z = {}^1E$$

such that ${}^iE = {}^iZ/{}^iB$ for each i. To do this, let ${}^1Z = {}^1E$, and ${}^1B = 0$, so that ${}^1E = {}^1Z/{}^1B$. Having defined ${}^iB$ and ${}^iZ$, for $i \leq r$ we define ${}^{r+1}Z$ as the kernel of the composite map

$${}^rZ \to {}^rZ/{}^rB = {}^rE \xrightarrow{d_r} {}^rE = {}^rZ/{}^rB,$$

and write the image of this map as ${}^{r+1}B/{}^rB$; clearly ${}^{r+1}Z/{}^{r+1}B = H({}^rE) = {}^{r+1}E$, and

$${}^rB \subset {}^{r+1}B \subset {}^{r+1}Z \subset {}^rZ,$$

as required. Having defined all the ${}^iZ$ and ${}^iB$, we set

$${}^\infty Z = \bigcap^r Z,$$

$${}^\infty B = \bigcup^r B;$$

[2]Spectral sequences = suites spectrales; and spectral sweets = ghost candy.

and finally we define the **limit** of the spectral sequence to be

$$^{\infty}E = {}^{\infty}Z/{}^{\infty}B.$$

We say that the spectral sequence **collapses at ^{r}E** if $^{r}E = {}^{\infty}E$, or equivalently if the differentials $d_r, d_{r+1}, d_{r+2}, \ldots$ are 0.

Where do interesting spectral sequences come from? Most of the applications in algebra have to do with a spectral sequence that arises from a double complex in a way to be described shortly, a construction that generalizes the theory of the mapping cone that we have already used. There are also a few applications of the more general notion of the spectral sequence of a filtered complex. Still more general is a construction introduced by Massey [1952] that derives a spectral sequence from an object called an exact couple. There is an exact couple associated to any monomorphism from one complex (or differential module) to another, and it seems that most useful spectral sequences can be defined this way.

The subject of spectral sequences is elementary, but the notion of the spectral sequence of a double complex involves so many objects and indices that it seems at first repulsive. The approach via exact couples allows a much simpler view, postponing the indices until they are really needed; we shall follow this approach. First, however, we introduce the subject by recasting the theory of the mapping cone in the form it takes as a special case of the theory of the spectral sequence of a double complex.

A3.13.1 Mapping Cones Revisited

Suppose that $\alpha : F \to G$ is a map of complexes, and that we are interested in the homology of the mapping cone $M := M(\alpha)$. We shall show that the long exact sequence in homology of Proposition A3.19 can be interpreted as giving a filtration on the homology of M and a (very simple) spectral sequence whose $^{\infty}E$ term is the associated graded module of this filtration. This is a special case of the situation that holds more generally for (reasonable) double complexes.

The complex M contains a subcomplex M^1 isomorphic to G, with quotient $M/M^1 \cong F[-1]$. The resulting long exact sequence in homology has the form

$$\cdots \to H_iF \xrightarrow{\alpha_i} H_iG \to H_iM \to H_{i-1}F \xrightarrow{\alpha_{i-1}} H_{i-1}G \to \cdots,$$

where we have written α_i and α_{i-1} for the maps induced on homology. Saying that there is such an exact sequence is equivalent to saying that H_iM has a filtration, which we shall write as

$$H_iM = (H_iM)^0 \supset (H_iM)^1 \supset (H_iM)^2 = 0,$$

where $(H_iM)^1 = \operatorname{im} H_iG \to H_iM$, such that

$$(H_iM)^1/(H_iM)^2 = \operatorname{coker} \alpha_i,$$
$$(H_iM)^0/(H_iM)^1 = \ker \alpha_{i-1}.$$

We write $HF = \oplus_i H_iF$, and similarly for G and M. Write α_* for the direct sum of the maps α_i, so that $\alpha_* : HF \to HG$.

We can now define the spectral sequence: Let ${}^1E$ be the module $HF[-1] \oplus HG$. The module ${}^1E$ has a "differential" d_1 that is the composite

$${}^1E : HF[-1] \oplus HG \twoheadrightarrow HF[-1] \xrightarrow{\alpha_*} HG \rightarrowtail HF[-1] \oplus HG,$$

where the left-hand map is projection onto the first factor, and the right-hand map is injection into the second factor. It is clear that $\ker d_1 = \ker \alpha_* \oplus HG$, and im $d_1 = 0 \oplus \operatorname{im} \alpha_*$, so

$${}^2E := H({}^1E) = \ker \alpha_* \oplus \operatorname{coker} \alpha_*.$$

We give ${}^2E$ and all the succeeding ${}^rE$ the differential 0, so that the resulting spectral sequence collapses at ${}^2E$, and ${}^2E = {}^3E = \cdots = {}^\infty E$. The above relations may thus be written as

$$\operatorname{gr} HM = H^\infty E,$$

where gr HM is the associated graded module of HM, that is,

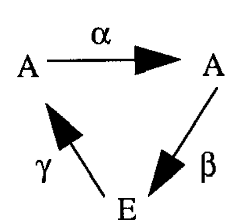
$$\operatorname{gr} HM := (HM)^0/(HM)^1 \oplus (HM)^1/(HM)^2.$$

This is the form that is generalized to arbitrary double complexes and beyond in the next section.

A3.13.2 Exact Couples

An **exact couple** is an **exact triangle**[3] of the form

(*)



[3]The reader who objects to defining an exact couple to be an exact triangle has my sympathy. Presumably the fact that there are only two distinct modules in the triangle, A and E, is the origin of the name.

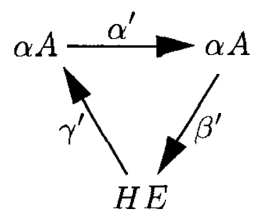
—that is, a diagram of modules and maps as above, which is exact in the obvious sense that $\ker\alpha = \operatorname{im}\gamma, \ker\gamma = \operatorname{im}\beta$, and $\ker\beta = \operatorname{im}\alpha$. Let $d : E \to E$ be the composite map $d = \beta\gamma$. Since $\gamma\beta = 0$, we see that $d^2 = 0$, so E is a differential module, and we write

$$HE = \ker d / \operatorname{im} d$$

for its homology.

Proposition–Definition A3.20. *If the diagram* $(*)$ *above is an exact couple, then there is a* ***derived exact couple***

(**)



where:

α' *is* α *restricted to* αA, *the image of* α*;*

β' *is* $\beta \circ \alpha^{-1} : \alpha A \to HE$, *taking* αa *to the homology class of* βa*;*

γ' *is the map induced by* γ *on* $\ker d$ *(which automatically kills* $\operatorname{im} d$*).*

Proof. Note that β' is well defined because $\ker\alpha = \operatorname{im}\gamma$ is taken to $\operatorname{im} d$ by β. The proof of exactness is completely straightforward, and we leave it to the reader. □

Given an exact couple $(*)$ we may form the derived exact couple $(**)$, and then repeat the process on $(**)$ $\ldots$. Thus we get the **spectral sequence of the exact couple,** defined by:

$^1E = E$ with differential $d_1 = d = \beta\gamma$, from the original couple;
$^2E = HE$ with differential $d_2 = \beta'\gamma'$, from the derived couple;
$^3E = HHE \cdots$ from the derived couple of the derived couple;

and so forth.

It is easy to check that with notation as in the definition of a spectral sequence we have

$$^{r+1}Z = \gamma^{-1}(\operatorname{im}\alpha^r)$$
$$^{r+1}B = \beta(\ker\alpha^r),$$

where α^r is the composite of α with itself r times. Thus

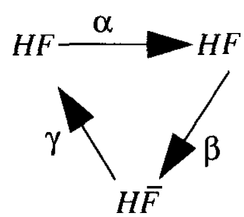
$$
\begin{aligned}
{}^{\infty}E &= {}^{\infty}Z/{}^{\infty}B \\
&= \frac{\gamma^{-1}\left(\bigcap_r \operatorname{im}\alpha^r\right)}{\beta\left(\bigcup_r \ker\alpha^r\right)}.
\end{aligned}
$$

Where do interesting exact couples come from? All of those treated here are instances of the following construction:

Let F be a differential module over a ring R, and let $\alpha : F \to F$ be a monomorphism. Set $\bar{F} = F/\alpha F$. The module $\bar{F}$ inherits a differential from F, so the short exact sequence of differential modules

$$0 \to F \xrightarrow{\alpha} F \to \bar{F} \to 0$$

gives rise to an exact triangle in homology



where we have written α again for the map on homology induced by $\alpha : F \to F$. The spectral sequence of this exact couple will be called the **spectral sequence of α on F**.

It is convenient to think of the map α as induced by multiplication with an element α of R that is a nonzerodivisor on F. Every case may be regarded this way—if necessary we adjoin a new variable α to R, and let it act as α on F (and thus also on HF), so that F and $\bar{F}$ become $R[\alpha]$-modules, with $\bar{F} = F/\alpha F$. If R is $\mathbf{Z}$, the ring of integers, and $\alpha \in \mathbf{Z}$ is an integer, then the spectral sequence above is widely known as the Bockstein spectral sequence, and the differentials as the Bockstein operators, but much of the theory is the same in the general case. With this in mind, we shall call $\ker\alpha^r : HF \to HF$ the **α^r-torsion of HF.** We shall also consider the intermediate complexes $F/\alpha^r F$; we say that a class in $H\bar{F}$ can be **lifted modulo α^r** if it is in the image of the natural map $H(F/\alpha^r F) \to H\bar{F}$; that is, if it has a representative in F (not necessarily a cycle) that becomes a cycle modulo α^r.

Proposition A3.21. *In the spectral sequence of α on F, the module ${}^{r+1}Z$ is the set of classes in $H\bar{F}$ that can be lifted modulo α^{r+1}, while ${}^{r+1}B$ is the image in $H\bar{F}$ of the α^r-torsion in HF. If $\bar{z}$ is a cycle in $\bar{F}$ with a representative $z \in F$ that is a cycle modulo α^r, then the differential $d_{r+1} : {}^{r+1}E \to {}^{r+1}E$ takes the class $\bar{z}$ to the class of $\alpha^{-(r+1)}dz$, where d is the differential of F.*

Proof. If z is any lifting to F of a cycle $\bar{z}$ in $H\bar{F}$, then $\gamma[\bar{z}] = [\alpha^{-1}dz] \in HF$. Further if $z \in F$ represents a cycle in $F/\alpha^r F$, then dz is divisible by α^r, so $\alpha^{-r}dz \in F$ makes sense; it is a cycle because α is a monomorphism on F. The rest is immediate from the definitions. □

A3.13.3 Filtered Differential Modules and Complexes

A **filtered differential module** is a differential module (G, d) together with a sequence of submodules G^p satisfying

$$G \supset \cdots \supset G^p \supset G^{p+1} \supset \cdots, \qquad p \in \mathbf{Z}$$

that are preserved by d —that is, $dG^p \subset G^p$ for all p. If in addition G is graded (for example, G might be a complex), say by upper degrees $G = \oplus_q G^q$, then we write $(G^q)^p$ for the pth level in the filtration of G^q. There are two examples that the reader should bear in mind. Recall that we write G_q for G^{-q}.

