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Characteristic classes of surface bundles

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Introduction

Let M be a C^∞-manifold. If one wants to classify all differentiable fibre bundles over a given manifold which have M as fibres, then one must study the topology of the corresponding structure group $\mathrm{Diff}\, M$, the group of all diffeomorphisms of M equipped with the C^∞ topology. It is easy to see that the connected component $\mathrm{Diff}_0\, M$ of the identity of $\mathrm{Diff}\, M$ is an open subgroup. Therefore the natural topology on the quotient group $\mathscr{D}(M) = \mathrm{Diff}\, M/\mathrm{Diff}_0\, M$, the diffeotopy group of M, is the discrete topology. Hence we have a fibration

$$B\,\mathrm{Diff}_0\, M \to B\,\mathrm{Diff}\, M \to K(\mathscr{D}(M), 1).$$

We may call cohomology classes of $B\,\mathrm{Diff}\, M$ *characteristic classes of differentiable M-bundles*. The above fibration shows that the cohomology $H^*(B\,\mathrm{Diff}\, M)$ can be approximated by that of the group $\mathscr{D}(M)$ with coefficients in the $\mathscr{D}(M)$-module $H^*(B\,\mathrm{Diff}_0\, M)$.

Recently some information has been discovered about the homotopy groups of $\mathrm{Diff}_0\, M$ for spheres and aspherical manifolds (see [FH]) and also there is a general result about $\mathscr{D}(M)$ for simply connected manifolds due to Sullivan [S]. However the problem to determine the topology of $B\,\mathrm{Diff}\, M$, in particular the problem to compute characteristic classes of differentiable M-bundles should be considered to be completely open.

The purpose of the present paper is to attack the above problem in the case where M is a closed orientable surface Σ_g of genus $g \geqq 2$. In this case Earle and Eells [EE] proved that $\mathrm{Diff}_0\, \Sigma_g$ is contractible. Hence $B\,\mathrm{Diff}_+\, \Sigma_g$ is a $K(\mathscr{M}_g, 1)$ where $\mathrm{Diff}_+\, \Sigma_g$ is the subgroup of $\mathrm{Diff}\, \Sigma_g$ consisting of all orientation preserving diffeomorphisms and $\mathscr{M}_g = \pi_0(\mathrm{Diff}_+\, \Sigma_g)$ is the mapping class group of Σ_g. Therefore characteristic classes of surface bundles can be naturally identified with the cohomology classes of the mapping class group. On the other hand $\mathscr{M}_g$ acts on the Teichmüller space of genus g properly discontinuously and the quotient space $\mathbf{M}_g$ is the moduli space for compact Riemann surfaces of genus g. Hence the rational cohomology of $\mathcal{M}_g$ is naturally isomorphic to that of $\mathbf{M}_g$. In this way characteristic classes of surface bundles are related to various branches of mathematics so that they should find many applications.

Now the contents of this paper are roughly as follows. Definitions of characteristic classes of surface bundles are given in § 1 and in § 2 we give known relations among them. § 3 is devoted to a study of the interplay between three types of mapping class groups. In § 4 we construct various surface bundles explicitly and in §§ 5, 6 we compute the characteristic classes of them. In § 7 we extend these results to the case of surface bundles with cross-sections. As a consequence it turns out that our characteristic classes are highly non-trivial (see Theorem 7.5). Finally as an application we give in § 8 a negative solution to the generalized Nielsen realization problem. More precisely we prove that the natural surjective homomorphism

$$\mathrm{Diff}_+ \Sigma_g \to \mathcal{M}_g$$

does not have a right inverse for all $g \geqq 18$. In fact our characteristic classes can be considered as obstructions for the existence of it. This should be compared with Kerckhoff's affirmative solution [Ke] to the original Nielsen realization problem that the homomorphism splits over *finite* subgroups.

We have defined characteristic classes of surface bundles as in § 1 inspired by Atiyah's paper [A]. After we had finished the main work, Mumford's paper [Mu] and Miller's paper [Mi] were published. In [Mu] Mumford initiated a new theory of the moduli space $\mathbf{M}_g$ and in [Mi] Miller obtained many of the results of §§ 1, 4 and 5 of this paper. It turns out that our theory corresponds to Mumford's one reduced to the rational cohomology via the natural isomorphism $H^*(\mathcal{M}_g; \mathbf{Q}) = H^*(\mathbf{M}_g; \mathbf{Q})$. This can be easily proved by passing to a torsion-free subgroup of $\mathcal{M}_g$ of finite index (cf. [Hav]). However we prefer to develop our theory in the framework of topology as far as possible.

The main part of the results of this paper have been announced in [Mo 1].

1. Definitions of characteristic classes of surface bundles

Let Σ_g be a closed orientable surface of genus g. In this paper we always assume that $g \geqq 2$. A differentiable fibre bundle $\pi\colon E \to X$ with fibre Σ_g is called a *surface bundle* or a *Σ_g-bundle*. Let ξ be the "tangent bundle" of π. Namely it is the subbundle of the tangent bundle of E consisting of those vectors which are tangent to the fibres of the bundle. It is a 2-plane bundle over E. If ξ is orientable and an orientation is given on it, we say that the surface bundle $\pi\colon E \to X$ is *oriented*. Henceforth we always assume this condition. Then we have the Euler class

$$e = e(\xi) \in H^2(E; \mathbf{Z})$$

which will be called the Euler class of the bundle. We define

$$e_i = \pi_*(e^{i+1}) \in H^{2i}(X; \mathbf{Z})$$

where $\pi_*: H^{2(i+1)}(E; \mathbf{Z}) \to H^{2i}(X; \mathbf{Z})$ is the Gysin homomorphism (of course $e_0 = 2-2g$).

Next we define a g-dimensional *complex* vector bundle η on X as follows. For each fibre $E_x = \pi^{-1}(x)$ $(x \in X)$, we consider the real cohomology $H^1(E_x; \mathbf{R})$. It is easy to see that the natural projection $\bigcup_{x\in X} H^1(E_x; \mathbf{R}) \to X$ admits a canonical structure of a $2g$-dimensional real vector bundle over X which we denote by $\eta_{\mathbf{R}}$. Now choose a fibre metric on ξ so that each fibre E_x inherits a Riemannian metric. Since E_x is a *two*-dimensional manifold, any Riemannian metric on it induces a complex structure and hence a hyperbolic structure by the Uniformization Theorem of Riemann surfaces. Hence we have another fibre metric on ξ such that the induced Riemannian metric on each fibre has constant negative curvature -1. For simplicity we assume henceforth that the fibre metric on ξ is such a metric. Now if we identify $H^1(E_x; \mathbf{R})$ with the space of harmonic 1-forms on E_x, then the $*$-operator on $H^1(E_x; R)$ satisfies $*^2 = -1$. Hence it induces a structure of g-dimensional complex vector bundle on $\eta_{\mathbf{R}}$. We define η to be this bundle. It is easy to see that the isomorphism class of η does not depend on the choice of the metric on ξ. Also the spaces of holomorphic differentials on E_x $(x \in X)$ define another g-dimensional complex vector bundle over X. However it is easy to see that this bundle is naturally isomorphic to the conjugate bundle $\bar{\eta}$ of η. Now we define

$$c_i = c_i(\eta) \in H^{2i}(X; \mathbf{Z})$$

where $c_i(\eta)$ is the i-th Chern class of η.

It is clear from the above definitions that the cohomology classes $e \in H^2(E; \mathbf{Z})$ and $e_i, c_i \in H^{2i}(X; \mathbf{Z})$ are functorial in the obvious sense. In fact we can define them at the classifying space level as follows.

The structure group of oriented Σ_g-bundles is the group $\mathrm{Diff}_+ \Sigma_g$ of all orientation preserving diffeomorphisms of Σ_g equipped with the C^∞ topology. Hence if we denote $B\,\mathrm{Diff}_+ \Sigma_g$ for the classifying space of it, there is a Σ_g-bundle

$$\Sigma_g \to E\,\mathrm{Diff}_+ \Sigma_g \to B\,\mathrm{Diff}_+ \Sigma_g$$

called the universal Σ_g-bundle such that any Σ_g-bundle $\pi: E \to X$ can be obtained as a pull back bundle of it by a certain continuous map $f: X \to B\,\mathrm{Diff}_+ \Sigma_g$, which is then called the classifying map of the bundle. Now, as already mentioned in the introduction, the result of Earle and Eells [EE] implies that $B\,\mathrm{Diff}_+ \Sigma_g$ is an Eilenberg-MacLane space $K(\mathcal{M}_g, 1)$ where $\mathcal{M}_g = \pi_0(\mathrm{Diff}_+ \Sigma_g)$ is the mapping class group. It follows that the isomorphism class of a surface bundle $\pi: E \to X$ is completely determined by a homomorphism

$$h: \pi_1(X) \to \mathcal{M}_g$$

which is induced from the classifying map. We call it the *holonomy homomorphism* and $\mathrm{Im}\, h$ the *holonomy group* of the bundle. The total space $E\,\mathrm{Diff}_+ \Sigma_g$ of the universal Σ_g-bundle is also an Eilenberg-MacLane space whose fundamental group is naturally isomorphic to the mapping class group $\mathcal{M}_{g,*} = \pi_0(\mathrm{Diff}_+ (\Sigma_g, *))$ where $\mathrm{Diff}_+ (\Sigma_g, *)$ is the subgroup of $\mathrm{Diff}_+ \Sigma_g$ consisting of all base point preserving diffeomorphisms. It is easy to see that $E\,\mathrm{Diff}_+\,\Sigma_g$ can be considered as the classifying space for Σ_g-bundles with cross sections. We identify cohomology classes of $B\,\mathrm{Diff}_+\,\Sigma_g$ and $E\,\mathrm{Diff}_+\,\Sigma_g$ with those of $\mathcal{M}_g$ and $\mathcal{M}_{g,*}$ as abstract groups.

Now if we apply the former constructions to the tangent bundle of the universal Σ_g-bundle, we obtain certain cohomology classes

$$e \in H^2(E\,\mathrm{Diff}_+\,\Sigma_g;\mathbf{Z}) = H^2(\mathcal{M}_{g,*};\mathbf{Z})$$
$$e_i, c_i \in H^{2i}(B\,\mathrm{Diff}_+\,\Sigma_g;\mathbf{Z}) = H^{2i}(\mathcal{M}_g;\mathbf{Z}).$$

We use the same letters e_i, c_i for the elements $\pi^*(e_i)$, $\pi^*(c_i) \in H^{2i}(\mathcal{M}_{g,*};\mathbf{Z})$ where $\pi\colon \mathcal{M}_{g,*} \to \mathcal{M}_g$ is the natural homomorphism. The cohomology class $c_i \in H^{2i}(\mathcal{M}_g;\mathbf{Z})$ can also be introduced in a different way as follows. The natural action of $\mathcal{M}_g$ on the cohomology $H^1(\Sigma_g;\mathbf{Z})$ preserves the symplectic form on it given by the cup product. Hence if we choose a symplectic basis for $H^1(\Sigma_g;\mathbf{Z})$, we obtain a homomorphism

$$\mathcal{M}_g \to \mathrm{Sp}(2g;\mathbf{Z})$$

where $\mathrm{Sp}(2g;\mathbf{Z})$ is the group of all $2g \times 2g$ symplectic matrices with integral entries. This induces a homomorphism $\mathcal{M}_g \to \mathrm{Sp}(2g;\mathbf{R})$. Now the maximal compact subgroup of $\mathrm{Sp}(2g;\mathbf{R})$ is isomorphic to $U(g)$. Hence passing to the classifying spaces we obtain a continuous map

$$B\,\mathrm{Diff}_+\,\Sigma_g = K(\mathcal{M}_g, 1) \to BU(g).$$

Now it can be checked that the cohomology class $c_i \in H^{2i}(\mathcal{M}_g;\mathbf{Z})$ coincides with the pull back of the universal Chern class $c_i \in H^{2i}(BU(g);\mathbf{Z})$ under the above map.

