CHAPTER 1
Some Homological Algebra

0 Review of Chain Complexes

We collect here for ease of reference some terminology and results concerning chain complexes. Much of this will be well-known to anyone who has studied algebraic topology. The reader is advised to skip this section (or skim it lightly) and refer back to it as necessary. We will omit some of the proofs; these are either easy or else can be found in standard texts, such as Dold [1972], Spanier [1966], or MacLane [1963].

Let R be an arbitrary ring. By a *graded R-module* we mean a sequence $C = (C_n)_{n \in \mathbb{Z}}$ of R-modules. If $x \in C_n$, then we say x has *degree n* and we write $\deg x = n$. A *map of degree p* from a graded R-module C to a graded R-module C' is a family $f = (f_n : C_n \to C'_{n+p})_{n \in \mathbb{Z}}$ of R-module homomorphisms; thus $\deg(f(x)) = \deg f + \deg x$. A *chain complex* over R is a pair (C, d) where C is a graded R-module and $d: C \to C$ is a map of degree -1 such that $d^2 = 0$. The map d is called the *differential* or *boundary operator* of C. We often suppress d from the notation and simply say that C is a chain complex. We define the *cycles* $Z(C)$, *boundaries* $B(C)$, and *homology* $H(C)$ by $Z(C) = \ker d$, $B(C) = \operatorname{im} d$, and $H(C) = Z(C)/B(C)$. These are all graded modules.

One often comes across graded modules C with an endomorphism d of square zero such that d has degree $+1$ instead of -1. In this case it is customary to use superscripts instead of subscripts to denote the grading, so that $C = (C^n)_{n \in \mathbb{Z}}$ and $d = (d^n : C^n \to C^{n+1})$. Such a pair (C, d) is called a *cochain complex*. There is no essential difference between chain complexes and cochain complexes, since we can always convert one to the other by setting $C_n = C^{-n}$. We will therefore confine ourselves, for the most part, to discussing chain complexes, it being understood that everything applies to cochain complexes by reindexing as above. [Note, however, that there *is* a difference when we consider *non-negative* complexes, i.e., complexes such that C_n

[or C^n] $= 0$ for $n < 0$; if the differential is thought of as going from left to right, then a non-negative chain complex extends indefinitely to the left, whereas a non-negative cochain complex extends indefinitely to the right.] In discussing cochain complexes, one often prefixes "co" to much of the terminology; thus d may be called a coboundary operator, and we have cocycles $Z(C)$, coboundaries $B(C)$, and cohomology $H(C) = (H^n(C))_{n \in \mathbb{Z}}$.

If (C, d) and (C', d') are chain complexes, then a *chain map* from C to C' is a graded module homomorphism $f: C \to C'$ of degree 0 such that $d'f = fd$. A *homotopy h* from a chain map f to a chain map g is a graded module homomorphism $h: C \to C'$ of degree 1 such that $d'h + hd = f - g$. We write $f \simeq g$ and say that f is *homotopic* to g if there is a homotopy from f to g.

(0.1) Proposition. *A chain map $f: C \to C'$ induces a map $H(f): H(C) \to H(C')$, and $H(f) = H(g)$ iff $f \simeq g$.* □

The abelian group of homotopy classes of chain maps $C \to C'$ will be denoted $[C, C']$. It is often useful to interpret $[C, C']$ as the 0-th homology group of a certain "function complex" $\mathscr{H}om_R(C, C')$, defined as follows: $\mathscr{H}om_R(C, C')_n$ is the set of graded module homomorphisms of degree n from C to C' [thus $\mathscr{H}om_R(C, C')_n = \prod_{q \in \mathbb{Z}} \operatorname{Hom}_R(C_q, C'_{q+n})$], and the boundary operator $D_n : \mathscr{H}om_R(C, C')_n \to \mathscr{H}om_R(C, C')_{n-1}$ is defined by $D_n(f) = d'f - (-1)^n fd$. [The sign here makes $D^2 = 0$. It is also consistent with other standard sign conventions, cf. exercise 3 below.] Note that the 0-cycles are precisely the chain maps $C \to C'$, and the 0-boundaries are the null-homotopic chain maps. Thus $H_0(\mathscr{H}om_R(C, C')) = [C, C']$. More generally, there is an interpretation of $H_n(\mathscr{H}om_R(C, C'))$ in terms of chain maps. Consider the complex $(\Sigma^n C, \Sigma^n d)$ defined by $(\Sigma^n C)_p = C_{p-n}$, $\Sigma^n d = (-1)^n d$; this complex is called the *n-fold suspension* of C. [If $n = 1$, we write ΣC instead of $\Sigma^1 C$.] Let $[C, C']_n = [\Sigma^n C, C']$. Then we have $H_n(\mathscr{H}om_R(C, C')) = [C, C']_n$. The elements of $[\ ,\ ]_n$ are called *homotopy classes of chain maps of degree n*.

A chain map $f: C \to C'$ is called a *homotopy equivalence* if there is a chain map $f': C' \to C$ such that $f'f \simeq \mathrm{id}_C$ and $ff' \simeq \mathrm{id}_{C'}$. And a chain map f is called a *weak equivalence* if $H(f): H(C) \to H(C')$ is an isomorphism.

(0.2) Proposition. *Any homotopy equivalence is a weak equivalence.* □

A chain complex C is called *contractible* if it is homotopy equivalent to the zero complex, or, equivalently, if $\mathrm{id}_C \simeq 0$. A homotopy from id_C to 0 is called a *contracting homotopy*. Any contractible chain complex is *acyclic*, i.e., $H(C) = 0$.

(0.3) Proposition. *C is contractible if and only if it is acyclic and each short exact sequence $0 \to Z_{n+1} \hookrightarrow C_{n+1} \xrightarrow{\bar{d}} Z_n \to 0$ splits, where $\bar{d}$ is induced by d.*

Proof. If h is a contracting homotopy, then $(h|Z): Z \to C$ splits the surjection $\bar{d}: C \to Z$. Conversely, suppose we have a splitting $s: Z \to C$, whence a graded module decomposition $C = \ker \partial \oplus \operatorname{im} s = Z \oplus \operatorname{im} s$. We then get a contracting homotopy $h: C \to C$ by setting $h|Z = s$ and $h|\operatorname{im} s = 0$. □

(0.4) Proposition. *A short exact sequence $0 \to C' \xrightarrow{i} C \xrightarrow{\pi} C'' \to 0$ of chain complexes gives rise to a long exact sequence in homology:*

$$\cdots \to H_n(C') \xrightarrow{H(i)} H_n(C) \xrightarrow{H(\pi)} H_n(C'') \xrightarrow{\partial} H_{n-1}(C') \to \cdots.$$

The "connecting homomorphism" ∂ is natural, in the sense that a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & E'' & \longrightarrow & 0 \end{array}$$

with exact rows yields a commutative square

$$\begin{array}{ccc} H_n(C'') & \longrightarrow & H_{n-1}(C') \\ \downarrow & & \downarrow \\ H_n(E'') & \longrightarrow & H_{n-1}(E'). \end{array}$$

□

(0.5) Corollary. *The inclusion $i: C' \to C$ is a weak equivalence if and only if C'' is acyclic.* □

This shows that the cokernel C'' of i is the appropriate object to consider if we want to measure the "difference" between $H(C)$ and $H(C')$. We now wish to define a "homotopy-theoretic" cokernel for an arbitrary chain map $f: C' \to C$, which plays the same role as the cokernel in the case of an inclusion: The *mapping cone* of $f: (C', d') \to (C, d)$ is defined to be the complex (C'', d'') with $C'' = C \oplus \Sigma C'$ (as a graded module) and $d''(c, c') = (dc + fc', -d'c')$. In matrix notation, we have

$$d'' = \begin{pmatrix} d & f \\ 0 & \Sigma d' \end{pmatrix}.$$

See exercise 2 below for the motivation for this definition.

(0.6) Proposition. *Let $f: C' \to C$ be a chain map with mapping cone C''. There is a long exact homology sequence*

$$\cdots \to H_n(C') \xrightarrow{H(f)} H_n(C) \to H_n(C'') \to H_{n-1}(C') \to \cdots.$$

In particular, f is a weak equivalence if and only if C'' is acyclic.

Proof. There is a short exact sequence $0 \to C \to C'' \to \Sigma C' \to 0$; now apply (0.4). By checking the definition of the connecting homomorphism $H_n(\Sigma C') \to H_{n-1}(C)$, one finds that it equals $H_{n-1}(f): H_{n-1}(C') \to H_{n-1}(C)$. □

The mapping cone is also useful for studying homotopy equivalences, not just weak equivalences:

(0.7) Proposition. *A chain map $f: C' \to C$ is a homotopy equivalence if and only if its mapping cone C'' is contractible.*

Proof. A straightforward computational proof can be found in the standard references (or can be supplied by the reader). For the sake of variety, we will sketch a conceptual proof. Suppose first that C'' is contractible. One then checks easily that the function complex $\mathscr{H}om_R(D, C'')$ is contractible for any complex D; in particular, it is acyclic. One also checks that $\mathscr{H}om_R(D, C'')$ is isomorphic to the mapping cone of $\mathscr{H}om_R(D, f): \mathscr{H}om_R(D, C') \to \mathscr{H}om_R(D, C)$. It therefore follows from (0.6) that $\mathscr{H}om_R(D, f)$ is a weak equivalence. Looking at H_0, we deduce that f induces an isomorphism $[D, C'] \to [D, C]$ for any D; hence f is a homotopy equivalence by a standard argument. Conversely, suppose f is a homotopy equivalence. Then one shows easily that $\mathscr{H}om_R(D, f): \mathscr{H}om_R(D, C') \to \mathscr{H}om_R(D, C)$ is a homotopy equivalence, so its mapping cone $\mathscr{H}om_R(D, C'')$ is acyclic by 0.6. In particular, $[D, C''] = 0$ for any D, and this implies that C'' is contractible. □

Finally, we recall briefly the Künneth and universal coefficient theorems. If (C, d) (resp. (C', d')) is a chain complex of right (resp. left) R-modules, then we define their *tensor product* $C \otimes_R C'$ by $(C \otimes_R C')_n = \bigoplus_{p+q=n} C_p \otimes_R C'_q$, with differential D given by $D(c \otimes c') = dc \otimes c' + (-1)^{\deg c} c \otimes d'c'$ for $c \in C$, $c' \in C'$. The sign here can be remembered by means of the following *sign convention*: When something of degree p is moved past something of degree q, the sign $(-1)^{pq}$ is introduced. [In the present case, the differential, which is of degree -1, is moved past c, so we get the sign $(-1)^{-\deg c} = (-1)^{\deg c}$.] Note that $C \otimes_R C'$ is simply a complex of abelian groups for general R, but it is a complex of R-modules if R is commutative.

