Math 541 : The Mumford Conjecture
Fall 2011

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1 Course Description

The Mumford conjecture identifies the rational cohomology ring of the moduli space of curves (or, equivalently, the mapping class group) in a stable range. Its proof by Madsen and Weiss in 2002 was a landmark in both algebraic geometry and geometric topology. The original 97 page proof is notoriously difficult. However, in the intervening years several alternate proofs have been found. In this course, we will cover a proof due to Galatius and Randal-Williams that is almost shockingly elementary. Almost all of its ingredients were known by 1970 or so, the only exceptions being some facts about the mapping class group that were developed in the 1980's.

As will be seen from a perusal of the course outline below, the actual proof will only take a small portion of the class. The majority of our time will be spent discussing background material. I view this course in part as a good excuse to lecture on a number of important topological and geometric topics that do not get covered in the first year courses but are nonetheless important to a wide range of people. I will attempt to make this course accessible to second year graduate students. Also, though most of our techniques will be topological, I will try to make the course understandable to algebraic geometers.

2 Outline of course

The course will have 28 lectures, each 75 minutes long. The tentative schedule for these lectures is as follows.

- Introduction to course (1 lecture)
- Teichmüller theory (and a little bundle theory) (5 lectures)
  - Riemann surfaces, almost complex structures, hyperbolic metrics
  - Teichmüller space and moduli space, Fenchel-Nielsen coordinates

  - The Vietoris-Begle mapping theorem, $\mathrm{H}^*(\mathcal{M}_g;\mathbb{Q}) \cong \mathrm{H}^*(\mathrm{Mod}_g;\mathbb{Q})$, fiber bundles.
  - Principal G-bundles.
  - The contractibility of $\mathrm{Diff}_0(\Sigma)$.

- Classifying spaces for bundles (3 lectures)
  - Definition of a classifying space. Work out key examples explicitly : Grassmannian for $GL_n(\mathbb{C})$, Eilenberg-MacLane spaces for discrete groups, spaces of embeddings for $\mathrm{Diff}^+(M^n)$ with M^n an n-manifold (2 lectures)
  - Construction of classifying spaces in general.

- Characteristic classes (7 lectures)
  - General discussion; cohomology of classifying space vs explicit construction (0.5 lectures)
  - Chern classes (3.5 lectures)
    * Summary of their properties
    * The Leray-Hirsch theorem
    * Construction of Chern classes
    * Examples to show that they are nontrivial
    * Cell structures on the Grassmannian, proof that we have found all char classes of complex vector bundles
  - The Euler class (2 lectures)
    * Summary of their properties
    * The Gysin homomorphism
    * Construction and examples
  - The Miller-Mumford-Morita classes (just the construction)

- The group completion theorem (2 lectures)
  - Classifying spaces of categories, statement of group completion theorem
  - Homology fibrations, the proof of the group completion theorem

- The Barratt-Priddy-Quillen Theorem (3 lectures)

- The generalized Mumford conjecture (3 lectures)
  - Thom spaces, statement of the generalized Mumford conjecture
  - The Thom isomorphism
  - Proof that the generalized Mumford conjecture implies the classical Mumford conjecture

- Proof of the generalized Mumford conjecture (4 lectures)