Next we consider one more type of surface bundle. Let

$$S^1 \to \hat{E}\,\mathrm{Diff}_+\,\Sigma_g \xrightarrow{\hat{\pi}} E\,\mathrm{Diff}_+\,\Sigma_g$$

be the S^1-bundle defined by the cohomology class $e \in H^2(E\,\mathrm{Diff}_+\,\Sigma_g;\mathbf{Z})$. Then $\hat{E}\,\mathrm{Diff}_+\,\Sigma_g$ is again an Eilenberg-MacLane space and its fundamental group is naturally isomorphic to the mapping class group $\mathcal{M}_{g,1} = \pi_0(\mathrm{Diff}(\Sigma_g^0, \partial\Sigma_g^0))$ where $\Sigma_g^0 = \Sigma_g - \mathrm{Int}\,D^2$ and $\mathrm{Diff}(\Sigma_g^0, \partial\Sigma_g^0)$ is the group of all diffeomorphisms of Σ_g^0 which restrict to the identity on the boundary. $\hat{E}\,\mathrm{Diff}_+\,\Sigma_g$ is the classifying space for surface bundles with cross sections whose normal bundles are trivial and have a specific trivialization. As before we use the same letters e_i, c_i for the elements $\hat{\pi}^*(e_i)$, $\hat{\pi}^*(c_i) \in H^{2i}(\hat{E}\,\mathrm{Diff}_+\,\Sigma_g;\mathbf{Z}) = H^{2i}(\mathcal{M}_{g,1};\mathbf{Z})$.

2. Relations between characteristic classes

In this section we investigate relations among various cohomology classes of the mapping class groups defined in § 1.

To deduce the first relation, we follow Atiyah's argument in [A]. Thus let $\pi\colon E \to X$ be an oriented Σ_g-bundle, ξ the tangent bundle of it and let η be the g-dimensional complex vector bundle defined in §1. Then as in [A] the index theorem for families of elliptic operators [AS] applied to the signature operators on E_x $(x \in X)$ gives

$$\mathrm{ch}(\eta^* - \eta) = \pi_*(\tilde{L}(\xi))$$

where ch is the Chern character and for any real vector bundle ζ of dimension $2n$, $\tilde{L}(\zeta)$ is defined to be

$$\tilde{L}(\zeta) = \prod_{i=1}^{n} \frac{x_i}{\tanh x_i/2},$$

where the Pontrjagin classes $p_i(\zeta)$ of ζ are expressed as the elementary symmetric functions in $x_1^2, \ldots, x_n^2$ as usual so that the right hand side is a polynomial of $p_i(\zeta)$ because $\dfrac{x}{\tanh x/2}$ is an even function of x. We have

$$\begin{aligned}\mathrm{ch}(\eta^* - \eta) &= g + \sum (-1)^i \frac{s_i(\eta)}{i!} - \left(g + \sum \frac{s_i(\eta)}{i!}\right) \\ &= -2 \sum_{i:\,\mathrm{odd}} \frac{s_i(\eta)}{i!}\end{aligned}$$

where $s_i(\eta)$ is the characteristic class of η corresponding to the formal sum $\sum_j x_j^i$ (it is a polynomial in the Chern classes of η, see [MS]). On the other hand

$$\begin{aligned}\pi_*(\tilde{L}(\xi)) &= \pi_*\left\{2\left(1 + \sum (-1)^{i-1} \frac{2^{2i}}{(2i)!} B_{2i} \left(\frac{e}{2}\right)^{2i}\right)\right\} \\ &= 2 \sum (-1)^{i-1} \frac{B_{2i}}{(2i)!} e_{2i-1}\end{aligned}$$

where e is the Euler class of ξ and B_{2i} is the $2i$-th Bernoulli number. Hence we obtain

$$e_{2i-1} = (-1)^i \frac{2i}{B_{2i}} s_{2i-1}(\eta). \qquad \text{(R-1)}$$

Since the above argument holds for *any* surface bundle, we conclude that (R-1) gives a relation in $H^{4i-2}(\mathcal{M}_g; \mathbf{Q})$. This remark also applies to the arguments below.

Secondly it is clear from the definition that the bundle $\eta_{\mathbf{R}}$ is a flat bundle with structure group $\mathrm{Sp}(2g; \mathbf{Z})$. In fact it is defined by the homomorphism $\pi_1(X) \to \mathrm{Sp}(2g; \mathbf{Z})$ which is the composition of the holonomy homomorphism with the natural homomorphism $\mathcal{M}_g \to \mathrm{Sp}(2g; \mathbf{Z})$. Hence all the Pontrjagin classes of η vanish. Equivalently we have

$$s_{2i}(\eta) = 0 \quad \text{in } H^{4i}(\mathcal{M}_g; \mathbf{Q}) \quad (i = 1, 2, \ldots). \qquad \text{(R-2)}$$

Now it can be derived from a result of [BH] that if we impose the relation (R-2) to the universal Chern classes $c_1, \ldots, c_g$, we obtain exactly characteristic classes of *flat* $\mathrm{Sp}(2g; \mathbf{R})$-bundles which correspond to elements of the relative Lie algebra cohomology $H^*(\mathfrak{sp}(2g; \mathbf{R}), \mathfrak{u}(g))$. Therefore we can conclude that there exist natural isomorphisms

$$\mathbf{Q}[e_1, e_3, \ldots]/\underset{(\mathrm{R}\text{-}1)(\mathrm{R}\text{-}2)}{\text{relations}} \cong H^*(\mathfrak{sp}(2g; \mathbf{R}), \mathfrak{u}(g); \mathbf{Q})$$

$$\cong H^*(S^2 \times S^4 \times \ldots \times S^{2g}; \mathbf{Q})$$

where the second isomorphism is only additive and is again due to [BH]. Also it is clear from the above that the Chern classes of η, $c_i(\eta)$, are polynomials in e_j (j: odd). In short then we can say that the characteristic classes $e_1, e_3, \ldots$ are precisely those of η as a flat $\mathrm{Sp}(2g; \mathbf{R})$-bundle.

Next we describe the third relation which the author learned from Mumford [Mu]. Let $\pi: E \to X$ be an oriented surface bundle and suppose a fibre metric is given on the tangent bundle ξ of it as in §1. Let $\overline{\pi^*(\eta)}$ be the conjugate bundle of the pull back bundle $\pi^*(\eta)$. We define a map

$$b: \overline{\pi^*(\eta)} \to \xi^*$$

as follows. Let $x \in X$ and let ω be a harmonic 1-form on E_x with the corresponding real cohomology class $[\omega] \in H^1(E_x; \mathbf{R})$. Then we define

$$b([\omega])(v) = \omega(v) + \sqrt{-1} * \omega(v) \qquad (v \in TE_x),$$

where $*$ denotes the Hodge's $*$-operator. It can be checked that b is complex linear so that it is a bundle map. Moreover it is easy to see that b is surjective. Hence we have a short exact sequence of complex vector bundles over E:

$$0 \to \operatorname{Ker} b \to \overline{\pi^*(\eta)} \to \xi^* \to 0.$$

Therefore

$$c(\operatorname{Ker} b) = \pi^*(1 - c_1(\eta) + c_2(\eta) - \ldots + (-1)^g c_g(\eta))(1 + e + e^2 + \ldots).$$

Since $c_k(\operatorname{Ker} b) = 0$ for all $k \geqq g$, we have the following relation in $H^*(\mathcal{M}_{g,*}; \mathbf{Z})$:

$$e^k - e^{k-1} c_1 + \ldots + (-1)^g e^{k-g} c_g = 0 \qquad (k \geqq g) \tag{R-3}$$

(recall that we write simply c_i for $\pi^*(c_i) \in H^{2i}(\mathcal{M}_{g,*}; \mathbf{Z})$). Since c_g is divisible by e in $H^{2g}(\mathcal{M}_{g,*}; \mathbf{Z})$, Proposition 3.3 below implies

$$c_g = 0 \quad \text{in } H^{2g}(\mathcal{M}_{g,1}; \mathbf{Z}). \tag{R-4}$$

Applying the Gysin homomorphism π_* to (R-3), we obtain

$$e_{k-1} - e_{k-2} c_1 + \ldots + (-1)^g e_{k-g-1} c_g = 0 \quad \text{in } H^{2(k-1)}(\mathcal{M}_g; \mathbf{Z}) \quad \text{for all } k \geqq g \tag{R-5}$$

here we understand $e_{-1}=0$. Then the argument of Mumford ([Mu], Corollary 6.2) shows that the relations (R-1) and (R-5) imply the fact that $e_k \in H^{2k}(\mathcal{M}_g; \mathbf{Q})$ can be expressed as a polynomial in $e_1, \ldots, e_{g-2}$ for all $k \geqq g-1$. In our recent papers [Mo 2, 3], we have obtained more relations. For example we have

$$2^{g+1} e_{g+h+k-1} - \left\{ e_{g+k} e_{h-1} + \binom{g+1}{1} e_{g+k-1} e_h + \ldots \right.$$

$$\left. + \binom{g+1}{g} e_k e_{h+g-1} + e_{k-1} e_{h+g} \right\} = 0 \quad \text{in } H^{2(g+h+k-1)}(\mathcal{M}_g; \mathbf{Q})$$

$$\text{for all } h, k \geqq 0. \tag{R-6}$$

$$\{2g(2-2g)e - e_1\}^{g+1} = 0 \quad \text{in } H^{2(g+1)}(\mathcal{M}_g; \mathbf{Q}). \tag{R-7}$$

See [Mo 2, 3] for more relations and proofs (see also [Harr] in which Harris proved (R-7) by a method of algebraic geometry). Finally [Har 3] determined the virtual cohomological dimensions of the mapping class groups. In particular $\operatorname{vcd}(\mathcal{M}_g) = 4g-5$ and $\operatorname{vcd}(\mathcal{M}_{g,*}) = 4g-3$. Hence we have

(i) any polynomial in $e_1, e_2, \ldots$ of degree $> 4g-5$ vanishes

(ii) any polynomial in $e, e_1, e_2, \ldots$ of degree $> 4g-3$ vanishes. (R-8)

We sum up the above as

Theorem 2.1. *We have homomorphisms*

$$\mathbf{Q}[e_1, \ldots, e_{g-2}]/\text{relations} \to H^*(\mathcal{M}_g; \mathbf{Q})$$
$$\mathbf{Q}[e_1, \ldots, e_{g-2}]/\text{relations} \to H^*(\mathcal{M}_{g,1}; \mathbf{Q})$$
$$\mathbf{Q}[e, e_1, \ldots, e_{g-2}]/\text{relations} \to H^*(\mathcal{M}_{g,*}; \mathbf{Q}).$$

Observe that the first known relation between $e_1, \ldots, e_{g-2}$ occurs in degree $2g-2$. As Mumford says in [Mu] for the first homomorphism, it seems to be reasonable to conjecture that the above homomorphisms are all isomorphisms up to small degree. Harer's result [Har 1] shows that they are actually isomorphisms in degree 2 for all $g \geqq 2$. We prove in §§ 6, 7 that the above homomorphisms are all *injective* up to any given degree if the genus is accordingly sufficiently large. However in higher degrees there seem to be still more relations than the above. A certain study of the natural action of $\mathcal{M}_{g,*}$ on the lower central series of $\pi_1(\Sigma_g)$, along the lines of [Mo 2, 3], would probably provide new relations. We would like to pursue this in a near future.

3. Relations among $\mathcal{M}_g$, $\mathcal{M}_{g,*}$ and $\mathcal{M}_{g,1}$

In this section we collect certain relations between the cohomology groups of the three types of mapping class groups considered in § 1. We have the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbf{Z} & & \mathbf{Z} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \pi_1(T_1\Sigma_g) & \longrightarrow & \mathcal{M}_{g,1} & \longrightarrow & \mathcal{M}_g & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \pi_1(\Sigma_g) & \longrightarrow & \mathcal{M}_{g,*} & \longrightarrow & \mathcal{M}_g & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
$$

where $T_1\Sigma_g$ is the unit tangent bundle of Σ_g. The central extension $0 \to \mathbf{Z} \to \mathcal{M}_{g,1} \to \mathcal{M}_{g,*} \to 1$ is defined by the element $e \in H^2(\mathcal{M}_{g,*}; \mathbf{Z})$ (see § 1) and the center $\mathbf{Z}$ of $\mathcal{M}_{g,1}$ is generated by the Dehn twist along a simple closed curve on Σ_g^0 which is parallel to the boundary.