(0.8) Proposition (Künneth Formula). *Let R be a principal ideal domain and let C and C' be chain complexes such that C is dimension-wise free. There are natural exact sequences*

$$0 \to \bigoplus_{p \in \mathbf{Z}} H_p(C) \otimes_R H_{n-p}(C') \to H_n(C \otimes_R C')$$
$$\to \bigoplus_{p \in \mathbf{Z}} \operatorname{Tor}_1^R(H_p(C), H_{n-p-1}(C')) \to 0$$

and

$$0 \to \prod_{p \in \mathbf{Z}} \operatorname{Ext}_R^1(H_p(C), H_{p+n+1}(C')) \to H_n(\mathscr{H}om_R(C, C'))$$
$$\to \prod_{p \in \mathbf{Z}} \operatorname{Hom}_R(H_p(C), H_{p+n}(C')) \to 0,$$

and these sequences split. □

We will not recall the definitions of Tor and Ext at this point, since we will be defining them in much greater generality in §III.2.

An important special case of 0.8 is that where C' consists of a single module M, regarded as a complex concentrated in dimension 0 (i.e. $C'_0 = M$, $C'_n = 0$ for $n \neq 0$). In this case 0.8 is called the *universal coefficient theorem*, and the exact sequences take the following form:

$$0 \to H_n(C) \otimes_R M \to H_n(C \otimes_R M) \to \operatorname{Tor}_1^R(H_{n-1}(C), M) \to 0$$

and

$$0 \to \operatorname{Ext}_R^1(H_{n-1}(C), M) \to H^n(\mathcal{H}om_R(C, M)) \to \operatorname{Hom}_R(H_n(C), M) \to 0.$$

[Here we are following standard conventions in regarding $\mathcal{H}om_R(C, M)$ as a *co*chain complex, with $\mathcal{H}om_R(C, M)^n = \mathcal{H}om_R(C, M)_{-n} = \operatorname{Hom}_R(C_n, M)$; the last equality comes from the fact that the only non-zero component of a graded map $f: C \to M$ of degree $-n$ is $f_n: C_n \to M$.]

Exercises

1. Let $T: (R\text{-modules}) \to (S\text{-modules})$ be a covariant functor which takes the zero module to the zero module (or, equivalently, which takes zero maps to zero maps). For any chain complex (C, d) over R, there is then a chain complex (TC, Td) over S. If T is an exact functor, show that $H(TC) \approx TH(C)$. [Recall that T is *exact* if it carries exact sequences to exact sequences. It follows that T preserves injections, surjections, kernels, cokernels, etc.]

2. (Motivation for the definition of the mapping cone) Given a chain map $f: C' \to C$, the "homotopy theoretic cokernel" of f should fit into a diagram $C' \xrightarrow{f} C \xrightarrow{g} C''$ with $gf \simeq 0$. Thus C'' must receive a chain map g [of degree 0] from C and a homotopy h [of degree 1] from C'; this suggests setting $C'' = C \oplus \Sigma C'$ as a graded module, and taking g and h to be inclusions. Show that the definition of the boundary operator d'' is now forced on us by the requirement that g be a chain map and that h be a homotopy from gf to 0. [Using matrix notation, set

$$d'' = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

and note that

$$g = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Now write out the matrix equations $d''g = gd$ and $d''h + hd' = gf$ and solve for $\alpha, \beta, \gamma, \delta$.]

The remaining exercises are designed to illustrate various compatibility properties of our sign conventions in $\mathcal{H}om(-, -)$ and $- \otimes -$. Few readers will have the patience to do all of them, but you should at least do enough to convince yourself that all reasonable identities involving $\mathcal{H}om$ and $\otimes$ are true, provided one follows the sign convention stated above.

3. Given $u \in \mathcal{H}om_R(C, C')_p$ and $x \in C_q$, set $\langle u, x \rangle = u(x) \in C'_{p+q}$. Let d, d', and D be the differentials in C, C', and $\mathcal{H}om_R(C, C')$, respectively. Verify that $d'\langle u, x \rangle = \langle Du, x \rangle + (-1)^p \langle u, dx \rangle$. [In fact, this is nothing but a restatement of the definition of D, but it is a convenient form in which to remember that definition.] In other words, the *evaluation map* $\mathcal{H}om_R(C, C') \otimes C \to C'$, given by $u \otimes x \mapsto \langle u, x \rangle$, is a chain map.

4. Given $v \in \mathcal{H}om_R(C, C')_q$ and $u \in \mathcal{H}om_R(C', C'')_p$, their composite $u \circ v$ is in

$$\mathcal{H}om_R(C, C'')_{p+q}.$$

Verify that $D(u \circ v) = Du \circ v + (-1)^p u \circ Dv$, where D denotes the differential in any of the three $\mathcal{H}om$ complexes; in other words, composition of graded maps defines a chain map $\mathcal{H}om_R(C', C'') \otimes \mathcal{H}om_R(C, C') \to \mathcal{H}om_R(C, C'')$. [Hint: This and the remaining exercises are most conveniently done by taking the definition of D in the form given in exercise 3. From this point of view, one starts with the equation $\langle u \circ v, x \rangle = \langle u, \langle v, x \rangle \rangle$ $(x \in C)$, which is the definition of composition, and one applies d'' to both sides. Applying exercise 3 several times, one obtains

$$\langle D(u \circ v), x \rangle + (-1)^{p+q} \langle u \circ v, dx \rangle = \langle Du, \langle v, x \rangle \rangle + (-1)^p \langle u, \langle Dv, x \rangle \rangle + (-1)^{p+q} \langle u, \langle v, dx \rangle \rangle$$

which simplifies to

$$\langle D(u \circ v), x \rangle = \langle Du \circ v, x \rangle + (-1)^p \langle u \circ Dv, x \rangle,$$

as required.]

5. If C and C' are chain complexes over a commutative ring R, there is an isomorphism of graded modules $C \otimes_R C' \xrightarrow{\approx} C' \otimes_R C$ given by $c \otimes c' \mapsto (-1)^{\deg c \cdot \deg c'} c' \otimes c$. Prove that this isomorphism is a chain map.

6. Let C be a complex of right R-modules, C' a complex of left R-modules, and C'' a complex of $\mathbb{Z}$-modules.

(a) There is a canonical isomorphism of graded abelian groups

$$\varphi: \mathcal{H}om_{\mathbb{Z}}(C \otimes_R C', C'') \xrightarrow{\approx} \mathcal{H}om_R(C, \mathcal{H}om_{\mathbb{Z}}(C', C'')),$$

given by $\langle\langle \varphi(u), c \rangle, c' \rangle = \langle u, c \otimes c' \rangle$ for $u \in \mathcal{H}om_{\mathbb{Z}}(C \otimes_R C', C'')$, $c \in C$, $c' \in C'$. [Sketch of proof: An element of $\mathcal{H}om_{\mathbb{Z}}(C \otimes_R C', C'')_n$ is a family of $\mathbb{Z}$-module maps $C_p \otimes_R C'_q \to C''_{p+q+n}$. In view of the universal mapping property of the tensor product, this is the same as a family of R-module maps $C_p \to \operatorname{Hom}_{\mathbb{Z}}(C'_q, C''_{p+q+n})$, i.e., as a graded R-module map $C \to \mathcal{H}om_{\mathbb{Z}}(C', C'')$ of degree n. Note here that $\mathcal{H}om_{\mathbb{Z}}(C', C'')$ is indeed a complex of right R-modules via the left action of R on C' and the contravariance of Hom in the first variable: $(ur)(c') = u(rc')$ for $u \in \mathcal{H}om_{\mathbb{Z}}(C', C'')$, $r \in R$, $c' \in C'$.] Show that φ is a chain map. [Hint: Let d, d', and d'' be the differentials in C, C', and C'', and let D be the differential in any of the $\mathcal{H}om$ complexes under consideration. Apply d'' to both sides of the equation defining φ; using exercise 3 several times, you should obtain

$$\langle\langle D(\varphi(u)), c \rangle, c' \rangle + (-1)^p \langle\langle \varphi(u), dc \rangle, c' \rangle + (-1)^{p+q} \langle\langle \varphi(u), c \rangle, d'c' \rangle$$

on the left, where $p = \deg u$ and $q = \deg c$, and

$$\langle Du, c \otimes c' \rangle + (-1)^p \langle u, dc \otimes c' \rangle + (-1)^{p+q} \langle u, c \otimes d'c' \rangle$$

on the right. Remembering the definition of φ, you can conclude that

$$\langle\langle D(\varphi(u)), c\rangle, c'\rangle = \langle Du, c \otimes c'\rangle$$

and hence that $\langle\langle D(\varphi(u)), c\rangle, c'\rangle = \langle\langle\varphi(Du), c\rangle, c'\rangle$.]

(b) Deduce from (a) [or check directly] that a map $u: C \otimes_R C' \to C''$ of degree 0 is a chain map iff the corresponding map $C \to \mathscr{H}om_{\mathbb{Z}}(C', C'')$ is a chain map.

7. Let C and C' (resp. E and E') be complexes of right (resp. left) R-modules. Given $u \in \mathscr{H}om_R(C, C')$ and $v \in \mathscr{H}om_R(E, E')$, their *tensor product*

$$u \otimes v \in \mathscr{H}om_{\mathbb{Z}}(C \otimes_R E, C' \otimes_R E')$$

is defined by

$$\langle u \otimes v, c \otimes e\rangle = (-1)^{\deg v \cdot \deg c}\langle u, c\rangle \otimes \langle v, e\rangle \qquad \text{for} \quad c \in C, e \in E.$$

(a) Let D be the differential in any of the three $\mathscr{H}om$ complexes under consideration. Verify that $D(u \otimes v) = Du \otimes v + (-1)^{\deg u} u \otimes Dv$. In other words, the operation "tensor product of graded maps" defines a chain map

$$\mathscr{H}om_R(C, C') \otimes \mathscr{H}om_R(E, E') \to \mathscr{H}om_{\mathbb{Z}}(C \otimes_R E, C' \otimes_R E').$$

[Hint: Once again, you are advised to do this by differentiating both sides of the equation defining $u \otimes v$.]

(b) Deduce from (a) (or check directly) that if $u: C \to C'$ is a chain map (of degree 0), then there is a chain map $\mathscr{H}om_R(E, E') \to \mathscr{H}om_{\mathbb{Z}}(C \otimes_R E, C' \otimes_R E')$ given by $v \mapsto u \otimes v$.

(c) Deduce from (a) (or check directly) that the tensor product of chain maps is compatible with homotopy, in the following sense: Given chain maps and homotopies $u_0 \simeq u_1: C \to C'$ and $v_0 \simeq v_1: E \to E'$, one has $u_0 \otimes v_0 \simeq u_1 \otimes v_1: C \otimes_R E \to C' \otimes_R E'$. [Hint: u_0 and u_1 are homologous 0-cycles in $\mathscr{H}om_R(C, C')$, and similarly for v_0 and v_1; it now follows from the boundary formula in (a) that $u_0 \otimes v_0$ and $u_1 \otimes v_1$ are homologous 0-cycles in $\mathscr{H}om_{\mathbb{Z}}(C \otimes_R E, C' \otimes_R E')$.]