Example A. Let

$$G : \cdots \to G_{q+1} \xrightarrow{\varphi_{q+1}} G_q \xrightarrow{\varphi_q} G_{q-1} \to \cdots$$

be a complex of finitely generated modules over a Noetherian local ring $(R, \mathfrak{m})$, and let

$$G^p = \mathfrak{m}^p G : \cdots \to \mathfrak{m}^p G_{q+1} \xrightarrow{\varphi_{q+1}} \mathfrak{m}^p G_q \xrightarrow{\varphi_q} \mathfrak{m}^p G_{q-1} \to \cdots .$$

For the interesting applications we shall need more general filtrations

$$G^p : \cdots \to G^p_{q+1} \xrightarrow{\varphi_{q+1}} G^p_q \xrightarrow{\varphi_q} G^p_{q-1} \to \cdots,$$

satisfying only the property that $\cdots \supset G^p_q \supset G^{p+1}_q \supset \cdots$ is an $\mathfrak{m}$-stable filtration in the sense of Chapter 5 and Exercise A3.42.

We regard G as a filtered differential module by taking the direct sum over all q, as usual.

Example B. Let F be a double complex as in Figure A3.9 and let G be the total complex, $G = \operatorname{tot} F$. There are two natural filtrations on G—vertical filtration and horizontal filtration. The horizontal filtration is defined by subcomplexes ${}_{\text{hor}}G^p$, where ${}_{\text{hor}}G^p$ comes from the rows of F where the second index $\geq p$; that is ${}_{\text{hor}}G^p$ is made from the rows from $F^{*,p}$ up, shaded in Figure A3.13.

Similarly, ${}_{\text{vert}}G^p$ is the subcomplex coming from the columns where the first index is $\geq p$; in the picture, these are the columns $F^{p,*}$ and to the right. More formally, we let

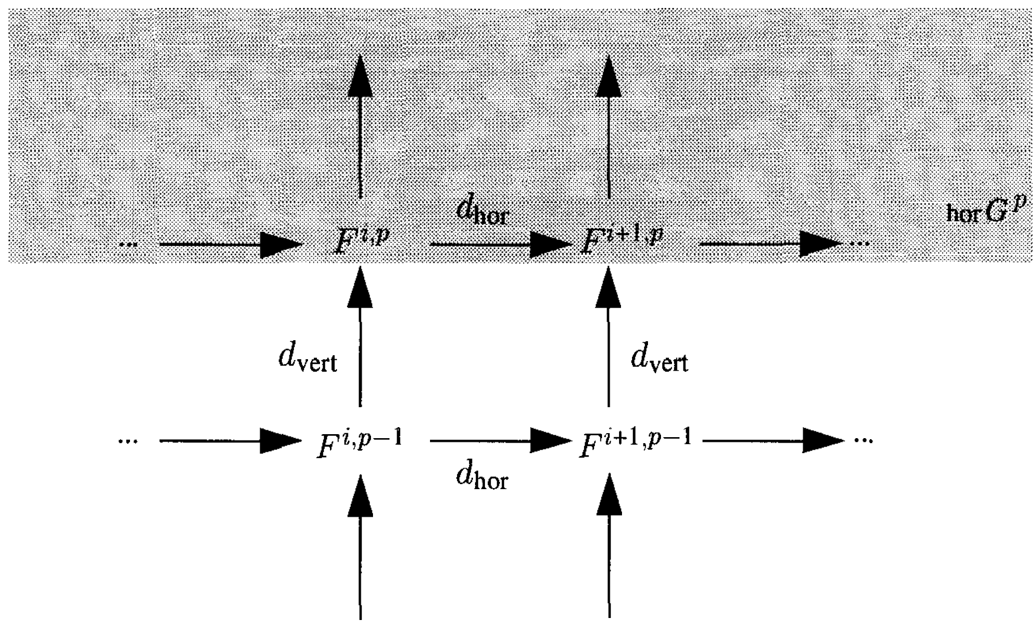


FIGURE A3.13.

$$({}_{\text{hor}}G^p)^k = \oplus_{i+j=k,\ j\geq p} F^{i,j}$$

with differential defined as the restriction of the differential of G, and similarly for ${}_{\text{vert}}G^p$.

In this section we shall give a general procedure for making a spectral sequence from a filtered differential module and we shall consider Example A. In the next section we shall consider Example B. In each of these cases we simply interpret Proposition A3.21; it is a new interpretation of the limit term that makes these situations interesting.

Let G be a filtered differential module as above, and let $F = \oplus_{p\in\mathbf{Z}} G^p$. The sum of the inclusion maps $G^{p+1} \to G^p$ defines a map $\alpha : F \to F$ that is obviously a monomorphism. Its cokernel $\bar{F}$ is obviously

$$\operatorname{gr} G := \oplus_p G^p/G^{p+1}.$$

Thus, setting ${}^1E^p = H(G^p/G^{p+1})$, we see that the spectral sequence of α on F starts with

$${}^1E = H(\operatorname{gr} G) = \oplus_p H(G^p/G^{p+1}) = \oplus_p {}^1E^p.$$

Since $F/\alpha^r F = \oplus_p G^p/G^{p+r}$, we may interpret Proposition A3.21 as saying in this case that ${}^{r+1}Z = \oplus_p {}^{r+1}Z^p$, with

$$\begin{aligned} {}^{r+1}Z^p &= \{[z] \in {}^1E^p | z \in G^p \text{ and } dz \in G^{p+r+1}\} \\ &= \{z \in G^p | dz \in G^{p+r+1}\} + G^{p+1}/G^{p+1} + dG^p; \\ {}^{r+1}B^p &= \{[z] \in {}^1E^p | z \in G^p \text{ and } z = dy \quad \text{for some } y \in G^{p-r}\} \\ &= (G^p \cap dG^{p-r}) + G^{p+1}/G^{p+1} + dG^p. \end{aligned}$$

So far, this is nothing but an application of Proposition A3.21. The new element is the following relation of ${}^{\infty}E$ with HG. The module HG is filtered by the submodules $(HG)^p = \operatorname{im} H(G^p) \to HG$. The associated graded module may be written as gr $HG = \oplus_p (HG)^p/(HG)^{p+1}$, and writing $K^p = \{z \in G^p | dz = 0\}$, we have

$$\begin{aligned}(HG)^p/(HG)^{p+1} &= K^p/(K^{p+1} + (dG \cap G^p)) \\ &= K^p + G^{p+1}/(G^p \cap dG) + G^{p+1}\end{aligned}$$

because $K^p \cap ((G^p \cap dG) + G^{p+1}) = K^{p+1} + (dG \cap G^p)$.

This last expression for $(HG)^p/(HG)^{p+1}$ is quite similar to the expression

$${}^{\infty}Z^p/{}^{\infty}B^p = \bigcap_r (\{z \in G^p | dz \in G^{p+r}\} + G^{p+1})/\bigcup_r ((G^p \cap dG^{p-r}) + G^{p+1}).$$

Writing the quotient on the right as M^p/N^p, we have

$$\begin{aligned}&M^p \supset K^p + G^{p+1}, \\ &N^p \subset (G^p \cap dG) + G^{p+1};\end{aligned}$$

so taking the direct sum over all p, we get

$$\operatorname{gr} HG \text{ is a quotient of a submodule of } {}^{\infty}Z/{}^{\infty}B.$$

Definition. *We say that the spectral sequence of the filtered differential module G* ***converges****, and for any term* ${}^rE$ *of the spectral sequence we write* ${}^r\boldsymbol{E} \Rightarrow$ **gr** $\boldsymbol{HG}$, *if gr* $HG = {}^{\infty}Z/{}^{\infty}B$*; that is, if for each p we have*

i. $\cap_r(\{z \in G^p | dz \in G^{p+r}\} + G^{p+1}) = \{z \in G^p | dz = 0\} + G^{p+1}$, *and*

ii. $\cup_r((G^p \cap dG^{p-r}) + G^{p+1}) = (G^p \cap dG) + G^{p+1}$.

Note that condition ii is relatively trivial; it will be satisfied as soon as $G = \cup_p G^p$. *Condition i, however, is much more subtle.*

Theorem A3.22. *The spectral sequence in Example A converges; further, the filtration induced on the homology of G is* $\mathfrak{m}$*-stable.*

Proof. We prove convergence for the $\mathfrak{m}$-adic filtration, leaving the important generalization to the reader in Exercise A3.42. In this spectral sequence, $G^p = G$ for $p \le 0$; thus convergence condition ii is trivially satisfied.

We now turn to condition i, which we may rewrite in the form

$$\begin{aligned}&\bigcap_r (\{z \in G^p | dz \in G^{p+r}\} + G^{p+1})/\{z \in G^p | dz = 0\} \\ &\quad = (\{z \in G^p | dz = 0\} + G^{p+1})/\{z \in G^p | dz = 0\}.\end{aligned}$$

The proof uses the Artin-Rees lemma (Lemma 5.1) and the Krull intersection theorem (Corollary 5.4).

Note that each G^p is a direct sum of the finitely generated modules $(G_q)^p = \mathfrak{m}^p G_q$, and the result we want may be checked for one of these summands at a time.

First, set $X = (G_q)^p/\{z \in (G_q)^p | dz = 0\}$. The differential d induces an inclusion $X \subset (G_{q-1})^p$, and for sufficiently large r',

$$\begin{aligned}\{z \in (G_q)^p \; | dz \in (G_{q-1})^{p+r'+r}\}/\{z \in G^p | dz = 0\} \\ \subset X \cap (G_{q-1})^{p+r} \\ = X \cap \mathfrak{m}^r (G_{q-1})^p.\end{aligned}$$

By the Artin-Rees lemma, there is a number s such that this is contained in $\mathfrak{m}^{r-s}X$ for all $r \geq s$, so that

$$\{z \in (G_q)^p | dz \in (G_{q-1})^{p+r'+r}\} \subset \mathfrak{m}^{r-s}(G_q)^p + \{z \in (G_q)^p | dz = 0\}.$$

Thus

$$\begin{aligned}&\{z \in (G_q)^p | dz \in (G_{q-1})^{p+r+r'}\} + (G_q)^{p+1}/\{z \in (G_q)^p | dz = 0\} \\ &\subset \mathfrak{m}^{r-s}(G_q)^p + \{z \in (G_q)^p | dz = 0\} + (G_q)^{p+1}/\{z \in (G_q)^p | dz = 0\},\end{aligned}$$

and the intersection of these for all r is

$$\{z \in G_u^p | dz = 0\} + G_u^{p+1}/\{z \in G_u^p | dz = 0\}$$

by the Krull intersection theorem. We leave the $\mathfrak{m}$-stability of the induced filtration on HG to the reader (see Exercise A3.42).