Proposition 3.1. *Let $\pi\colon E \to X$ be an oriented Σ_g-bundle and let $k = \mathbf{Q}$ or $\mathbf{Z}/p$ where p is a prime not dividing $(2g-2)$. Then the spectral sequence $\{E_r^{p,q}, d_r\}$ for the cohomology $H^*(E; k)$ collapses so that the homomorphism $\pi^*\colon H^*(X; k) \to H^*(E; k)$ is injective and we have*

$$H^p(E; k) \cong H^p(X; k) \oplus H^{p-1}(X; H^1(\Sigma_g; k)) \oplus eH^{p-2}(X; k)$$

where $e \in H^2(E; k)$ is the Euler class of the bundle.

Proof. The E_2-term is given by $E_2^{p,q} = H^p(X; H^q(\Sigma_g; k))$. Since the bundle is orientable, we have

$$E_2^{p,0} = H^p(X; k) \quad \text{and} \quad E_2^{p,2} = H^p(X; k).$$

Now consider the following commutative diagram:

$$
\begin{array}{ccc}
H^2(E; k) & \xrightarrow{i^*} & H^2(\Sigma_g; k) \\
\downarrow & & \| \\
E_\infty^{0,2} & \subset \ldots \subset & E_2^{0,2}
\end{array}
$$

where $i\colon \Sigma_g \to E$ is the inclusion. Since $i^*(e)$ is the generator of $H^2(\Sigma_g; k)$ by the assumption, we conclude that

$$E_2^{0,2} = E_3^{0,2} = \ldots = E_\infty^{0,2} \cong k.$$

Hence $d_2=0$ on $E_2^{0,2}$ and $d_3=0$ on $E_3^{0,2}$. Now the cup product defines an isomorphism

$$E_2^{p,0}\otimes E_2^{0,2}\xrightarrow{\ \sim\ } E_2^{p,2}.$$

Hence any element of $E_2^{p,2}$ can be written as $u\cup v$ for some $u\in E_2^{p,0}$ ($v\in E_2^{0,2}$ is the generator). Then

$$\begin{aligned} d_2(u\cup v) &= d_2(u)\cup v+(-1)^p u\cup d_2(v)\\ &=0 \end{aligned}$$

because $d_2(u)\in E_2^{p+1,-1}=0$ and $d_2(v)=0$. Therefore we have $E_3^{p,2}=E_2^{p,2}$. It follows that the surjective homomorphism $E_2^{p,0}\to E_3^{p,0}$ is actually an isomorphism because otherwise let $u\neq 0$ be in the kernel. Then $u\cup v\neq 0$ in $E_2^{p,2}$ while $[u]\cup[v]=0$ in $E_3^{p,2}$, a contradiction. Therefore the differentials

$$d_2: E_2^{p,2}\to E_2^{p+2,1}, \qquad d_2: E_2^{p-2,1}\to E_2^{p,0}$$

are zero. Next consider $d_3: E_3^{p,2}\to E_3^{p+3,0}$. Since the cup product defines an isomorphism

$$E_3^{p,0}\otimes E_3^{0,2}\xrightarrow{\ \sim\ } E_3^{p,2}$$

the same argument as before shows $d_3=0$. It follows that all the differentials d_r vanish by the dimension reasons. Hence

$$\begin{aligned} H^p(E;k) &\cong E_\infty^{p,0}\oplus E_\infty^{p-1,1}\oplus E_\infty^{p-2,2}\\ &= H^p(X;k)\oplus H^{p-1}(X;H^1(\Sigma_g;k))\oplus eH^{p-2}(X;k). \end{aligned}$$

This completes the proof.

If we apply the above Proposition to the universal Σ_g-bundle, we obtain

Corollary 3.2. *The homomorphism $\pi^*: H^*(\mathcal{M}_g;k)\to H^*(\mathcal{M}_{g,*};k)$ is injective and we have*

$$H^p(\mathcal{M}_{g,*};k)\cong H^p(\mathcal{M}_g;k)\oplus H^{p-1}(\mathcal{M}_g;H^1(\Sigma_g;k))\oplus eH^{p-2}(\mathcal{M}_g;k).$$

From this follows that the homomorphism $\pi_*: H_*(\mathcal{M}_{g,*};\mathbf{Z})\to H_*(\mathcal{M}_g;\mathbf{Z})$ is surjective modulo $(2g-2)$-torsions. It is actually a surjection for $*=2$ as proved first by Harer [Har 1].

Next we consider the Gysin exact sequence of the central extension $0\to\mathbf{Z}\to\mathcal{M}_{g,1}\to\mathcal{M}_{g,*}\to 1$:

$$\ldots\to H^k(\mathcal{M}_{g,*})\xrightarrow{\ \cup e\ } H^{k+2}(\mathcal{M}_{g,*})\xrightarrow{\ \hat{\pi}^*\ } H^{k+2}(\mathcal{M}_{g,1})\to H^{k+1}(\mathcal{M}_{g,*})\to\ldots.$$

From this we conclude

Proposition 3.3. *Let $\hat{\pi}^*: H^*(\mathcal{M}_{g,*})\to H^*(\mathcal{M}_{g,1})$ be the natural homomorphism (coefficients are in $\mathbf{Z}$ or $\mathbf{Q}$). Then for an element $u\in H^*(\mathcal{M}_{g,*})$, $\hat{\pi}^*(u)=0$ if and only if u is divisible by e.*

Finally let $g, g_1, \ldots, g_k$ be natural numbers such that $\Sigma g_j \leqq g$. Choose an embedding of the disjoint union $\coprod_{j=1}^{k} \Sigma_{g_j}^0$ of compact surfaces $\Sigma_{g_j}^0$, each with one boundary component, into Σ_g^0. This induces a homomorphism

$$\iota\colon \mathcal{M}_{g_1,1} \times \ldots \times \mathcal{M}_{g_k,1} \to \mathcal{M}_{g,1}.$$

Proposition 3.4. $\iota^*(e_i) = \sum_{j=1}^{k} p_j^*(e_i)$ *for all* $i \geqq 1$, *where* $p_j\colon \mathcal{M}_{g_1,1} \times \ldots \times \mathcal{M}_{g_k,1} \to \mathcal{M}_{g_j,1}$ *is the j-th projection.*

Proof. Let $\pi_j\colon E_j \to X_j$ be a Σ_{g_j}-bundle with holonomy group G_j lying in $\mathcal{M}_{g_j,1}$ $(j=1, \ldots, k)$. Namely there is given a cross section $s_j\colon X_j \to E_j$ and an identification $N(s_j(X_j)) = X_j \times D^2$, where $N(s_j(X_j))$ is a tubular neighborhood of $s_j(X_j)$ in E_j. Let $X = X_1 \times \ldots \times X_k$ and let $\bar{\pi}_j\colon \bar{E}_j \to X$ be the pull back of the Σ_{g_j}-bundle $\pi_j\colon E_j \to X_j$ by the j-th projection $X \to X_j$. We have a cross section $\bar{s}_j\colon X \to \bar{E}_j$ and an identification $N(\bar{s}_j(X)) = X \times D^2$ (henceforth we identify them). Now let $\hat{\Sigma} = \Sigma_g - \coprod_{j=1}^{k} \operatorname{Int} \Sigma_{g_j}^0$ and also we write $\hat{E}_j$ for $\bar{E}_j - X \times \operatorname{Int} D^2$. We attach the manifold $X \times \hat{\Sigma}$ to the disjoint union $\coprod_{j=1}^{k} \hat{E}_j$ along their boundaries by identifying $X \times \partial \Sigma_{g_j}^0$ with $\partial \hat{E}_j = X \times S^1$. We denote the resulting manifold by E. Then it is easy to see that the canonical projection $\pi\colon E \to X$ admits a natural structure of a Σ_g-bundle whose holonomy group is $\iota(G_1 \times \ldots \times G_k)$. Now let ξ (resp. ξ_j) be the tangent bundle of $\pi\colon E \to X$ (resp. $\pi_j\colon E_j \to X_j$). Choose a canonical framing of the tangent bundle of Σ_g restricted to the subset $\cup \partial \Sigma_{g_j}^0$. This induces framings of $\xi \mid \cup \partial \hat{E}_j$ and of $\xi_j \mid X_j \times S^1$. Therefore we have the corresponding *relative* Euler classes $\hat{\chi}_j \in H^2(\hat{E}_j, \partial \hat{E}_j; \mathbf{Z})$, $\hat{\chi}_0 \in H^2(X \times \hat{\Sigma}, X \times \partial \hat{\Sigma}; \mathbf{Z})$ and $\chi_j \in H^2(E_j - \operatorname{Int} D^2, X_j \times S^1; \mathbf{Z})$, $\chi_{j,0} \in H^2(X_j \times D^2, X_j \times S^1; \mathbf{Z})$. Of course we have

$$e(\xi) = \hat{\chi}_0 + \Sigma \hat{\chi}_j \quad \text{and} \quad e(\xi_j) = \chi_{j,0} + \chi_j$$

(here we omit symbols for the homomorphisms induced from inclusions and excisions). Now clearly $\hat{\chi}_0^i$ and $\chi_{j,0}^i$ vanish for all $i \geqq 2$ because $\hat{\chi}_0$ (resp. $\chi_{j,0}$) is a pull back of an element of $H^2(\hat{\Sigma}, \partial \hat{\Sigma}; \mathbf{Z})$ (resp. $H^2(D^2, S^1; \mathbf{Z})$). Also if we denote $p_j\colon (\hat{E}_j, \partial \hat{E}_j) \to (E_j - X_j \times \operatorname{Int} D^2, X_j \times S^1)$ for the natural projection, we have $\hat{\chi}_j = p_j^*(\chi_j)$. Hence for all $i \geqq 2$ we have

$$\begin{aligned}(e(\xi))^i &= \sum \hat{\chi}_j^i \\ &= \sum p_j^*(\chi_j^i) \\ &= \sum p_j^*((e(\xi_j))^i).\end{aligned}$$

The required assertion follows from this, completing the proof.

Remark 3.5. It might be interesting to compare the above result to Wolpert's result that the Weil-Petersson form of the moduli spaces restricts to subsurfaces in a similar fashion.

4. A method of constructing surface bundles

In this section we give a method of constructing a new surface bundle from a given one which is a generalization of the constructions of [A] and [Ko]. By this method we can construct surface bundles whose characteristic classes are highly non-trivial. First we prepare a few facts.