8. Prove that the tensor product operation of the previous exercise is compatible with composition of graded maps, i.e., that

$$(u \otimes v) \circ (u' \otimes v') = (-1)^{\deg v \cdot \deg u'}(u \circ u') \otimes (v \circ v')$$

whenever the composites are defined.

1 Free Resolutions

Let R be a ring (associative, with identity) and M a (left) R-module. A *resolution* of M is an exact sequence of R-modules

$$\cdots \to F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} M \to 0.$$

If each F_i is free, then this is called a *free resolution*. Free resolutions exist for any module M and can be constructed by an obvious step-by-step

procedure: Choose a surjection $\varepsilon: F_0 \to M$ with F_0 free, then choose a surjection $F_1 \to \ker \varepsilon$ with F_1 free, etc. Note that the initial segment $F_1 \to F_0 \to M \to 0$ of a free resolution is simply a *presentation* of M by generators and relations.

There are two useful ways to interpret a resolution in terms of chain complexes.[1] The first is to regard the resolution itself as an acyclic chain complex, with M in degree -1. We will refer to this as the *augmented chain complex* associated to the resolution. The second way is to consider the nonnegative chain complex $F = (F_i, \partial_i)_{i \geq 0}$ and to view $\varepsilon: F \to M$ as a chain map, where M is regarded as a chain complex concentrated in dimension 0. The exactness hypothesis, from this point of view, simply says that ε is a weak equivalence.

In case there is an integer n such that $F_i = 0$ for $i > n$, we will say that the resolution has *length* $\leq n$. In this case we will simply write

$$0 \to F_n \to \cdots \to F_0 \to M \to 0,$$

it being understood that the resolution continues to the left with zeroes.

Examples

1. A free module F admits the free resolution

$$0 \to F \xrightarrow{\text{id}} F \to 0$$

of length 0.

2. If $R = \mathbb{Z}$ (or any principal ideal domain) then submodules of a free module are free, hence any module M admits a free resolution

$$0 \to F_1 \to F_0 \to M \to 0$$

of length ≤ 1. For example, the $\mathbb{Z}$-module $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ admits the resolution

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}_2 \to 0.$$

3. Let M again be $\mathbb{Z}_2$, regarded now as a module over the polynomial ring $R = \mathbb{Z}[T]$, with T acting as 0 (i.e., $f(T)$ acts as multiplication by $f(0)$). Then M can be regarded as the quotient of R by the ideal generated by T and 2. Using unique factorization in R and the fact that T and 2 are relatively prime, one easily obtains the free resolution of length 2

$$0 \to R \xrightarrow{\partial_2} R \oplus R \xrightarrow{\partial_1} R \xrightarrow{\varepsilon} \mathbb{Z}_2 \to 0$$

where $\varepsilon(f) = f(0) \bmod 2$, and ∂_1 and ∂_2 are given by the matrices $(T\ 2)$ and $\binom{2}{-T}$, respectively.

4. Let $R = \mathbb{Z}[T]/(T^2 - 1)$ and let t be the image of T in R. Let M be the R-module $R/(t - 1)$. (Equivalently, $M = \mathbb{Z}$, with t acting as the identity.)

[1] See §0 for terminology and basic facts concerning chain complexes.

Since $T^2 - 1 = (T - 1)(T + 1)$, it is clear that an element of R is annihilated by $t - 1$ (resp. $t + 1$) if and only if it is divisible by $t + 1$ (resp. $t - 1$). One therefore has a free resolution

$$\cdots R \xrightarrow{t-1} R \xrightarrow{t+1} R \xrightarrow{t-1} R \to M \to 0.$$

Note that, unlike the previous examples, this resolution is of *infinite* length. We will see, in fact, that M admits no free resolution of finite length, cf. §II.3, exercise 2.

In this book we will be primarily interested in the case where R is a group ring, so we digress now to recall what a group ring is.

2 Group Rings

Let G be a group, written multiplicatively. Let $\mathbb{Z}G$ (or $\mathbb{Z}[G]$) be the free $\mathbb{Z}$-module generated by the elements of G. Thus an element of $\mathbb{Z}G$ is uniquely expressible in the form $\sum_{g \in G} a(g)g$, where $a(g) \in \mathbb{Z}$ and $a(g) = 0$ for almost all g. The multiplication in G extends uniquely to a $\mathbb{Z}$-bilinear product $\mathbb{Z}G \times \mathbb{Z}G \to \mathbb{Z}G$; this makes $\mathbb{Z}G$ a ring, called the *integral group ring* of G.

Note that G is a subgroup of the group $(\mathbb{Z}G)^*$ of units of $\mathbb{Z}G$ and that we have the following *universal mapping property*:

(2.1) Given a ring R and a group homomorphism $f: G \to R^*$, there is a unique extension of f to a ring homomorphism $\mathbb{Z}G \to R$. Thus we have the "adjunction formula"

$$\mathrm{Hom}_{(\mathrm{rings})}(\mathbb{Z}G, R) \approx \mathrm{Hom}_{(\mathrm{groups})}(G, R^*).$$

EXAMPLES

1. Let G be cyclic of order n and let t be a generator. Then the powers t^i $(0 \le i \le n - 1)$ form a $\mathbb{Z}$-basis for $\mathbb{Z}G$, and one has $t^n = 1$. It follows that $\mathbb{Z}G \approx \mathbb{Z}[T]/(T^n - 1)$.

2. If G is infinite cyclic with generator t, then $\mathbb{Z}G$ has a $\mathbb{Z}$-basis $\{t^i\}_{i \in \mathbb{Z}}$. Hence $\mathbb{Z}G$ can be identified with the ring (usually denoted $\mathbb{Z}[t, t^{-1}]$) of Laurent polynomials $\sum_{i \in \mathbb{Z}} a_i t^i$ $(a_i \in \mathbb{Z}, a_i = 0$ for almost all $i)$.

EXERCISES

1. For any group G we define the *augmentation map* to be the ring homomorphism $\varepsilon: \mathbb{Z}G \to \mathbb{Z}$ such that $\varepsilon(g) = 1$ for all $g \in G$. The kernel of ε is called the *augmentation ideal* of $\mathbb{Z}G$ and is denoted I or IG.

   (a) Show that the elements $g - 1$ $(g \in G, g \neq 1)$ form a basis for I as a $\mathbb{Z}$-module.

   (b) If S is a set of generators for G, show that the elements $s - 1$ $(s \in S)$ generate I as a left ideal.

   (c) Conversely, if S is a set of elements of G such that the elements $s - 1$ $(s \in S)$ generate I as a left ideal, show that S generates G. [Hint: The hypothesis on S implies that every element of I is a sum of elements of the form $g - g'$, where $g = g's^{\pm 1}$ for some $s \in S$. Writing $g - 1$ in this way, where $g \in G$ is arbitrary, deduce that there is a sequence $g_1, g_2, \ldots, g_n$ such that $g_1 = g$, $g_n = 1$, and $g_i = g_{i+1}s_i^{\pm 1}$ for some $s_i \in S$. See exercise 2 of §3 below for an alternative proof.]

   (d) Show that G is a finitely generated group if and only if I is finitely generated as a left ideal.

2. Let G be cyclic with generator t and let M be the G-module $\mathbb{Z}$, with t acting as the identity. (Equivalently, $M = \mathbb{Z}G/I = \mathbb{Z}G/(t - 1)$.) Write down a free resolution of M. [See §1, example 4, for the case where $|G| = 2$.]

3 G-modules

A (left) $\mathbb{Z}G$-module, also called a *G-module*, consists of an abelian group A together with a homomorphism from $\mathbb{Z}G$ to the ring of endomorphisms of A. In view of 2.1, such a ring homomorphism corresponds to a group homomorphism from G to the group of automorphisms of A. Thus a G-module is simply an abelian group A together with an action of G on A. For example, one has for any A the *trivial* module structure, with $ga = a$ for $g \in G$, $a \in A$. (Thus $ra = \varepsilon(r) \cdot a$ for $r \in \mathbb{Z}G$, where $\varepsilon: \mathbb{Z}G \to \mathbb{Z}$ is the augmentation map discussed in §2, exercise 1.)

One way of constructing G-modules is by linearizing permutation representations. More precisely, if X is a G-set (i.e., a set with G-action), then one forms the free abelian group $\mathbb{Z}X$ (also denoted $\mathbb{Z}[X]$) generated by X and one extends the action of G on X to a $\mathbb{Z}$-linear action of G on $\mathbb{Z}X$. The resulting G-module is called a *permutation module*. In particular, one has a permutation module $\mathbb{Z}[G/H]$ for every subgroup H of G, where G/H is the set of cosets gH and G acts on G/H by left translation.

The operation of disjoint union in the category of G-sets corresponds to the direct sum operation in the category of G-modules:

$$\mathbb{Z}[\textstyle\coprod X_i] = \bigoplus \mathbb{Z}X_i.$$

It follows that every permutation module $\mathbb{Z}X$ admits a decomposition

$$\mathbb{Z}X \approx \bigoplus \mathbb{Z}[G/G_x],$$

where x ranges over a set of representatives for the G-orbits in X and G_x is the isotropy subgroup of G at x. In particular, if X is a *free* G-set (i.e., if all isotropy groups are trivial), then $G/G_x = G$, and we obtain:

(3.1) Proposition. *Let X be a free G-set and let E be a set of representatives for the G-orbits in X. Then $\mathbb{Z}X$ is a free $\mathbb{Z}G$-module with basis E.*

Remark. If k is an arbitrary commutative ring, then one can form the *group algebra* kG of G over k. This is the free k-module generated by G, with the unique k-bilinear product extending the group multiplication on G. Everything we have done in this section generalizes in an obvious way to this situation. For example, a kG-module is simply a k-module A together with a (k-linear) action of G on A.

EXERCISES

1. Let H be a subgroup of G and let E be a set of representatives for the *right* cosets Hg. Show that $\mathbb{Z}G$, regarded as a left-module over its subring $\mathbb{Z}H$, is free with basis E.

2. Use permutation modules to give a non-computational solution of exercise 1c of §2. [Let H be the subgroup of G generated by S and consider $\mathbb{Z}[G/H]$. It has an element fixed by H and hence annihilated by I. But then the element is fixed by G.]

4 Resolutions of $\mathbb{Z}$ over $\mathbb{Z}G$ via Topology

In this section we will consider $\mathbb{Z}$ as a G-module with trivial G-action, and we will see that free resolutions of this module arise from free actions of G on contractible complexes.

By a *G-complex* we will mean a CW-complex X together with an action of G on X which permutes the cells. Thus we have for each $g \in G$ a homeomorphism $x \mapsto gx$ of X such that the image $g\sigma$ of any cell σ of X is again a cell. For example, if X is a simplicial complex on which G acts simplicially, then X is a G-complex.