The following corollary contains two simple applications.

Corollary A3.23.

a. *Let $x_1, \ldots, x_r$ be a sequence of elements in the maximal ideal $\mathfrak{m}$ of a local ring $(R, \mathfrak{m})$, and write x_i^* for the leading form of x_i in $\operatorname{gr}_{\mathfrak{m}} R$. If the x_i^* form a regular sequence on $\operatorname{gr}_{\mathfrak{m}} R$, then the x_i form a regular sequence on R.*

b. *Let M and N be finitely generated modules over a local ring $(R, \mathfrak{m})$. There is a spectral sequence*

$$\operatorname{Tor}^{\operatorname{gr} R}(\operatorname{gr} M, \operatorname{gr} N) \Rightarrow \operatorname{Tor}^R(M, N).$$

Thus, for example, if $\operatorname{Tor}_u^{\operatorname{gr} R}(\operatorname{gr} M, \operatorname{gr} N) = 0$ then $\operatorname{Tor}_u^r(M, N) = 0$.

Proof. Follow the hints in Exercises A3.43 and A3.44. □

We emphasize that the filtration induced on $\operatorname{Tor}^R(M, N)$ in this corollary may not be the $\mathfrak{m}$-adic filtration, but will be $\mathfrak{m}$-stable.

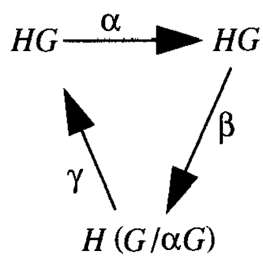
A3.13.4 The Spectral Sequence of a Double Complex

We now take up Example B, which is arguably the most important for algebraists. For this discussion we keep the example's notation, summarized in Figure A3.9, with $G = \operatorname{tot} F$.

We consider HG and $H(G/\alpha G)$ as bigraded modules by setting

$$(HG)^{p,q} := H^{p+q}(G^p) \text{ and } H(G/\alpha G)^{p,q} := H^{p+q}(G^p/G^{p+1}).$$

The maps α, β, and γ in the exact triangle



are then bigraded of degrees $(-1,1)$, $(0,0)$, and $(1,0)$, respectively. Thus the differential $d_r : {}^rE \to {}^rE$ is bihomogeneous of degree r in the p grading and $-(r-1)$ in the q grading; that is, d_r is the direct sum of maps

$$d_r : {}^rE^{p,q} \to {}^rE^{p+r,q-r+1}.$$

More graphically, representing ${}^1E$ as an array of the ${}^1E^{p,q}$,

$$\begin{array}{lll} {}^1E^{p,q} & {}^1E^{p+1,q} & {}^1E^{p+2,q} \\ {}^1E^{p,q-1} & {}^1E^{p+1,q-1} & {}^1E^{p+2,q-1} \\ {}^1E^{p,q-2} & {}^1E^{p+1,q-2} & {}^1E^{p+2,q-2} \end{array}$$

the differential d_r goes "r steps to the right and $r-1$ steps down" as in Figure A3.14.

Of course, this picture needs some interpretation: d_2 is actually defined on the kernel of d_1 (a quotient of which is ${}^2E^{*,*}$); d_3 is actually defined on the kernel of d_2; and so on. To describe the d_i, suppose for definiteness that we are working with the spectral sequence of the horizontal filtration. Then d_1 is simply the map induced by d_{hor} on the homology of d_{vert}. An element of the kernel of d_1 is represented by a "vertical cycle" $z \in F^{p,q}$ (that is, an element of the kernel of d_{vert}) that is mapped by d_{hor} to 0 in homology —that is, such that $d_{\text{hor}}(z)$ is a "vertical boundary," an element of the form $d_{\text{vert}}(z')$. The map d_2 takes (the homology class of) z to the homology class of $d_{\text{hor}}(z')$. For this to be zero means that $d_{\text{hor}}(z') = d_{\text{vert}}(z'')$ for some z'', and in this case d_3 carries (the homology class of) z to $d_{\text{hor}}(z'')$; and so on.

Here is our main result.

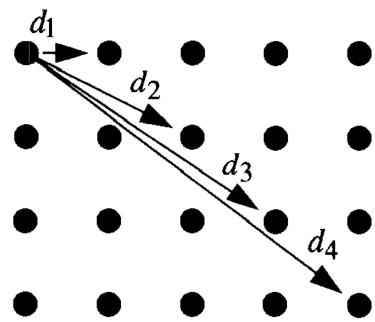


FIGURE A3.14.

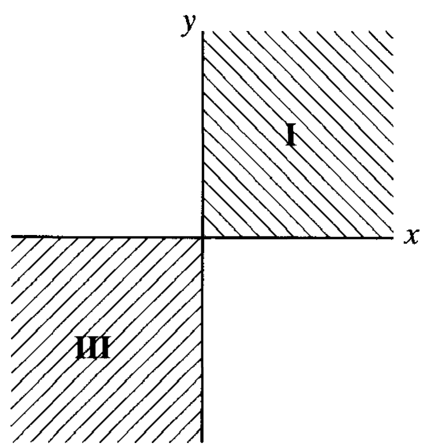


FIGURE A3.15.

Theorem A3.24. *Associated with the double complex F are two spectral sequences, ${}^r_{\text{hor}}E$ and ${}^r_{\text{vert}}E$, corresponding, respectively, to the horizontal and vertical filtrations of* $\text{tot}(F) = G$. *The ${}^1E$ terms are bigraded with the components given by*

$${}^1_{\text{hor}}E^{p,q} = H^q(F^{*,p}), \qquad {}^1_{\text{vert}}E^{p,q} = H^q(F^{p,*}).$$

If $F^{i,j} = 0$ for all $i < 0$ or for all $j > 0$, then the horizontal spectral sequence converges; that is,

$${}^{\infty}_{\text{hor}}E = \text{gr}_{\text{hor}}\, H(\text{tot}\, F).$$

Symmetrically, if $F^{i,j} = 0$ for all $i > 0$ or for all $j < 0$, then the vertical spectral sequence converges.

Terminology: Theorem A3.24 implies that both spectral sequences converge either if $F^{i,j} = 0$ for all $i < 0$ and for all $j < 0$ or if $F^{i,j} = 0$ for all $i > 0$ and for all $j > 0$. In the former case, the nonzero terms are all in the first quadrant of the i, j-plane, and we call F a **first-quadrant double complex**. In the second case the nonzero terms are all in the third quadrant, and we call F a **third-quadrant double complex** (see Figure A3.15).

Proof. The proof of the first formula is immediate from the definitions. For example, we have

$$\begin{aligned} {}_{\text{hor}}^{\;1}E^{p,q} &= H^{p+q}(\text{gr}_{\text{hor}}(\text{tot}\, F)^p) \\ &= H^q(F^{*,p}), \end{aligned}$$

whence the formula for ${}_{\text{hor}}^{\;1}E^{p,q}$; the case of the vertical spectral sequence is similar.

The proof of convergence uses the bigrading. Writing G for the total complex tot F, we must show that

i. $\cap_r(\{z \in G^p | dz \in G^{p+r}\} + G^{p+1}) = \{z \in G^p | dz = 0\} + G^{p+1}$

and

ii. $\cup_r((G^p \cap dG^{p-r}) + G^{p+1}) = (G^p \cap dG) + G^{p+1}$.

Since $G = \cup_p G^p$ with respect to either filtration, condition ii is trivially satisfied (this does not use any conditions on the double complex). Condition i means that if $z \in G^p$ and for each r there is an element $y_r \in G^{p+1}$ such that $d(z - y_r) \equiv 0 \bmod G^{p+r}$, then there is some $y \in G^{p+1}$ such that $d(z - y) = 0$. In our case, since G is a complex, it is enough to check this for $z \in (G^q)^p$, for some q. For definiteness, consider again the case of the horizontal filtration. The element $d(z - y_r)$ is then in

$$(G^{q+1})^{p+r} = \bigoplus_{i+j=q+1,\, j \geq p+r} F^{i,j}.$$

If $F^{i,j} = 0$ for $j > 0$, then $(G^*)^{p+r} = 0$ for $r > -p$. If, on the other hand, $F^{i,j} = 0$ for $i < 0$, then $(G^{q+1})^{p+r} = 0$ if $r > q + 1 - p$. Thus in either case $d(z - y_r) = 0$ for suitable r, and we may take $y = y_r$ for this value of r.

A refinement of the notation for convergence is useful: We write

$${}^rE^{p,q} \Rightarrow_p H^{p+q}(\text{tot}\, F)$$

to mean that the spectral sequence containing the terms ${}^rE^{p,q}$ converges, and that writing $H^{p+q}(\text{tot}\, F)^p$ for the pth level in the associated filtration of $H^{p+q}(\text{tot}\, F)$,

$${}^\infty E^{p,q} = H^{p+q}(\text{tot}\, F)^p / H^{p+q}(\text{tot}\, F)^{p+1}.$$

We now give two simple applications. More will be found in the exercises. The first involves a double complex both of whose spectral sequences degenerate at ${}^2E$.

i. Balanced Tor. We shall show that $\text{Tor}_i^R(M, N)$ may be computed from a free resolution of either M or N and is in fact a "balanced" functor in the

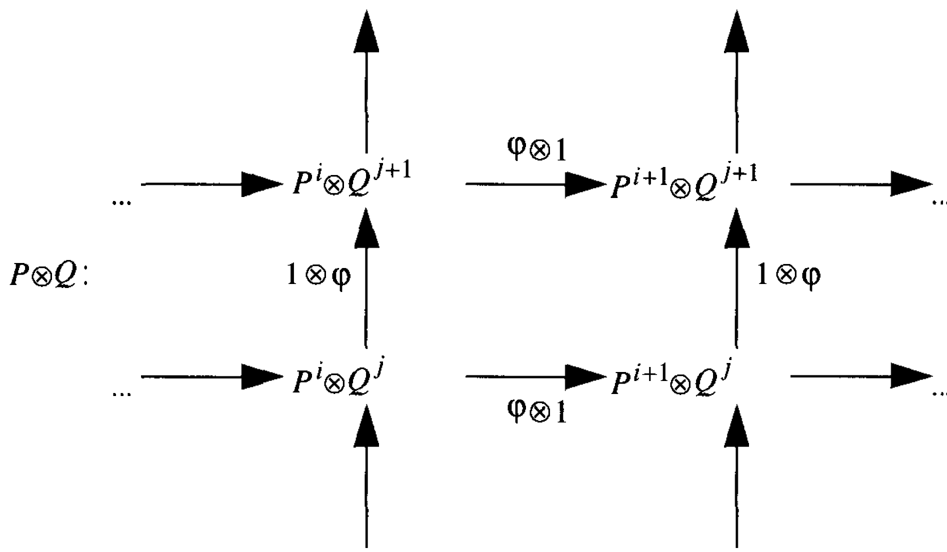


FIGURE A3.16.

sense that if $a \in R$, then multiplication by a on M induces the same map on $\operatorname{Tor}_i^R(M, N)$ as does multiplication by a on N—that is, the R-module structure on $\operatorname{Tor}_i^R(M, N)$ may be induced from the module structure of either M or N. To see this let

$$P: \cdots \to P_i \xrightarrow{\varphi_i} P_{i-1} \to \cdots \to P_0$$

and

$$Q: \cdots \to Q_i \xrightarrow{\psi_i} Q_{i-1} \to \cdots \to Q_0$$

be free resolutions of M and N, respectively. We shall show that

$$H(P \otimes_R N) \cong H(\operatorname{tot}(P \otimes_R Q)) \cong H(M \otimes_R Q),$$

as R-modules. Since "$\operatorname{Tor}_i^R(M, N)$ computed from a free resolution of M" is the first of these, and "$\operatorname{Tor}_i^R(M, N)$ computed from a free resolution of N" is the last, this will suffice.

