Notes in Math. 386, Springer-Verlag (1974).

[52] J. Tits. On buildings and their applications, Proc. Int. Congress Math., Vancouver (1974), t. I, 209-20.

[53] L. Vaseršteĭn. On the group SL_2 over Dedekind rings of arithmetic type, Math. USSR Sbornik 18 (1972), 321-32.

[54] M.-F. Vignéras. Invariants numériques des groupes de Hilbert, Math. Ann. 224 (1976), 189-215.

[55] E. B. Vinberg. Discrete groups generated by reflections in Lobačevskii space, Mat. Sbornik 72 (1967), 471-88; correction, ibid. 73 (1967), 303.

[56] E. B. Vinberg. Some examples of crystallographic groups on Lobačevskii spaces, Mat. Sbornik 78 (1969), 633-9.

[57] E. B. Vinberg. Some arithmetical discrete groups in Lobačevskii spaces, Proc. Int. Coll., Bombay, (1973), 323-48.

[58] C. T. C. Wall. The topological space-form problems, pp. 319-31 in Topology of Manifolds (ed. J. C. Cantrell and C. H. Edwards Jr.), Markham (1970).

[59] A. Weil. On discrete subgroups of Lie groups, Ann. of Math. 72 (1960), 369-84; II, ibid., 75 (1962), 578-602.

[60] A. Weil. Remarks on the cohomology of groups, Ann. of Math. 80 (1964), 149-57.

[61] C. W. Wilkerson. Applications of minimal simplicial groups, Topology 15 (1976), 111-30. Corrections: Topology (to appear).

5 · Topological methods in group theory

PETER SCOTT and TERRY WALL

University of Liverpool

Introduction

This article is a revised version of notes on an advanced course given in Liverpool from January to March 1977 in preparation for the symposium. The lectures given by Terry Wall at the symposium were mainly taken from Sections 3 and 4, and much of the material in John Stallings' lectures is in Sections 5 and 6. It seemed worth publishing the whole, as a rather full introduction to the area for those with a background in topology. Originality is not claimed for the results in the earlier sections (though full references have not always been given), but the uniqueness results in Section 7 and most of Section 8 are due to Peter Scott.

1. BASIC NOTIONS

The link between topology and group theory comes from the fundamental group. I shall make no attempt to present this: almost every introductory topology text does so. Particularly suitable for this course is Massey's book [18]. An equivalent account, from a different viewpoint, is given by Brown [2]. Let us recall the basic properties of the fundamental group.

(1) For every topological space X and point $x \in X$ we have a group $\pi_1(X; x)$. This depends only on the path component of X containing x. A path from x to y induces an isomorphism $\pi_1(X; x) \to \pi_1(X; y)$; a closed path induces an inner automorphism. A map $f : X \to Y$ with $f(x) = y$ induces a homomorphism $f_* : \pi_1(X; x) \to \pi_1(Y; y)$, and this assignment is functorial: in fact we have a homotopy functor.

(2) A map $\pi : Z \to X$ is a covering if it is locally trivial, with discrete fibres - i.e. every $x \in X$ has a neighbourhood U, and a homeo-

morphism $\pi^{-1}(U) \xrightarrow{h} U \times D$ with D discrete, such that $pr_1 \circ h = \pi$. For a covering π, and $z \in Z$, the map $\pi_* : \pi_1(Z; z) \to \pi_1(X; f(z))$ is injective. If X is reasonably nice (the minimal technical conditions are path-connected, locally path-connected, and weakly locally 1-connected), the correspondence between triples ($\pi : Z \to X$ a connected covering, $z \in \pi^{-1}(x)$) and subgroups $\pi_*\pi_1(Z; z)$ of $\pi_1(X; x)$ induces an isomorphism of categories. Hence, in particular, the coverings are isomorphic if and only if the corresponding subgroups are conjugate.

(3) The third basic fact we need to recall is the technique for calculating fundamental groups, due to van Kampen. First suppose X_1, X_2 are path connected open subsets of X, with path-connected intersection X_0. Then for any base point $x \in X_0$, we have the following commutative diagram in which all the maps are induced by inclusions of spaces.