If X is a G-complex then the action of G on X induces an action of G on the cellular chain complex $C_*(X)$, which thereby becomes a chain complex of G-modules. Moreover, the canonical augmentation $\varepsilon\colon C_0(X) \to \mathbb{Z}$ (defined by $\varepsilon(v) = 1$ for every 0-cell v of X) is a map of G-modules.

We will say that X is a *free* G-complex if the action of G freely permutes the cells of X (i.e., $g\sigma \neq \sigma$ for all σ if $g \neq 1$). In this case each chain module $C_n(X)$ has a $\mathbb{Z}$-basis which is freely permuted by G, hence by 3.1 $C_n(X)$ is a *free* $\mathbb{Z}G$-module with one basis element for every G-orbit of cells. (Note that to obtain a specific basis we must choose a representative cell from each orbit and we must choose an orientation of each such representative.)

Finally, if X is contractible, then $H_*(X) \approx H_*(\text{pt.})$; in other words, the sequence

$$\cdots \to C_n(X) \xrightarrow{\partial} C_{n-1}(X) \to \cdots \to C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

is exact. We have, therefore:

(4.1) Proposition. *Let X be a contractible free G-complex. Then the augmented cellular chain complex of X is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.*

The reader who has studied covering spaces has, of course, seen many examples of free G-complexes. Indeed, suppose $p\colon \tilde{Y} \to Y$ is a regular covering map with G as group of deck transformations. (See the appendix to this chapter for a review of regular covers.) If Y is a CW-complex, then it is an elementary fact that $\tilde{Y}$ inherits a CW-structure such that the G-action permutes the cells, cf. Schubert [1968], III.6.9. Explicitly, the open cells of $\tilde{Y}$ lying over an open cell σ of Y are simply the connected components of $p^{-1}\sigma$; these cells are permuted freely and transitively by G, and each is mapped homeomorphically onto σ under p. Thus $\tilde{Y}$ is a free G-complex and $C_* \tilde{Y}$ is a complex of free $\mathbb{Z}G$-modules with one basis element for each cell of Y.

In view of 4.1, it is natural now to consider CW-complexes Y satisfying the following three conditions:

(i) Y is connected.
(ii) $\pi_1 Y = G$.
(iii) The universal cover X of Y is contractible.

Such a Y is called an *Eilenberg–MacLane complex* of type $(G, 1)$, or simply a *$K(G, 1)$-complex*. Condition (ii) above is to be interpreted as meaning that we are given a basepoint $y \in Y$ and a specific isomorphism $\pi_1(Y, y) \xrightarrow{\approx} G$ by which we identify $\pi_1(Y, y)$ with G.

It is sometimes convenient to note that condition (iii) above can be replaced by:

(iii') $H_i X = 0$ for $i > 2$.

Indeed, we clearly have (iii) $\Rightarrow$ (iii'). Conversely, (iii') implies that X is *acyclic*, i.e., that $H_* X \approx H_*(\text{pt.})$; and it is shown in elementary homotopy theory that simply-connected acyclic CW-complexes are contractible. [The reader who is not familiar with this fact can simply take (i), (ii), and (iii') as the *definition* of a $K(G, 1)$-complex, since we will never make serious use of the fact that X is contractible. Note, for instance, that 4.1 remains true if "contractible" is replaced by "acyclic."]

It also follows from elementary homotopy theory that (iii) can be replaced by

(iii'') $\pi_i Y = 0$ for $i \geq 2$.

Thus the Eilenberg–MacLane complexes are precisely the *aspherical* complexes studied by Hurewicz (cf. Introduction). Once again, we will not need to use this fact.

If Y is a $K(G, 1)$ then the universal cover $p\colon X \to Y$ is a regular cover whose group is isomorphic to $\pi_1 Y = G$. [This depends on a choice of basepoint in X, cf. Appendix.] We therefore obtain from 4.1:

(4.2) Proposition. *If Y is a $K(G, 1)$ then the augmented cellular chain complex of the universal cover of Y is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.*

We now give one example. There will be many other examples in Chapter II and later in the book.

(4.3) EXAMPLE. Let $G = F(S)$, the free group generated by a set S. Let $Y = \bigvee_{s \in S} S^1_s$, a bouquet of circles S^1_s in 1-1 correspondence with S. Thus Y is a 1-dimensional CW-complex with exactly one vertex and with one 1-cell for each element of S. One knows that $\pi_1 Y \approx F(S)$. Moreover, Y is a $K(F(S), 1)$ since condition (iii') above holds for trivial reasons. [X is 1-dimensional.] As basepoint in X we take a vertex x_0; this then represents the unique G-orbit of vertices of X and hence generates the free $\mathbb{Z}G$-module $C_0(X)$. As basis for $C_1(X)$ we take, for each $s \in S$, an oriented 1-cell e_s of X which lies over S^1_s (traversed in the positive sense). Replacing e_s by ge_s for suitable $g \in G$, we can assume that the initial vertex of e_s is the basepoint x_0; the endpoint of e_s is then sx_0 (cf. Appendix, A1) so that $\partial e_s = sx_0 - x_0 = (s - 1)x_0$. We obtain, therefore, the free resolution

(4.4) $$0 \to \mathbb{Z}G^{(S)} \xrightarrow{\partial} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where $\mathbb{Z}G^{(S)}$ is a free $\mathbb{Z}G$-module with basis $(e_s)_{s \in S}$, $\partial e_s = s - 1$, and $\varepsilon(g) = 1$ for all $g \in G$.

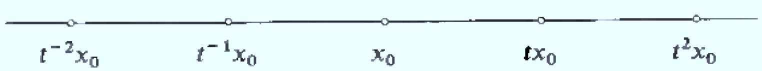
Remarks

1. Note that ε coincides with the natural augmentation of the group ring $\mathbb{Z}G$, as defined in §2, exercise 1. The exactness of 4.4 says, therefore, that the augmentation ideal of $\mathbb{Z}[F(S)]$ is a free left $\mathbb{Z}[F(S)]$-module with basis $(s - 1)_{s \in S}$. [We will later indicate a purely algebraic proof of this fact, cf. §IV.2, exercise 3b.] The reader who is not used to working with modules over non-commutative rings may find it surprising that $\mathbb{Z}G$, the free module of rank 1, can contain a submodule which is free of rank > 1; this sort of thing cannot happen over a *commutative* ring R, since then any two elements $a, b \in R$ are linearly dependent: $(-b) \cdot a + a \cdot b = 0$.

2. If S contains a single element t (i.e., G is infinite cyclic) then (4.4) reduces to the "obvious" resolution which the reader probably wrote down in §2, exercise 2:

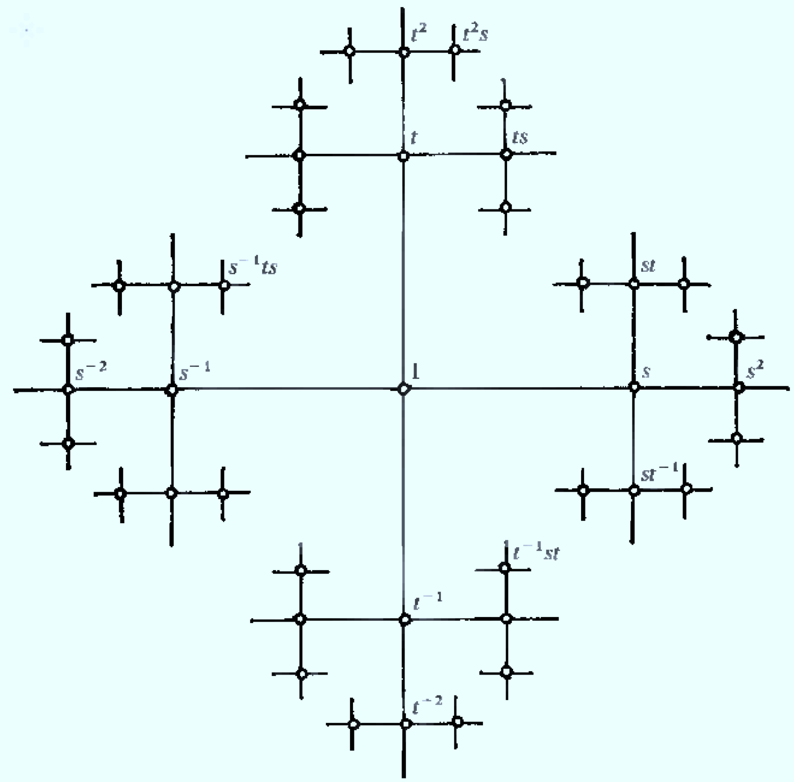
(4.5) $$0 \to \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0.$$

Note that X, in this case, is simply $\mathbb{R}^1$, with t acting by $x \mapsto x + 1$:



3. The contractible free $F(S)$-complex X, and hence the resolution 4.4, can easily be constructed directly, without appeal to the theory of covering complexes, or to the fact that $\pi_1(\bigvee S^1) \approx F(S)$. Namely, X can be defined as the 1-dimensional simplicial complex whose vertices are the elements of $G = F(S)$ and whose 1-simplices are the pairs $\{g, gs\}$ $(g \in G, s \in S)$. The action of G on itself by left translation induces a simplicial action of G on X, which is easily seen to be free. Finally, one can construct an explicit contracting homotopy which contracts X to the vertex 1 along paths of the form

where $g = s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n}$ is a reduced word of length n ($s_i \in S$, $\varepsilon_i = \pm 1$, $\varepsilon_i = \varepsilon_{i+1}$ if $s_i = s_{i+1}$) and $g_i = s_1^{\varepsilon_1} \cdots s_i^{\varepsilon_i}$ $(0 \le i \le n)$. In case S is a two-element set $\{s, t\}$, for example, X is the tree pictured below:



EXERCISES

1. If X is an arbitrary G-complex, is $C_n(X)$ necessarily a permutation module?

*2. Let X be a free G-complex, where G is an arbitrary group. Show that every point of X has a neighborhood U such that $gU \cap U = \varnothing$ for all $g \neq 1$ in G. Deduce that $X \to X/G$ is a regular covering map with G as group of covering transformations. In particular, if X is contractible then X/G is a $K(G, 1)$ and X is its universal cover.

[Hint: A CW-complex X, with explicitly given characteristic maps $(B^n, S^{n-1}) \to (\bar{\sigma}, \partial\sigma)$ for its cells, admits a canonical open cover $\{U_\sigma\}$ indexed by the cells, such that $U_\sigma \cap U_\tau = \emptyset$ if σ and τ are distinct cells of the same dimension. If X is simplicial, for example, we can take U_σ to be the open star of the barycenter of σ in the barycentric subdivision of X.]

3. Let G be a free abelian group of rank 2. Use the methods of this section to construct a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

*4. (This exercise requires some homotopy theory.)

(a) For any group G, show that there exists a $K(G, 1)$. [Start with a connected 2-complex with $\pi_1 = G$; then attach cells to kill the higher homotopy.]