Let ${}_{\text{vert}}E$ be the vertical spectral sequence associated with the third-quadrant double complex $F = P \otimes_R Q$, which may be written with upper indices, using the convention that $P^i = P_{-i}$, in the form shown in Figure A3.16. We have ${}_{\text{vert}}^{\ 1}E_{i,j} = {}_{\text{vert}}^{\ 1}E^{-i,-j} = H_j(P_i \otimes Q)$. Since P_i is free, the complex $P_i \otimes Q$ is just a direct sum of copies of Q; more invariantly, we have $H_j(P_i \otimes Q) = P_i \otimes H_j(Q)$. This is 0 for $j > 0$, while $P_i \otimes H_0(Q) = P_i \otimes N$. Thus the only nonzero ${}_{\text{vert}}^{\ 1}E_{i,j}$ are those with $j = 0$. The differential d_1 is induced by $d_{\text{hor}} = \varphi \otimes 1$. Thus ${}_{\text{vert}}^{\ 1}E$ is the complex $P \otimes N$, and

$${}_{\text{vert}}^{\ 2}E_{i,j} = \begin{cases} H_i(P \otimes N) & \text{for } j = 0 \\ 0 \quad \text{for } j > 0. \end{cases}$$

It follows that the spectral sequence degenerates at ${}^2E$; that is, ${}_{\text{vert}}^{\infty}E = {}_{\text{vert}}^{2}E$. Since all the nonzero terms have $j = 0$,

$$\bigoplus_{i+j=k} {}_{\text{vert}}^{\infty}E_{i,j} = {}^{\infty}E_{k,0},$$

and the filtration of $H(\text{tot}(P \otimes Q))$ has only one nonzero piece. Thus we get $H(P \otimes N) = H(\text{tot}(P \otimes Q))$. By symmetry, $H(\text{tot}(P \otimes Q)) = H(M \otimes Q)$ as well; we could also deduce this from the horizontal spectral sequence of $P \otimes Q$.

ii. Change of Rings. Let $R \to S$ be a ring homomorphism, let A be an S-module, and let B be an R-module. We shall derive one of the "change of rings spectral sequences" (see Exercise A3.45 for others), whose ${}^2E^{i,j}$ term is $\text{Ext}_S^i(A, \text{Ext}_R^j(S, B))$, converging to $\text{Ext}_R^{i+j}(A, B)$; that is,

$$\text{Ext}_S^i(A, \text{Ext}_R^j(S, B)) \Rightarrow_i \text{Ext}_R^{i+j}(A, B).$$

Let

$$P: \ \cdots \to P_i \to P_{i-1} \to \cdots \to P_0$$

be an S-free resolution of A as an S-module, and let

$$Q : Q^0 \to \cdots \to Q^j \to Q^{j+1} \to \cdots$$

be an R-injective resolution of B as an R-module, respectively. We regard $\text{Hom}_S(P, \text{Hom}_R(S, Q))$ as a first-quadrant double complex, with $\text{Hom}_S(P_i, \text{Hom}_R(S, Q^j))$ as the i, j term. We first claim that the horizontal spectral sequence degenerates, as in Example i. We have

$${}_{\text{hor}}^{1}E^{j,i} = H^i(\text{Hom}_S(P_*, \text{Hom}_R(S, Q^j))).$$

Since Q^j is R-injective, $\text{Hom}_R(S, Q^j)$ is S-injective, and

$$H^i(\text{Hom}_S(P_*, \text{Hom}_R(S, Q^j)) = \text{Hom}_S(H_i(P_*), \text{Hom}_R(S, Q^j)).$$

Since P_* is a resolution of A, this vanishes except for $i = 0$, and when $i = 0$ it is $\text{Hom}_S(A, \text{Hom}_R(S, Q^j)) \cong \text{Hom}_R(A, Q^j)$. Since the ${}_{\text{hor}}^{1}E$ differential is induced from the differential in Q, we see that

$${}_{\text{hor}}^{2}E^{j,i} = \begin{cases} H^j(\text{Hom}_R(A, Q_*)) = \text{Ext}_R^j(A, B) & \text{for } i = 0 \\ 0 \quad \text{for } i > 0. \end{cases}$$

Thus, as in the last example, the spectral sequence degenerates at the ${}^2E$ term, so $H^j(\text{tot}(\text{Hom}_S(P, \text{Hom}_R(S, Q))) \cong \text{Ext}_R^j(A, B)$.

However, the vertical spectral sequence does not degenerate in this case! We have

$$ {}_{\text{vert}}^{1}E^{i,j} = H^j(\mathrm{Hom}_S(P_i, \mathrm{Hom}_R(S, Q^*)), $$

and since P_i is free over S, this may be written as

$$ \mathrm{Hom}_S(P_i, H^j(\mathrm{Hom}_R(S, Q^*))) = \mathrm{Hom}(P_i, \mathrm{Ext}_R^j(S, B)). $$

The ${}_{\text{vert}}^{1}E$ differential is the map induced by the differential of P, and thus the ${}^2E$ term has the form

$$ {}_{\text{vert}}^{2}E^{i,j} = \mathrm{Ext}_S^i(A, \mathrm{Ext}_R^j(S, B)). $$

The ${}^2E$ differential d_2 maps this term to $\mathrm{Ext}_S^{i+2}(A, \mathrm{Ext}_R^{j-1}(S, B)) = {}_{\text{vert}}^{2}E^{i+2,j-1}$. Since ${}_{\text{hor}}E$ has the same limit as ${}_{\text{vert}}E$, we get

$$ {}_{\text{vert}}^{2}E = \mathrm{Ext}_S(A, \mathrm{Ext}_R(S, B)) \Rightarrow \mathrm{Ext}_R(A, B) $$

as required.

The change of rings spectral sequence is a special case of the **spectral sequence of a composite functor**; we have only used the fact that $\mathrm{Hom}_R(A, B)$ is the composite of the functor $\mathrm{Hom}_S(A, -)$ with the functor $\mathrm{Hom}_R(S, -)$ and the fact that the functor $\mathrm{Hom}_R(S, -)$ takes injectives to injectives. The general construction plays an important role in algebraic geometry, beginning with the Leray spectral sequence. See Exercise A3.50 below.

A3.13.5 Exact Sequence of Terms of Low Degree

In general, the relation between the ${}^rE$ term and the ${}^\infty E$ term of a spectral sequence is somewhat tenuous, but there is often a simple relation between the $H_k(\mathrm{tot}\, F)$ and some of the ${}^2E^{p,q}$. For the sake of definiteness we treat the vertical spectral sequence of a third-quadrant double complex F only. Of course, similar remarks will hold for the horizontal spectral sequence, and also for a first-quadrant double complex; they may be extended to other ${}^rE$ as well (see Exercise A3.48).

Proposition A3.25 (5-term exact sequence). *If $F^{i,j}$ is a third-quadrant double complex, then writing H_i for H^{-i} $(\mathrm{tot}\, F)$, and E for ${}_{\text{vert}}E$, we have*

a. $H_0 \cong {}^2E^{0,0}$.

b. For every i there is a pair of natural maps

$$ {}^2E^{0,-i} \xrightarrow{\iota} H_i \xrightarrow{\kappa} {}^2E^{-i,0}. $$

c. There is a 5-term exact sequence

$$ H_2 \xrightarrow{\kappa} {}^2E^{-2,0} \xrightarrow{d_2} {}^2E^{0,-1} \xrightarrow{\iota} H_1 \xrightarrow{\kappa} {}^2E^{-1,0} \to 0, $$

where d_2 is the differential of the spectral sequence.

Proof. We use the fact that E converges to $H(\text{tot } F)$, together with the fact that ${}^2E^{p,q} = 0$ for $p > 0$ and for $q > 0$. For example, to prove part c, look at Figure A3.17 where we have shown some of the ${}^2E$ differentials.

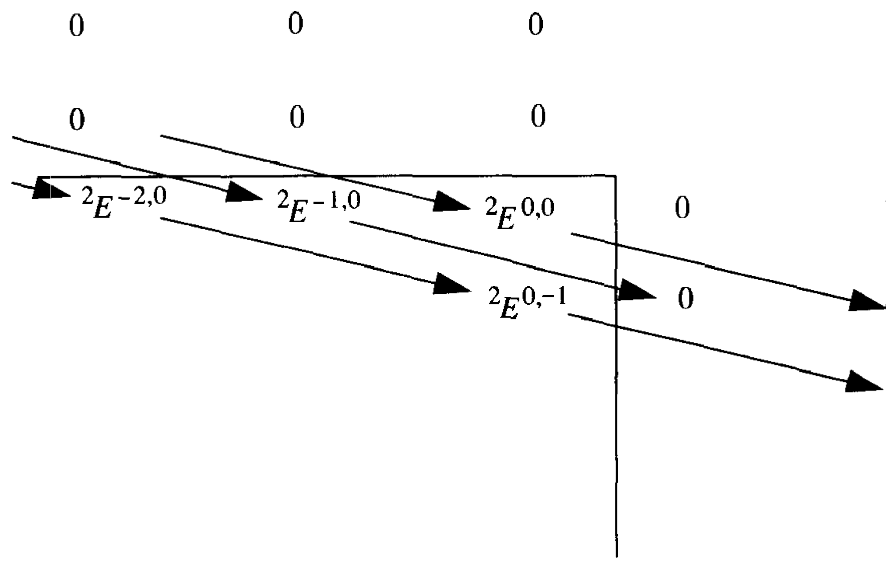


FIGURE A3.17.