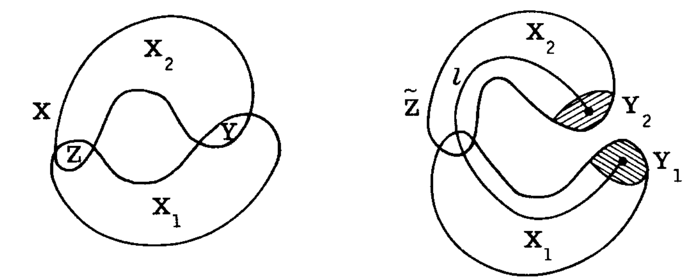
$$\begin{array}{ccc} \pi_1(X_0; x) & \xrightarrow{i_1} & \pi_1(X_1; x) \\ \downarrow i_2 & & \downarrow j_1 \\ \pi_1(X_2; x) & \xrightarrow{j_2} & \pi_1(X; x) \end{array}$$

Proposition 1.1. <u>This is a pushout diagram in the category of groups. In other words, for any group</u> G <u>we have a bijection, induced by</u> (j_1, j_2), $\mathrm{Hom}(\pi_1(X; x), G) \cong \{(f_1, f_2) \in \mathrm{Hom}(\pi_1(X_1; x), G) \times \mathrm{Hom}(\pi_1(X_2; x), G) : f_1 i_1 = f_2 i_2 \in \mathrm{Hom}(\pi_1(X_0; x), G)\}$.

The standard argument with universals shows that the pushout is uniquely determined; existence is provided by the proposition. The proof, involving breaking up a path in X into subpaths each lying in X_1 or X_2, is somewhat messy. The restriction that the X_i are open can be relaxed if each is a deformation retract of some neighbourhood, as is usually the case in practice.

The restriction that X_0 be connected is less desirable. One may reformulate the proposition to cover this case by using groupoids [2]. More naively, suppose X_0 has just two path-components Y and Z; define $\tilde{Z}$ by identifying X_1 and X_2 along Z, and $\tilde{X}$ by attaching $Y \times I$ to $\tilde{Z}$ by identifying $Y \times i$ to the copy Y_{i+1} of Y in X_{i+1} $(i = 0, 1)$.

There is an obvious map $\phi : \tilde{X} \to X$ (identify $Y \times I$ to Y) which is usually a homotopy equivalence, and induces isomorphisms of π_1. We can calculate $\pi_1(\tilde{Z})$ by Proposition 1.1.



Now choose a base point $y \in Y$; let y_i be the corresponding point in Y_i, and l a path in $\tilde{Z}$ joining y_1 to y_2. We have homomorphisms

$$\alpha_1 : \pi_1(Y; y) \cong \pi_1(Y_1; y_1) \to \pi_1(\tilde{Z}; y_1),$$
$$\alpha_2 : \pi_1(Y; y) \cong \pi_1(Y_2; y_2) \to \pi_1(\tilde{Z}; y_2) \overset{l}{\cong} \pi_1(\tilde{Z}; y_1).$$

Proposition 1.2. <u>For any group</u> G, <u>we have a bijection</u>

$$\mathrm{Hom}(\pi_1(\tilde{X}; y_1), G) \cong \{(f_1, t) \in \mathrm{Hom}(\pi_1(\tilde{Z}; y_1), G) \times G :$$
$$f_1\alpha_2(p) = t^{-1} f_1 \alpha_1(p) t \text{ for all } p \in \pi_1(Y; y)\}.$$

Here, f_1 is the composite $\pi_1(\tilde{Z}; y_1) \to \pi_1(\tilde{X}; y_1) \to G$, and t is the image of the class in $\pi_1(\tilde{X}; y_1)$ of the loop $l \cup y \times I$.

We will show that Proposition 1.2 follows from Proposition 1.1.

In order to start our study of fundamental groups, we need to know that the circle S^1 has infinite cyclic fundamental group. This is easy to prove using the covering of S^1 by the real line $\mathbf{R}$. One might think of deducing this result from Proposition 1.2 by taking X to be S^1 and X_1 and X_2 to be open intervals. For then $\mathrm{Hom}(\pi_1(X), G) \cong G$ for any group G. However our proof of Proposition 1.2 uses the fact that S^1 has infinite cyclic fundamental group. Here is a quick sketch of that proof.

First, the intermediate space $W = \tilde{Z} \cup (y \times I)$ can be considered as the union of $\tilde{Z}$ and the circle $l \cup (y \times I)$ intersecting in the arc l.

