Lectures on the Madsen–Weiss theorem

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This is a draft of the lecture notes for my four lectures at the Park City Math Institute, Utah 2011. The lectures form a continuation of Nathalie Wahl's lectures, and I refer to her lectures for context and motivation.

My goal for the lectures will be to give a precise statement of the Madsen-Weiss theorem and give (most of) a proof. The theorem was first proved by Madsen and Weiss in [4]. The lectures will concern the proof given in [1] (parts of which were outlined in [2]), and the reader is referred there for more details. The exposition is influenced by Hatcher's survey [3].

1 Spaces of submanifolds and the Madsen–Weiss theorem

The main goal of this lecture is to give a precise statement of the Madsen–Weiss theorem ([4]). In the process, we will introduce several objects which will be useful in the proof.

1.1 Spaces of manifolds

Let us start with a definition of a topological space. The relation to Madsen–Weiss' theorem will become clear during this lecture. We first define the underlying set. First, let us point out a potential cause of confusion about words: For a submanifold $W \subseteq \mathbb{R}^n$, being "closed" could have two very different meanings. It could either mean that W is a closed manifold, i.e. it is compact and has no boundary (this property does not depend on how W sits in $\mathbb{R}^n$), or it could mean that W is a closed subset of $\mathbb{R}^n$. To avoid this confusion, we shall say that $W \subseteq \mathbb{R}^n$ is *topologically closed* to mean that W is merely a closed subset.

Definition 1.1. Let $\Psi(\mathbb{R}^n)$ be the set of pairs (W, ω), where $W \subseteq \mathbb{R}^n$ is a smooth 2-manifold without boundary which is topologically closed in $\mathbb{R}^n$, and ω is an orientation of W.

We emphasize that there are no further conditions required, and in particular W is not required to be path connected. In general $H_i(W)$ could have infinite rank for all $i \leq 2$. This set has something akin to a "smooth structure" (although it is not in any good sense an infinite dimensional manifold).

Definition 1.2. Let X be a smooth manifold. A map $f : X \to \Psi(\mathbb{R}^n)$ is *smooth* if the graph

$$\Gamma_f = \{(x, v) \in X \times \mathbb{R}^n | v \in f(x)\}$$

is a smooth topologically closed submanifold of $X \times \mathbb{R}^n$ such that the projection $p_f : \Gamma_f \to X$ is a submersion and the orientations of $f(x)$ vary continuously (i.e. assemble to an orientation on the kernel of $Dp_f : T\Gamma_f \to TX$).

In [2] and [1], a natural topology on $\Psi(\mathbb{R}^n)$ was defined, and the following property was proved.

Lemma 1.3. *Any smooth map $X \to \Psi(\mathbb{R}^n)$ is continuous. Any continuous map $f : X \to \Psi(\mathbb{R}^n)$ can be perturbed to a smooth map; the perturbation can be assumed constant near any closed set on which f is already smooth.* □

"Perturbation" means a continuous extension of f to $X \times [0, 1)$, smooth on $X \times (0, 1)$. In these lectures we shall not prove in detail that a topology with this property exists, but let us say some words about what the open sets are. A neighborhood basis at $\emptyset \in \Psi(\mathbb{R}^n)$ is given by the sets

$$\mathcal{U}(K) = \{W \in \Psi(\mathbb{R}^n) | W \cap K = \emptyset\},$$

where K runs through compact subsets of $\mathbb{R}^n$. In particular, a sequence of points $W_i \in \Psi(\mathbb{R}^n)$ converges to $\emptyset$ if and only if the following condition holds: For each compact K, there exists an $N = N_K$ such that $W_i \cap K = \emptyset$ for $i > N$. A neighborhood basis at an arbitrary $W \in \Psi(\mathbb{R}^n)$ is defined similarly: It consists of manifolds which near a large compact subset of $\mathbb{R}^n$ looks like a small perturbation of W (where "small" is in a C^∞ sense.)

For example, the function $f : \mathbb{R} \to \Psi(\mathbb{R}^3)$ given by $f(t) = \{t^{-1}\} \times \mathbb{R}^2$ and $f(0) = \emptyset$ is continuous (in fact smooth). As another example of the behavior of the topology, let us prove that $\Psi(\mathbb{R}^n)$ is path connected for $n \geq 3$. In that case, given any $W \in \Psi(\mathbb{R}^n)$, we can pick $p \in \mathbb{R}^n - W$. Then the path $t \mapsto (1 - t)^{-1}(W - tp)$ starts at W and ends at $\emptyset$, proving that there is a path from any element to the base point $\emptyset \in \Psi(\mathbb{R}^n)$.

The space $\Psi(\mathbb{R}^n)$ is related to surface bundles through the subspace

$$B_n = \{W \in \Psi(\mathbb{R}^n) | W \subseteq (0, 1)^n\}.$$

The exact relation to surface bundles is given by the following consequence of lemma 1.3.

Proposition 1.4. *Let X be a smooth k-dimensional manifold. For $n > 2k+4$, there is a bijection between $[X, B_n]$ and the set of isomorphism classes of surface bundles $E \to X$.*

Proof. By lemma 1.3, any map $f : X \to B_n$ is homotopic to a smooth map, which is given by a smooth, topologically closed $\Gamma_f \subseteq X \times \mathbb{R}^n$, which is also contained in $X \times (0, 1)^n$. An exercise in point-set topology shows that this implies that the projection $\Gamma_f \to X$ is a surface bundle. If two maps $f_0, f_1 : X \to B_n$ are smoothly homotopic, there is an induced surface bundle over $X \times I$, proving that the two surface bundles are isomorphic. This gives the map in one direction.

Conversely, if $E \to X$ is a surface bundle and $n > 2k + 4$, we can pick an embedding $j : E \to X \times (0,1)^n$ (by Whitney's embedding theorem), and we can define a smooth map $f : X \to B_n$ by the formula $j(E_x) = \{x\} \times f(x)$. If j' is another embedding of the same surface bundle, then j and j' are isotopic (smoothly homotopic through embeddings), so the resulting maps $X \to B_n$ become homotopic. This gives the map in the other direction, and it is easy to see that they are each other's inverse. □

Corollary 1.5. *For any smooth X, there is a natural bijection between $[X, B_\infty]$ and the set of isomorphism classes of surface bundles $E \to X$.* □

Corollary 1.6. *$H^*(B_\infty)$ is the ring of characteristic classes of surface bundles.*

Proof sketch. If we extend the definition of surface bundle a little, to allow the identity map to classify a surface bundle over B_∞, this follows from Yoneda's lemma in the usual way. In our setup, where the base space of a surface bundle is required to be a (finite dimensional) manifold, a rigorous proof uses that B_∞ is weakly equivalent to a directed limit of spaces which are homotopy equivalent to manifolds (e.g. the system of finite subcomplexes of a CW approximation to B_∞). □

A slightly different point of view on this last result is seen by noting a bijection

$$B_n = \coprod_W \mathrm{Emb}(W, (0,1)^n)/\mathrm{Diff}(W),$$

where the disjoint union is over oriented closed surfaces W, one of each diffeomorphism class. Giving the right hand side the quotient topology, this becomes a homeomorphism, and it is known that the quotient map $\mathrm{Emb}(W, (0,1)^n) \to \mathrm{Emb}(W, (0,1)^n)/\mathrm{Diff}(W)$ is a principal $\mathrm{Diff}(W)$-bundle for all $n \leq \infty$. A strong version of Whitney's embedding theorem implies that $\mathrm{Emb}(W, (0,1)^\infty)$ is contractible, and hence a model for $E\mathrm{Diff}(W)$. It follows that we have models for $B\mathrm{Diff}(W)$, namely the subspace of B_∞ consisting of manifolds diffeomorphic to W, in which

$$B_\infty = \coprod_W B\mathrm{Diff}(W). \tag{1}$$

We have now explained how the subspace $B_n \subseteq \Psi(\mathbb{R}^n)$ is related to surface bundles and their characteristic classes, but we have not yet seen why the full space $\Psi(\mathbb{R}^n)$ is useful. Its relevance comes through the important construction in definition 1.7 below. We shall consider $\emptyset \in \Psi(\mathbb{R}^n)$ the basepoint, and as usual we let $\Omega^n\Psi(\mathbb{R}^n)$ be the n-fold loop space, i.e. the space of continuous based maps $S^n \to \Psi(\mathbb{R}^n)$. We shall consider S^n as the one-point compactification of $\mathbb{R}^n$.