Because the terms outside the third quadrant are 0, we see that

$$^2E^{-1,0} = {}^\infty E^{-1,0} = H_1/(H_1)^0,$$

while

$$\operatorname{coker} d_2 = {}^3E^{0,-1} = {}^\infty E^{0,-1} = (H_1)^0$$

and

$$\ker d_2 = {}^3E^{-2,0} = {}^\infty E^{-2,0} = H_2/(H_2)^{-1}.$$

Putting these facts together, we get the five-term sequence. The other parts are similar, but even easier. □

A3.13.6 Exercises on Spectral Sequences

Exercise A3.35: Check the exactness of the derived couple in Proposition A3.20. Check the formulas for ${}^{r+1}Z$ and ${}^{r+1}B$.

Exercise A3.36: Let (F, d) be any differential module, and filter F by

$$F^0 := F \supset F^1 := \ker d \supset F^2 := \operatorname{im} d \supset F^3 := 0.$$

Writing $({}^rE, d_r)$ for the associated spectral sequence, show that ${}^3E = {}^\infty E = HF$.

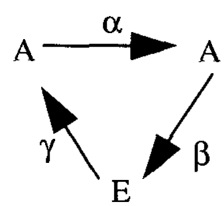
Exercise A3.37: Let $p \in \mathbf{Z}$ be an integer. Explicitly construct the Bockstein spectral sequence associated with the complex

$$F : 0 \to \mathbf{Z}^2 \xrightarrow{(p,0)} \mathbf{Z} \to 0$$

with respect to the endomorphism that is multiplication by p; that is, compute all the ${}^rE$ and d_r, and compute ${}^\infty E$.

Exercise A3.38:

a. Show that for the spectral sequence of the exact couple



there are short exact sequences

$$0 \to A/(\operatorname{im}\alpha + \ker\alpha^r) \to {}^rE \to (\ker\alpha) \cap (\operatorname{im}\alpha^r) \to 0,$$

by showing that the left- and right-hand terms are the images of the appropriate maps in the rth derived exact couple.

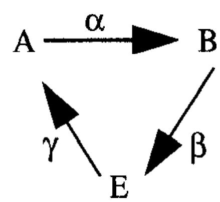
b. Show that if $\ker\alpha^{r+1} = \ker\alpha^r$ for some r, then the spectral sequence collapses at ${}^rE$ and ${}^rE = {}^\infty E = A/(\operatorname{im}\alpha + \ker\alpha^r)$.

c. Show that if A is a finitely generated module and the ground ring is Noetherian, then for some r the condition of part b is satisfied. Give a version that holds for the spectral sequence of a monomorphism $\alpha : F \to F$ of a (not necessarily finite) complex F of finitely generated modules over a Noetherian ring.

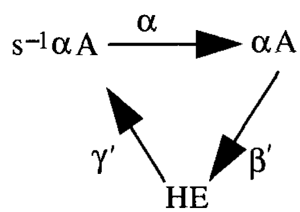
d. If F is a finite complex of finitely generated, torsion-free Abelian groups and p is an integer, then show that the Bockstein spectral sequence for p on F (that is, the spectral sequence for the endomorphism of F that is multiplication by p) has limit $HF/(T + pHF)$, where T is the p-torsion submodule of HF (the set of elements killed by some power of p). For what r is ${}^rE$ equal to ${}^\infty E$?

e. Generalize the argument in the case where F is an infinite complex of finitely generated, torsion-free Abelian groups; again, show that the Bockstein spectral sequence for p has limit $HF/(T + pHF)$.

Exercise A3.39: Generalize the construction of the spectral sequence of an exact couple to the following: Suppose we are given an exact triangle



together with an epimorphism $s : A \to B$. Define a differential $d : E \to E$ by $d = \beta s \gamma$. Show that there is a "derived triangle"



where $\beta'(\alpha a)$ is the class of $\beta s a$, and α', γ' are induced from α, γ; there is also a natural "derived epimorphism" $s : s^{-1}\alpha A \to \alpha A$. Thus the process may be repeated, and we get a spectral sequence.

Exercise A3.40: Let R be the local ring $k[x,y]_{(x,y)}$. Work out all the terms and differentials of the spectral sequence

$$\mathrm{Tor}^{k[x,y]}(k[x,y]/x^2, k[x,y]/xy) \Rightarrow \mathrm{Tor}^R(R/x^2, R/xy + y^3)$$

of Corollary A3.23b.

Exercise A3.41 (Comparison Theorem): Suppose that $F \cdots \supset F^p \supset \cdots$ and $G \cdots \supset G^p \supset \cdots$ are filtered complexes, and that $\alpha : F \to G$ is a morphism of filtered complexes—that is, a morphism of complexes carrying F^p into G^p. Writing ${}^rE(F)$ and ${}^rE(G)$ for the associated spectral sequences, show that there are induced maps ${}^rE(F) \to^r E(G)$ for every r. Show that if one of these maps is an isomorphism, and the spectral sequence converges, then α induces an isomorphism on homology, $H(F) \cong H(G)$.

Exercise A3.42: Let R be a ring, and let $\mathfrak{m}$ be any ideal of R. Recall from Chapter 5 that a filtration

$$\cdots \supset G^p \supset G^{p+1} \supset \cdots$$

of an R-module G is called **$\mathfrak{m}$-stable** if

i. $\mathfrak{m}G^p \subset G^{p+1}$ for all p; and

ii. $\mathfrak{m}G^p = G^{p+1}$ for all sufficiently large p.

In Chapter 5 it is shown that if condition i is satisfied then $\operatorname{gr} G$ is naturally a module over the ring $\operatorname{gr}_{\mathfrak{m}} R$, and that if G is a finitely generated module and both conditions are satisfied, then $\operatorname{gr} G$ is a finitely generated $(\operatorname{gr}_{\mathfrak{m}} R)$-module. Show conversely that if condition i is satisfied and $\operatorname{gr} G$ is a finitely generated $(\operatorname{gr}_{\mathfrak{m}} R)$-module, then ii holds.

Assume that $(R, \mathfrak{m})$ is a local Noetherian ring, and that G is a finitely generated module with $\mathfrak{m}$-stable filtration

$$G \supset \cdots \supset G^p \supset G^{p+1} \supset \cdots .$$

a. If the associated graded module $\operatorname{gr} G = \oplus G^p/G^{p+1}$ is zero, then G is zero.

b. If F is any submodule of G, then the filtration of F by $F^p := F \cap G^p$ is $\mathfrak{m}$-stable. Similarly, the filtration of G/F by $(G/F)^p := (F^p + G)/G$ is $\mathfrak{m}$-stable.

c. Suppose that $(R, \mathfrak{m})$ is a local Noetherian ring. If

$$G : \ \cdots \to G_q \to G_{q-1} \to \cdots$$

is a filtered complex of finitely generated R-modules such that the filtration on each G_q is $\mathfrak{m}$-stable, show that the induced filtration on the homology $H_i G$ is also $\mathfrak{m}$-stable. Prove that the spectral sequence of the filtered complex G converges to HG, that is, $H(\operatorname{gr} G) \Rightarrow HG$.

Exercise A3.43: Prove assertion a of Corollary A3.23 by giving an $\mathfrak{m}$-stable filtration of the Koszul complex $K(x_1, \ldots, x_r)$ as follows: Let δ_i be the degree of the leading form x_i^* of x_i, that is, δ_i is the largest integer δ such that $x_i \in \mathfrak{m}^\delta$. In the Koszul complex, the ith free module may be written as $\wedge^i R^r$. This module has a basic consisting of elements of the form $e_{j_1} \wedge \cdots \wedge e_{j_i}$, where e_j is the basis vector of $\wedge^1 R^r = R^r$ that maps to x_j in R. We filter $\wedge^i R^r$ by submodules

$$(\wedge^i R^r)^p = \oplus(\mathrm{Re}_{j_1} \wedge \cdots \wedge e_{j_i})^p,$$

where $(Re_{j_1} \wedge \cdots \wedge e_{j_i})^p = \mathfrak{m}^{p - \Sigma_t \delta_{j_t}}(Re_{j_1} \wedge \cdots \wedge e_{j_i})$; here $\mathfrak{m}^k$ is interpreted as R for $k \leq 0$.

Show that the filtration $\cdots \supset (\wedge^i R^r)^p \supset (\wedge^i R^r)^{p+1} \supset \cdots$ of $\wedge^i R^r$ is $\mathfrak{m}$-stable, and that with this filtration

$$\operatorname{gr} K(x_1, \ldots, x_r) = K(x_1^*, \ldots, x_r^*),$$

the Koszul complex of the leading forms of the x_i, over the ring $\operatorname{gr}_{\mathfrak{m}} R$. Now deduce assertion a of the corollary from the convergence of the associated spectral sequence

$$H(K(x_1^*, \ldots, x_r^*)) \Rightarrow \operatorname{gr} H(K(x_1, \ldots, x_r)).$$

Exercise A3.44: Prove the assertion of part b of Corollary A3.23 as follows.

a.* First find a free resolution G of M and an $\mathfrak{m}$-stable filtration $\cdots \supset G^p \supset G^{p+1} \supset \cdots$ of it such that the associated graded complex is a free resolution of $\mathrm{gr}_{\mathfrak{m}} M$ over $\mathrm{gr}_{\mathfrak{m}} R$.

b. Define a filtration on the complex $G \otimes N$ by taking

$$(G \otimes N)^p = \text{image in } G \otimes N \text{ of } G^p \otimes N.$$

Show that with respect to this filtration, $\mathrm{gr}(G \otimes N) = \mathrm{gr}\, G \otimes \mathrm{gr}\, N$. Now consider the spectral sequence of the filtered complex $G \otimes N$.