Hence $\pi_1(\tilde{W}, y_1)$ is the pushout of $\mathbb{Z} \leftarrow 1 \rightarrow \pi_1(\tilde{Z}, y_1)$, i.e. (see later) the free product $\mathbb{Z} * \pi_1(\tilde{Z}, y_1)$. Now $\tilde{X} = W \cup (Y \times I)$, and $W \cap (Y \times I) = (Y \times 0) \cup (y \times I) \cup (Y \times 1)$. We deduce, after a little manipulation, a pushout diagram

$$\begin{array}{ccc} \pi_1(Y, y) * \pi_1(Y, y) & \xrightarrow{\phi} & \mathbb{Z} * \pi_1(\tilde{Z}, y_1) \\ \downarrow & & \downarrow \\ \pi_1(Y, y) & \longrightarrow & \pi_1(\tilde{X}, y_1) \end{array}$$

where ϕ is given by α_1 on the first factor and by $c \mapsto t \cdot \alpha_2(c) \cdot t^{-1}$ on the second. This is equivalent to Proposition 1.2. We have given the proposition independently, however, since it introduces a construction which will be important below.

Example 1.3. $X = S^1 \vee S^1$, the one-point union. Now apply Proposition 1.1, taking X_1, X_2 as the two circles. Thus $\mathrm{Hom}(\pi_1(X;y), G) \cong G \times G$. The group $\pi_1(X, y)$ is called the free group on generators t_1, t_2 the classes of the circles.

To put some bones into this abstraction, we next give a concrete description of this free group on t, u. A letter is any one of $t, u, \bar{t}, \bar{u}$. A word is a finite (perhaps empty) sequence of letters. The word is reduced if none of $t\bar{t}, \bar{t}t, u\bar{u}, \bar{u}u$ occurs as a pair of consecutive letters.

Theorem 1.4. There is a bijection between elements of the free group F on t, u and the set W of reduced words. Each word defines an element of G by forming the product of various of t, u, t^{-1}, u^{-1} in the indicated order.

Proof. (i) Observe that F contains the elements t, u and hence the set H of products of finite ordered sequences of elements t, u, t^{-1}, u^{-1}. Clearly H is closed under products and inverses, hence is a subgroup. There is a homomorphism $F \rightarrow H$ such that $t \rightarrow t$, $u \rightarrow u$. The composite $F \rightarrow H \subset F$ coincides with the identity on t, u hence is the identity. Thus $F = H$.

By definition, each element of H is represented by a word. If the word is not reduced, we can cancel products tt^{-1} etc. Thus each element is represented by a reduced word and we have a surjection $\alpha : W \rightarrow F$. It remains to prove α bijective.

(ii) Write S for the symmetric group on the set W of reduced words. Define permutations $\tau, \vartheta \in S$ as follows: If the word w ends in $\bar{t}$ (resp. $\bar{u}$), then $w\tau$ (resp. $w\vartheta$) is obtained from w by deleting the last letter. Otherwise, $w\tau$ (resp. $w\vartheta$) consists of w followed by t (resp. u). We see at once that these are permutations with inverses $\tau^{-1}, \vartheta^{-1}$ defined similarly but interchanging the roles of t and $\bar{t}$ (resp. u and $\bar{u}$).

By definition of F, there is a unique homomorphism $\phi : F \rightarrow S$ such that $\phi(t) = \tau$, $\phi(u) = \vartheta$. We define a map $\beta : F \rightarrow W$ by $\beta(g) = \phi(g)(1)$. For any reduced word w, we see by induction on the length of w that $\beta(\alpha(w)) = w$. Thus α is injective, hence bijective.

We used the example of $S^1 \vee S^1$ to demonstrate the existence of a free group F of rank two. Note that the proof above does this quite independently for it shows that the set W has a natural group structure which makes it a free group of rank two.

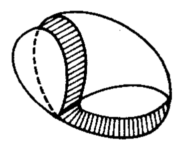
There is an obvious analogue to the above for the free group F(X) on any set X of generators. If X is finite, existence is seen by induction. We observe that if $X_1 \subset X_2$, the natural map $F(X_1) \rightarrow F(X_2)$ is injective. Now for X infinite, define

$$F(X) = \cup \{F(Y) : Y \subset X, \ Y \text{ finite}\}.$$

If $y_i \in F(Y_i) \subset F(X)$ for $i = 1, 2, 3$ we define $y_1 y_2$ to be the product in $F(Y_1 \cup Y_2)$. Associativity follows by considering $F(Y_1 \cup Y_2 \cup Y_3)$. It is immediately verified that for any G, restriction to X yields a bijection $\mathrm{Hom}(F(X), G) = \mathrm{Map}(X, G)$, so F(X) is th free group on X.

Now consider any group G, set X and map $\phi : X \rightarrow G$. By the above, ϕ has a unique extension $\psi : F(X) \rightarrow G$ to a homomorphism whose image is then a subgroup $\bar{X}$. Any element of $\bar{X}$ can be written as a word in the elements $\phi(x)$, and thus lies in any subgroup of G containing $\phi(X)$. Thus $\bar{X}$ is the intersection of the subgroups containing $\phi(X)$: it is called the subgroup generated by $\phi(X)$ (or by ϕ). If $\overline{X} = G$, we say G is generated by $\phi(X)$.