Lemma 4.1. *Let $\pi: E \to X$ be an orientable Σ_g-bundle with $g \geqq 2$. Then for each natural number m, there exists a finite covering map $X_1 \to X$ such that the pull back Σ_g-bundle $E_1 \to X_1$ over X_1 admits a fibre-wise m-fold covering space. More precisely there is a finite covering map $E'_1 \to E_1$ of degree m such that the composed map $E'_1 \to X_1$ is a surface bundle with fibre $\Sigma_{g'}$ which is an m-fold covering space of Σ_g (hence $g' = m(g-1)+1$).*

Proof. First we recall two well-known results about the mapping class group $\mathcal{M}_g$. One is the classical result that $\mathcal{M}_g$ is naturally isomorphic to the *proper* outer automorphism group $\mathrm{Aut}^+ \pi_1(\Sigma_g)/\mathrm{Inn}\, \pi_1(\Sigma_g)$ of $\pi_1(\Sigma_g)$ (see [Bi] for the terminology as well as more informations). Second is the fact that $\mathcal{M}_g$ is virtually torsion-free, namely it has a torsion-free subgroup of finite index (see [Hav]). Now fix an m-fold regular covering map $\Sigma_{g'} \to \Sigma_g$. In other words choose a normal subgroup $\pi_1(\Sigma_{g'})$ of $\pi_1(\Sigma_g)$ of index m. There exist only a finite number of subgroups of $\pi_1(\Sigma_g)$ of index m so that it is easy to choose a normal subgroup Γ_1 of $\mathrm{Aut}^+ \pi_1(\Sigma_g)$ of finite index such that the action of Γ_1 preserves $\pi_1(\Sigma_{g'})$. Consider the natural homomorphism

$$r: \Gamma_1 \to \mathrm{Aut}^+ \pi_1(\Sigma_{g'}) \to \mathcal{M}_{g'}$$

where $\mathrm{Aut}^+ \pi_1(\Sigma_{g'}) \to \mathcal{M}_{g'}$ is the natural projection. It is easy to see that for any element $\gamma \in \mathrm{Inn}\, \pi_1(\Sigma_g) \cap \Gamma_1$, $r(\gamma)$ is a torsion element. Now choose a torsion-free normal subgroup Γ_2 of $\mathcal{M}_{g'}$ of finite index. Then we have $r^{-1}(\Gamma_2) \cap \mathrm{Inn}\, \pi_1(\Sigma_g) \subset \mathrm{Ker}\, r$. Let $\pi: \mathrm{Aut}^+ \pi_1(\Sigma_g) \to \mathcal{M}_g$ be the projection. Then $\pi(r^{-1}(\Gamma_2))$ has finite index in $\mathcal{M}_g$, let Γ_3 be the intersection of its conjugates so that it is a normal subgroup of $\mathcal{M}_g$ of finite index. Now let $h: \pi_1(X) \to \mathcal{M}_g$ be the holonomy homomorphism of the given surface bundle $\pi: E \to X$. We define the finite covering map $X_1 \to X$ to be the one defined by the kernel of the homomorphism $\pi_1(X) \to \mathcal{M}_g \to \mathcal{M}_g/\Gamma_3$. Then the $\Sigma_{g'}$-bundle $E'_1 \to X_1$ defined by the homomorphism $\pi_1(X_1) \to \Gamma_3 \to \mathcal{M}_{g'}$ satisfies the required condition.

Definition 4.2. *We define a class $\mathscr{C}_n$ of C^∞ $2n$-manifolds recursively as follows. $\mathscr{C}_0$ is the class {one point}, $\mathscr{C}_1$ is the class consisting of Σ_g's with $g \geqq 2$ and in general $\mathscr{C}_{n+1}$ is the one consisting of those connected C^∞-manifolds which are total spaces of orientable fibre bundles whose base spaces belong to $\mathscr{C}_n$ and whose fibres are disjoint union of closed orientable surfaces of genus $\geqq 2$. Let $\mathscr{C}$ denote the class of manifolds which are type $\mathscr{C}_n$ for some n. We call elements of $\mathscr{C}$ iterated surface bundles.*

It is clear from the definition that the class $\mathscr{C}$ is closed under the operation of taking finite coverings.

Proposition 4.3. *Let M_0 be a manifold belonging to the class $\mathscr{C}_n$ and let m be a natural number. Then for any cohomology class $u_0 \in H^2(M_0; \mathbf{Z}/m)$, there exists a finite covering space $p: \tilde{M}_0 \to M_0$ such that $p^*(u_0)=0$.*

Proof. We use the induction on n. The assertion is clear for the case $n=1$. Thus assume that M_0 fibres over a manifold $X_0 \in \mathscr{C}_{n-1}$ with fibre Σ which is a disjoint union of orientable surfaces. $\pi_1(X_0)$ acts on the finite set $\pi_0(\Sigma)$ of connected component of Σ. Let $X \to X_0$ be a finite covering map which kills this action and let $M'_0 \to X$ be the pull back bundle. We have $X \in \mathscr{C}_{n-1}$. Now choose a connected component M of M'_0. It is a finite covering space of M_0 and we have the cohomology class $u \in H^2(M; \mathbf{Z}/m)$ with is the pull back of $u_0 \in H^2(M_0; \mathbf{Z}/m)$. Also it is clear that the projection $M \to X$ is a Σ_g-bundle for some g, because its fibre is connected. Hence by Lemma 4.1, there is a finite covering map $X_1 \to X$ such that the pull back Σ_g-bundle $M_1 \to X_1$ admits a fibre-wise m-fold covering map $M'_1 \to M_1$. Namely the composed map $M'_1 \to X_1$ is a $\Sigma_{g'}$-bundle, where $\Sigma_{g'}$ is an m-fold covering of Σ_g. $\pi_1(X_1)$ acts on $H^1(\Sigma_{g'}; \mathbf{Z}/m)$ which is a finite group. Let $X'_1 \to X_1$ be a finite covering which kills this action and let $X_2 \to X'_1$ be another finite covering such that the induced homomorphism $H^1(X'_1; \mathbf{Z}/m) \to H^1(X_2; \mathbf{Z}/m)$ is trivial. Let $M_2 \to X_2$ be the pull back $\Sigma_{g'}$-bundle. We summarize the above construction as the following commutative diagram:

$$\begin{array}{ccccccc}
M_2 & \longrightarrow & M'_1 & \longrightarrow & M_1 & \longrightarrow & M \\
\pi \downarrow \Sigma_{g'} & & \downarrow \Sigma_{g'} & & \downarrow \Sigma_g & & \downarrow \Sigma_g \\
X_2 & \longrightarrow & X_1 & \longrightarrow & X_1 & \longrightarrow & X.
\end{array}$$

Let us write $p: M_2 \to M$ for the composition of the three maps in the above top sequence. We claim that there is an element $v \in H^2(X_2; \mathbf{Z}/m)$ such that $p^*(u) = \pi^*(v)$. To show this, consider the Serre spectral sequence $\{E_r^{p,q}, d_r\}$ for the cohomology $H^*(M_2; \mathbf{Z}/m)$. The E_2-term is given by $E_2^{p,q} = H^p(X_2; H^q(\Sigma_{g'}; \mathbf{Z}/m))$. $p^*(u)$ is an element of $H^2(M_2; \mathbf{Z}/m)$ and we have short exact sequences

$$0 \to K \to H^2(M_2; \mathbf{Z}/m) \to E_\infty^{0,2} \to 0$$
$$0 \to E_\infty^{2,0} \to K \to E_\infty^{1,1} \to 0$$

where $K = \operatorname{Ker}(H^2(M_2; \mathbf{Z}/m) \to H^2(\Sigma_{g'}; \mathbf{Z}/m))$. The image of $p^*(u)$ in $E_\infty^{0,2}$ is zero because $\Sigma_{g'} \to \Sigma_g$ is an m-fold covering. Hence $p^*(u)$ is contained in K. Then it follows from the constructions of the finite coverings $X'_1 \to X_1$ and $X_2 \to X'_1$ that the image of $p^*(u)$ in $E_\infty^{1,1}$ vanishes. Hence $p^*(u)$ is contained in $E_\infty^{2,0} = \operatorname{Im}(H^2(X_2; \mathbf{Z}/m) \to H^2(M_2; \mathbf{Z}/m))$. Therefore there is an element $v \in H^2(X_2; \mathbf{Z}/m)$ such that $\pi^*(v) = p^*(u)$ as required. Now observe that $X_2 \in \mathscr{C}_{n-1}$. Hence by the induction assumption there is a finite covering $p: X_3 \to X_2$ such that $p^*(v)=0$. Finally let $\tilde{M}_0 \to X_3$ be the pull back $\Sigma_{g'}$-bundle. Then it is clear from the above construction that the finite covering $\tilde{M}_0 \to M_0$ satisfies the required condition. This completes the proof.

Now we recall a well-known criterion for the existence of a ramified covering.

Proposition 4.4 (see Hirzebruch [Hi]). *Let M be an oriented closed $2n$-manifold and let N be a (possibly disconnected) oriented submanifold of M of codimension 2. Assume that the cohomology class $v \in H^2(M; \mathbf{Z})$, which is the Poincaré dual of the fundamental homology class of N, is divisible by a natural number m as an integral class. Then there exists an m-fold cyclic covering $p: \tilde{M} \to M$ ramified along N.*

Now we can describe our main construction. Let $\pi: E \to X$ be a given oriented Σ_g-bundle such that the total space E belongs to the class $\mathscr{C}$. For each natural number $m \geqq 2$, we define an *m-construction* on it as follows. Roughly speaking it constructs a new surface bundle from the old one by a combination of taking pull backs and ordinary or ramified finite coverings. More precisely consider first the following pull back diagram:

$$\begin{array}{ccc} E^* & \xrightarrow{q} & E \\ {\scriptstyle \pi'}\downarrow {\scriptstyle \Sigma_g} & & \downarrow {\scriptstyle \Sigma_g} \\ E & \xrightarrow[\pi]{} & X. \end{array}$$

Here $E^* = \{(z, z') \in E \times E;\ \pi(z) = \pi(z')\}$, $\pi'(z, z') = z$ and $q(z, z') = z'$. The Σ_g-bundle $\pi': E^* \to E$ admits a cross section $s: E \to E^*$ defined by $s(z) = (z, z)$. We write D for $\operatorname{Im} s$. Let $v \in H^2(E^*; \mathbf{Z})$ be the cohomology class defined by it and let $v_m \in H^2(E^*; \mathbf{Z}/m)$ be the mod m reduction of v. Now we consider the following commutative diagram of surface bundles:

$$\begin{array}{ccccccccccc} \tilde{E}^* & \longrightarrow & E_3^* & \longrightarrow & E_2^* & \longrightarrow & 'E_1^* & \longrightarrow & E_1^* & \longrightarrow & E^* \\ \downarrow {\scriptstyle \Sigma_{g''}} & & \downarrow {\scriptstyle \Sigma_{g'}} & & \downarrow {\scriptstyle \Sigma_{g'}} & & \downarrow {\scriptstyle \Sigma_{g'}} & & \downarrow {\scriptstyle \Sigma_g} & & \downarrow {\scriptstyle \Sigma_g} \\ \tilde{E} & = & \tilde{E} & \longrightarrow & E_2 & \longrightarrow & E_1 & = & E_1 & \longrightarrow & E. \end{array}$$

Here the four surface bundles on the right of the above diagram are the ones obtained by applying the construction of Proposition 4.3 (see especially the diagram in its proof) to the Σ_g-bundle $E^* \to E$ and the cohomology class $v_m \in H^2(E^*; \mathbf{Z}/m)$ replacing $M \to X$ and $u \in H^2(X; \mathbf{Z}/m)$. In particular $'E_1^* \to E^*$ is an m-fold fibre-wise covering (see also Lemma 4.1). By the construction, the image in $H^2(E_2^*; \mathbf{Z}/m)$ of the cohomology class v_m comes from an element $v \in H^2(E_2; \mathbf{Z}/m)$. Observe that E_2 belongs to the class $\mathscr{C}$. Hence again by Proposition 4.3, there is a finite covering $p: \tilde{E} \to E_2$ such that $p^*(v) = 0$. $E_3^* \to \tilde{E}$ is the pull back $\Sigma_{g'}$-bundle. Clearly the image of v_m in $H^2(E_3^*; \mathbf{Z}/m)$ vanishes. This means the following. If we write D^* for the inverse image of D under the map $E_3^* \to E^*$, the pair (E_3^*, D^*) satisfies the condition of Proposition 4.4. Therefore there exists an m-fold cyclic covering $\tilde{E}^* \to E_3^*$ ramified along D^*. The projection $\tilde{E}^* \to \tilde{E}$ is a $\Sigma_{g''}$-bundle where $\Sigma_{g''}$ is an m-fold covering of $\Sigma_{g'}$ ramified along m points on it (hence $g'' = m^2 g - \frac{1}{2} m(m+1) + 1$). This is the surface bundle obtained by applying an *m-construction* on the original bundle $\pi: E \to X$. It is clear that $\tilde{E}^*$ belongs to the class $\mathscr{C}$ so that we can apply an m'-construction on $\tilde{E}^* \to \tilde{E}$. In this way starting from a surface bundle whose total space belongs to $\mathscr{C}$, we can apply m_j-constructions successively ($j=1, 2, \ldots$) to obtain various surface bundles. The 2-construction on the trivial surface bundle $\Sigma_g \to \text{pt.}$ is nothing but Atiyah's method in [A] (see also Kodaira [Ko]). The above is therefore a generalization of their constructions.