Definition 1.7. Let $\alpha : B_n \to \Omega^n \Psi(\mathbb{R}^n)$ be the map given by

$$\alpha(W)(v) = \begin{cases} W + v & \text{if } v \in \mathbb{R}^n \\ \emptyset & \text{if } v = \infty. \end{cases} \tag{2}$$

Since $W + v$ leaves any compact set as $|v| \to \infty$, $\alpha(W)$ is continuous at the basepoint of S^n. It is seen in a similar way that α is in fact a continuous map.

We can then let $n \to \infty$ (using the map $\Psi(\mathbb{R}^n) \to \Omega\Psi(\mathbb{R}^{n+1})$ given by $W \mapsto (t \mapsto W \times \{t\})$) and get a map

$$\alpha : B_\infty \to \Omega^\infty \mathbf{\Psi} = \operatorname*{colim}_{n\to\infty} \Omega^n \Psi(\mathbb{R}^n).$$

Neither the source nor the target of α are path connected. By (1), the path components of B_∞ are precisely the $B\mathrm{Diff}(W)$, and there is one path component for each diffeomorphism type of oriented 2-manifold W. The map α sends $B\mathrm{Diff}(W)$ to a path component of $\Omega^\infty \mathbf{\Psi}$; we shall write $\Omega^\infty_{[W]} \mathbf{\Psi}$ for that path component. We can now formulate Madsen–Weiss' theorem in a slightly unconventional form.

Theorem 1.8 (Madsen-Weiss)**.** *If W is a surface of genus g, the restricted map*

$$B\mathrm{Diff}(W) \to \Omega^\infty_{[W]} \mathbf{\Psi}$$

induces an isomorphism in integral cohomology through degree $2g/3$.

In the remaining part of this lecture, we will say more about the homotopy type of $\Psi(\mathbb{R}^n)$. To understand the homotopy type of the space, let us consider the Grassmannian $\mathrm{Gr}_2^+(\mathbb{R}^n)$ of oriented 2-planes in $\mathbb{R}^n$. There are two interesting vector bundles over that space, the *canonical bundle* γ_n and its orthogonal complement $\gamma_n^\perp$. A point in the total space of $\gamma_n^\perp$ is a pair (V, v) where $V \subseteq \mathbb{R}^n$ is an oriented 2-plane, and $v \in V^\perp$. We can think of V as an oriented 2-manifold and since it's topologically closed, it defines a point $V \in \Psi(\mathbb{R}^n)$. Translating it by the vector v, we get a map

$$\begin{aligned} \gamma_n^\perp &\to \Psi(\mathbb{R}^n) \\ (V, v) &\mapsto V + v. \end{aligned}$$

The manifold $V + v$ leaves all compact sets as $|v| \to \infty$, so the map extends to a continuous map

$$\begin{aligned} \mathrm{Th}(\gamma_n^\perp) &\xrightarrow{q} \Psi(\mathbb{R}^n) \\ (V, v) &\mapsto V + v \\ \infty &\mapsto \emptyset, \end{aligned}$$

where $\mathrm{Th}(\gamma_n^\perp)$ denotes the *Thom space* of $\gamma_n^\perp$, i.e. the one-point compactification of its total space.

Theorem 1.9. *The map q is a weak equivalence.*

(Recall that a map $f : X \to Y$ is a weak equivalence if the induced map $\pi_k(X) \to \pi_k(Y)$ is an isomorphism for all k and all basepoints. Homotopy equivalence implies weak equivalence which implies that $f_* : H_*(X) \to H_*(Y)$ is an isomorphism. For CW complexes, weak equivalence also implies homotopy equivalence.)

Proof. This will be sketched in the exercises, using Lemma 1.3. □

The theorem above relates our definition of $\Omega^\infty \mathbf{\Psi}$ to the definition that "usually" appears in Madsen–Weiss' theorem, namely the space

$$\Omega^\infty MTSO(2) = \operatorname*{colim}_{n\to\infty} \Omega^n \mathrm{Th}(\gamma_n^\perp).$$

Theorem 1.9 above implies the following.

Corollary 1.10. *There is a weak equivalence*

$$\Omega^\infty MTSO(2) \to \Omega^\infty \mathbf{\Psi}.$$

Proof. The theorem implies that $\Omega^n \mathrm{Th}(\gamma_n^\perp) \to \Omega^n \Psi(\mathbb{R}^n)$ induces an isomorphism in all homotopy groups. The direct systems on both sides consists of injective maps, and in this case homotopy groups commute with direct limit. □

The Madsen–Weiss theorem is usually stated in terms of the "Thom spectrum" $MTSO(2)$, instead of the weakly equivalent $\mathbf{\Psi}$. (The notation $\Omega^\infty MTSO(2)$ might seem a little heavy for one space, but it has the advantage that it generalizes easy to manifolds of other dimensions and other structures, e.g. there's a space called $\Omega^\infty MT\mathrm{Spin}(3)$.)

1.2 Exercises for lecture 1

1. Recall that for a space X with base point $x \in X$ and $k \geq 0$, we define $\pi_k(X, x)$ is defined as the set of basepoint preserving maps $S^k \to X$ modulo basepoint preserving homotopies. Recall also that a map $f : X \to Y$ is called a weak equivalence if the induced map $\pi_k(X, x) \to \pi_k(Y, f(x))$ is a bijection for all $k \geq 0$ and all $x \in X$. Prove that f is a weak equivalence if and only if it satisfies the following condition: For any pair of maps $g_0 : \partial D^k \to X$ and $h_0 : D^k \to Y$ making the following diagram commute

$$\begin{array}{ccc} \partial D^k & \xrightarrow{g_0} & X \\ \big\downarrow & & \big\downarrow f \\ D^k & \xrightarrow[h_0]{} & Y, \end{array}$$

there exist homotopies $h_s : D^k \to Y$ and $g_s : \partial D^k \to X$, $s \in [0, 1]$ with $f \circ g_s = h_s | \partial D^k$, such that $g_1 : \partial D^k \to X$ extends to a map $G : D^k \to X$ with $f \circ G = h_1$. (In words: any such commutative diagram is homotopic, through commutative diagrams, to a diagram admitting a diagonal $D^k \to X$.)

2. Prove that if X is a k-dimensional manifold, with $k < n-2$, and $f : X \to \Psi(\mathbb{R}^n)$ is smooth, then there exists a point $p \in \mathbb{R}^n$, not contained in $f(x)$ for any $x \in X$. Use this to deduce that $\pi_k(\Psi(\mathbb{R}^n)) = 0$ for $k < n-2$.

3. Use theorem 1.9 to prove that $\pi_{n-2}(\Psi(\mathbb{R}^n)) = H_{n-2}(\Psi(\mathbb{R}^n)) = H_{n-2}(\mathrm{Th}(\gamma_n^\perp))$. Then use Thom isomorphism to deduce $\pi_{n-2}(\Psi(\mathbb{R}^n)) = \mathbb{Z}$.

4. The next exercises works some examples with $n = 3$: First, prove that $\mathrm{Gr}_2^+(\mathbb{R}^3)$ is homeomorphic to S^2.

5. Use theorem 1.9 to prove that $\Psi(\mathbb{R}^3)$ is weakly equivalent to $S^3 \vee S^1$, deduce that $\pi_1\Psi(\mathbb{R}^3) = \mathbb{Z}$ and find an explicit generator. Deduce that $\pi_3\Psi(\mathbb{R}^3)$ is a free abelian group of infinite rank. Describe an explicit map $S^1 \vee S^3 \to \Psi(\mathbb{R}^3)$.

6. Reversing orientation gives map $\Psi(\mathbb{R}^n) \to \Psi(\mathbb{R}^n)$. Describe the induced maps on $\pi_1\Psi(\mathbb{R}^3)$ and $\pi_3\Psi(\mathbb{R}^3)$.

7. Prove that if $T \subseteq (0,1)^3$ is a torus, then $\alpha(T) \in \Omega^3\Psi(\mathbb{R}^3)$ is in the same path component as the basepoint. (Hint: Use that the Gauss map $T \to \mathrm{Gr}_2^+(\mathbb{R}^3)$ is null homotopic.)

8. Let $f_1 : E_1 \to X$ and $f_2 : E_2 \to X$ be two surface bundles. Then $f_3 : E_3 = E_1 \amalg E_2 \to X$ is again a surface bundle. By proposition 1.4, they are represented by maps $g_i : X \to B_n$ for some n. How is g_3 related to g_1 and g_2? What about the maps $\alpha \circ g_i : X \to \Omega^n\Psi(\mathbb{R}^n)$?