Exercise A3.45 (More Change-of-Rings Spectral Sequences): Suppose that $R \to S$ is a homomorphism of rings, and A is an S-module, B an R-module.

a. Show that there is a spectral sequence whose ${}^2E$ term is $\mathrm{Ext}_S(\mathrm{Tor}^R(S, B), A)$, and that converges to $\mathrm{Ext}_R(B, A)$.

b. Similarly, show that there is a spectral sequence

$$\mathrm{Tor}_q^S(\mathrm{Tor}_p^R(S, B), A) \Rightarrow_p \mathrm{Tor}_{p+q}^R(B, A).$$

Exercise A3.46 (The Two-Row and Two-Column Cases):

a. Let F be a double complex whose vertical spectral sequence $E =_{\text{vert}} E$ converges to $H := H(\mathrm{tot}\, F)$. Suppose that for some r only two columns of ${}^rE$ are nonzero—that is, suppose that the ${}^rE^{p,q}$ are nonzero for only two distinct values of p, say $p = s$ and $p = t$, with $s > t$. Show that there is a long exact sequence

$$\cdots \to {}^rE^{s,i-s} \to H^i \to {}^rE^{t,i-t} \xrightarrow{\delta} {}^rE^{s,i-s+1} \to H^{i+1} \to \cdots,$$

where $\delta = d_{s-t}$ if $r \le s - t$, and $\delta = 0$ if $r > s - t$.

b. A similar result holds if the ${}^rE^{p,q}$ are nonzero for only two values of q. Apply this to the change-of-rings spectral sequence in the text: For example, assume that $\mathrm{Ext}_R^j(S, B) = 0$ unless $j = s$ or $j = s + 1$, and derive the isomorphism

$$\mathrm{Ext}_R^s(A, B) \cong \mathrm{Hom}_S(A, \mathrm{Ext}_R^s(S, B))$$

and the long exact sequence

$$0 \to \mathrm{Ext}_S^1(A, \mathrm{Ext}_R^s(S, B)) \to \mathrm{Ext}_R^{s+1}(A, B) \to \mathrm{Hom}_S(A, \mathrm{Ext}_R^{s+1}(S, B)) \to$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\mathrm{Ext}_S^u(A, \mathrm{Ext}_R^s(S, B)) \to \mathrm{Ext}_R^{s+u}(A, B) \to \mathrm{Ext}_S^{u-1}(A, \mathrm{Ext}_R^{s+1}(S, B))$$
$$\to \mathrm{Ext}_S^{u+1}(A, \mathrm{Ext}_R^s(S, B)) \to \cdots .$$

c.* Suppose that R is a regular ring of dimension d (for example, a polynomial ring in d variables over a field) and $S = R/I$ is a two-dimensional domain (for example the homogeneous coordinate ring of an irreducible projective curve). Let $B = R$, and let A be any S-module. Show that part b above applies, with $s = d - 2$.

Exercise A3.47: Suppose that $R \to S \twoheadrightarrow k$ are maps of rings. Using the change-of-rings spectral sequence of Exercise A3.45b, show that there is a five-term exact sequence

$$\operatorname{Tor}_2^R(k,k) \to \operatorname{Tor}_2^S(k,k) \to \operatorname{Tor}_1^R(S,k) \to \operatorname{Tor}_1^R(k,k) \to \operatorname{Tor}_1^S(k,k) \to 0.$$

This sequence is particularly interesting when R is a local ring, S is a factor ring of R, and k is the residue field of R and S. Interpret the sequence in this case in terms of minimal free resolutions. See, for example, Gulliksen and Levin [1969] for information on the resolution of the residue class field of a local ring.

Exercise A3.48: Find explicit analogues and generalizations for Proposition A3.25 for all ${}^rE$, for horizontal spectral sequences, and for the case where $F^{i,j} = 0$ for $i < 0$ and $j < 0$.

Exercise A3.49 Resolutions of complexes:* If

$$F : 0 \to F^0 \to F^1 \to F^2 \to \cdots$$

is a complex of modules, show that there is a double complex $I^{j,k}$ and maps

$$\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & I^{0,2} & \to & I^{1,2} & \to & I^{2.2} & \to & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & I^{0,1} & \to & I^{1,1} & \to & I^{2,1} & \to & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & I^{0,0} & \to & I^{1,0} & \to & I^{2,0} & \to & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & F^0 & \to & F^1 & \to & F^2 & \to & \cdots
\end{array}$$

such that

i. Each column $I^{j,0} \to I^{j,1} \to I^{j,2} \to \cdots$ is an injective resolution of F_j.

ii. In the rows, the kernel of each $I^{j,k} \to I^{j+1,k}$ is an injective summand of $I^{j,k}$, and thus the image of $I^{j,k} \to I^{j+1,k}$ and the homology of $I^{j-1,k} \to I^{j,k} \to I^{j+1,k}$, which is ${}_{\text{hor}}^{1}E^{j,k}$, are injective modules.

iii. The spectral sequence ${}_{\text{hor}}E$ degenerates at ${}^2E$ to $H(F)$; that is, the term ${}_{\text{hor}}^{\;\,1}E$ of the spectral sequence, with differential d_1 induced by the vertical maps in the diagram above, forms injective resolutions

$$0 \to H^j(F) \to {}_{\text{hor}}^{\;\,1}E^{j,0} \to {}_{\text{hor}}^{\;\,1}E^{j,1} \to {}_{\text{hor}}^{\;\,1}E^{j,2} \to \cdots$$

of the homology of F.

Such a double complex is called an injective resolution of the complex F.

Exercise A3.50 (Grothendieck's spectral sequence of a composite functor): Suppose that $\mathcal{A}$ and $\mathcal{B}$ are categories of modules over some rings and that $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{B} \to \mathcal{C}$ are left-exact functors. What is the relation between the derived functors $R^i\mathcal{F}$, $R^i\mathcal{G}$, and $R^i(\mathcal{G}\mathcal{F})$? Under favorable circumstances, it is given by a spectral sequence. Prove this as follows:

a. We say that an object B of $\mathcal{B}$ is $\mathcal{G}$-acyclic if $R^i\mathcal{G}(B) = 0$ for all $i > 0$. Show that if B is any object of $\mathcal{B}$ and $0 \to B \to B^0 \to B^1 \to \cdots$ is an exact sequence of objects of $\mathcal{B}$ with each B^i $\mathcal{G}$-acyclic, then

$$R^i\mathcal{G}(B) = H^i(0 \to \mathcal{G}B^0 \to \mathcal{G}B^1 \to \mathcal{G}B^2 \to \cdots).$$

b.* Now suppose that A has a resolution by $\mathcal{F}$-acyclic objects that are carried by $\mathcal{F}$ to $\mathcal{G}$-acyclic objects. Show that there is a spectral sequence

$${}^2E^{p,q} = R^p\mathcal{F}(R^q\mathcal{G}(A)) \Rightarrow_p R^{p+q}(\mathcal{F}\mathcal{G})(A).$$

c. Show that the change-of-rings spectral sequence given in the text is of this form, where the composite functor is

$$\operatorname{Hom}_S(A, \operatorname{Hom}_R(S, -)) = \operatorname{Hom}_R(A, -).$$

d.* If you know enough about sheaves, derive the Leray spectral sequence $H^i(R^j\pi_*(A)) \Rightarrow H^{i+j}(A)$ for a sheaf of Abelian groups A on a topological space X and a continuous map of spaces $\pi : X \to Y$.

A3.14 Derived Categories

> ... le manque de fondements adéquats d'Algebre Homologique m'avait empêché.... Cette lacune de fondements est sur le point d'être comblée par la thèse de Verdier....
>
> [... the lack of an adequate foundation for homological algebra hindered me.... This gap in the foundations has just been filled by the thesis of Verdier... .]
>
> —Alexandre Grothendieck, 1963

We have given a somewhat primitive view of derived functors simply as things constructed from projective or injective resolutions. Various more axiomatic definitions have been used, but the most complete and powerful seems to be Verdier's formulation by means of his notion of the **derived category** [1977]. We give a very brief sketch of the derived category and the picture of derived functors to which it leads, in the hope that this will help orient the reader. More complete pictures may be found in Hartshorne [1966b, Chapter I] Iversen [1986], Grivel [1987], or Lipman [1995].

As we have seen, the central idea in homological algebra is to replace a module by a projective resolution or an injective resolution: for simplicity we shall stick with projective resolutions for this description, and leave the dualization to the reader. There are two desiderata addressed by the construction of the derived category: First, one would like the association of a module to one of its projective resolutions to be a functor of some kind. Second, one would like to be able to replace a module, or a complex of modules, with a complex of projective modules having the same homology, as in some ways these are easier to manipulate. This leads to a construction in two steps, which we now explain. We shall ignore some set-theoretic points (coming for example from the fact that the "set of all modules" is not a set) that would form a part of a careful treatment.

A3.14.1 Step One: The Homotopy Category of Complexes

The association of a module to its projective resolution is not a functor, because projective resolutions are not unique, and neither are the maps induced on projective resolutions by maps of modules. The first of these problems is easy to cure: We simply choose a fixed projective resolution $P(M)$ for each module M (other, more canonical solutions would be to make a "canonical projective resolution", with each module free on the elements of the kernel of the map before; or to take some direct limit over all projective resolutions). Unfortunately, the nonuniqueness of maps induced on projective resolutions keeps P from being a functor. However, we have already seen that every map of modules lifts to a map of projective resolutions that is unique up to homotopy. Thus P becomes a functor from the category $\mathcal{M}$ of R-modules to the category $K(\mathcal{M})$ whose objects are complexes of R-modules and whose morphisms are the homotopy classes of maps between complexes.

Because we would like to have projective resolutions for any object in our category, and because it is not so clear how to make projective resolutions for unbounded complexes, we restrict ourselves at this point to the category $K^+(\mathcal{M})$ of "bounded-below complexes," that is complexes

$$F: \cdots \to F_{i+1} \to F_i \to \cdots$$

with $F_i = 0$ for $i \ll 0$. See Exercise A3.53 for the meaning of projective resolutions in this setting. (Recent developments suggest that there are good resolutions for unbounded complexes too—see Avramov and Halperin [1986]. In any case, a more thorough treatment of derived categories would contain parallel constructions with bounded-below complexes, unbounded complexes, and bounded-above complexes, the last for the purpose of using injective resolutions and constructing right derived functors. We shall systematically ignore all but the first of these.) Because homotopic maps induce the same map on homology, one still can speak of the "nth homology module" $H_i(X)$ of an object X of $K(\mathcal{M})$, even though one cannot speak of the "term of degree n" in X.

Now the category $K^+(\mathcal{M})$ is no longer an Abelian category. For example, if F and G are the complexes of Abelian groups

$$F: \cdots 0 \to 0 \to \mathbf{Z} \to 0 \to 0 \to \cdots,$$
$$G: \cdots 0 \to 0 \to \mathbf{Z}/(2) \to 0 \to 0 \to \cdots,$$

and $\pi : F \to G$ is the natural map of complexes mapping $\mathbf{Z}$ onto $\mathbf{Z}/(2)$, then no image for π exists in $K^+(\mathcal{M})$; see Exercise A3.51.