(b) Show that the construction in (a) can be made functorial in G. [Given the $(n-1)$-skeleton Y^{n-1}, attach an n-cell for every possible map $S^{n-1} \to Y^{n-1}$.]

5 The Standard Resolution

Let X be a contractible simplicial complex on which a group G acts simplicially. It may happen that the G-action is free on the vertices but not on the higher-dimensional simplices. [Note: This cannot happen if G is torsion-free.] In this case we can still obtain a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ by using, instead of the usual chain complex $C_*(X)$, the *ordered* chain complex $C'_*(X)$ (cf. Spanier [1966], ch. 4, §3). Thus $C'_n(X)$ has a $\mathbb{Z}$-basis consisting of the ordered $(n+1)$-tuples $(v_0, \ldots, v_n)$ of vertices of X such that $\{v_0, \ldots, v_n\}$ is a simplex of X (of dimension $<n$ if the v_i are not all distinct). Since G clearly acts freely on these $(n+1)$-tuples, we obtain a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

The most obvious example of such an X is the "simplex" spanned by G; i.e., the vertices of X are the elements of G (with G acting by left translation), and every finite subset of G is a simplex of X. [If G is finite, this really is a simplex; otherwise it is an infinite dimensional analogue. In any case it is contractible by a straight-line contracting homotopy.] The corresponding free resolution $F_* = C'_*(X)$ is called the *standard resolution* of $\mathbb{Z}$ over $\mathbb{Z}G$. Explicitly, F_n is the free $\mathbb{Z}$-module generated by the $(n+1)$-tuples $(g_0, \ldots, g_n)$ of elements of G, with the G-action given by $g \cdot (g_0, \ldots, g_n) = (gg_0, \ldots, gg_n)$. The boundary operator $\partial: F_n \to F_{n-1}$ is given by $\partial = \sum_{i=0}^{n} (-1)^i d_i$, where

(5.1) $$d_i(g_0, \ldots, g_n) = (g_0, \ldots, \hat{g}_i, \ldots, g_n).$$

The augmentation $\varepsilon: F_0 \to \mathbb{Z}$ is given by $\varepsilon(g_0) = 1$. Note that the acyclicity (i.e., exactness) of this augmented complex has been deduced from the contractibility of X, but one can also verify it directly by writing down a contracting homotopy $h: F_n \to F_{n+1}$ for the underlying augmented complex of $\mathbb{Z}$-modules (i.e., h will *not* be a map of G-modules). Namely, we define h by $h(g_0, \ldots, g_n) = (1, g_0, \ldots, g_n)$ if $n \geq 0$ and $h(1) = (1)$ if $n = -1$.

As basis for the free $\mathbb{Z}G$-module F_n we may take the $(n+1)$-tuples whose first element is 1, since these represent the G-orbits of $(n+1)$-tuples. It is often convenient to write such an $(n+1)$-tuple in the form $(1, g_1, g_1g_2, \ldots, g_1g_2 \cdots g_n)$ and to introduce the "bar notation"

$$[g_1|g_2|\cdots|g_n] = (1, g_1, g_1g_2, \ldots, g_1g_2\cdots g_n).$$

(If $n = 0$ there is only one such basis element, denoted $[\ \ ]$; if we identify F_0 with $\mathbb{Z}G$ in the obvious way, then $[\ \ ] = 1$.) It is easy to compute $\partial: F_n \to F_{n-1}$ in terms of this $\mathbb{Z}G$-basis $\{[g_1|\cdots|g_n]\}$; one finds $\partial = \sum_{i=0}^{n} (-1)^i d_i$, where d_i is the $\mathbb{Z}G$-homomorphism given by

(5.2) $$d_i[g_1|\cdots|g_n] = \begin{cases} g_1[g_2|\cdots|g_n] & i = 0 \\ [g_1|\cdots|g_{i-1}|g_ig_{i+1}|g_{i+2}|\cdots|g_n] & 0 < i < n \\ [g_1|\cdots|g_{n-1}] & i = n. \end{cases}$$

This standard resolution F_* is often called the *bar resolution*. In low dimensions it has the form

$$F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

where $\partial_2([g|h]) = g[h] - [gh] + [g]$, $\partial_1([g]) = g[\ \ ] - [\ \ ] = g - 1$, and $\varepsilon(1) = 1$.

Finally, we mention the *normalized* standard (or bar) resolution $\bar{F}_* = F_*/D_*$, where D_*, the "degenerate" subcomplex of F_*, is generated by the elements $(g_0, \ldots, g_n)$ such that $g_i = g_{i+1}$ for some i. In terms of the bar notation, D_* can be described as the G-subcomplex of F_* generated by the elements $[g_1|\cdots|g_n]$ such that $g_i = 1$ for some i. Thus $\bar{F}_n$ is a free $\mathbb{Z}G$-module with one basis element (still denoted $[g_1|\cdots|g_n]$) for every n-tuple of *non-trivial* elements of G. The fact that $\bar{F}_*$ is acyclic over $\mathbb{Z}$, and hence a resolution, follows from general facts about normalization (cf. MacLane [1963], VIII.6); alternatively, one can simply observe that the contracting homotopy h defined above carries D_* into itself and hence induces a contracting homotopy of the quotient $\bar{F}_*$.

Exercises

1. Write down the homotopy operator h in terms of the $\mathbb{Z}$-basis $g[g_1|\cdots|g_n]$ for F_n.

2. Write out the normalized bar resolution in case G is the group of order 2; compare it with the resolution given in §1, example 4.

*3. (a) Show that the standard resolution is the cellular chain complex of a free contractible G-complex X. Note that this reproves the result of exercises 4a and 4b of §4. [For each $(n+1)$-tuple $\sigma = (g_0, \ldots, g_n)$, let Δ_σ be a copy of the standard n-simplex with vertices $v_0, \ldots, v_n$. Let $d_i\sigma = (g_0, \ldots, \hat{g}_i, \ldots, g_n)$ and let $\delta_i: \Delta_{d_i\sigma} \to \Delta_\sigma$ be the linear embedding which sends $v_0, \ldots, v_{n-1}$ to $v_0, \ldots, \hat{v}_i, \ldots, v_n$. Form the disjoint union $X_0 = \coprod_\sigma \Delta_\sigma$ and obtain X by identifying $\Delta_{d_i\sigma}$ with its image under δ_i

for all σ and all i. Use the quotient map $C(X_0) \to C(X)$ to compute the boundary operator on $C(X)$ and see that $C(X) \approx F$. To prove X contractible, consider $h\sigma = (1, g_0, \ldots, g_n)$ for each σ, and use the straight-line homotopy between δ_0: $\Delta_\sigma \to \Delta_{h\sigma}$ and the constant map $\Delta_\sigma \to \Delta_{h\sigma}$ at v_0.]

(b) Do the same for the normalized standard resolution. [Use the same method as in (a), but make further identifications in X_0 to collapse degenerate simplices. Explicitly, for each $\sigma = (g_0, \ldots, g_n)$ let $s_i\sigma = (g_0, \ldots, g_i, g_i, \ldots, g_n)$ and collapse $\Delta_{s_i\sigma}$ to Δ_σ via the linear map which sends $v_0, \ldots, v_{n+1}$ to $v_0, \ldots, v_i, v_i, \ldots, v_n$.]

6 Periodic Resolutions via Free Actions on Spheres

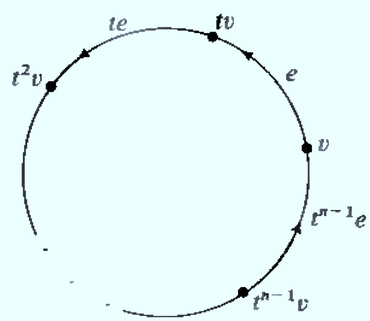
Let X be a free G-complex which is homeomorphic to an odd dimensional sphere S^{2k-1}. (G is then necessarily finite since X is compact.) By an easy special case of the Lefschetz fixed-point theorem (cf. exercise 1 below), G acts trivially on $H_{2k-1}X = \mathbb{Z}$. Writing $C_* = C_*(X)$, we have then an exact sequence of G-modules

(6.1) $$0 \to \mathbb{Z} \xrightarrow{\eta} C_{2k-1} \to \cdots \to C_1 \to C_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

where each C_i is free. Splicing together an infinite number of copies of 6.1, we obtain a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ which is periodic of period $2k$:

(6.2) $$\cdots \to C_1 \to C_0 \xrightarrow{\eta\varepsilon} C_{2k-1} \to \cdots \to C_1 \to C_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0.$$

Example. Let G be a finite cyclic group of order n with generator t. Then G acts freely as a group of rotations of S^1, regarded as a CW-complex with n vertices and n 1-cells:



Note that H_1S^1 is generated by the cycle $e + te + t^2e + \cdots + t^{n-1}e = Ne$, where N is the "norm element" $1 + t + \cdots + t^{n-1}$ of $\mathbb{Z}G$, so 6.1 takes the form

$$0 \to \mathbb{Z} \xrightarrow{\eta} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

where $\varepsilon(1) = 1$ and $\eta(1) = N$. We therefore obtain the following periodic resolution of period 2, which the reader has probably already seen (§2, exercise 2):

(6.3) $$\cdots \to \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0.$$

We will see other examples of groups with periodic resolutions in Chapter VI.

Exercises

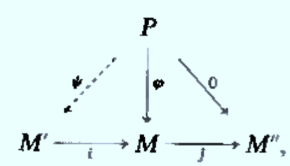
1. Prove the following special case of the Lefschetz fixed-point theorem: Let X be a finite CW-complex and $f: X \to X$ a map such that, for every cell σ,
$$f(\sigma) \subseteq \bigcup_{\substack{\tau \neq \sigma \\ \dim \tau \leq \dim \sigma}} \tau.$$
(In particular, f has no fixed-points.) Then the Lefschetz number $\sum (-1)^i \operatorname{trace} f_i$ is zero, where f_i is the endomorphism induced by f on the free abelian group H_iX/torsion. [Hint: The Lefschetz number can be computed on the chain level, where the matrix of f has zeroes on the diagonal.] If $H_*(X) \approx H_*(S^{2n-1})$, deduce that f_*: $H_{2n-1}X \to H_{2n-1}X$ must be the identity.

2. Prove that the group of order 2 is the only non-trivial group which can act freely on an even-dimensional sphere.

7 Uniqueness of Resolutions

We return to the generality of §1, i.e., we work over an arbitrary ring R. It is obvious that a given R-module M admits many different free resolutions; the purpose of this section is to show that all such resolutions are homotopy equivalent. In the course of proving this we will be faced with mapping problems which can be put in the form

(7.1)

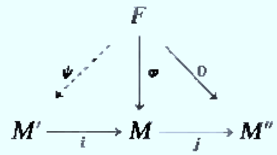


where the solid arrows represent given maps (with $j\varphi = 0$) and the dotted arrow represents a map we would like to construct. A solution to this problem, then, consists of a map $\psi: P \to M'$ such that $i\psi = \varphi$. A module P is called *projective* if a solution exists for every mapping problem 7.1 in which the row is exact. More concisely, P is projective if the functor $\mathrm{Hom}_R(P, -)$ is exact.