9. Let X be a compact manifold and $f : X \to \Psi(\mathbb{R}^n)$ a smooth map. Prove that if $B(0,\varepsilon) \subseteq \mathbb{R}^n$ denotes the open ε ball, then $\{x \in X | f(x) \cap B(0,\varepsilon) = \emptyset\}$ is a closed subset of X. Let $U_\varepsilon \subseteq X$ denote its complement and prove that for small enough ε, there exist a smooth function $p : U_\varepsilon \to B(0,\varepsilon)$ such that $p(x) \in f(x)$ and that $|y| > |p(x)|$ for all $y \in f(x) - \{p(x)\}$. (I.e. $p(x)$ is the point in $f(x)$ which is closest to 0. Hint: use the tubular neighborhood theorem.) Use this to prove theorem 1.9.

2 Rational cohomology and outline of proof

The goals of this lecture are the following.

1. Define $\kappa_i \in H^{2i}(\Omega^\infty \mathbf{\Psi}; \mathbb{Z})$.

2. Explain why induced map $\mathbb{Q}[\kappa_1, \kappa_2, \dots] \to H^*(\Omega^\infty_\bullet \mathbf{\Psi}; \mathbb{Q})$ is an isomorphism, where $\Omega^\infty_\bullet \mathbf{\Psi} \subseteq \Omega^\infty \mathbf{\Psi}$ denotes a path component.

3. Outline the steps in the proof of Madsen–Weiss' theorem: $B\mathrm{Diff}(W) \to \Omega^\infty_\bullet \mathbf{\Psi}$ induces an isomorphism in H_k for $k < 2(g-1)/3$.

2.1 Cohomology of $\Omega^\infty \mathbf{\Psi}$

The starting point in calculating the cohomology of $\Omega^\infty \mathbf{\Psi}$ is that we understand the cohomology of $\Psi(\mathbb{R}^n) \simeq \mathrm{Th}(\gamma_n^\perp)$. Firstly, the inclusion $\mathrm{Gr}_2^+(\mathbb{R}^n) \to \mathrm{Gr}_2^+(\mathbb{R}^\infty) \simeq \mathbb{C}P^\infty$ induces a map

$$\mathbb{Z}[e] = H^*(\mathrm{Gr}_2^+(\mathbb{R}^\infty)) \to H^*(\mathrm{Gr}_2^+(\mathbb{R}^n)), \tag{3}$$

which is an isomorphism in degrees $< (n-2)$. Secondly, the Thom isomorphism theorem gives an isomorphism

$$H^k(\mathrm{Gr}_2^+(\mathbb{R}^n)) \to H^{k+(n-2)}(\mathrm{Th}(\gamma_n^\perp), *) \tag{4}$$
$$x \mapsto x.u, \tag{5}$$

so at least we have a full understanding of $H^*(\mathrm{Th}(\gamma_n^\perp))$ in degrees $< 2n-4$.

To understand the effect of the functor Ω^n, we need the *suspension homomorphism*, a natural homomorphism

$$\sigma : H^{k+1}(X, *) \to H^k(\Omega X, *)$$

defined for any pointed space $(X, *)$. To define it, we use the evaluation map

$$\Sigma\Omega X \xrightarrow{\mathrm{ev}} X$$
$$(t, \gamma) \mapsto \gamma(t),$$

where Σ denotes (unreduced) suspension. Then the suspension homomorphism is defined as the composition

$$H^{k+1}(X, *) \xrightarrow{\mathrm{ev}^*} H^{k+1}(\Sigma\Omega X, *) \simeq H^k(\Omega X, *).$$

Letting $\sigma^n : H^{k+n}(X, *) \to H^k(\Omega^n X, *)$ denote the n-fold iteration, we can now give a definition of the κ-classes using (3) and (4).

Definition 2.1. Let $\kappa_i \in H^{2i}(\Omega^n \Psi(\mathbb{R}^n))$ be the class defined as

$$\kappa_i = \sigma^n(e^{i+1}.u) \in H^{2i}(\Omega^n \mathrm{Th}(\gamma_n^\perp)) = H^{2i}(\Omega^n \Psi(\mathbb{R}^n)). \tag{6}$$

Lemma 2.2. *There is a unique class $\kappa_i \in H^{2i}(\Omega^\infty \mathbf{\Psi})$ which restricts to the classes in definition 2.1 for all n.*

Proof. This follows from the following two facts, which we leave to the reader as an exercise.

(i) The map $\Omega^n \mathrm{Th}(\gamma_n^\perp) \to \Omega^{n+1}\mathrm{Th}(\gamma_{n+1}^\perp)$ is $(n-4)$-connected. Hence $H^{2i}(\Omega^\infty \mathbf{\Psi}) \to H^{2i}(\Omega^n \Psi(\mathbb{R}^n))$ is an isomorphism for $n > 2i+4$.

(ii) The classes $\kappa_i \in H^{2i}(\Omega^n \Psi(\mathbb{R}^n))$ in definition 2.1 are compatible with the map $\Omega^n \Psi(\mathbb{R}^n) \to \Omega^{n+1}\Psi(\mathbb{R}^{n+1})$.

□

We have defined classes κ_i in $\Omega^\infty \mathbf{\Psi}$, which can be pulled back to classes in B_∞. By corollary 1.6, these give rise to characteristic classe of surface bundles, which we shall also denote κ_i. The following lemma, whose proof we leave as an exercise, shows that our definition agrees with the "usual" definition of the κ classes.

Lemma 2.3. *Let $\pi : E \to X$ be a surface bundle with X a compact oriented n-manifold. Let $\pi_! : H^{k+2}(E) \to H^k(X)$ be the map Poincare dual to $\pi_* : H_{n-k}(E, \partial E) \to H_{n-k}(X, \partial X)$, and let $T_\pi E$ be the fiberwise tangent bundle, i.e. the oriented 2-dimensional bundle* $\mathrm{Ker}(D\pi : TE \to TX)$. *Then $\kappa_i = \pi_!(e^{i+1}(T_\pi(E)))$.* □

To calculate the entire ring $H^*(\Omega^\infty_\bullet \mathbf{\Psi}; \mathbb{Q})$, it is helpful to again work more generally. For a *graded vector space* $V = \oplus_{n\geq 1} V_n$, we shall write $\mathbb{Q}[V]$ for the free graded-commutative $\mathbb{Q}$-algebra generated by V. If X is space, and $\phi : V \to H^*(X;\mathbb{Q})$ is a homomorphism ($\mathbb{Q}$-linear and grading preserving), the cup product gives a unique extension to a $\mathbb{Q}$-algebra map

$$\mathbb{Q}[V] \to H^*(X;\mathbb{Q}). \tag{7}$$

We can also compose ϕ with the suspension to get a map $\sigma \circ \phi : V_{n+1} \to H^{n+1}(X, *) \to H^*(\Omega X, *)$, and then the cup product in ΩX induces a $\mathbb{Q}$-algebra homomorphism

$$\mathbb{Q}[s^{-1}V] \to H^*(\Omega X;\mathbb{Q}), \tag{8}$$

where $(s^{-1}V)_n = V_{n+1}$. We will use the following general result about this situation.

Theorem 2.4. *Assume $\pi_1(X)$ is abelian and acts trivially on the rational cohomology of the universal cover. Let $V = \bigoplus_{n\geq 1} V_n$ be a graded vector space and $V \to H^*(X, *)$ a homomorphism such that* (7) *is an isomorphism in degrees $\leq n$. Then* (8) *restricts to an isomorphism $\mathbb{Q}[s^{-1}V] \to H^*(\Omega_\bullet X;\mathbb{Q})$ in degrees $\leq (n-1)$.*

Proof. This can be proved using the Serre spectral sequence for the path-loop fibration over the universal cover $\tilde{X}$. Alternatively, one can use the Eilenberg-Moore spectral sequence. □

Lemma 2.5. *If $X = \Omega Y$, then $\pi_1(X)$ is abelian and acts trivially on the cohomology of the universal cover.*

Proof. We leave it to the reader to check that any deck transformation $\tilde{X} \to \tilde{X}$ is homotopic to the identity. □

We can now apply the general theory to calculate $H^*(\Omega^n \mathrm{Th}(\gamma_n); \mathbb{Q})$. To this end, set $X = \mathrm{Th}(\gamma_n)$ and let V be the graded vector space with basis $e^{i+1}.u$ for all $i \geq -1$. Then the induced map

$$\mathbb{Q}[V] \to H^*(\mathrm{Th}(\gamma_n^\perp); \mathbb{Q})$$

is an isomorphism in degrees $< 2(n-4)$. (This is the range in which we previously calculated the right hand side. Also, in this range there are no products on the left hand side.) If we apply theorem 2.4 n times, we see that we get

$$\begin{aligned} \mathbb{Q}[s^{-n}V] &\mapsto H^*(\Omega^n \mathrm{Th}(\gamma_n^\perp); \mathbb{Q}) \\ e^{i+1}.u &\mapsto \sigma^n(e^{i+1}.u) \end{aligned}$$

which is an isomorphism in degrees $< (n-4)$. By definition of the κ classes, the map can be rewritten as

$$\mathbb{Q}[\kappa_1, \kappa_2, \dots] \to H^*(\Omega^n \mathrm{Th}(\gamma_n^\perp); \mathbb{Q}).$$

2.2 Outline of proof

Finally, we will outline the steps in the proof of the Madsen–Weiss theorem. At the heart of the theorem is the map $\alpha : B_n \to \Omega^n \Psi(\mathbb{R}^n)$, obtained by moving compact manifolds around in all directions in $\mathbb{R}^n$. An essential step of the proof is to do this process in multiple steps, each of which "moves manifolds around" in only one direction in $\mathbb{R}^n$. To make this idea precise, we first need some definitions.