Because $K^+(\mathcal{M})$ is not Abelian, we cannot speak of exact sequences in this category. However, the category $K^+(\mathcal{M})$ has a new structure, called a **triangulation**, which can be used as a substitute for exact sequences. First, we have a "translation functor" T on complexes that takes a complex F to the complex $F[-1]$. Given a translation functor T on a category, a triangulation is a distinguished collection of diagrams of the form

$$A \to B \to C \to TA,$$

satisfying certain axioms, which we shall not state. In the case of the category $K^+(\mathcal{M})$, we may take the triangles to be the diagrams made from the mapping cones of maps $\alpha : A \to B$ of complexes, that is the diagrams of the form

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} M(\alpha) \xrightarrow{\gamma} TA,$$

where for simplicity we have written α for the homotopy class of α, and β and γ are the homotopy classes of the standard inclusion of B in $M(\alpha)$ and the projection of $M(\alpha)$ to $A[-1] = TA$, respectively. If we apply the homology functor to such a triangle, we get a long exact sequence in homology, as explained in the section on mapping cones.

The reason for making this choice instead of taking the triangles to be (or at least include) the short exact sequences of complexes is that, with the definition above, any additive functor on the category of modules induces, in an obvious way, a functor on $K^+(\mathcal{M})$ that preserves triangles.

A3.14.2 Step Two: The Derived Category

Following our outline, we wish to be able to replace any complex by a projective complex with the same homology. The construction of projective resolutions of complexes, Exercise A3.53, shows that given any bounded-below complex F there is a bounded-below complex F' of projective modules and a map $F' \to F$ of complexes that induces an isomorphism on homology. Thus we may attain our goal by formally inverting every morphism that is an isomorphism on homology—such a morphism is called a **quasi-isomorphism**, or **quism**.

Now, quite generally, given a category $\mathcal{A}$ and a set $\mathcal{S}$ of morphisms, there is a universal solution to the problem of finding a category $\mathcal{B}$ and a functor $\mathcal{A} \to \mathcal{B}$ taking all the elements of $\mathcal{S}$ to isomorphisms; the resulting category $\mathcal{B}$ is unique up to equivalence of categories and is called $\mathcal{A}[\mathcal{S}^{-1}]$. The objects of $\mathcal{A}[\mathcal{S}^{-1}]$ may be taken to be the same as the objects of $\mathcal{A}$, and the morphisms are "words" whose letters are morphisms of $\mathcal{A}$ and formal inverses s^{-1} of morphisms $s \in \mathcal{S}$, subject to the condition of composability and the equivalence relation generated by composition in $\mathcal{A}$ and the rule that s^{-1} is inverse to s. (The construction is directly analogous to localization of rings, which is actually the special case where $\mathcal{A}$ is an additive category having just one object X—the ring in question is $\mathrm{Hom}(X, X)$.) However, these localized categories are in general quite awkward. For example, there may be no simple criterion to tell whether two morphisms from $\mathcal{A}$ become equal in $\mathcal{B}$. (The same phenomenon occurs in the special case of localizations of general noncommutative rings. In logical terms, the "word problem" may be recursively insoluble.)

In the case of the category $K^+(\mathcal{M})$ we are lucky (the recognition of this luck seems to have been one of Verdier's fundamental insights): The localization of $K^+(\mathcal{M})$ with respect to the set of quasi-isormorphisms has a nice form. The fundamental point is that the maps in the localized category can all be represented in the form $\alpha^{-1}a$, where a is a morphism of $K^+(\mathcal{M})$ and α is a quasi-isomorphism, so that we have a sort of "calculus of fractions." The crucial point that must be checked is that we can rewrite any composition $b\beta^{-1}$ with β a quism in the form $\alpha^{-1}a$, with α a quism. Rewriting this without using inverses, one must check that given a map $a : B \to C$ and a quasi-isomorphism $\alpha : B \to A$, there exists, for some complex B', a quasi-isomorphism $\beta : C \to B'$ and a map $b : A \to B'$ such that $b\alpha = \beta a$ (see, for example, Hartshorne [1977, p. 30].)

We now define the **derived category** $D^+(\mathcal{M})$ to be the category $K^+(\mathcal{M})$ with the quasi-isomorphisms formally inverted. The objects of $D^+(\mathcal{M})$ are complexes, and the maps are things of the form $a\alpha^{-1} : A \to C$, where $a : B \to C$ is a morphism and $\alpha : B \to A$ is a quasi-isomorphism in $K^+(\mathcal{M})$, modulo an equivalence relation effectively saying that one can cancel a quasi-isomorphism from a product. We write $P : K^+(\mathcal{M}) \to D^+(\mathcal{M})$ for the localization functor.

The derived category inherits from $K^+(\mathcal{M})$ the structure of a triangulated category: Since the translation functor on $K^+(\mathcal{M})$ preserves quasi-isomorphisms, it induces a functor, called again translation, on $D^+(\mathcal{M})$, and we take as a triangle anything quasi-isomorphic to a triangle in $K^+(\mathcal{M})$.

It is interesting to note that any exact sequences of complexes becomes a triangle of $D^+(\mathcal{M})$; this follows from Exercise A3.33.

Just as using the localization of a ring is conceptually simpler than working in the original ring but "as if" elements of a multiplicative system were invertible, working in the derived category has proved simpler for certain applications than working with the category of complexes directly. However, every quasi-isomorphism between bounded-below projective complexes is actually a homotopy equivalence, so that if we define $\mathcal{P}$ to be the category of projective R-modules, the derived category may be described simply as $K^+(\mathcal{P})$; see Exercise A3.54.

With these ideas in place we can describe left-derived functors (for right-derived functors one would use bounded-above complexes and injective resolutions). If F is an additive functor from $\mathcal{M}$ to $\mathcal{M}$, say, then as we have already noted, F induces a functor that we may call $K(F) : K(\mathcal{M}) \to K(\mathcal{M})$. The left-derived functor LF of F is a functor $LF : D^+(\mathcal{M}) \to D^+(\mathcal{M})$, together with a natural transformation $\eta : LF \circ P \to KF$, which gives the "best possible approximation to KF" in the sense that for any functor $G : D^+(\mathcal{M}) \to D^+(\mathcal{M})$ and natural transformation $\nu : G \circ P \to KF$, there is a unique map $G \to LF$ such ν is the composite $G \circ P \to LF \circ P \to F$. The old derived functors L_iF are obtained by composing LF with the "ith homology functor" $H_i : D^+(\mathcal{M}) \to \mathcal{M}$.

The first hint of the simplification that is obtained by all this comes when one considers composite functors. Previously we saw that under good conditions the derived functors of a composite functor fit into a spectral sequence (we did this in the text in the context of "change of rings" and in the exercises in general). But in terms of the derived category, the derived functor of a composite functor is (under the same favorable circumstances) simply the composition of derived functors! For example, if S is an R-algebra, M is an S-module,

$$F = S \otimes_R - : R - \text{modules} \ \to S - \text{modules},$$

and

$$G = M \otimes_S - : S - \text{modules} \ \to S - \text{modules},$$

then the spectral sequence

$$L_iG(L_jF) = \operatorname{Tor}_i^S(M, \operatorname{Tor}_j^R(S, N)) \Rightarrow \operatorname{Tor}_{i+j}^R(M, N) = L_{i+j}(G \circ F)(N),$$

where N is an R-module, is replaced by the much simpler

$$LG \circ LF = L(G \circ F).$$

When there are many functors and compositions around, this simplification can be decisive. Of course, when one wants to make computations one must fall back to the more concrete language of spectral sequences.

A3.14.3 Exercises on the Derived Category

Exercise A3.51 (The category $K^+(\mathcal{M})$ is not Abelian):* In an Abelian category, every morphism $A \to C$ can be factored into an epimorphism followed by a monomorphism $A \twoheadrightarrow B \rightarrowtail C$. Show that the natural map $\mathbf{Z} \to \mathbf{Z}/(p)$ gives rise to a map of complexes

$$A = \{\cdots \to 0 \to \mathbf{Z} \to 0 \to \cdots\} \to \{\cdots \to 0 \to \mathbf{Z}/(p) \to 0 \to \cdots\} = C$$

that cannot be factored in this way in $K^+(\mathcal{M})$.

Exercise A3.52: If A and B are bounded-below complexes of projective modules, and $\alpha : A \to B$ is a quasi-isomorphism, show that α is a homotopy equivalence.

Exercise A3.53: Let F be a bounded-below complex of R-modules. Imitate Exercise A3.49 to show that there is a bounded-below complex of projective R-modules P and a quasi-isomorphism $P \to F$. Such a P is a projective resolution of F.

Exercise A3.54: Let $K^+(\mathcal{P})$ be the category whose objects are bounded-below complexes of projective R-modules, and whose morphisms are homotopy classes of morphisms of complexes. Define a "projective resolution functor" $K^+(\mathcal{M}) \to K^+(\mathcal{P})$. Show that it sends quasi-isomorphisms to isomorphisms (that is, to homotopy equivalences), and thus induces a functor $D^+(\mathcal{M}) \to K^+(\mathcal{P})$. Show that together with the composite functor $K^+(\mathcal{P}) \to K^+(\mathcal{M}) \to D^+(\mathcal{M})$, this defines an equivalence of categories $D^+(\mathcal{M}) \cong K^+(\mathcal{P})$.

Appendix 4
A Sketch of Local Cohomology

As we have often seen, there is a tight analogy between local and graded rings. We have generally started from things that we proved for the local case and adapted them for the graded case. But the analogy flows in the other direction too. A graded domain R gives rise to a subvariety $X = \operatorname{Proj} R$ of projective space, and a module M over R gives rise to a sheaf $\tilde{M}$ on X. One of the most important tools available in this context is the cohomology $H^*(X, \tilde{M})$. If we take the local-global analogy seriously, we should ask whether there is a good local analogue of this cohomology.

The answer is yes, and the corresponding construction is called local cohomology. We will state some of the most basic definitions and results pertaining to it but omit the proofs for the sake of brevity. The reader can find more information in Grothendieck [1967] and Brodmann and Sharp [1996].

First a general definition: If R is a ring, I an ideal of R, and M an R-module, then we define the zeroeth local cohomology module of M with supports in I to be simply the set of all elements of M which are annihilated by some power of I:

$$H_I^0(M) = \cup_n (0 :_M I^n) = \lim_{n\to\infty} \operatorname{Hom}(R/I^n, M),$$

where $(0 :_M I^n)$ denotes the set of elements of M annihilated by I^n. We define the higher local cohomology groups as the right-derived functors of H_I^0—that is, $H_I^i(M)$ is the i^{th} cohomology module of the complex obtained by applying H_I^0 to an injective resolution of M.

Geometrically, if we think of elements of M as global sections of the sheaf on Spec R associated to M, then the elements of $H_I^0(M)$ are just the sections with support on the closed subscheme Spec $R/I \subset$ Spec R. It is clear that a similar definition could be made for any closed subscheme of any scheme, and indeed the theory is most naturally developed in this context—see, for example, Grothendieck [1967].