5. Non-triviality of the characteristic classes

In this section we compute the characteristic classes of surface bundles constructed in § 4. For a given oriented Σ_g-bundle $\pi: E \to X$ with the total space E being a member of $\mathscr{C}$, we consider an m-construction on it:

$$\begin{array}{ccccc} \tilde{E}^* & \xrightarrow{r} & E^* & \xrightarrow{q} & E \\ {\scriptstyle \tilde{\pi}} \downarrow {\scriptstyle \Sigma_{g''}} & & {\scriptstyle \pi'} \downarrow {\scriptstyle \Sigma_g} & & {\scriptstyle \pi} \downarrow {\scriptstyle \Sigma_g} \\ \tilde{E} & \xrightarrow[\tilde{r}]{} & E & \xrightarrow[\pi]{} & X. \end{array}$$

Let $e \in H^2(E; \mathbf{Z})$ (resp. $\tilde{e} \in H^2(\tilde{E}^*; \mathbf{Z})$) be the Euler class of $\pi: E \to X$ (resp. $\tilde{\pi}: \tilde{E}^* \to \tilde{E}$). Recall that we write $D \subset E^*$ for the image of the "diagonal" cross section which is an oriented submanifold of E^* of codimension 2. Set $\tilde{D} = r^{-1}(D)$. We write $v \in H^2(E^*; \mathbf{Z})$ (resp. $\tilde{v} \in H^2(\tilde{E}^*; \mathbf{Z})$) for the cohomology class defined by the codimension two submanifold D (resp. $\tilde{D}$).

Lemma 5.1. (i) *The Euler class of* $\pi': E^* \to E$ *is* $q^*(e)$.
(ii) $v^2 = q^*(e)\, v = (\pi')^*(e)\, v$ *in* $H^4(E^*; \mathbf{Z})$.

Proof. (i) is clear. To prove (ii), let $i: D \to E^*$ be the inclusion. It is easy to see that the cohomology class $i^* q^*(e) = i^* (\pi')^*(e)$ is equal to the Euler class of the normal bundle of D in E^*. Then the required assertion follows from a standard argument using the Thom isomorphism theorem.

Lemma 5.2. $r^*(v) = m\, \tilde{v}$ *and hence* $\tilde{v} = \frac{1}{m} r^*(v)$ in $H^2(\tilde{E}^*; \mathbf{Q})$.

Proof is easy and omitted.

Lemma 5.3. $\tilde{e} = r^* q^*(e) - (m-1)\, \tilde{v}$ *in* $H^2(\tilde{E}^*; \mathbf{Z})$.

Proof. Let ξ (resp. $\tilde{\xi}$) be the tangent bundle of $E^* \to E$ (resp. $\tilde{E}^* \to \tilde{E}$). Then the induced bundle $r^*(\xi)$ is canonically isomorphic to $\tilde{\xi}$ on $\tilde{E}^* - \tilde{D}$. Also near the locus $\tilde{D}$, the map r is locally the identity of $\tilde{D}$ times the map $\mathbf{C} \to \mathbf{C}$ given by $z \to z^m$. It is easy to deduce the required assertion from these facts.

Now we write $\tilde{e}_k$ and e_k for the characteristic classes of $\tilde{\pi}: \tilde{E}^* \to \tilde{E}$ and $\pi: E \to X$ corresponding to the k-th class e_k.

Proposition 5.4. (i) $\tilde{e} = r^* \left\{ q^*(e) - \frac{m-1}{m} v \right\}$ in $H^2(\tilde{E}^*; \mathbf{Q})$.

(ii) $\tilde{e}_k = m^2\, \tilde{r}^* \{\pi^*(e_k) - (1 - m^{-(k+1)})\, e^k\}$.

Proof. (i) follows from Lemma 5.2 and Lemma 5.3. We have

$$\tilde{e}^{k+1} = r^* \left\{ \left(q^*(e) - \frac{m-1}{m} v \right)^{k+1} \right\}$$

$$= r^* \{ q^*(e^{k+1}) - (1 - m^{-(k+1)}) \pi^*(e^k) v \}$$

here we have used Lemma 5.1. If we apply the Gysin homomorphism $\pi_*: H^{2(k+1)}(E^*; Q) \to H^{2k}(E; Q)$ to the above equation, we obtain

$$\tilde{e}_k = m^2 \bar{r}^* \{ \pi^*(e_k) - (1 - m^{-(k+1)}) e^k \}$$

here we have used the fact that the map $\Sigma_{g''} \to \Sigma_g$ is a ramified covering of degree m^2. This completes the proof.

If the base manifold X of a surface bundle is a closed oriented $2n$-manifold, then we can evaluate cohomology classes of X of degree $2n$, which are polynomials in the characteristic classes of the bundle, on the fundamental cycle of X to obtain various numbers. More precisely for each partition $I = \{i_1, \ldots, i_r\}$ of n, we have the corresponding number

$$e_I[X] = e_{i_1} \ldots e_{i_r}[X].$$

We call them *characteristic numbers* of the bundle. For each subset $J = \{j_1, \ldots, j_s\}$ of a partition $I = \{i_1, \ldots, i_r\}$ of some natural number, we express the complement $J^c = I \setminus J$ as $J^c = \{k_1, \ldots, k_t\}$ $(s + t = r)$. With these notations we have

Proposition 5.5. *Let $\pi: E \to X$ be an oriented Σ_g-bundle over an oriented closed $2n$-dimensional manifold X and let $\tilde{\pi}: \tilde{E}^* \to \tilde{E}$ be a surface bundle obtained by applying an m-construction on it. Then for each partition $I = \{i_1, \ldots, i_r\}$ of $n+1$, the I-th characteristic number of $\tilde{\pi}: \tilde{E}^* \to \tilde{E}$ is given by*

$$\tilde{e}_I[\tilde{E}] = dm^{2r} \sum_J (-1)^t (1 - m^{-(k_1+1)}) \ldots (1 - m^{-(k_t+1)}) e_J e_{k_1 + \ldots + k_t - 1}[X]$$

where d is the degree of the finite covering $\tilde{E} \to E$ and J runs through all the subsets of I. In particular $\tilde{e}_{n+1}[E] = d(m^{-n} - m^2) e_n[X]$ so that if $e_n[X]$ is different from zero and $m \neq 1$, then so is $e_{n+1}[E]$.

Proof. Using Proposition 5.4, (ii) we compute

$$\begin{aligned} \tilde{e}_I[\tilde{E}] &= \tilde{e}_{i_1} \ldots \tilde{e}_{i_r}[\tilde{E}] \\ &= m^2 \bar{r}^* \{ \pi^*(e_{i_1}) - (1 - m^{-(i_1+1)}) e^{i_1} \} \ldots m^2 \bar{r}^* \{ \pi^*(e_{i_r}) - (1 - m^{-(i_r+1)}) e^{i_r} \} [\tilde{E}] \\ &= dm^{2r} \{ \pi^*(e_{i_1}) - (1 - m^{-(i_1+1)}) e^{i_1} \} \ldots \{ \pi^*(e_{i_r}) - (1 - m^{-(i_r+1)}) e^{i_r} \} [E] \\ &= dm^{2r} \sum_J (-1)^t (1 - m^{-(k_1+1)}) \ldots (1 - m^{-(k_t+1)}) e_J e_{k_1 + \ldots + k_t - 1}[X]. \end{aligned}$$

Here the last equality follows from the fact that the Gysin homomorphism $\pi_*: H^{2(n+1)}(E; \mathbf{Z}) \to H^{2n}(X; \mathbf{Z})$ is an isomorphism. This completes the proof.

Now we apply our m-constructions successively to obtain a "tower" of surface bundles. More precisely we define a $\Sigma_{g(n)}$-bundle $\pi_n: E_n \to X_n$ inductively as follows. π_0 is the trivial Σ_g-bundle $\Sigma_g \to \mathrm{pt.}$ with $g \geqq 2$ and for $n \geqq 1$ we define $\pi_n: E_n \to X_n$ to be the surface bundle obtained by applying an m_n-construction on $\pi_{n-1}: E_{n-1} \to X_{n-1}$. Thus we have a commutative diagram:

$$\begin{array}{ccccc}
E_n & \xrightarrow{r_n} & E^*_{n-1} & \xrightarrow{q_{n-1}} & E_{n-1} \\
{\scriptstyle \pi_n}\downarrow & & \downarrow{\scriptstyle \pi'_{n-1}} & & \downarrow{\scriptstyle \pi_{n-1}} \\
X_n & \xrightarrow[\bar{r}_n]{} & E_{n-1} & \xrightarrow[\pi_{n-1}]{} & X_{n-1}
\end{array}$$

where $\pi'_{n-1}: E^*_{n-1} \to E_{n-1}$ is the pull back of $\pi_{n-1}: E_{n-1} \to X_{n-1}$ by the map π_{n-1} so that $E^*_{n-1} = \{(z, z') \in E_{n-1} \times E_{n-1};\ \pi_{n-1}(z) = \pi_{n-1}(z')\}$ (see § 4). Let $(\Sigma_g)^n$ be the Cartesian product of n-copies of Σ_g. We define maps $p_n: E_n \to (\Sigma_g)^{n+1}$ and $\bar{p}_n: X_n \to (\Sigma_g)^n$ which make the following diagram commutative

$$\begin{array}{ccc}
E_n & \xrightarrow{p_n} & (\Sigma_g)^{n+1} \\
{\scriptstyle \pi_n}\downarrow & & \downarrow{\scriptstyle \pi} \\
X_n & \xrightarrow[\bar{p}_n]{} & (\Sigma_g)^n
\end{array}$$

inductively as follows, where $\pi: (\Sigma_g)^{n+1} \to (\Sigma_g)^n$ is the projection onto the first n-factors. First p_0 and $\bar{p}_0$ are the identities. For $n \geqq 1$, $\bar{p}_n$ is defined to be the composed map $X_n \xrightarrow{\bar{r}_n} E_{n-1} \xrightarrow{p_{n-1}} (\Sigma_g)^n$. To define p_n, first define a map

$$p^*_{n-1}: E^*_{n-1} \to (\Sigma_g)^{n+1}$$

as follows. For each element $(z, z') \in E^*_{n-1}$, consider $p_{n-1}(z)$ and $p_{n-1}(z') \in (\Sigma_g)^n$. By the induction assumption, the first $(n-1)$-components in $(\Sigma_g)^n$ of them coincide because $\pi_{n-1}(z) = \pi_{n-1}(z')$. We set $p^*_{n-1}(z, z') = (p_{n-1}(z)$, the last component of $p_{n-1}(z')) \in (\Sigma_g)^{n+1}$. Finally we define p_n to be the composed map $E_n \xrightarrow{r_n} E^*_{n-1} \xrightarrow{p^*_{n-1}} (\Sigma_g)^{n+1}$. From the definition it is easy to see that $\pi p_n = \bar{p}_n \pi_n$ so that the induction assumption is satisfied. The above construction shows that the $\Sigma_{g(n)}$-bundle $\pi_n: E_n \to X_n$ is a "ramified covering" of the trivial Σ_g-bundle $(\Sigma_g)^{n+1} \to (\Sigma_g)^n$.

If we apply Proposition 5.5 to the surface bundles $\pi_n: E_n \to X_n$ constructed above, we can inductively determine all the characteristic numbers of them and we obtain

Proposition 5.6. *For any non-negative integer n, there exists no non-trivial linear relation between the characteristic numbers of surface bundles whose base spaces are iterated surface bundles of dimension 2n.*

Proof. The assertion is clear for the case $n = 0$. We use the induction on n. Thus we assume that the assertion holds for n and prove it for $n+1$. Suppose that some linear relation $\sum_I a_I e_I = 0$ holds for $n+1$. Let us recall here that we can make the operations of our m-constructions for *any* m. Then in view of the form of the formula of Proposition 5.5, it is easy to deduce from the induction assumption that all the a_I must vanish. This completes the proof.

In the above we have proved in particular that the n-th characteristic class e_n of $\pi_n: E_n \to X_n$ is non-trivial (provided $m_i > 1$ for all i). In the next section we prove a stronger statement (Proposition 6.4). For that we look into the surface bundles $\pi_n: E_n \to X_n$ more closely.