Definition 2.6. (i) Let $\psi(n,k) \subseteq \Psi(\mathbb{R}^n)$ be the subspace consisting of those $W \in \Psi(\mathbb{R}^n)$ which satisfy $W \subseteq \mathbb{R}^k \times (0,1)^{n-k}$.

(ii) Let $\alpha_k : \psi(n,k) \to \Omega\psi(n,k+1)$ be the map given by

$$\alpha_k(W)(t) = \begin{cases} W + te_{k+1} & \text{for } t \in \mathbb{R} \\ \emptyset & \text{for } t = \infty. \end{cases}$$

Each map α_k is continuous, and it is clear that the map (2) can be decomposed as

$$B_n = \psi(n,0) \xrightarrow{\alpha_0} \psi(n,1) \xrightarrow{\Omega\alpha_1} \dots \xrightarrow{\Omega^{n-1}\alpha_{n-1}} \psi(n,n) = \Psi(\mathbb{R}^n).$$

We shall prove the Madsen–Weiss theorem in the following steps.

- For $1 \leq k \leq n-1$, the map $\alpha_k : \psi(n,k) \to \Omega\psi(n,k+1)$ is a weak equivalence.
- The map $\alpha_0 : B_n \to \Omega\psi(n,1)$ is compatible with letting $n \to \infty$. Restricting to a path component of the resulting map $B_\infty \to \Omega\psi(\infty,1)$, we get a map $B\mathrm{Diff}(W) \to \Omega\psi(\infty,1)$.
- For $W = \Sigma_g$ a surface of genus g, the resulting map

  $$B\mathrm{Diff}(\Sigma_g) \to \Omega\psi(\infty,1)$$

  induces an isomorphism in homology through degree $2g/3$.

The last step actually has several sub-steps. An important such is to introduce surfaces with boundary, in order to take a limit as $g \to \infty$, cf. Nathalie's talks.

2.3 Exercises for lecture 2

1. Let $c : B_\infty \to B_\infty$ be the map which reverses orientation. Prove that $c^*\kappa_i = (-1)^i\kappa_i$.
2. Prove the assertion in lemma 2.5.
3. Let X be a based space and let $\mu : \Omega X \times \Omega X \to \Omega X$ be the map which concatenates loops. For $x \in H^{k+1}(X,*)$ with $k \geq 1$, prove that the class $y = \sigma(x) \in H^k(\Omega X,*)$ has the property that $\mu^*(y) = y \otimes 1 + 1 \otimes y$.
4. Deduce that the classes $\kappa_i \in H^{2i}(\Omega^n\Psi(\mathbb{R}^n))$ satisfy $\mu^*(\kappa_i) = \kappa_i \otimes 1 + 1 \otimes \kappa_i$. What does this imply about the characteristic classes $\kappa_i \in H^{2i}(B_\infty)$? (Hint: the question is about the behavior with respect to fiberwise disjoint union of surface bundles.)
5. Use lemma 2.3 to prove that the value of $\kappa_0 \in H^0(\Omega^n\Psi(\mathbb{R}^n))$ at the point $\alpha(W) \in \Omega^n\Psi(\mathbb{R}^n)$ is the Euler characteristic $\chi(W)$.
6. Prove the following property of the suspension homomorphism. Let $f : X \to \Omega Y$ have "adjoint" $g : \Sigma X \to Y$ (i.e. $g(t,x) = f(x)(t)$) and let $c \in H^{k+1}(Y)$. Then $f^*(\sigma c) \in H^k(X)$ and $g^*(c) \in H^{k+1}(\Sigma X)$ agree under the isomorphism $H^k(X) = H^{k+1}(\Sigma X)$.
7. Let $u : \mathrm{Th}(\gamma_n^\perp) \to K(\mathbb{Z}/2, n-2)$ be a map representing the mod 2 Thom class. Prove that in mod 2 cohomology, $u^*(\mathrm{Sq}^2) = e.u$. Deduce that the image of the Hurewicz map $\pi_n(\mathrm{Th}(\gamma_n^\perp)) \to H_n(\mathrm{Th}(\gamma_n))$ vanishes in mod 2 homology. Then deduce that $\kappa_0 \in H^0(\Omega^n\Psi(\mathbb{R}^n))$ is divisible by 2.
8. With notation as the previous exercise, prove that $u^*(\mathrm{Sq}^{2(i+1)}) = e^{i+1}.u$ and deduce that κ_i vanishes in mod 2 cohomology for all $i \geq 1$.
9. Prove the assertions in lemma 2.2.
10. Prove the assertion in lemma 2.3.

3 Topological monoids and the first part of the proof

The goal of this lecture is to prove that the map $\alpha_k : \phi(n,k) \to \Omega\phi(n,k+1)$ is a weak equivalence for $1 \leq k \leq n-1$. As a corollary, we get that the iterated map $\psi(n,1) \to \Omega^{n-1}\psi(n,n)$ is a weak equivalence, and hence

$$\Omega\psi(\infty,1) \simeq \Omega^\infty \mathbf{\Psi}.$$

This will reduce the Madsen–Weiss theorem to a property of the map $\alpha_0 : B_\infty \to \Omega\psi(\infty,1)$, which we shall study in the fourth lecture. For both lectures, a very important tool is *topological monoids* and their *classifying spaces*. We will discuss these now, omitting many proofs.

3.1 Topological monoids

A topological monoid is a space M with a multiplication $M \times M \to M$ which is associative but not necessarily commutative. We will not assume that M has a unit (although all our monoids will at least have homotopy units). Associated to such M is a space BM, called the classifying space of M. This is usually defined as the "geometric realization of the nerve of M"; we shall use the following more explicit (but equivalent) definition, which fits well with our setup.

Definition 3.1. Let BM be the set of pairs (A,f), where $A \subseteq \mathbb{R}$ is a finite subset, and $f : A \to M$ is a function. For the purpose of defining a topology on this set, we shall think of its points as "configurations of points in $\mathbb{R}$, labeled by M": Points in BM can be depicted as finitely many points on the real line, each labeled by an element of M.

Topologize this set by allowing the labels to move continuously in M and the points to move continuously in $\mathbb{R}$. Points are allowed to collide, in which case we multiply the labels (in the order they appear on the line), and to tend to ∞ or $-\infty$, in which case we forget the labels.

The point where $A = \emptyset$ gives the basepoint $\emptyset \in BM$.

To define the topology more rigorously, let $K \subseteq \mathbb{R}$ be a compact set, $V \subseteq M$ an open set, and $a < b \in \mathbb{R}$. Let $\mathcal{U}(K) \subseteq BM$ be the set of points satisfying $A \cap K = \emptyset$, and let $\mathcal{U}(a,b,V) \subseteq BM$ consist of those (A,f) such that if $A \cap (a,b) = (a_1 < \cdots < a_k)$, then $k \geq 1$ and $f(a_1)f(a_2)\dots f(a_k) \in V$. Declare the collection of sets $\mathcal{U}(K)$ and $\mathcal{U}(a,b,V)$ a subbasis for the topology.

There is a natural map $\beta : M \to \Omega BM$, given by

$$\beta(m)(t) = \begin{cases} (\{t\}, (t \mapsto m)) & \text{for } t \in \mathbb{R} \\ \emptyset & \text{for } t = \infty, \end{cases}$$

where we regard ΩBM as the space of pointed maps from the one-point compactification of $\mathbb{R}$. The following well known theorem shall be used without proof.