For our present purposes, the main interest in projectivity is provided by:

(7.2) Lemma. *Free modules are projective.*

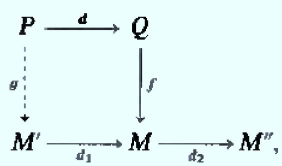
PROOF. Let F be free with basis (e_α), and consider a mapping problem



with exact row. Then $\varphi(e_\alpha) \in \ker j = \operatorname{im} i$, so we can choose $x_\alpha \in M'$ with $i(x_\alpha) = \varphi(e_\alpha)$. Now let ψ be the unique R-module map with $\psi(e_\alpha) = x_\alpha$. □

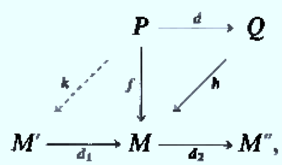
The next lemma treats the particular mapping problems which one has to solve in trying to construct chain maps and homotopies inductively:

(7.3) Lemma. (a) *Suppose given a diagram*



where $d_2 fd = 0$ and it is desired to find a g such that $d_1 g = fd$. If P is projective and the bottom row is exact, then such a g exists.

(b) *Suppose given a diagram (not necessarily commutative)*



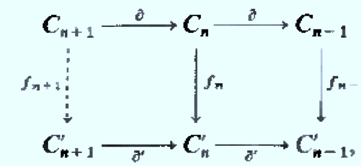
where $d_2 hd = d_2 f$ and it is desired to find a k such that $d_1 k + hd = f$. If P is projective and the bottom row is exact, then such a k exists.

PROOF. (a) is obvious, since the given mapping problem is of the form 7.1 with $\varphi = fd$. Similarly, (b) is a problem of the form 7.1 with $\varphi = f - hd$. □

We can now prove the "fundamental lemma of homological algebra," which says, roughly speaking, that it is easy to construct chain maps and homotopies from a projective complex to an acyclic one:

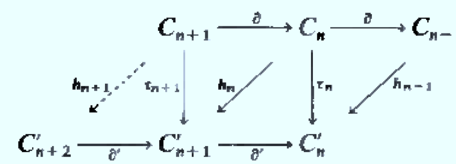
(7.4) Lemma. *Let (C, ∂) and (C', ∂') be chain complexes and let r be an integer. Let $(f_i: C_i \to C'_i)_{i \le r}$ be a family of maps such that $\partial'_i f_i = f_{i-1}\partial_i$ for $i \le r$. If C_i is projective for $i > r$ and $H_i(C') = 0$ for $i \ge r$, then $(f_i)_{i \le r}$ extends to a chain map $f: C \to C'$, and f is unique up to homotopy. More precisely, any two extensions are homotopic by a homotopy h such that $h_i = 0$ for $i \le r$.*

PROOF. Assume inductively that f_i has been defined for $i \le n$, where $n \ge r$, and that $\partial'_i f_i = f_{i-1}\partial_i$ for $i \le n$. We then have a mapping problem



where $\partial' f_n \partial = f_{n-1}\partial\partial = 0$. The desired f_{n+1} therefore exists by 7.3a.

Suppose now that g is a second extension of $(f_i)_{i \le r}$. We wish to find a homotopy h between f and g. Assume inductively that $h_i: C_i \to C'_{i+1}$ has been defined for $i \le n$, where $n \ge r$, and that $\partial' h_i + h_{i-1}\partial = f_i - g_i$. (Note that we can start the induction by setting $h_i = 0$ for $i \le r$.) Setting $\tau_i = f_i - g_i$, we then have a mapping problem



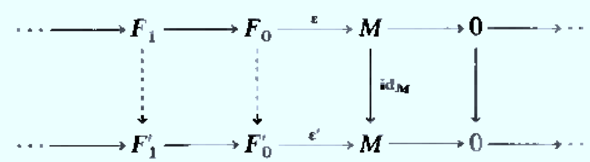
with

$$\begin{aligned} \partial' h_n \partial &= (\tau_n - h_{n-1}\partial)\partial && \text{by the inductive hypothesis} \\ &= \tau_n \partial && \text{since } \partial^2 = 0 \\ &= \partial' \tau_{n+1} && \text{since } \tau \text{ is a chain map.} \end{aligned}$$

The desired h_{n+1} with $\partial' h_{n+1} + h_n \partial = \tau_{n+1}$ therefore exists by 7.3b. □

Remark. The proof of 7.4 should have looked familiar to anyone who has seen the method of acyclic models in algebraic topology. We will explain this "similarity" in exercise 3 below.

Now let $\varepsilon: F \to M$ and $\varepsilon': F' \to M$ be two free (or projective) resolutions of a module M. We can then form the augmented chain complexes with M in dimension -1 and apply 7.4 with $r = -1$:



We conclude that there is a chain map $f: F \to F'$ which is *augmentation-preserving*, i.e., which satisfies $\varepsilon' f = \varepsilon$. Moreover, f is unique up to homotopy. (Note that the homotopy h given by 7.4 on the level of *augmented* chain complexes yields a homotopy $F \to F'$, because $h_{-1} = 0$.) Similarly, there is an augmentation-preserving map $f': F' \to F$, and we have $f'f \simeq \mathrm{id}_F$ and $ff' \simeq \mathrm{id}_{F'}$, again by the uniqueness part of 7.4. This proves:

(7.5) Theorem. *Given projective resolutions F and F' of a module M, there is an augmentation-preserving chain map $f: F \to F'$, unique up to homotopy, and f is a homotopy equivalence.*

We will express this informally by saying that projective resolutions are "unique up to canonical homotopy equivalence." At the moment, of course, we are mainly interested in *free* resolutions, but later (beginning in Chapter VIII) we will need to consider projective resolutions which are not known to be free.

For future reference we record two special cases of 7.5:

(7.6) Corollary. *Let $\varepsilon: F \to \mathbb{Z}$ be a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}$-module. Then ε is a homotopy equivalence.*

PROOF. $\mathrm{id}_{\mathbb{Z}}: \mathbb{Z} \to \mathbb{Z}$ is also a free resolution, and $\varepsilon: F \to \mathbb{Z}$ can be viewed as an augmentation-preserving chain map from one resolution to another. □

(7.7) Corollary. *If F is a non-negative acyclic chain complex of projective modules (over an arbitrary ring R), then F is contractible.*

PROOF. F and the zero complex are projective resolutions of 0, so they are homotopy equivalent. See also exercise 3 of §8 for an alternative proof. □

Remarks

1. In practice, a free resolution usually comes equipped with an explicit basis in each dimension, and one usually proves that it is a resolution by exhibiting a contracting homotopy for the underlying augmented chain complex of $\mathbb{Z}$-modules. In this case, the proof of 7.5 yields a specific map $f: F \to F'$. Namely, if k is a contracting homotopy for the augmented complex associated to F', then f is determined inductively by $f_{n+1}(e_\alpha) = k_n f_n \partial e_\alpha$, where (e_α) is a basis for F_{n+1}.

2. The uniqueness theorem 7.5 can be thought of as the algebraic analogue of Hurewicz's theorem, which we quoted in the introduction, asserting the uniqueness up to homotopy of an aspherical space with a given fundamental group.

EXERCISES

1. Let G be a finite cyclic group. Let F be the free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ given in 6.3 and let F' be the bar resolution. Write down an augmentation preserving chain map $f: F \to F'$.

2. The method of proof of 7.4 applies to many situations, some of which will arise later in this book. It is therefore useful to axiomatize 7.4, as follows. By an *additive category* we mean a category $\mathscr{A}$ in which $\mathrm{Hom}(A, B)$ is endowed with an abelian group structure for any two objects A, B, in such a way that the composition law $\mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$ is $\mathbb{Z}$-bilinear. In particular, since we have a zero element $0 \in \mathrm{Hom}(A, B)$ it is clear what we mean by a *chain complex* in $\mathscr{A}$. The usual definitions of *chain map* and *homotopy* also go through with no change. [Note that additivity is needed to define "homotopy."] Suppose now that we are given a class $\mathscr{E}$ of sequences $M' \to M \to M''$ in $\mathscr{A}$ called *exact sequences*. We then say that an object P of $\mathscr{A}$ is *projective* relative to $\mathscr{E}$ if every mapping problem 7.1 whose row is in $\mathscr{E}$ admits a solution. Verify that the proof of 7.4 is valid in this situation and leads to an analogue of 7.4 for chain complexes in $\mathscr{A}$. [Note: In the statement of this analogue, the "acyclicity" hypothesis on C' should be replaced by the assumption that $C'_{i+1} \to C'_i \to C'_{i-1}$ is in $\mathscr{E}$ for $i \geq r$.] Deduce an analogue of 7.5.

3. Let $\mathscr{C}$ be an arbitrary category and let $\mathscr{A}$ be the additive category whose objects are covariant functors $\mathscr{C} \to$ (abelian groups) and whose maps are natural transformations of functors. Fix a subclass $\mathscr{M}$ of the class of objects of $\mathscr{C}$. A sequence $T' \to T \to T''$ in $\mathscr{A}$ will be called *$\mathscr{M}$-exact* if the resulting sequence $T'(M) \to T(M) \to T''(M)$ of abelian groups is exact for all $M \in \mathscr{M}$. The purpose of this exercise is to show that when exercise 2 is applied to $\mathscr{A}$ (with $\mathscr{E}$ equal to the class of $\mathscr{M}$-exact sequences), the result is the acyclic models theorem as given, for instance, in Spanier [1966], Theorem 4.2.8, or Dold [1972], Proposition VI.11.7. The crucial step is to prove an analogue in $\mathscr{A}$ of Lemma 7.2; this is done in (a) and (b) below.

(a) (Yoneda's lemma) Let A be an object of $\mathscr{C}$ and let $h_A: \mathscr{C} \to$ (sets) be the covariant functor *represented* by A, i.e., $h_A = \mathrm{Hom}_{\mathscr{C}}(A, -)$. Let $u_A \in h_A(A)$ be the identity map $A \to A$. Let $T: \mathscr{C} \to$ (sets) be an arbitrary covariant functor. For any $v \in T(A)$, show that there is a unique natural transformation $\varphi: h_A \to T$ such that $\varphi(u_A) = v$. Thus $\mathrm{Hom}_{\mathscr{F}}(h_A, T) \approx T(A)$, where $\mathscr{F}$ is the category of functors $\mathscr{C} \to$ (sets). [This can be thought of as saying that h_A is "freely generated" by u_A. The proof is straightforward definition-checking. To prove uniqueness, for example, note that we must have $\varphi(f) = T(f)(v)$ for any $f \in h_A(B) = \mathrm{Hom}_{\mathscr{C}}(A, B)$, in view of the diagram

$$\begin{array}{ccc} h_A(A) & \xrightarrow{h_A(f)} & h_A(B) \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\ T(A) & \xrightarrow{T(f)} & T(B). \end{array}$$

To prove existence, take this equation as a definition and check that it works.]