We have the "diagonal" $D_k \subset E^*_{k-1}$ ($k=1, \ldots, n$) which is a codimension two submanifold of E^*_{k-1}. Let us write $v_k \in H^2(E^*_{k-1}; \mathbf{Z})$ for the corresponding cohomology class. Next consider the codimension two submanifold $\tilde{D}_k = r_k^{-1}(D_k)$ of E_k. We have the corresponding cohomology class $\tilde{v}_k \in H^2(E_k; \mathbf{Z})$. Of course we have $r_k^*(v_k) = m_k \tilde{v}_k$ (see Lemma 5.2). Now we define a map $Q_{n,k}: E_n \to E_k$ to be the composition

$$Q_{n,k}: E_n \xrightarrow{r_n} E^*_{n-1} \xrightarrow{q_{n-1}} E_{n-1} \to \ldots \to E^*_k \xrightarrow{q_k} E_k$$

and set $u_k^{(n)} = Q^*_{n,k}(\tilde{v}_k) \in H^2(E_n; \mathbf{Z})$ ($u_n^{(n)} = \tilde{v}_n$). Next let $v_0 \in H^2(\Sigma_g; \mathbf{Z})$ be the Euler class of the *cotangent* bundle of Σ_g (so that $v_0 = 2g - 2 \in \mathbf{Z} \cong H^2(\Sigma_g; \mathbf{Z})$). We define $u_0^{(n)} = Q^*_{n,0}(v_0) \in H^2(E_n; \mathbf{Z})$. With these notations we have

Proposition 5.7. *The Euler class $e^{(n)}$ of the $\Sigma_{g(n)}$-bundle $\pi_n: E_n \to X_n$ is given by*

$$e^{(n)} = -\{u_0^{(n)} + (m_1 - 1)\, u_1^{(n)} + \ldots + (m_n - 1)\, u_n^{(n)}\}.$$

Proof. A simple inductive argument using Lemma 5.3 yields the result.

Remark 5.8. We can define the operation of our m-constructions in the category of algebraic varieties (apply the argument of [A][Hi]) so that if one wants, we may assume that E_n, X_n and E^*_n are all non-singular algebraic varieties and maps between them are holomorphic. In such a situation, the cohomology class $u_k^{(n)}$ is the one corresponding to the divisor $Q_{n,k}^{-1}(\tilde{D}_k)$ of E_n which turn out to be *non-singular.*

Lemma 5.9. (i) $(u_0^{(n)})^2 = 0$.

(ii) $$(u_k^{(n)})^2 = -\frac{1}{m_k}\, u_k^{(n)} \{u_0^{(n)} + (m_1 - 1)\, u_1^{(n)} + \ldots + (m_{k-1} - 1)\, u_{k-1}^{(n)}\} \quad (k = 1, \ldots, n).$$

Proof. (i) is clear. To prove (ii), we apply Lemma 5.1 to the following commutative diagram:

$$\begin{array}{ccc} E^*_{k-1} & \xrightarrow{q_{k-1}} & E_{k-1} \\ \downarrow{\scriptstyle \pi'_{k-1}} & & \downarrow \\ E_{k-1} & \xrightarrow[\pi_{k-1}]{} & X_{k-1} \end{array}$$

we obtain

$$v_k^2 = -v_k\, q^*_{k-1} \{u_0^{(k-1)} + (m_1 - 1)\, u_1^{(k-1)} + \ldots + (m_{k-1} - 1)\, u_{k-1}^{(k-1)}\}$$

here we have used Proposition 5.7 applied to the bundle $\pi_{k-1}: E_{k-1} \to X_{k-1}$.

If we pull back the above equation to E_k by the map $r_k\colon E_k \to E^*_{k-1}$, we obtain

$$m_k\, \tilde{v}_k^2 = -\tilde{v}_k \{u_0^{(k)} + (m_1 - 1)\, u_1^{(k)} + \ldots + (m_{k-1} - 1)\, u_{k-1}^{(k)}\}$$

here we have used Lemma 5.2. The required assertion follows from this immediately.

6. Non-triviality of the characteristic classes (continued)

In this section we prove the following theorem which shows that there are no relations between polynomials in e_i's in small degrees.

Theorem 6.1. *For any natural number n, there exists a number $g(n)$ such that the homomorphisms*

$$\mathbf{Q}[e_1, \ldots, e_{g-2}]/\text{relations} \to H^*(\mathcal{M}_g; \mathbf{Q})$$
$$\mathbf{Q}[e_1, \ldots, e_{g-2}]/\text{relations} \to H^*(\mathcal{M}_{g,1}; \mathbf{Q})$$

(*see Theorem* 2.1) *are injective up to degree $2n$ for all $g \geqq g(n)$.*

Remark 6.2. The above theorem was first proved by Miller [Mi] by constructing surface bundles with non-zero characteristic number e_n for all n and then using Harer's stability theorem on the homology of the mapping class groups [Har 2], which states that the homology groups $H_k(\mathcal{M}_g; \mathbf{Q})$ and $H_k(\mathcal{M}_{g,1}; \mathbf{Q})$ are all naturally isomorphic to each other in the range $k \leqq \frac{1}{3} g$. (In particular we can take $6n$ for $g(n)$.) In the following we would like to give a more constructive proof of Theorem 6.1. Our proof will play an essential role when we later generalize Theorem 6.1 to surface bundles with cross sections (Theorem 7.5) because Miller's argument does not apply to them.

To prove Theorem 6.1 we will first show (Corollary 6.5) that each class e_i is non-trivial. Then we will use Proposition 3.4 to show the e_i are independent in the range of the Theorem. Now we prove

Proposition 6.3. *Let $\pi\colon E \to X$ be an oriented Σ_g-bundle and let $e \in H^2(E; \mathbf{Q})$ (resp. $e_i = \pi_*(e^{i+1}) \in H^{2i}(X; \mathbf{Q})$) be its Euler class (resp. the i-th characteristic class) with rational coefficients. Suppose that $\pi^*(e_i)$ is not divisible by e. Then the universal i-th characteristic class e_i is non-trivial in $H^{2i}(\mathcal{M}_{g,1}; \mathbf{Q})$.*

Proof. Let $\hat{\pi}\colon \hat{E} \to E$ be the S^1-bundle defined by the cohomology class $e \in H^2(E; \mathbf{Z})$. Then we have the following commutative diagram:

$$\begin{array}{ccccc}
\hat{E} & \xrightarrow{\hat{\pi}} & E & \xrightarrow{\pi} & X \\
\big\downarrow{\scriptstyle \hat{f}} & & \big\downarrow & & \big\downarrow{\scriptstyle f} \\
\hat{E}\,\mathrm{Diff}_+\,\Sigma_g & \longrightarrow & E\,\mathrm{Diff}_+\,\Sigma_g & \longrightarrow & B\,\mathrm{Diff}_+\,\Sigma_g
\end{array}$$

where $f: X \to B\,\mathrm{Diff}_+ \Sigma_g$ is the classifying map. Hence $\hat{f}^*(e_i) = \hat{\pi}^* \pi^*(e_i)$. The Gysin exact sequence for the rational cohomology group of the S^1-bundle $\hat{E} \to E$ shows that if $\pi^*(e_i)$ is not divisible by e, then $\hat{\pi}^* \pi^*(e_i) \neq 0$ (cf. Proposition 3.3). The result follows from this.

Now we consider the $\Sigma_{g(n)}$-bundle $\pi_n: E_n \to X_n$ constructed in § 5. In this section we assume that $m_i = 2$ $(i = 1, \ldots, n)$. We claim

Proposition 6.4. *Let $e \in H^2(E_{2i-1}; \mathbf{Q})$ (resp. $e_i \in H^{2i}(E_{2i-1}; \mathbf{Q})$) be the Euler class (resp. the i-th characteristic class) of the $\Sigma_{g(2i-1)}$-bundle $\pi_{2i-1}: E_{2i-1} \to X_{2i-1}$. Then $\pi_{2i-1}^*(e_i)$ is not divisible by e in $H^*(E_{2i-1}; \mathbf{Q})$.*

If we combine Proposition 6.3 and Proposition 6.4, we obtain

Corollary 6.5. *For any i, there is a number g_i such that the i-th characteristic class e_i is non-trivial in $H^{2i}(\mathscr{M}_{g_i, 1}; \mathbf{Q})$.*

To prove Proposition 6.4, we prepare a few Lemmas. We have $(n+1)$ cohomology classes $u_0, \ldots, u_n \in H^2(E_n; \mathbf{Z})$. Here and henceforth we simply write u_k for $u_k^{(n)}$. For each number k with $n > k \geqq 0$, we consider the cohomology classes

$$u_{i_1} \ldots u_{i_l} \pi_n^*(e_k) \in H^{2(n+1)}(E_n; \mathbf{Z}) \cong \mathbf{Z}$$

as *numbers*, where $l = (n+1) - k$ and $e_k \in H^{2k}(X_n; \mathbf{Z})$ is the k-th characteristic class of $\pi_n: E_n \to X_n$. In the above expression we always assume that i_j's are all different. For a finite ramified covering $r: \tilde{E} \to E$, we denote $\deg r$ for the degree of it. With these conventions and notations we have

Lemma 6.6. (i) *Assume that $i_j < n$ for all $j = 1, \ldots, l$. Then*

$$u_{i_1} \ldots u_{i_l} \pi_n^*(e_k) = -2^2(1 - 2^{-(k+1)}) \deg r_n\, u_{i_1} \ldots u_{i_l} \pi_{n-1}^*(e'_{k-1})$$

where $e'_{k-1} \in H^{2(k-1)}(X_{n-1}; \mathbf{Z})$ is the $(k-1)$-st characteristic class of $\pi_{n-1}: E_{n-1} \to X_{n-1}$.

(ii) *We have*

$$\begin{aligned} u_{i_1} \ldots u_{i_{l-1}} u_n \pi_n^*(e_k) = {} & 2 \deg r_n\, u_{i_1} \ldots u_{i_{l-1}} \pi_{n-1}^*(e'_k) \\ & - 2(1 - 2^{-(k+1)}) \deg r_n\, u_{i_1} \ldots u_{i_{l-1}} (e')^k \end{aligned}$$

where $e' \in H^2(E_{n-1}; \mathbf{Z})$ is the Euler class of $\pi_{n-1}: E_{n-1} \to X_{n-1}$.

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccc} E_n & \xrightarrow{r_n} & E_{n-1}^* & \xrightarrow{q_{n-1}} & E_{n-1} \\ \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi'_{n-1}} & & \downarrow{\scriptstyle \pi_{n-1}} \\ X_n & \xrightarrow[\bar{r}_n]{} & E_{n-1} & \xrightarrow[\pi_{n-1}]{} & X_{n-1}. \end{array}$$

We prove (i). According to Proposition 5.4, (ii), we have

$$e_k = 2^2 \bar{r}_n^* \{\pi_{n-1}^*(e'_k) - (1 - 2^{-(k+1)})(e')^k\}.$$

Hence we have

$$u_{i_1}\dots u_{i_l}\,\pi_n^*(e_k)=-2^2(1-2^{-(k+1)})\deg r_n\,q_{n-1}^*(u_{i_1}\dots u_{i_l})(\pi'_{n-1})^*\{(e')^k\}$$

here we have used the obvious fact that $u_{i_1}\dots u_{i_l}\,e'_k=0$ (we can consider $u_{i_1}\dots u_{i_l}$ as an element of $H^*(E_{n-1};\mathbf{Z})$ because of the assumption that $i_j<n$ for all j). Now let $\iota\colon E_{n-1}^*\to E_{n-1}^*$ be the involution defined by $\iota(z,z')=(z',z)$. Clearly it preserves the orientation of E_{n-1}^* so that the induced homomorphism $\iota^*\colon H^{2(n+1)}(E_{n-1}^*;\mathbf{Z})\to H^{2(n+1)}(E_{n-1}^*;\mathbf{Z})$ is the identity. Obviously $\pi'_{n-1}\,\iota=q_{n-1}$ and $q_{n-1}\,\iota=\pi'_{n-1}$. Hence we have

$$q_{n-1}^*(u_{i_1}\dots u_{i_l})(\pi'_{n-1})^*\{(e')^k\}=(\pi'_{n-1})^*(u_{i_1}\dots u_{i_l})\,q_{n-1}^*\{(e')^k\}.$$

Now if we apply the Gysin homomorphism $(\pi'_{n-1})_*\colon H^{2(n+1)}(E_{n-1}^*;\mathbf{Z})\to H^{2n}(E_{n-1};\mathbf{Z})$, which is clearly an isomorphism, we obtain

$$(\pi'_{n-1})^*(u_{i_1}\dots u_{i_l})\,q_{n-1}^*\{(e')^k\}=u_{i_1}\dots u_{i_l}\,\pi_{n-1}^*(e'_{k-1}).$$

The result follows.