Theorem 3.2. *$\beta : M \to \Omega BM$ is a weak equivalence if and only if M is* group-like *(i.e. that M has a homotopy unit and the monoid $\pi_0 M$ is a group).* □

This theorem suggests a very useful strategy for proving weak equivalences of the form $X \simeq \Omega Y$: If X admits a monoid structure, we prove $BX \simeq Y$ instead; in general, we find a monoid $M \simeq X$ and prove $BM \simeq Y$. This strategy is often easier than working directly with the loop space of Y. We shall apply this strategy to prove that $\alpha_k : \psi(n,k) \to \Omega\psi(n,k+1)$ is a weak equivalence. More precisely, we shall define a topological monoid M and construct a commutative diagram like the following

$$\begin{array}{ccc} M & \xrightarrow{\beta} & \Omega BM \\ \simeq \uparrow & & \downarrow \simeq \\ \psi(n,k) & \xrightarrow[\alpha_k]{} & \Omega\psi(n,k+1). \end{array} \tag{9}$$

The monoid M will of course depend on k and n, but we shall omit this from the notation. Let us define it.

Definition 3.3. For $0 \leq k < n$, let M denote the space

$$M = \{(t,W) \in (0,\infty) \times \psi(n,k+1) | W \subseteq \mathbb{R}^k \times (0,t) \times (0,1)^{n-k-1}\},$$

equipped with the multiplication

$$(t,W)(t',W') = (t+t', W \cup (W' + te_{k+1})).$$

The following lemma is obvious.

Lemma 3.4. *The inclusion $\psi(n,k) \to M$ (as the subspace with $t = 1$) is a homotopy equivalence.* □

Lemma 3.5. *The monoid M is grouplike.*

Proof. Let $m = (t,W) \in M$ with $W \subseteq \mathbb{R}^k \times (0,t) \times (0,1)^{n-k-1}$ with $k \geq 1$. We define $W' \subseteq \mathbb{R}^k \times (0,t) \times (0,1)^{n-k-1}$ by rotating W in the (e_k, e_{k+1}) plane around the point $(0, \frac{t}{2})$ and let $m' = (t, W')$. To see that mm' and $m'm$ are both in the path component of the empty set, we draw the following cartoon:
[picture] □

Next, we define the map $BM \to \psi(n,k+1)$, giving the right hand vertical map in the diagram (9). To this end, let us start with a point in $(A,m) \in BM$ with $A = (a_1 < \cdots < a_p) \subseteq \mathbb{R}$ and labels $m_1 = (t_1, W_1)$, …, $m_p = (t_p, W_p)$. Let $b_i \geq a_i$ be the smallest possible numbers such that the intervals $(b_i, b_i + t_i)$ are disjoint (i.e. set $b_1 = a_1$ and inductively $b_{i+1} = \max(a_{i+1}, b_i + t_i)$). Define a subset $W \subseteq \mathbb{R}^n$ as the union

$$W = (W_1 + b_1 e_{k+1}) \cup \cdots \cup (W_p + b_p e_{k+1}).$$

Since $W_i \subseteq \mathbb{R}^k \times (0,t_i) \times (0,1)^{n-k-1}$ and the intervals (b_i, b_i+t_i) are disjoint, W is the union of disjoint elements of $\psi(n,k+1)$, and hence $W \in \psi(n,k+1)$. It is easy to see that the resulting map $BM \to \psi(n,k+1)$ makes the diagram (9) commutative. Let us sketch a proof of its continuity.

Lemma 3.6. *W depends continuously on $(A,m) \in BM$.*

Proof sketch. There are three interesting events to check. The first is what happens when $a_i \in \mathbb{R}$ collides with a_{i+1}. It follows from the definition that W is independent of a_{i+1} as long as $a_{i+1} \leq a_i + t_i$, an that in this case, the value agrees with that of $(a_0 < \dots < \widehat{a_{i+1}} < \dots a_p)$ and $(m_1, \dots, m_i m_{i+1}, \dots, m_p)$.

The second is what happens when $a_1 \to -\infty$. In this case, W is eventually constant near any compact subset of $\mathbb{R}^n$, and converges to the value at $(a_2 < \dots < a_p)$ and $(m_2, \dots, m_p)$. The third interesting event is what happens when $a_p \to \infty$, but this is similar. $\square$

We have defined all maps in the diagram (9), and it is easy to see that the diagram is commutative. We have proved that two of them are weak equivalences. The main result of this lecture is the following.

Theorem 3.7. *The resulting map $BM \to \psi(n,k+1)$ is a weak equivalence for $k \geq 2$. For $k=1$, it is a weak equivalence onto the path component of $\psi(n,2)$ containing the empty manifold.*

Before embarking on the proof, let us point out the main consequence.

Corollary 3.8. *The map $\alpha_k : \psi(n,k) \to \Omega\psi(n,k+1)$ is a weak equivalence for $k \geq 1$. Consequently we get a weak equivalence $\Omega\psi(n,1) \simeq \Omega^n\psi(n,n)$ and hence a weak equivalence*

$$\Omega\psi(\infty,1) \simeq \Omega^\infty \mathbf{\Psi}.$$

Proof. By the diagram, α_k is the composition of three weak equivalences. $\square$

To motivate the proof of theorem 3.7, let us contemplate what we wish to achieve. To prove surjectivity in π_k, for example, we need to prove that any map $f: S^k \to \psi(n,k+1)$ is homotopic to a map which lifts to BM. In particular, for each $x \in S^k$, we need a path from $W = f(x) \in \psi(n,k+1)$ to a point in the image of $BM \to \psi(n,k+1)$. This is easy, provided W satisfies the condition that there *exists* a real number a such that

$$(\mathbb{R}^k \times \{a\} \times \mathbb{R}^{n-k-1}) \cap W = \emptyset. \tag{10}$$

Namely, in that case we can pick a finite set $(a_1 < \dots < a_{p+1})$ of real numbers with this property, let $t_i = a_{i+1} - a_i$, and let W_i be the part of W which is contained in $\mathbb{R}^k \times (a_i, a_{i+1}) \times \mathbb{R}^{n-k-1}$ (translated by $-a_i e_{k+1}$). Then the finite subset $A = (a_1 < \dots < a_p)$, labeled by the elements $m_i = (t_i, W_i) \in M$, gives a point $(A,m) \in BM$, and there is an obvious path from $W \in \psi(n,k+1)$ to the image of (A,m) (the path translates $W \cap \mathbb{R}^k \times (a_{p+1}, \infty) \times \mathbb{R}^{n-k-1}$ by a bigger and bigger multiple of e_{k+1} and $W \cap \mathbb{R}^k \times (-\infty, a_1) \times \mathbb{R}^{n-k-1}$ by a bigger and bigger multiple of $-e_{k+1}$). With slightly more care, this process can be performed for all $f(x)$, $x \in S^k$ at once, giving a continuous lift up to homotopy. Injectivity is similar: we wish to lift a map $f : [0,1] \times S^k \to \psi(n, k+1)$, with a prescribed lift over $\{0,1\} \times S^k$.

This discussion focuses attention on the condition (10), and it is convenient to have a name for it. For a map $f : X \to \psi(n, k+1)$, let us write $X_a \subseteq X$ for the set of elements $x \in X$ such that $W = f(x)$ satisfies (10). Let us say that a smooth map $f : X \to \psi(n, k+1)$ is *good* if $X = \cup_a \mathrm{int}(X_a)$. The following result is the main technical result underlying theorem 3.7.

Lemma 3.9. *For $1 \leq k < n$ and X a compact manifold (possibly with boundary), any map $f : X \to \psi(n, k+1)$ with image in the basepoint component is homotopic to a good map. The homotopy can be taken constant near any closed set on which f is already good.*

Let us postpone the proof of this lemma until the end of the section. In order to formalize the above discussion of how this implies that $BM \to \psi(n, k+1)$ is a weak equivalence (to the basepoint component if $k = 1$), we define yet another space BP.

Definition 3.10. Let BP be the space whose points are triples (A, C, W), where $A = (a_1 < a_2 < \cdots < a_p) \subseteq \mathbb{R}$, $C = (c_0 < \cdots < c_p) \subseteq \mathbb{R}$, and $W \in \psi(n, k+1)$ satisfies

$$W \cap (\mathbb{R}^k \times C \times \mathbb{R}^{n-k-1}) = \emptyset.$$

We think of c_i as a label on the interval $(a_i, a_{i+1}) \subseteq \mathbb{R} - A$, and topologize BP so that the a_i's are allowed to collide and go to $\pm\infty$. (Both processes decrease the number of intervals in $\mathbb{R} - A$, and we forget the corresponding c_i.)