(b) Let $\mathbb{Z}h_A: \mathscr{C} \to$ (abelian groups) be the composite of h with the functor (sets) $\to$ (abelian groups) which associates to a set the free abelian group it generates. A functor $F: \mathscr{C} \to$ (abelian groups) will be called *$\mathscr{M}$-free* if it is isomorphic to a direct sum $\bigoplus_\alpha \mathbb{Z}h_{A_\alpha}$, where $A_\alpha \in \mathscr{M}$ for all α. Deduce from (a) that $\mathscr{M}$-free functors are projective relative to the class of $\mathscr{M}$-exact sequences.

(c) Using (b) and exercise 2, state a theorem about chain maps in $\mathscr{A}$ from $\mathscr{M}$-free complexes to "$\mathscr{M}$-acyclic" complexes. [Note: This theorem is precisely the acyclic models theorem cited above.]

4. Another important example of exercise 2 is obtained by taking $\mathscr{A}$ to be the *dual* of the category of R-modules. Thus $\mathscr{A}$ has one object M° for every R-module M and one map $f^\circ: M_1^\circ \to M_2^\circ$ for every R-module map $f: M_2 \to M_1$. Composition is defined by $f^\circ g^\circ = (gf)^\circ$. As exact sequences in $\mathscr{A}$ we take those sequences $M_1^\circ \to M_2^\circ \to M_3^\circ$ such that the corresponding sequence $M_3 \to M_2 \to M_1$ of R-modules is exact. Applying exercise 2, we get an analogue of 7.4 for chain complexes in $\mathscr{A}$, which can obviously be restated as a theorem about cochain complexes of R-modules. Explicitly state this theorem, as well as the theorem corresponding to 7.5. [Note: An R-module Q is called *injective* if Q° is projective in $\mathscr{A}$, or, equivalently, if the functor $\mathrm{Hom}_R(-, Q)$ is exact. Your theorem should therefore be stated in terms of maps of an acyclic cochain complex into a cochain complex of injectives. To give this theorem substance, of course, we should have an analogue of 7.2, so that we will have examples of injectives. We will provide such an analogue later, in §III.4.]

8 Projective Modules

The reader may be curious at this point to know more about projective modules, other than the fact that free modules are projective. We give in this section, therefore, some miscellaneous results and examples concerning projective modules and complexes. We will not make serious use of these results (except as they apply to free modules) until Chapter VIII. One may therefore skip this section now and return to it later.

The first observation is that to prove a module P is projective one need only consider mapping problems 7.1 in which $M'' = 0$; for 7.1 can be replaced by

$$\begin{array}{ccccc} & & P & & \\ & \swarrow & \downarrow & & \\ M' & \longrightarrow & \ker j & \longrightarrow & 0. \end{array}$$

Thus we have:

(8.1) Proposition. *P is projective if and only if for every surjection $\pi: M \to \overline{M}$ and every map $\varphi: P \to \overline{M}$ there is a map $\psi: P \to M$ such that $\varphi = \pi\psi$:*

$$\begin{array}{ccc} & & P \\ & \overset{\psi}{\swarrow} & \downarrow \varphi \\ M & \underset{\pi}{\longrightarrow} & \overline{M}. \end{array}$$

One also has the following characterization of projectivity:

(8.2) Proposition. *The following conditions on an R-module P are equivalent:*

(i) *P is projective.*
(ii) *Every exact sequence $0 \to M' \to M \to P \to 0$ splits.*
(iii) *P is a direct summand of a free module.*
(iv) *There are elements $e_i \in P$ and $f_i \in \mathrm{Hom}_R(P, R)$ (where i ranges over some index set I) such that for every $x \in P$, $f_i(x) = 0$ for almost all i and $x = \sum_{i \in I} f_i(x) e_i$.*

PROOF. If P is projective and we are given an exact sequence as in (ii), then we can split the sequence by lifting id: $P \to P$ to a map $P \to M$. Hence (i) ⇒ (ii). Choosing such an exact sequence with M free, we see that (ii) ⇒ (iii). It is immediate from the definitions that any direct summand of a projective is projective; so (iii) ⇒ (i). Finally, (iv) is simply a restatement of (iii). □

EXAMPLES

1. Let $e \in R$ be *idempotent*, i.e., $e^2 = e$. Then right multiplication by e is a projection operator of R onto the direct summand Re. So the left ideal Re is projective.

2. Let R be a (commutative) integral domain and I an invertible ideal. [This means that there is an R-submodule J of the field of fractions of R such that $IJ = R$, where IJ is the set of finite sums $\sum a_i b_i$, $a_i \in I$, $b_i \in J$.] Then I is a projective module. For if we write $1 = \sum e_i f_i$ where $e_i \in I$, $f_i \in J$, then the criterion of 8.4(iv) is satisfied. [Each f_i gives rise, by multiplication, to a homomorphism $I \to R$ which plays the role of the f_i in 8.4(iv).] On the other hand, I is free only if it is a principal ideal, since any two elements $a, b \in I$ are linearly dependent.

3. Let $R = \mathbb{Z}[\zeta]$, where ζ is a primitive twenty-third root of unity. It is known from algebraic number theory that R has an ideal I which is not principal and that every non-zero ideal in R is invertible. Hence I is projective but not free.

4. Let G be a cyclic group of prime order p. There is a theorem due to Rim which relates projective modules over $\mathbb{Z}G$ to projective modules over $\mathbb{Z}[\zeta]$, where ζ is a primitive p-th root of unity, cf. Milnor [1971], §3. In particular, if $p = 23$, we deduce from example 3 that $\mathbb{Z}G$ has non-free projectives.

5. If R is the rational group algebra $\mathbb{Q}G$ of a finite group G, then one can show that *every* R-module is projective, cf. exercise 5 below. As an illustration we will prove that $\mathbb{Q}$, with trivial G-action, is projective. Note first that the functor $\mathrm{Hom}_{\mathbb{Q}G}(\mathbb{Q}, -)$ is simply the "invariants" functor $M \mapsto M^G$, where

$$M^G = \{m \in M : gm = m \text{ for all } g \in G\}.$$

Thus we need to show that any surjection $M \to \overline{M}$ of $\mathbb{Q}G$-modules gives rise to a surjection $M^G \to \overline{M}^G$. This is shown by *averaging*: if $\overline{m} \in \overline{M}^G$, lift $\overline{m}$ to $m \in M$; then $(1/|G|)\sum_{g \in G} gm$ is also a lifting of $\overline{m}$ and is in M^G.

We turn now to the *duality theory* for finitely generated projectives, analogous to that for finite dimensional vector spaces over a field. For any left R-module M, let $M^* = \mathrm{Hom}_R(M, R)$. Here R is regarded as a left R-module in forming Hom, but it also has a right R-module structure which we can use to make M^* a right R-module; namely, we set $(ur)(m) = u(m)r$ for $u \in M^*$, $r \in R$, $m \in M$. Similarly, we can define the dual of a right module, and it is a left module.

The main facts about duality are given in the following proposition. Parts (b) and (c) are the most important ones for our purposes; they allow one to use duality to convert Hom to $\otimes$ and vice-versa.

(8.3) Proposition. *Let P be a finitely generated projective left R-module.*

(a) *P^* is a finitely generated projective right R-module.*

(b) *For any left R-module M, there is an isomorphism*

$$\varphi: P^* \otimes_R M \xrightarrow{\approx} \mathrm{Hom}_R(P, M)$$

of abelian groups, given by $\varphi(u \otimes m)(x) = u(x) \cdot m$ for $u \in P^*$, $m \in M$, $x \in P$.

(c) *For any right R-module M, there is an isomorphism*

$$\varphi': M \otimes_R P \xrightarrow{\approx} \mathrm{Hom}_R(P^*, M),$$

given by $\varphi'(m \otimes x)(u) = m \cdot u(x)$ for $m \in M$, $x \in P$, $u \in P^*$.

(d) *There is an isomorphism*

$$\varphi'': P \xrightarrow{\approx} P^{**}$$

of left R-modules, given by $\varphi''(x)(u) = u(x)$ for $x \in P$, $u \in P^*$.

(In connection with (b) and (c), the reader should recall that one can form $M \otimes_R N$ whenever M is a right module and N is a left module; the tensor products written down above therefore make sense.)

Proof. It is clear from the proof of 8.2 that P can be written as a direct summand of a finitely generated free module F. By additivity, then, it suffices to prove the proposition for F. In more detail: If $F = P \oplus Q$, then $F^* = P^* \oplus Q^*$, $F^* \otimes_R M = (P^* \otimes_R M) \oplus (Q^* \otimes_R M)$, etc., and the maps φ, φ', and φ'' preserve these decompositions. So (a)–(d) for P will follow from (a)–(d) for F. [In order to use this argument, of course, one must first check that φ and φ' are well-defined.] By additivity again, it suffices to consider the case where F is free of rank 1, i.e., we may assume $F = R$. In this case $R^* \approx R$, whence (a), and it is easy to verify (b)–(d). To prove (b), for instance, one need only check that φ is the composite of the canonical isomorphisms $R^* \otimes_R M \xrightarrow{\approx} M \xrightarrow{\approx} \mathrm{Hom}_R(R, M)$. ☐

Next we give some properties of chain complexes of projectives.

(8.4) Theorem. *If $f: P' \to P$ is a weak equivalence between non-negative complexes of projectives, then f is a homotopy equivalence.*

Proof. The mapping cone of f is non-negative, projective, and acyclic (cf. 0.6). It is therefore contractible by 7.7, so f is a homotopy equivalence by 0.7. ☐

Using similar methods, we will prove a closely related mapping property of projective complexes, from which we could have deduced 8.4:

(8.5) Theorem. *Let $f: C' \to C$ be a weak equivalence between arbitrary complexes. If P is a non-negative complex of projectives, then*

$$\mathscr{H}om_R(P, f): \mathscr{H}om_R(P, C') \to \mathscr{H}om_R(P, C)$$

is a weak equivalence. In particular, the map $[P, C] \to [P, C']$ induced by f is an isomorphism.

Proof. Let C'' be the mapping cone of f. It is acyclic, and the mapping cone of $\mathscr{H}om_R(P, f)$ is $\mathscr{H}om_R(P, C'')$; so it suffices to show that $\mathscr{H}om_R(P, C'')$ is acyclic, i.e., that $[P, C'']_n = 0$ for all $n \in \mathbb{Z}$. Now $[P, C'']_n = [P, \Sigma^{-n}C'']$, and the latter is zero by the uniqueness part of the fundamental lemma 7.4, since any map on P is zero in negative dimensions. ☐

Finally, we prove an analogue of 8.5 for tensor products. Projectivity is unnecessarily strong for this purpose and can be replaced by a "flatness" hypothesis. Recall that a (left) R-module F is *flat* if the functor $- \otimes_R F$ is exact. Free modules are flat, for example, and hence projectives are flat by 8.2(iii).