Next we prove (ii). Recall that $u_n=\frac{1}{2}r_n^*(v_n)$ (see Lemma 5.2). Hence by a similar argument as above, we have

$$u_{i_1}\dots u_{i_{l-1}}u_n\,\pi_n^*(e_k)=2\deg r_n\,u_{i_1}\dots u_{i_{l-1}}\,v_n(\pi'_{n-1})^*\{\pi_{n-1}^*(e'_k)-(1-2^{-(k+1)})(e')^k\}.$$

Now it is easy to see that for any cohomology class $v\in H^*(E_{n-1};\mathbf{Z})$, we have the equality $v_n(\pi'_{n-1})^*(v)=v_n\,q_{n-1}^*(v)$ (cf. the proof of Lemma 5.1, (ii)). Hence we have

$$u_{i_1}\dots u_{i_{l-1}}u_n\,\pi_n^*(e_k)=2\deg r_n\,v_n(\pi'_{n-1})^*\{u_{i_1}\dots u_{i_{l-1}}(\pi_{n-1}^*(e'_k)-(1-2^{-(k+1)})(e')^k)\}.$$

As before if we apply the Gysin homomorphism $(\pi'_{n-1})_*$ to the above, we obtain the result because $(\pi'_{n-1})_*(v_n)=1$. This completes the proof.

Next for each i, consider the cohomology class

$$u=(u_0-2u_1)(u_2-2u_3)\dots(u_{2i-2}-2u_{2i-1})\in H^{2i}(E_{2i-1};\mathbf{Z}).$$

We claim

Lemma 6.7. *The cohomology class $u\in H^{2i}(E_{2i-1};\mathbf{Z})$ defined above is contained in* Ker $(\cup e\colon H^{2i}(E_{2i-1};\mathbf{Z})\to H^{2(i+1)}(E_{2i-1};\mathbf{Z}))$.

Proof. We use the induction on i. For $i=1$, $e=-(u_0+u_1)$ and $u_1^2=-\frac{1}{2}u_0u_1$ (see Proposition 5.7 and Lemma 5.9). The equation $ue=0$ follows from this easily. Next we assume that the assertion is true for $i-1$ and prove it for i $(i>1)$. We have

$$\begin{aligned}-ue&=(u_0-2u_1)\dots(u_{2i-2}-2u_{2i-1})(u_0+\dots+u_{2i-1})\\&=(u_0-2u_1)\dots(u_{2i-2}-2u_{2i-1})(u_{2i-2}+u_{2i-1})\\&=(u_0-2u_1)\dots(u_{2i-4}-2u_{2i-3})(u_0+\dots+u_{2i-3})(u_{2i-1}-\tfrac{1}{2}u_{2i-2})\\&=0\end{aligned}$$

here we have used the induction assumption and Lemma 5.9.

Proof of Proposition 6.4. Assume that $\pi_{2i-1}^*(e_i)$ is divisible by e in $H^*(E_{2i-1};\mathbf{Q})$.

Then in view of Lemma 6.7, we should have $u\pi_{2i-1}^*(e_i)=0$. Hence to prove Proposition 6.4, it is enough to show that the number

$$R_i=u\pi_{2i-1}^*(e_i)$$

is non-zero. We claim

$$R_i=2^3(1-2^{-(i+1)})\deg r_{2i-1}\deg r_{2i-2}R_{i-1}\qquad (i\geqq 2).$$

To prove this, consider the following commutative diagram:

$$\begin{array}{ccccccccc}
E_{2i-1} & \xrightarrow{r_{2i-1}} & E_{2i-2}^* & \longrightarrow & E_{2i-2} & \xrightarrow{r_{2i-2}} & E_{2i-3}^* & \longrightarrow & E_{2i-3}\\
\downarrow{\scriptstyle \pi_{2i-1}} & & \downarrow{\scriptstyle \pi'_{2i-2}} & & \downarrow{\scriptstyle \pi_{2i-2}} & & \downarrow{\scriptstyle \pi'_{2i-3}} & & \downarrow{\scriptstyle \pi_{2i-3}}\\
X_{2i-1} & \longrightarrow & E_{2i-2} & \longrightarrow & X_{2i-2} & \longrightarrow & E_{2i-3} & \longrightarrow & X_{2i-3}.
\end{array}$$

We denote $e'\in H^2(E_{2i-2};\mathbf{Z})$ (resp. $e''\in H^2(E_{2i-3};\mathbf{Z})$) for the Euler class of $\pi_{2i-2}: E_{2i-2}\to X_{2i-2}$ (resp. $\pi_{2i-3}: E_{2i-3}\to X_{2i-3}$) and similarly we denote $e'_j\in H^{2j}(E_{2i-2};\mathbf{Z})$ (resp. $e''_j\in H^{2j}(E_{2i-3};\mathbf{Z})$) for the corresponding characteristic class. We also set $u'=(u_0-2u_1)\dots(u_{2i-4}-2u_{2i-3})$ so that $u=u'u_{2i-2}-2u'u_{2i-1}$. Now if we apply Lemma 6.6 to R_i, we obtain

$$\begin{aligned}R_i= &-2^2(1-2^{-(i+1)})\deg r_{2i-1}\,u'u_{2i-2}\,\pi_{2i-2}^*(e'_{i-1})\\ &-2^2\deg r_{2i-1}\,u'\pi_{2i-2}^*(e'_i)+2^2(1-2^{-(i+1)})\deg r_{2i-1}\,u'(e')^i.\end{aligned}$$

Now we have

$$\begin{aligned}u'(e')^i&=u'(u_0+\dots+u_{2i-3}+u_{2i-2})^i\\ &=u'u_{2i-2}^i\\ &=-\tfrac{1}{2}u'u_{2i-2}^{i-1}(u_0+\dots+u_{2i-3})\\ &=0\end{aligned}$$

here we have used Lemma 6.7 and Lemma 5.9. Next applying Lemma 6.6 again we obtain

$$\begin{aligned}&u'u_{2i-2}\,\pi_{2i-2}^*(e'_{i-1})\\ &\quad=2\deg r_{2i-2}\,u'\pi_{2i-3}^*(e''_{i-1})-2(1-2^{-i})\deg r_{2i-2}\,u'(e'')^{i-1}\\ &\quad=2\deg r_{2i-2}\,R_{i-1}\end{aligned}$$

because $u'(e'')=0$ (Lemma 6.7). Similarly we have

$$\begin{aligned}u'\pi_{2i-2}^*(e'_i)&=-2^2(1-2^{-(i+1)})\deg r_{2i-2}\,u'\pi_{2i-3}^*(e''_{i-1})\\ &=-2^2(1-2^{-(i+1)})\deg r_{2i-2}\,R_{i-1}.\end{aligned}$$

Summing up we obtain

$$R_i=2^3(1-2^{-(i+1)})\deg r_{2i-1}\deg r_{2i-2}R_{i-1}$$

as claimed. Now again Lemma 6.6 yields

$$\begin{aligned} R_1 &= (u_0 - 2u_1)\, \pi_1^*(e_1) \\ &= -2^2(1-2^{-2}) \deg r_1\, u_0\, \pi_0^*(e'_0) + 2^2(1-2^{-2}) \deg r_1\, e' \\ &= 3 \deg r_1 (2g-2)(2g-3) \end{aligned}$$

because $u_0 = 2g-2$ and $e'_0 = e' = 2-2g$. Now in view of the above computations clearly we can conclude that $R_i \neq 0$ for all i. This completes the proof.

Proof of Theorem 6.1. In view of Proposition 3.4 (for the case when $k=1$), we have only to prove the existence of a number $g(n)$, depending on the given natural number n, such that the homomorphism

$$\mathbf{Q}[e_1, \ldots, e_{g-2}]/\text{relations} \to H^*(\mathcal{M}_{g(n),1}; \mathbf{Q})$$

is injective up to degree $2n$. According to Corollary 6.5, for each i there is a number g_i and a homomorphism $h_i: \pi_1(X_i) \to \mathcal{M}_{g_i, 1}$, where X_i is a certain closed orientable $2i$-manifold, such that $h_i^*(e_i)$ is non-zero in $H^{2i}(X_i; \mathbf{Z}) \cong \mathbf{Z}$. Now for each $i \leqq n$, choose a natural number d_i such that $d_i \geqq n/i$ and set $g(n) = \sum_{i=1}^{n} d_i g_i$. Consider the manifold X given by

$$X = X_1^{d_1} \times \ldots \times X_n^{d_n}$$

and define a homomorphism

$$h: \pi_1(X) \xrightarrow{h'} (\mathcal{M}_{g_1,1})^{d_1} \times \ldots \times (\mathcal{M}_{g_n,1})^{d_n} \xrightarrow{\iota} \mathcal{M}_{g(n),1}$$

where $h' = h_1^{d_1} \times \ldots \times h_n^{d_n}$ and ι is the homomorphism defined as in Proposition 3.4. Let π be the resulting $\Sigma_{g(n)}$-bundle over X. Then by virtue of Proposition 3.4, each class $e_j(\pi) \in H^{2j}(X; \mathbf{Z})$ can be expressed in terms of e_j of surface bundles over X_i which are defined by the homomorphisms h_i. By the choices of X_i and h_i, for any non-trivial element $\sum a(I)\, e_{i_1} \ldots e_{i_r}$ of degree $\leqq 2n$, the cohomology class $\sum a(I)\, e_{i_1}(\pi) \ldots e_{i_r}(\pi)$ is non-zero. Hence we can conclude that the homomorphism

$$\mathbf{Q}[e_1, \ldots, e_{g-2}]/\text{relations} \to H^*(\mathcal{M}_{g(n),1}; \mathbf{Q}) \xrightarrow{h^*} H^*(X; \mathbf{Q})$$

is injective up to degree $2n$. This completes the proof of Theorem 6.1.

7. Surface bundles with cross sections

In this section we extend the results of §§ 5, 6 to the case of surface bundles with cross sections. Thus let $\pi: E \to X$ be an oriented Σ_g-bundle and let ξ be its tangent bundle as before. Assume now that there are given cross sections

$$s_i: X \to E \qquad (i = 1, \ldots, p, p+1, \ldots, p+q)$$

such that the images $s_i(X)$ are mutually disjoint. We also assume that the normal bundles of the images of the last q cross sections are trivialized, namely there are given trivializations

$$N(s_i(X)) = X \times D^2 \qquad (i = p+1, \ldots, p+q)$$

where $N(s_i(X))$ denotes a tubular neighborhood of $s_i(X)$. As in §1 let $e = e(\xi) \in H^2(E; \mathbf{Z})$ be the Euler class of the surface bundle $\pi: E \to X$. We set

$$\sigma_i = s_i^*(e) \in H^2(X; \mathbf{Z}) \qquad (i = 1, \ldots, p).$$

It is clear that the cohomology classes σ_i of the base space X behave naturally with respect to bundle maps of surface bundles with cross sections.

There is one natural way to obtain a surface bundle with a cross section out of a given surface bundle $\pi: E \to X$. Namely as in § 4, let

$$\begin{array}{ccc} E^* & \xrightarrow{q} & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ E & \xrightarrow[\pi]{} & X \end{array}$$

be the pull back of the bundle $\pi: E \to X$ by the map π itself so that $E^* = \{(z, z')\, E \times E; \pi(z) = \pi(z')\}$, $\pi'(z, z') = z$ and $q(z, z') = z'$. Then the surface bundle $\pi': E^* \to E$ has a natural cross section $s: E \to E^*$ given by $s(z) = (z, z)$. The Euler class of π' is clearly equal to $q^*(e)$. Hence the σ-class of it is given by

$$\sigma = s^* q^*(e) = e \in H^2(E; \mathbf{Z}).$$

This fact will play an important role in the proof of our main result of this section (Theorem 7.5).