It is easy to factor the map $BM \to \psi(n, k+1)$ through BP, viz. we set $c_i = b_{i+1}$ and $c_p = b_p + t_p$.

Lemma 3.11. *The forgetful map $BP \to \psi(n, k+1)$ is a weak equivalence.*

Proof sketch. To prove that the induced map in π_i is surjective, we use Lemma 3.9 to represent an element of $\pi_i(n, k+1)$ by a good map $S^i \to \psi(n, k+1)$. By compactness of S^i and by definition of goodness, we can find finitely many numbers $c \in \mathbb{R}$ such that the open sets

$$U_c = \mathrm{int}\{x \in S^i | f(x) \cap \mathbb{R}^k \times \{c\} \times \mathbb{R}^{n-k-1}\}$$

cover S^i. Then we use a partition of unity to define a map $S^i \to BP$ in the following way. Let $\lambda_c : S^i \to [0,1]$, $c \in \mathbb{R}$ be a (smooth, locally finite) partition of unity subordinate to the open sets U_c. For $x \in S^i$, we let $C = \{c \in \mathbb{R} | \lambda_c(x) > 0\}$ and $A = \{\lambda\big(\sum_{c \leq t} \lambda_c(x)\big) | t \in \mathbb{R}\}$, where $\lambda : (0,1) \to \mathbb{R}$ is an increasing homeomorphism. If we then define $S^i \to BP$ by mapping x to $(A, C, W) \in BP$, we have lifted our original good map $S^i \to \psi(n, k+1)$.

Injectivity is similar, using that if two good maps $S^i \to \psi(n, k+1)$ are homotopic, then there exists a homotopy $[0,1] \times S^i \to \psi(n, k+1)$ which is a good map. □

Lemma 3.12. *The map $BM \to BP$ is a homotopy equivalence.*

Proof. We first note that BM deformation retracts onto the subspace $B'M \subseteq BM$, defined by the inequalities $a_i + t_i \geq a_{i+1}$. The deformation increases t_i linearly until the inequality holds.

Secondly, we note that BP deformation retracts onto the subspace $B'P \subseteq BP$, defined by the requirements $a_1 = c_0$, $a_i \leq c_{i-1}$, and $W \subseteq \mathbb{R}^k \times (c_0, c_p) \times \mathbb{R}^{n-k-1}$. To see the deformation retraction, we write $t_i = c_i - c_{i-1}$, and let $c'_{i-1} \geq a_i$ be the smallest numbers such that $c'_i - c'_{i-1} \geq t_i$. The deformation retraction of BP onto $B'P$ deforms c_i to c'_i in a linear fashion. At the same time, it moves the part of W which lies in $\mathbb{R}^k \times (c_{i-1}, c_i) \times \mathbb{R}^{n-k-1}$ to $\mathbb{R}^k \times (c'_{i-1}, c'_{i-1} + t_i) \times \mathbb{R}^{n-k-1}$ in a linear fashion. The part of W which is in $\mathbb{R}^k \times (-\infty, c_0) \times \mathbb{R}^{n-k-1}$ is pushed away in the direction of $-e_{k+1}$, and the part of W which is in $\mathbb{R}^k \times (c_p, \infty) \times \mathbb{R}^{n-k-1}$ is pushed away in the direction of e_{k+1}.

Finally, we note that the map $BM \to BP$ restricts to a homeomorphism $B'M \to B'P$. The inverse is given by setting $t_i = c_i - c_{i-1}$ and letting W_i be the part of W which lies in $\mathbb{R}^k \times (c_{i-1}, c_i) \times \mathbb{R}^{n-k-1}$. □

Proof of lemma 3.9. We first give the proof in the case $f : X \to \psi(n, k+1)$ is a smooth map satisfying the following assumption: for all $x \in X$, the submanifold $f(x) \subseteq \mathbb{R}^n$ does *not* surject onto the first $(k+1)$ coordinates. (This hypothesis is of course automatic in the case $k \geq 2$ since $f(x) \subseteq \mathbb{R}^n$ is a 2-dimensional manifold.) In that case, we pick for all x a point $q_x \in \mathbb{R}^{k+1}$ not in the image of the projection $f(x) \to \mathbb{R}^{k+1}$. We can also pick an $\varepsilon_x > 0$ so that the (closed) ε_x-disk around q_x is disjoint from the image of that projection. The same choices will work in a neighborhood U_x of x, so by compactness of X we can find $\varepsilon > 0$ and finitely many open sets $U_i \subseteq X$ with corresponding $q_i \in \mathbb{R}^{k+1}$ such that the ε-ball around q_i consists of points not in the image of $f(x) \to \mathbb{R}^{k+1}$ for all $x \in U_i$. Writing $q_i = (p_i, t_i) \in \mathbb{R}^k \times \mathbb{R}$, we can assume that the t_i are distinct and that the intervals $[t_i - \varepsilon, t_i + \varepsilon]$ are disjoint (after possibly shrinking ε). Then, we can find an isotopy of diffeomorphisms of $\mathbb{R}^k \times \mathbb{R}$ supported in $\cup_i \mathbb{R}^k \times (t_i - \varepsilon, t_i + \varepsilon)$ which starts at the identity and ends at a map that sends $(p_i, t_i) \mapsto (0, t_i)$. Using this isotopy to deform each $f(x) \subseteq \mathbb{R}^n$, we may assume that all $p_i = 0$, and hence that all $f(x) \subseteq \mathbb{R}^n$ is disjoint from some $B(0, \varepsilon) \times \{t_i\} \times \mathbb{R}^{n-k-1}$. Finally, we may pick an isotopy of embeddings $e_s : \mathbb{R}^k \to \mathbb{R}^k$ that starts at the identity and ends in a map with $e_1(\mathbb{R}^k)$ contained in the ε-ball. Then the isotopy of embeddings $\phi_s = e_s \times \mathrm{id} : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k \times \mathbb{R}^{n-k}$ gives a path $f_s(x) = \phi_s^{-1}(f(x)) \in \psi(n, k+1)$ from $f_0(x)$ to an element satisfying that $f_1(x) \subseteq \mathbb{R}^n$ is disjoint from $\mathbb{R}^k \times \{t_i\} \times \mathbb{R}^{n-k-1}$ for all $t \in U_i$. Thus, f_1 is good, and we have finished the proof in the case $k \geq 2$.

This finishes the proof in the case $k \geq 2$. In the remaining case $k = 1$, we will prove that f can be homotoped to a map satisfying the extra assumption in the first part of the proof. First we use the hypothesis that f maps to the basepoint component of $\psi(n,2)$ to pick, for each $x \in X$, a contractible open neighborhood $U_x \subseteq X$ and a null homotopy of $f|U_x$, given by a smooth map $h_x : U_x \times \mathbb{R} \to \psi(n,2)$. For $y \in U_x$, we shall identify the smooth map $h_x(y,-) : \mathbb{R} \to \psi(n,2)$ with its graph, which is a smooth submanifold $E_x(y) \subseteq \mathbb{R} \times \mathbb{R}^n$ submersing to $\mathbb{R}$. The coordinates in $\mathbb{R}\times\mathbb{R}^n$ shall be written $(s, x_1, \dots, x_n)$, and we consider the restriction $(x_1, x_2) : W_x(y) \to \mathbb{R}^2$. We can pick $q_x = (p_x, t_x) \in \mathbb{R}^2$ which is a regular value of $(x_1, x_2) : W_x(x) \to \mathbb{R}^2$ and an $\varepsilon_x > 0$ so that, after possibly shrinking U_x, no critical points of $W_x(y) \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^2$ have $x_1 \in [p_x - \varepsilon_x, p_x + \varepsilon_x]$, $x_2 \in [t_x - \varepsilon_x, t_x + \varepsilon_x]$, and all other coordinates in $[-1,1]$. We can then refine the U_x's to a cover of X by finitely many U_i's with corresponding $q_i = (p_i, t_i)$ and $\varepsilon_i = \varepsilon > 0$. As before, we can arrange that the t_i's are distinct, the intervals $[t_i - \varepsilon, t_i + \varepsilon]$ are disjoint, and $q_i = 0$.