It is easy to see that the functor H_I^0 is left-exact, and so for any short exact sequence of modules we get a long exact sequence in local cohomology. Since local cohomology is the derived functor, it is universal among sequences of functors with this property. On the other hand, the functors

$$\lim_{n\to\infty} \mathrm{Ext}_R^i(R/I^n, M)$$

behave in a similar way, taking short exact sequences to long exact sequences. A careful inspection shows that they have the same universal property as the local cohomology, so they are in fact naturally isomorphic:

$$H_I^i(M) \cong \lim_{n\to\infty} \mathrm{Ext}_R^i(R/I^n, M).$$

Besides $\cup_n(0 :_M I^n)$ and $\lim_{n\to\infty} \mathrm{Hom}(R/I^n, M)$, we can express the zeroeth local cohomology in another way in terms of familiar objects. If $I = (x_1, \ldots, x_s)$, then the elements of M annihilated by some power of I are the same as the elements annihilated by some power of each of the x_i. Hence we have

$$H_I^0(M) = \lim_{n\to\infty} H^0(M \otimes K(x_1^n, \ldots, x_s^n)),$$

where the maps

$$H^0(M \otimes K(x_1^n, \ldots, x_s^n)) \to H^0(M \otimes K(x_1^{n+1}, \ldots, x_s^{n+1}))$$

over which the limit is taken are the inclusions

$$\begin{aligned} H^0(M \otimes K(x_1^n, \ldots, x_s^n)) &= (0 :_M (x_1^n, \ldots, x_s^n)) \subset (0 :_M (x_1^{n+1}, \ldots, x_s^{n+1})) \\ &= H^0(M \otimes K(x_1^{n+1}, \ldots, x_s^{n+1})). \end{aligned}$$

Equivalently, and more usefully, we may think of these maps as induced by the natural maps of Koszul complexes

$$K(x_1^n, \ldots, x_s^n) \to K(x_1^{n+1}, \ldots, x_s^{n+1})$$

which in degree 1 are given by the map $f : R^n \to R^n$ multiplying the ith component by x_i, and in degree d are $\wedge^d f$, which acts by multiplying a basis vector $e_{i_1} \wedge \cdots \wedge e_{i_d}$ by $x_{i_1} \cdot \ldots \cdot x_{i_d}$. Thus we may take the limit in each of the Koszul homology groups, and arguing as before we get

$$H_I^i(M) \cong \lim_{n\to\infty} H^i(M \otimes K(x_1^n, \ldots, x_s^n)).$$

A4.1 Local Cohomology and Global Cohomology

The last isomorphism above provides the means to relate local cohomology to the cohomology of coherent sheaves on a projective variety or scheme.

Suppose that R is graded, with maximal ideal P generated by $x_1, \ldots, x_s$, and having degree 0 part, R_0, a field. We write $\tilde{M}$ for the sheaf induced by M on the scheme $\operatorname{Proj} R$.

First a general remark which will help to identify the limit of the Koszul complexes: If we take a sequence of modules $M_n \cong M$, and maps $M_n \to M_{n+1}$ induced by multiplication by some fixed element $a \in R$, then

$$\varinjlim M_n = M[a^{-1}],$$

the localization of M with respect to the multiplicative set generated by a. Of course if a is homogeneous, then the degree 0 part of $M[a^{-1}]$ is the module of sections of $\tilde{M}$ on the open set $a \neq 0$ of $\operatorname{Proj} R$, and $M[a^{-1}]$ itself is the sum over all ν of the global sections of $\tilde{M}(\nu)$ on the open set $a \neq 0$. If $a = x_{i_1} \cdot \ldots \cdot x_{i_d}$, then writing U_i for the open set $x_i \neq 0$, the open set $a \neq 0$ is the intersection $U_{i_1} \cap \cdots \cap U_{i_d}$.

Thus with f as above, the limit of modules isomorphic to $M \otimes \wedge^d R^s$ under the maps induced by $\wedge^d f$ is

$$\lim_{n\to\infty} M \otimes \wedge^d R^s = \oplus_{i_1 \cdots i_d} \sum_\nu H^0(U_{i_1} \cap \cdots \cap U_{i_d}, \tilde{M}(\nu)|_{U_{i_1}\cap\cdots\cap U_{i_d}}).$$

Since taking homology commutes with direct limits over directed sets, we see that the local cohomology of M is the cohomology of the complex

$$0 \to M \to \oplus_i \sum_\nu H^0(U_i, \tilde{M}(\nu)_{|U_i}) \to \ldots$$
$$\to \oplus_{i_1\cdots i_d} \sum_\nu H^0(U_{i_1} \cap \cdots \cap U_{i_d}, \tilde{M}(\nu)|_{U_{i_1}\cap\cdots\cap U_{i_d}}) \to \ldots,$$

and except for the first term, this is the Čech complex, whose ith homology is the ordinary Čech cohomology H^i $(\operatorname{Proj} R, \tilde{M})$. This shows that local and global cohomologies are related in the following way:

Theorem A4.1. *If M is a graded R-module, then there is a natural exact sequence*

$$0 \to H^0_P(M) \to M \to \sum_\nu H^0(\operatorname{Proj} R, \tilde{M}(\nu)) \to H^1_P(M) \to 0$$

and for every $i > 0$ a natural isomorphism

$$\sum_\nu H^i(\operatorname{Proj} R, \tilde{M}(\nu)) \cong H^{i+1}_P(M),$$

where the sums extend over all positive and negative integers.

One reason that ordinary cohomology is so useful is that each of the $H^i(\operatorname{Proj} R, \tilde{M}(\nu))$ is a finite-dimensional vector space over the field R_0. The local cohomology modules, being infinite-direct sums of these, are not in general finite-dimensional, and in the case where R is local rather than graded, and P is the maximal ideal, they do not break up into such convenient finite pieces. However, if M is finitely generated one can show directly that the $H^i_P(M)$ are at least Artinian modules. (*Reason:* The ith step in an injective resolution of M consists of a direct sum of injective envelopes $E(R/Q)$ of modules R/Q, with Q a prime ideal, and only finitely many of each $E(R/Q)$ occur. Applying the zeroeth local cohomology functor to one of these $E(R/Q)$ gives 0 unless $Q = P$, in which case it gives $E(R/P)$, so the local cohomology is the homology of a complex of finite direct sums of copies of $E(R/P)$. Since each of these is an Artinian module, the local cohomology is too.)

A4.2 Local Duality

One of the most important results about cohomology is the duality theorem, which for a sheaf $\mathcal{F}$ on a d-dimensional projective space X says

$$H^i(X, \mathcal{F}) \cong \mathrm{Ext}_X^{d-i}(\mathcal{F}, \mathcal{O}_X(-d-1))^*,$$

where * represents Hom into the ground field. Of course if $\mathcal{F}$ is invertible, this degenerates to the more familiar

$$H^i(X, \mathcal{F}) \cong H^{d-i}(X, \mathcal{F}^{-1} \otimes \omega_X)^*.$$

The local form of this is:

Theorem A4.2. *If (S, Q) is a regular local ring of dimension d, and M is a finitely generated R-module, then*

$$H^i_Q(M) \cong \mathrm{Ext}_S^{d-i}(M, R)^*,$$

where * denotes the duality functor $\mathrm{Hom}_R(-, E(R/P))$.

If (R, P) is a factor ring of a regular local ring ring (S, Q), and M is an R-module, then just as in the case of ordinary cohomology it is easy to see that the corresponding local cohomology modules agree:

$$H^i_Q(M) = H^i_P(M),$$

so the theorem reduces all local cohomology questions to questions about Ext modules, at least for rings which are factor rings of regular local rings (virtually every ring of geometrical interest).

A4.3 Depth and Dimension

In particular, one can deduce from Theorem A4.2, Theorem 18.20, and the Auslander-Buchsbaum formula that the functor $\mathrm{Ext}_R^*(-, R)$, and thus also the local cohomology, measures both the depth and the dimension of a module.

Theorem A4.3. *Let (R, P) be a local ring, and let M be a finitely generated R-module. Let $d = \dim M$, and let $\delta = \mathrm{depth}(P, M)$. We have:*

a. $H^i_P(M) = 0$ *for* $i < \delta$ *and for* $i > d$.

b. $H^i_P(M) \neq 0$ *for* $i = \delta$ *and for* $i = d$.

Of course it follows that $\delta \leq d$; in the case where M is a factor ring, this is a consequence of Proposition 18.2, and it can be proved in the same way in general.

Exercise A4.1:

a: Let (R, P) be a local ring such that R_Q is Cohen-Macaulay for every prime ideal $Q \neq P$. Show that $H^i_P(R)$ has finite length for every $i < \dim R$.

b: Let $R = \oplus_{d \geq 0} R_d$ be a Noetherian positively graded ring and suppose that R_0 is a field. Suppose that R_Q is Cohen-Macaulay for each homogeneous prime $Q \neq P$. Show that $R_{Q'}$ is Cohen-Macaulay for every prime Q' of R, homogeneous or not. Then show that $H^i_P(R)$ has finite length for every $i < \dim R$.

Exercise A4.2: Let $R = \oplus_{d \geq 0} R_d$ be a Noetherian positively graded ring and suppose that R_0 is a field. Let $P = \oplus_{d > 0} R_d$ be the maximal homogeneous ideal. Let $R_{(e)} = \oplus_{d \geq 0} R_{de}$ be the eth Veronese subring of R, and let $P_{(e)}$ be its maximal homogeneous ideal.

a: Show that
$$H^i_{P_{(e)}}(R_{(e)}) = H^i_P(R)_{(e)} := \oplus_d H^i_P(R)_{de}.$$

b: Deduce from Exercises A4.1b and A4.2a that if R_Q is Cohen-Macaulay for each homogeneous prime $Q \neq P$, then $H^i_{P_{(e)}}(R_{(e)})$ is concentrated in degree 0 (that is $H^i_{P_{(e)}}(R_{(e)})_d = 0$ for $d \neq 0$) for all sufficiently large integers e and all $i < \dim R$.

Exercise A4.3: A local ring (R, P) is said to be **Buchsbaum** if the natural map $\operatorname{Ext}^i_R(R/P, R) \to \lim_{d\to\infty} \operatorname{Ext}^i_R(R/P^d, R) = H^i_P(R)$ is an isomorphism for every $i < \dim R$. It turns out that this somewhat unappetizing definition leads to a rich and surprising theory; see Stückrad and Vogel [1986]. Show that a sufficiently high Veronese embedding of any projective scheme has Buchsbaum homogeneous coordinate ring as follows. Let $R = \oplus_{d \geq 0} R_d$ be a Noetherian positively graded ring and suppose that R_0 is a field. Suppose that $H^i_P(R)$ is concentrated in degree 0 for $i < d$ as in Exercise A4.2b. Show that R is Buchsbaum.