(8.6) Theorem. *Let $f: C' \to C$ be a weak equivalence between complexes of right R-modules. If P is a non-negative complex of flat left R-modules, then $f \otimes_R P: C' \otimes_R P \to C \otimes_R P$ is a weak equivalence.*

Proof. Let C'' be the mapping cone of f. It is acyclic, and $C'' \otimes_R P$ is the mapping cone of $f \otimes P$; so it suffices to show that $C'' \otimes_R P$ is acyclic. Let $P^{(n)}$ be the n-skeleton of P, i.e., the truncation $(P_i)_{i \le n}$. We will show inductively that $C'' \otimes_R P^{(n)}$ is acyclic. Note first that $C'' \otimes_R F$ is acyclic for any complex F consisting of a flat module concentrated in a single dimension, since the exact sequences $C''_{i+1} \to C''_i \to C''_{i-1}$ remain exact when tensored with F. But $P^{(n)}/P^{(n-1)}$ is such a complex F. So if we assume inductively that $C'' \otimes_R P^{(n-1)}$ is acyclic, it follows from the exact sequence $0 \to C'' \otimes_R P^{(n-1)} \to C'' \otimes_R P^{(n)} \to C'' \otimes_R (P^{(n)}/P^{(n-1)}) \to 0$ that $C'' \otimes_R P^{(n)}$ is acyclic. Finally, $C'' \otimes_R P$ is the increasing union of the acyclic complexes $C'' \otimes_R P^{(n)}$, hence it is acyclic. □

Exercises

1. For what groups G is $\mathbb{Z}$ a projective $\mathbb{Z}G$-module? [Hint: When does $\varepsilon: \mathbb{Z}G \to \mathbb{Z}$ split as a map of G-modules?]

2. If P is a projective $\mathbb{Z}G$-module, show that P is also projective as $\mathbb{Z}H$-module for any $H \subset G$. [Use criterion 8.2(iii).]

3. (a) Use 8.2 to give another proof of 7.7. [According to 0.3, it suffices to show that the surjection $\bar{\partial}_n: P_n \to Z_{n-1}$ induced by ∂_n splits for all n. Assuming inductively that $\bar{\partial}_{n-1}$ splits, $Z_{n-1} = \ker \bar{\partial}_{n-1}$ is a direct summand of P_{n-1}, hence it is projective. Therefore $\bar{\partial}_n$ splits.]

   (b) Use the same method to show that the non-negativity hypothesis in 7.7 can be dropped for certain rings R, e.g., for principal ideal domains. [If R is a principal ideal domain, then submodules of a projective module are projective (in fact, free). So Z_{n-1} above is automatically projective and we do not need to use induction.]

4. If G is a group and X is a G-set such that all isotropy groups G_x are finite, show that the permutation module $\mathbb{Q}X$ is a projective $\mathbb{Q}G$-module.

5. If G is finite and k is a field of characteristic zero, show that every kG-module is projective. [Given an exact sequence as in 8.2(ii), choose a splitting $f: P \to M$ for the underlying sequence of k-vector spaces. Then $x \mapsto (1/|G|) \sum_{g \in G} gf(g^{-1}x)$ is a kG-splitting.]

6. Prove the following converse of 8.3b: If P is a module such that $\varphi: P^* \otimes_R P \to \mathrm{Hom}_R(P, P)$ is surjective, then P is finitely generated and projective. [Write $\mathrm{id}_P = \varphi(\sum f_i \otimes e_i)$ and show that 8.2(iv) is satisfied.]

7. Let P be finitely generated and projective. For any $z \in P^* \otimes_R P$ and any module M there is a map $\psi_z: \mathrm{Hom}_R(P, M) \to P^* \otimes_R M$, defined as follows: $\psi_z(f)$ is the image of z under $P^* \otimes f: P^* \otimes_R P \to P^* \otimes_R M$. Show that the inverse of the canonical isomorphism $\varphi: P^* \otimes_R M \xrightarrow{\approx} \mathrm{Hom}_R(P, M)$ of 8.3b is a map of the form ψ_z for some fixed z (independent of M). [Method 1: View φ^{-1} as a natural transformation $\mathrm{Hom}_R(P, -) \to P^* \otimes_R -$. By Yoneda's lemma (exercise 3a of §7), φ^{-1} is uniquely determined by what it does to id_P. Moreover, the proof of Yoneda's lemma tells you how to describe φ^{-1} in terms of $z = \varphi^{-1}(\mathrm{id}_P)$, and this description says precisely that $\varphi^{-1} = \psi_z$. Method 2: Choose (e_i) and (f_i) as in 8.2(iv) and set $z = \sum f_i \otimes e_i$. By directly checking definitions, verify that $\psi_z \circ \varphi = \mathrm{id}$ and/or that $\varphi \circ \psi_z = \mathrm{id}$.]

8. If P is finitely presented and flat, show that P is projective. [Take a finite presentation $F_1 \to F_0 \to P \to 0$ with F_0 and F_1 free of finite rank. This gives an exact sequence $0 \to P^* \to F_0^* \to F_1^*$ of right R-modules. Tensor with P and deduce that $P^* \otimes_R P \xrightarrow{\approx} \mathrm{Hom}_R(P, P)$. Now apply exercise 6.]

Appendix. Review of Regular Coverings

The material summarized in this appendix can be found in many algebraic topology texts, such as Massey [1967] or Spanier [1966].

Let $p: \tilde{X} \to X$ be a covering map of connected, locally path-connected spaces. A *deck transformation* of p is a homeomorphism $g: \tilde{X} \to \tilde{X}$ such that $pg = p$. The group G of all deck transformations acts *freely* on $\tilde{X}$, in the sense that $g\tilde{x} \neq \tilde{x}$ for all $\tilde{x} \in \tilde{X}$ and $g \neq 1$ in G.

The cover p is said to be *regular* if it satisfies the following conditions, which are equivalent:

(i) G acts transitively on $p^{-1}x$ for all $x \in X$. [Hence $X \approx \tilde{X}/G$.]
(ii) The image of $\pi_1\tilde{X} \to \pi_1 X$ is normal in $\pi_1 X$ for some (and hence every) choice of basepoints.
(iii) For any closed loop ω in X, if one lift of ω to $\tilde{X}$ is closed, then all lifts of ω are closed.

In this case we have $G \approx \pi_1 X/\pi_1\tilde{X}$. In particular, if $\tilde{X}$ is simply connected (so that p is the *universal cover* of X), then $G \approx \pi_1 X$.

To get an explicit isomorphism $G \approx \pi_1 X/\pi_1\tilde{X}$ above, we must choose basepoints $x \in X$ and $\tilde{x} \in p^{-1}x$. We then have a homomorphism $\varphi: \pi_1(X, x) \to G$, defined as follows: Let $\omega: [0, 1] \to X$ represent $[\omega] \in \pi_1(X, x)$ and let $\tilde{\omega}: [0, 1] \to \tilde{X}$ be the lift of ω with $\tilde{\omega}(0) = \tilde{x}$. Then $\tilde{\omega}(1) \in p^{-1}x$, and $\varphi([\omega])$ is defined to be the unique element of G such that

$$\varphi([\omega])\tilde{x} = \tilde{\omega}(1). \qquad \text{(A1)}$$

One verifies that φ is a homomorphism, is surjective, and has kernel $\pi_1\tilde{X}$, so it induces the required isomorphism.

Finally, we mention a slightly different point of view which is sometimes useful. [For our purposes this will be needed only in exercise 2 of §II.7.] Fix an abstract group G and a connected, locally path-connected space X. By a *regular G-cover* of X (also called a *principal G-bundle* over X) we mean a covering map $p: \tilde{X} \to X$, where $\tilde{X}$ is not necessarily connected, together with a free G-action on $\tilde{X}$ satisfying condition (i) above.

In case $\tilde{X}$ is connected, such a p is simply a regular cover in the usual sense, together with an isomorphism of G with the group of deck transformations.

We will assume that a basepoint $x \in X$ has been chosen and that all covering spaces $\tilde{X}$ come equipped with a basepoint $\tilde{x} \in p^{-1}x$. We can then define a homomorphism $\varphi: \pi_1 X \to G$ exactly as in A1 above, the only difference being that φ will not be surjective if $\tilde{X}$ is disconnected. In fact, one checks easily that $G/\mathrm{im}\,\varphi \approx \pi_0\tilde{X}$ (isomorphism of G-sets).

The main theorem on regular G-covers (with basepoint) says that they are completely classified by $\varphi \in \mathrm{Hom}(\pi_1 X, G)$:

(A2) Theorem. *Let $\mathscr{C}_G(X)$ be the set of isomorphism classes of pointed, regular G-covers of X. The assignment of φ to p gives a bijection*

$$\mathscr{C}_G(X) \approx \mathrm{Hom}(\pi_1 X, G).$$

(Note: Isomorphisms are required to preserve basepoints, commute with the G action, and commute with the projection onto X.)

SKETCH OF PROOF. Using the usual classification of connected covering spaces in terms of subgroups of $\pi_1 X$, one easily sees that connected, pointed, regular G-covers correspond to surjections $\varphi: \pi_1 X \to G$. The study of disconnected covers is easily reduced to the connected case by considering the connected components of $\tilde{X}$. □

CHAPTER II
The Homology of a Group

1 Generalities

In homological algebra one constructs homological invariants of algebraic objects by the following process, or some variant of it:

Let R be a ring and T a covariant *additive* functor from R-modules to abelian groups. Thus the map $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_{\mathbb{Z}}(TM, TN)$ defined by T is a homomorphism of abelian groups for all R-modules M, N. For any R-module M, choose a free (or projective) resolution $\varepsilon: F \to M$ and consider the chain complex TF of abelian groups obtained by applying T to F termwise. Now T, being additive, preserves chain homotopies; so we can apply the uniqueness theorem for resolutions (I.7.5) to deduce that the complex TF is independent, up to canonical homotopy equivalence, of the choice of resolution. Passing to homology, we obtain groups $H_n(TF)$ which depend only on T and M (up to canonical isomorphism).

This construction is of no interest, of course, if T is an *exact* functor; for then the augmented complex

$$\cdots \to TF_1 \to TF_0 \to TM \to 0$$

is acyclic, so that $H_n(TF) = 0$ for $n > 0$ and $H_0(TF) = TM$. Thus we can regard the groups $H_n(TF)$ in the general case as a measure of the failure of T to be exact.

In this chapter we will apply this construction with $R = \mathbb{Z}G$, $M = \mathbb{Z}$, and T equal to the "co-invariants" functor which we will describe in §2 below. This particular choice of R, M, and T is not arbitrary, as we will see, but rather it is a reflection of the topology which motivates the homology theory of groups.