We now deform the paths $h_i(y,-) : \mathbb{R} \to \psi(n,2)$ in the following way. Pick a smooth function $\lambda : \mathbb{R} \to [0,1]$ which is 1 near 0 and has support in $(-\varepsilon, \varepsilon)$, and consider the maps $\phi_{i,s} : \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$, given by

$$\begin{aligned} &\mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n \\ &\quad x \mapsto (x_1 \sin(\tfrac{s\pi}{2}\lambda(x_2 - t_i)), x_1 \cos(\tfrac{s\pi}{2}\lambda(x_2 - t_i)), x_2, \dots, x_n) \end{aligned}$$

When $s = 0$, this is just the inclusion as $\{0\}\times\mathbb{R}^n$, but as $s \in [0,1]$, it "rotates" in the first two coordinates, dampened by the bump function so that the rotation happens only near $x_2^{-1}(t_i)$, and that $\phi_{i,s}$ is indepent of s unless $x_2 \in (t_i - \varepsilon, t_i + \varepsilon)$. Define subsets

$$W_i(y,s) = \phi_{i,s}^{-1}(W_i(y)) \subseteq \mathbb{R}^n.$$

These are closed subsets contained in $\mathbb{R}^{k+1} \times (0,1)^{n-k-1}$ and $W_i(y,0) = f(y)$, but they are not necessarily submanifolds of $\mathbb{R}^n$, because $\phi_{i,s}$ need not be transverse to the submanifold $W_i(y) \subseteq \mathbb{R} \times \mathbb{R}^n$. By construction it is transverse near $x_1^{-1}(0)$. At time $s = 1$, it is transverse for all x_1 near $x_2^{-1}(t_i)$. Moreover, the subset is independent of i and s outside $x_2^{-1}(t_i - \varepsilon, t_i + \varepsilon)$, and since these are disjoint open sets, we can define a subset $W(y,s) \subseteq \mathbb{R}^n$ which agrees with $W_i(y,s)$ inside $x_{k+1}(t_i - \varepsilon, t_i + \varepsilon)$ and with $f(y)$ outside these sets. This process gives a closed subset $W(y,s) \subseteq \mathbb{R}^n$ which is a submanifolds near $x_1^{-1}(0)$ and, if $y \in U_i$ and $s = 1$, near $(x_2^{-1}(t_i) \cap x_1^{-1}([0,1]))$. Finally, we can push the singularities of W to infinity in the x_1 direction and achieve a family of manifolds, which assemble to a smooth function $X \times [0,1] \to \psi(n, k+1)$. More precisely, "push to infinity in the x_1 direction" means that we replace the subsets $W(y,s) \subseteq \mathbb{R}^n$ with $(e_s \times \mathrm{id})^{-1}(W(y,s))$, where $e_s : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ is an isotopy of embeddings such that $e_0 = \mathrm{id}$, $e_s(\mathbb{R}^2)$ contains $\{0\} \times \mathbb{R}^k$ and is disjoint form the projection of the singularities of W for all s, and $e_1(\mathbb{R}^2)$ contains $[0,1] \times \{0\} \times \{t_i\}$.

The result is a smooth homotopy $W : X \times [0,1] \to \psi(n,2)$ which starts at f and ends at a map which satisfies the hypothesis in the first part of the proof. □

This finishes our proof of the weak equivalence $\psi(n,1) \to \Omega^{n-1}\Psi(\mathbb{R}^n)$.

3.2 Exercises for lecture 3

1. Use theorem 3.2 to prove that if $f : M \to M'$ is a map of group-like topological monoids, and f is a weak equivalence (of the underlying topological spaces), then $Bf : BM \to BM'$ is a weak equivalence.

2. Prove that if M is commutative, then BM is a commutative monoid (define a product on BM which takes union of finite subsets of $\mathbb{R}$, possibly multiplying labels). Explain why $B(BM)$ is the "space of configurations of points in $\mathbb{R}^2$, labeled by elements of M".

3. Let $N = \{1, 2, 3, \dots\}$ have monoid structure given by addition and pick any homeomorphism $\lambda : S^1 = \mathbb{R} \cup \{\infty\} \to U(1)$ with $\lambda(\infty) = 1$. (For example $\lambda(t) = (t + i)(t - i)$.) Prove that there is a unique monoid map $BN \to U(1)$ which takes the point $(\{t\}, 1)$ to $\lambda(t) \in U(1)$. (The following two exercises proves that this map is a homotopy equivalence.)

4. With N as in the previous exercise, let $B'N \subseteq BN$ denote the subspace where the labeled configuration is either disjoint from $(0, 1) \subseteq \mathbb{R}$, or that interval contains one points labeled 1. Prove that $B'N \simeq S^1$. (Hint: "push to infinity".)

5. With $B'N$ as in the previous exercise, prove that the inclusion $B'N \to BN$ is a homotopy equivalence. (Hint: There are several ways to construct a homotopy inverse, one is as follows: If the number $a_i \in \mathbb{R}$ have label n_i, we let $b_i \geq a_i$ be the smallest numbers such that the intervals $(b_i, b_i + n_i)$ are disjoint. If we give each point $b_i, b_i + 1, \dots, b_i + n_i - 1$ the label 1, we have a point in $B'N$.)

6. Prove that the inclusion $N \to \mathbb{Z}$ induces a weak equivalence $BN \to B\mathbb{Z}$.

7. Prove that diagram (9) is commutative.

8. The "Moore loop space" of a based space X is the space of pairs (t, γ), where $t \geq 0$ and $\gamma : [0, t] \to X$ is a loop. This space is naturally a topological monoid (add the t's, concatenate the loops), and we denote it $\Omega' X$. Prove that the inclusion $\Omega X \to \Omega' X$ is a homotopy equivalence and that $\Omega' X$ is a grouplike topological monoid and hence (by theorem 3.2) that $\beta : \Omega' X \to \Omega B(\Omega' X)$ is a weak equivalence. Then construct a map $B\Omega' X \to X$ which is a weak equivalence onto the basepoint component of X.

9. Let $\lambda : (0, 1) \to \mathbb{R}$ be a homeomorphism and M a topological monoid. For $t = (t_0, \dots, t_p) \in \Delta^p$ we have points $a_n = \lambda(\sum_{i=0}^{n-1} t_i) \in [-\infty, \infty]$ for $1 \leq n \leq p$. If we are also given $(m_1, \dots, m_p) \in M^p$ and label a_n by m_n, we have defined a map $\phi_p : \Delta^p \times M^p \to BM$ (if some a_i's coincide, we multiply the labels; if some a_i are infinite, we forget their labels). Prove that ϕ_p is continuous. (Remark: BM is often defined as the "geometric realization of the nerve" of M. The maps defined in this exercise glue

to a map from the thick realization of $N_\bullet M$ to the space BM defined in the text; the resulting map is a continuous bijection and a homotopy equivalence.)

4 Final step of the proof

It remains to study the map $\alpha_0 : \psi(n,0) \to \Omega\psi(n,1)$. Let us contemplate applying the same methods as we did for $k \geq 1$ and see where it goes wrong. The exact same proof as for $k > 0$ shows that $\psi(n,0)$ is homotopy equivalent to a monoid M. However, the monoid $\pi_0 M$ is the set of diffeomorphism classes of closed oriented 2-manifolds (at least for $n \geq 5$), where the monoid operation is disjoint union. Firstly, this monoid is not a group, so we won't have $M \simeq \Omega BM$. Secondly, the natural map $BM \to \psi(n,1)$ has no chance of being a weak equivalence, because BM is much too large: We will show in the exercises that $\pi_1\psi(n,1) = \mathbb{Z}$ while $\pi_1(BM)$ contains an abelian group of infinite rank.

In retrospect, it is also clear that $\psi(n,0)$ is not quite the right object: The Madsen–Weiss theorem concerns surfaces that are *connected* and have *high genus*, whereas $\psi(n,0)$ contain all surfaces. If we restrict to the subspace consisting of connected surfaces, we no longer have that $\psi(n,0)$ is homotopy equivalent to a monoid (the monoid operation is essentially "disjoint union"). The solution to these problems is to modify $\psi(n,0)$ in a way that we only have path connected surfaces, but still have a monoid operation. To achieve this, we will consider surfaces with boundary, and construct a monoid operation which glues surfaces along their boundary. More precisely, we make the following definition.