Next we describe the classifying spaces for surface bundles with cross sections. Let $\Sigma_{g,q}$ be a compact orientable surface of genus g with q boundary components and let $x_1, \ldots, x_p$ be p fixed points on $\operatorname{Int} \Sigma_{g,q}$. Let $D^p_{g,q}$ be the group of all orientation preserving diffeomorphisms of $\Sigma_{g,q}$ such that they restrict to the identity on the boundary $\partial \Sigma_{g,q}$ and also fix the p points $x_1, \ldots, x_p$. We denote $\mathcal{M}^p_{g,q}$ for the group of path components of $D^p_{g,q}$. It is usually called the (pure) mapping class group of genus g with p punctures and q boundary components. Let $E^p_{g,q}$ be the connected component of the identity of $D^p_{g,q}$.

Proposition 7.1. *$E^p_{g,q}$ is contractible.*

Proof. The cases where $p = 0$ are nothing but the main theorems of Earle and Eells [EE] ($q = 0$) and Earle and Schatz [EE] (q: arbitrary). The general cases follow from them by an inductive argument using the natural sequence

$$D^p_{g,q} \to D^{p-1}_{g,q} \to \Sigma_{g,q} \setminus \{x_1, \ldots, x_{p-1}\}$$

which is a locally trivial fibration.

Corollary 7.2. *The classifying space $BD^p_{g,q}$ of the topological group $D^p_{g,q}$ is an Eilenberg-MacLance space $K(\mathcal{M}^p_{g,q}, 1)$.*

Now it is easy to see that the space $BD^p_{g,q}$ classifies those oriented Σ_g-bundles which have $p+q$ disjoint cross sections the normal bundles of the last q of which are trivialized. Therefore we can consider elements e_i and σ_i as cohomology classes of the space $BD^p_{g,q}$ and hence cohomology classes of the group $\mathcal{M}^p_{g,q}$ by Corollary 7.2. Thus we obtain a homomorphism

$$\phi: \mathbf{Q}[e_1, \ldots, e_{g-2}, \sigma_1, \ldots, \sigma_p] \to H^*(\mathcal{M}^p_{g,q}; \mathbf{Q}).$$

We prove that the above homomorphism is injective up to degree $\frac{1}{3}g$. For that let $\pi: E \to X$ be an oriented surface bundle over an oriented closed $2n$-dimensional manifold X which is assumed to be an iterated surface bundle. We consider an m-construction on it:

$$\begin{array}{ccccc} \tilde{E}^* & \xrightarrow{r} & E^* & \xrightarrow{q} & E \\ \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ \tilde{E} & \xrightarrow[\tilde{r}]{} & E & \xrightarrow[\pi]{} & X \end{array}$$

(see § 4). For each non-negative integer i and a partition $I=\{i_1, \ldots, i_r\}$ of $n+1-i$, we consider the number

$$\tilde{e}^i\, \tilde{\pi}^*(\tilde{e}_I)[\tilde{D}] = \tilde{e}^i\, \tilde{\pi}^*(\tilde{e}_{i_1} \ldots \tilde{e}_{i_r})[\tilde{D}]$$

where $\tilde{D} \subset \tilde{E}^*$ is the ramification locus. For each subset $J=\{j_1, \ldots, j_s\}$ of I, we write $J^c=\{k_1, \ldots, k_t\}$ $(s+t=r)$ as before.

Proposition 7.3. *The number $\tilde{e}^i\, \tilde{\pi}^*(\tilde{e}_I)[\tilde{D}]$ is given by*

$$\tilde{e}^i\, \tilde{\pi}^*(\tilde{e}_I)[\tilde{D}] = dm^{2r+1-i} \sum_J (-1)^t (1-m^{-(k_1+1)}) \ldots (1-m^{-(k_t+1)}) \cdot e_J\, e_{i+k_1+\ldots+k_t-1}[X]$$

where d is the degree of the finite covering $\tilde{E} \to E$ and J runs through all the subsets of I.

Proof. First we recall the fact that the restriction of the cohomology class $q^*(e)$ to D is equal to that of the cohomology class v. Also it is easy to see that the degree of the natural map $\tilde{D} \to D$ is equal to dm. Using these facts together with Proposition 5.4, we compute

$$\begin{aligned} \tilde{e}^i\, \tilde{\pi}^*(\tilde{e}_I)[\tilde{D}] &= \left\{ r^*\left(q^*(e) - \frac{m-1}{m} v\right)\right\}^i m^2 r^*(\pi')^* \{\pi^*(e_{i_1}) - (1-m^{-(i_1+1)})\, e^{i_1}\} \ldots \\ &\quad m^2 r^*(\pi')^* \{\pi^*(e_{i_r}) - (1-m^{-(i_r+1)})\, e^{i_r}\} [\tilde{D}] \\ &= dm^{2r+1-i} v^i (\pi')^* \{\pi^*(e_{i_1}) - (1-m^{-(i_1+1)})\, e^{i_1}\} \ldots \\ &\quad (\pi')^* \{\pi^*(e_{i_r}) - (1-m^{-(i_r+1)})\, e^{i_r}\} [D] \\ &= dm^{2r+1-i} \sum_J (-1)^t (1-m^{-(k_1+1)}) \ldots \\ &\quad (1-m^{-(k_t+1)})\, e_J\, e_{i+k_1+\ldots+k_t-1}[X]. \end{aligned}$$

The completes the proof.

Proposition 7.4. *For any natural number n, there exists no non-trivial linear relation between the numbers $\tilde{e}^i\tilde{\pi}^*(\tilde{e}_I)[\tilde{D}]$ where $\tilde{D}$ runs through all the ramification loci of m-constructions on surfaces bundles $\pi: E \to X$ with the base space X being iterated surface bundles of dimension 2n.*

Proof. This follows from Proposition 5.6 and Proposition 7.3 by a similar argument as that of the proof of Proposition 5.6.

The following is the main result of this section.

Theorem 7.5. *The homomorphism*

$$\phi: \mathbf{Q}[e_1, \ldots, e_{g-2}, \sigma_1, \ldots, \sigma_p] \to H^*(\mathcal{M}^p_{g,q}; \mathbf{Q})$$

is injective up to degree $\frac{1}{3}g$ for all g, p and q.

To prove this we use a fundamental result of Harer mentioned in § 6, which states that the homology group $H_k(\mathcal{M}^p_{g,q}; \mathbf{Q})$ is independent of g and q provided $3k \leqq g$. In terms of the cohomology group, for a fixed p the groups $H^*(\mathcal{M}^p_{g,q}; \mathbf{Q})$ are all isomorphic each other in the above range. Moreover it is easy to see that under these isomorphisms, our characteristic classes are preserved.

Next let $g_1, \ldots, g_p$ be integers greater than one and let x_j be the base point of $\Sigma_{g_j,1}$ $(j=1, \ldots, p)$. We write $\sigma^{(j)} \in H^2(\mathcal{M}^1_{g_j,1}; \mathbf{Z})$ for the σ-class defined by the base point x_j.

Now let g be a natural number such that $\Sigma g_j \leqq g$ and let $y_1, \ldots, y_p$ be p fixed points on $\operatorname{Int}\Sigma_{g,1}$. We have the characteristic classes $\sigma_j \in H^2(\mathcal{M}^p_{g,1}; \mathbf{Z})$ $(j=1, \ldots, p)$ which are defined by the fixed points $y_1, \ldots, y_p$. Choose any embedding of the disjoint union $\coprod_{j=1}^{p} \Sigma_{g_j,1}$ of compact surfaces $\Sigma_{g_j,1}$ with boundaries into $\Sigma_{g,1}$ such that the point x_j goes to y_j $(j=1, \ldots, p)$. This induces a homomorphism

$$\iota: \mathcal{M}^1_{g_1,1} \times \ldots \times \mathcal{M}^1{}_{g_p,1} \to \mathcal{M}^p_{g,1}.$$

The following is immediate.

Lemma 7.6. $\iota^*(\sigma_j) = p_j^*(\sigma^{(j)})$ *for all* $j = 1, \ldots, p$.

Proof of Theorem 7.5. First recall that the total space E of any surface bundle $\pi: E \to X$ serves as the base space of the associated pull back surface bundle $\pi': E^* \to E$ which has a canonical cross section. Moreover the σ-class of π' is equal to the Euler class $e \in H^2(E; \mathbf{Z})$ of π. Then in view of Harer's result mentioned above, the required assertion for the case $p=1$ follows from Proposition 7.4. The general cases follow from this by an easy argument using Proposition 3.4 and Lemma 7.6 (cf. the proof of Theorem 6.1). This completes the proof.

Remark 7.7. According to a recent result of Harer and Zagier [HZ], the homomorphism ϕ is far from being surjective, because the Euler characteristic of $\mathcal{M}^p_{g,q}$ is extremely large. However in view of the fact that the Euler characteristic of the Siegel modular group $\mathrm{Sp}(2g; \mathbf{Z})$ is even larger than that of $\mathcal{M}_g$, yet the stable cohomology is a polynomial ring on $(4k+2)$-dimensional generators, it seems to be still reasonable to conjecture that our characteristic classes exhaust all the "stable" characteristic classes of surface bundles. Namely the natural homomorphism

$$\mathbf{Q}[e_1, e_2, \ldots; \sigma_1, \ldots, \sigma_p] \to \lim_{g \to \infty} H^*(\mathcal{M}_{g,q}^p; \mathbf{Q})$$

would be an isomorphism. Theorem 7.5 shows that it is in fact injective.

8. The generalized Nielsen realization problem

Recall that the Nielsen realization problem is the one to determine whether any *finite* subgroup of the mapping class group $\mathcal{M}_g$ lifts to $\mathrm{Diff}_+ \Sigma_g$ (or $\mathrm{Homeo}_+ \Sigma_g$). This problem was solved affirmatively by Kerckhoff [Ke]. We can also ask the same question for *infinite* subgroups of $\mathcal{M}_g$. However we have

Theorem 8.1. *There are cohomological obstructions to the existence of liftings of infinite subgroups of* $\mathcal{M}_g$ *to* $\mathrm{Diff}_+ \Sigma_g$. *More precisely if we denote* $p: \mathrm{Diff}_+ \Sigma_g \to \mathcal{M}_g$ *for the natural homomorphism, then we have*

$$p^*(e_i) = 0 \quad \text{in } H^{2i}(\mathrm{Diff}_+ \Sigma_g; \mathbf{Q}) \qquad \text{for all } i \geqq 3$$

(here we understand $\mathrm{Diff}_+ \Sigma_g$ *to be a discrete group). It follows that the homomorphism p does not have a right inverse for all* $g \geqq 18$.

Proof. Let $\pi: E \to X$ be an oriented Σ_g-bundle with holonomy homomorphism $h: \pi_1(X) \to \mathcal{M}_g$. Assume that h lifts to $\mathrm{Diff}_+ \Sigma_g$, namely there is a homomorphism $h': \pi_1(X) \to \mathrm{Diff}_+ \Sigma_g$ such that $ph' = h$. This means that there is a codimension two foliation $\mathcal{F}$ on E whose leaves are all transverse to the fibres of π. Now the normal bundle $\nu(\mathcal{F})$ of $\mathcal{F}$ is canonically isomorphic to the tangent bundle ξ of π. According to Bott's vanishing theorem [Bo], $p_1^k(\xi) = 0$ for all $k \geqq 2$, where $p_1(\xi)$ is the first *rational* Pontrjagin class of ξ. Since $p_1(\xi) = (e(\xi))^2$, we conclude that $(e(\xi))^4$ vanishes in $H^8(E; \mathbf{Q})$. In view of Theorem 6.1, the required assertion follows from this.

Remark 8.2. The above proof does not work for homeomorphisms. In fact Thurston [T] proved that the natural homomorphism

$$\mathrm{Homeo}_+ \Sigma_g \to \mathcal{M}_g$$

induces an isomorphism on homology. Therefore the problem to determine whether the above homomorphism splits or not remains to be open.

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