Definition 4.1. Write $L_t = [0,t] \times [0,1] \subseteq \mathbb{R}^2$ and let M be the set of pairs (t, W) where $t > 0$ and $W \subseteq L_t \times (-1,1)^{n-2}$ is a compact, connected, oriented 2-dimensional submanifold which agrees with $L_t \times \{0\}$ near $(\partial L_t) \times \mathbb{R}^{n-2}$. Define a product operation on M as

$$(t, W)(t', W') = (t + t', W \cup (W' + te_1)).$$

In order to describe the topology, we note that M is in bijection with the set of (t, W), where $W \in \Psi(\mathbb{R}^n)$ agrees with $\mathbb{R}^2 \times \{0\}$ outside $L_t \times \mathbb{R}^{n-2}$ and has $W \subseteq \mathbb{R}^2 \times (-1,1)^{n-2}$. Then we topologize as a subspace of $\Psi(\mathbb{R}^n)$.

Lemma 4.2. *M is a homotopy commutative topological monoid. If $n \geq 5$, $\pi_0 M = \mathbb{N}$.*

Proof. Homotopy commutativity is proved using the same picture as one uses to prove that π_2 of a space is abelian.

The map $M \to \mathbb{N}$ which maps a connected surface to its genus gives a monoid map $\pi_0 M \to \mathbb{N}$ which is surjective for $n \geq 3$ (because a genus 1 surface can be embedded in $L_t \times \mathbb{R}$) and injective for $n \geq 5$ (because any two embeddings of a genus g surface are isotopic in that case). □

In fact, we can say a bit more when n is large. Indeed, Lemma 1.3 can again be used to interpret M as a classifying space for smooth surface bundles (at least in the limit where $n \to \infty$), but now it classifies bundles of connected surfaces with one parametrized boundary component. Thus, for $n = \infty$ we have

$$M = \coprod_{g \geq 0} B\mathrm{Diff}^{\partial}(\Sigma_{g,1}),$$

where $B\mathrm{Diff}^{\partial}(\Sigma_{g,1})$ classifies smooth surface bundles $E \to X$ whose fibers are connected genus g surfaces and has $\partial E = X \times S^1$.

In this last lecture we shall prove the following result, which is the proper replacement of theorem 3.7 for $k = 0$.

Theorem 4.3. *There is a map $BM \to \psi(n,1)$ making the diagram*

$$\begin{array}{ccc} M & \xrightarrow{\beta} & \Omega BM \\ \downarrow & & \downarrow \\ \psi(n,0) & \xrightarrow[\alpha_0]{} & \Omega\psi(n,1) \end{array}$$

homotopy commutative. For $n \geq 5$, the map $BM \to \psi(n,1)$ is a weak equivalence.

Using this, we are *almost* ready to copy the steps involved in the case $k > 0$. The only caveat is that the monoid M is not group-like, so $\beta : M \to \Omega BM$ is not a weak equivalence. A striking result, known as the "group completion theorem," describes the induced map $H_*(M) \to H_*(\Omega BM)$ as a localization. More precisely, let $m_0 \in M$ be a surface of genus 1, and let M_∞ be the mapping telescope of the direct system

$$M \xrightarrow{\cdot m_0} M \xrightarrow{\cdot m_0} \dots.$$

The mapping telescope is defined as the quotient space

$$(M \times \mathbb{N} \times [0,1])/((m,n,1) \sim (mm_0, n+1, 0)).$$

and in our case, it can be rewritten as $\mathbb{Z} \times B\mathrm{Diff}_\infty$, where $B\mathrm{Diff}_\infty$ is the mapping telescope of of the direct system

$$\cdots \to B\mathrm{Diff}(\Sigma_{g,1}) \to B\mathrm{Diff}(\Sigma_{g+1,1}) \to \dots.$$

We regard $M \subset M_\infty$ as the subspace $M \times \{0\} \times \{0\}$. In this case, the group completion theorem implies that the canonical map $\beta : M \to \Omega BM$ extends to a map

$$\beta_\infty : M_\infty \to \Omega BM$$

which induces an isomorphism in integral homology. In the limit $n \to \infty$, we have

$$H_*(M) = \bigoplus H_*(B\mathrm{Diff}^{\partial}(\Sigma_{g,1}))$$

and

$$H_*(M_\infty) = H_*(M)[m_0^{-1}] = H_*(\mathbb{Z} \times B\mathrm{Diff}_\infty).$$

Combining with what we proved previously, we have a map

$$B\mathrm{Diff}_\infty \to \Omega_0^\infty \mathbf{\Psi},$$

inducing an isomorphism in integral homology. Together with homological stability, this proves the Madsen–Weiss theorem.

4.1 Proof of theorem 4.3

We now want to prove the weak equivalence $BM \simeq \psi(n,1)$. The natural map ends up in a modified space $\psi'(n,1)$, defined as follows.

Definition 4.4. Let $\psi'(n,1)$ be the space of topologically closed submanifolds $W \subseteq \mathbb{R} \times [0,1] \times \mathbb{R}^{n-2}$ which are contained in $\mathbb{R} \times [0,1] \times (-1,1)^{n-2}$ and agree with $\mathbb{R} \times [0,1] \times \{0\}$ near $\mathbb{R} \times \partial[0,1] \times \mathbb{R}^{n-2}$.

Lemma 4.5. *The map $\psi(n,k+1) \to \psi'(n,k+1)$ which maps W to $W \cup \mathbb{R} \times [0,1] \times \{0\}$ is a homotopy equivalence.*

Proof. [picture] □

With M the monoid of connected surfaces, we define $BM \to \psi'(n,1)$ analogously to what we did for $k > 0$. In that case, an important notion was that of a "good map" $X \to \psi(n,k+1)$. It is key to the case $k = 0$ to have the right notion of good map $X \to \psi'(n,1)$ in this case.

Definition 4.6. Let X be a smooth manifold and $f : X \to \psi'(n,1)$ a smooth map. For $a \in \mathbb{R}$, let $X_a \subseteq X$ be the set of points x satisfying

$$f(x) \cap (\{a\} \times \mathbb{R}^{n-1}) = \{a\} \times [0,1] \times \{0\}.$$

Furthermore, let $X^{\mathrm{nc}} \subseteq X$ be the set of points x satisfying that no path component of $f(x) \subseteq \mathbb{R} \times [0,1] \times \mathbb{R}^{n-2}$ is compact. Let us say that f is good provided $X = X^{\mathrm{nc}} = \cup_a \mathrm{int}(X_a)$.

Proposition 4.7. *Any smooth map $f : X \to \psi'(n,1)$ is smoothly homotopic to a good map. The homotopy can be taken constant near a closed set on which f is already good.*

Proof. We first deform f to achieve that no path component of any $f(x) \subseteq \mathbb{R} \times (-1,1)^{n-1}$ is contained in $\{0\} \times (-1,1)^{n-1}$. Then we pick an isotopy of embeddings $e_s : \mathbb{R} \to \mathbb{R}$ which has $e_0 = \mathrm{id}$ and $e_1(\mathbb{R}) \subseteq (-\varepsilon, \varepsilon)$. Then we deform $f(x)$ through the path $f_s(x) = (e_s \times \mathrm{id})^{-1}(f(x))$. After this deformation, $f(x)$ has no compact path components.

Next, we want to change f in order to have that for each x there exists t such that $f(x) \cap x_1^{-1}(t)$ is diffeomorphic to an interval.

[picture] □

Using this, we finish the proof in the same way as for $k > 0$.

4.2 Exercises for lecture 4

1. Let $N = \{1, 2, \dots\}$ with addition as monoid structure. Describe the map $\beta : N \to \Omega BN$ up to homotopy, both explicitly (using the homotopy equivalence $BN \simeq S^1$ from yesterday) and using the "group completion theorem".

2. Let M be the monoid which is homotopy equivalent to $\psi(n, 0)$ (i.e. pairs (t, W) with $W \subseteq (0, t) \times (0, 1)^{n-1}$). For each $g \geq 0$ construct a map $BM \to BN$ which "counts the number of genus g components" (i.e. the composition $M \to \Omega BM \to \Omega BN$ sends $[W] \in \pi_0 M$ to the element of $\pi_0 \Omega BN = \pi_1 BN = \mathbb{Z}$ which is the number of path components of W which have genus g. Use these maps to prove that $\pi_1 BM$ surjects to a free abelian group of countable rank.

3. We proved earlier that $\kappa_0 \in H^0(\Omega^n \Psi(\mathbb{R}^n)) = H^0(\Omega\psi(n, 1))$ is divisible by 2. Prove that for large n, $\kappa_0/2$ gives rise to an isomorphism $\pi_1\psi(n, 1) \to \mathbb{Z}$. (Hint: One method is to use the Serre spectral sequence to calculate H_n of the homotopy fiber of the map $u : \mathrm{Th}(\gamma_n^\perp) \to K(\mathbb{Z}, n-2)$ representing the Thom class.)

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