Now consider a finite CW-complex K with one vertex x. By induction we see that the 1-skeleton K^1 has free fundamental group generated by the classes g_i of the 1-cells. As K^1 is a deformation retract of a neighbourhood, we can apply van Kampen's theorem to calculate the effect on the fundamental group of attaching a 2-cell e^2. Now e^2 is contractible, and a suitable neighbourhood of K^1 meets it in a copy of $S^1 \times \mathbb{R}$. The map $\alpha : S^1 \to N(K^1) \to K^1$ is homotopic to the attaching map of the cell, and determines $\alpha_* : \mathbb{Z} = \pi_1 S^1 \to \pi_1 K^1$ with $\alpha_*(1) = r$, say.



Then $\pi_1(K^1 \cup e^2)$ is the pushout of

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\alpha_*} & \pi_1(K^1) \\ \downarrow & & \\ \{1\} & & \end{array}$$

Arguing similarly with the other 2-cells e_j, we find that if their attaching maps yield classes $r_j \in \pi_1(K^1)$, then the 2-skeleton K^2 has fundamental group $\pi_1(K^2)$ characterized by the property that for any group G,

$$\mathrm{Hom}(\pi_1(K^2), G) = \{f \in \mathrm{Hom}(\pi_1(K^1), G): f(r_j) = 1 \text{ for each } j\}.$$

Since $\pi_1(K^1)$ is free on $\{g_i : i \in I\}$, f is determined by the $f(g_i)$.

The sequence $\{g_i | r_j\}$ where the g_i are abstract symbols and $r_j \in F\{g_i\}$ is called a presentation of π if, for any G, $\mathrm{Hom}(\pi, G)$ is given as above. The same argument as in (i) of the proof above shows that the images of the g_i are generators of π. The r_j are called

relators. Let N be the subgroup of $F\{g_i\}$ generated by the r_j and all their conjugates: N is called the normal closure of the r_j. Clearly it is the least normal subgroup of $F\{g_i\}$ containing them all. Hence $F\{g_i\}/N$ has the universal property defining $\{g_i | r_j\}$: this yields a construction of this group. Again, the restriction to finite sets of generators and relators is easily seen to be irrelevant.

Of course you have all seen generators and relators before: here it is the relation with two dimensional CW complexes that I wish to stress. (Incidentally, adding cells of dimension > 2 does not affect π_1, as we see on applying van Kampen's theorem again.) For example, let X^2 be any such - say having one vertex x - and Y any space.

Lemma 1.5. For any homomorphism $\phi : \pi_1(X^2, x) \to \pi_1(Y, y)$ there is a map $\alpha : X^2 \to Y$ with $\alpha_* = \phi$.

Proof. The 1-cells and 2-cells of X^2 give a presentation $\pi_1(X^2) = \{g_i | r_j\}$. The image of g_i by ϕ is an element of $\pi_1(Y)$, represented by a map $(S^1, x) \to (Y, y)$. Use these maps to define $\alpha^1 : X^1 \to Y$. Then we have a diagram

$$\begin{array}{ccc} F\{g_i\} \cong \pi_1(X^1, x) & \xrightarrow{\alpha^1_*} & \\ \downarrow & & \pi_1(Y, y) \\ \{g_i | r_j\} \cong \pi_1(X^2, x) & \xrightarrow{\phi} & \end{array}$$

which commutes, by construction of α^1. Hence $\alpha^1_*(r_j) = 1$. For each 2-cell of X, with characteristic map

$$\chi_j : (D^2, S^1) \to (X^2, X^1, x)$$

the class of $\chi_j | S^1$ is r_j (by definition) so the class of $\alpha^1 \circ \chi_j | S^1$ is $\alpha^1_*(r_j) = 1$. Thus $\alpha^1 \circ \chi_j$ is nullhomotopic, so there is a continuous extension

$$\psi_j : D^2 \to Y$$

with $\psi_j | S^1 = \alpha^1 \circ (\chi_j | S^1)$. Now by definition of the topology of X as an identification space, the diagram

$$\begin{array}{ccc} X^1 \cup (\bigcup_j (D^2 \times j)) & \xrightarrow{\alpha^1 \cup \{\psi_j\}} & Y \\ \downarrow \text{incl} \cup \{\chi_j\} & \nearrow & \\ X^2 & & \end{array}$$

defines a map $\alpha : X^2 \to Y$ such that $\alpha | X^1 = \alpha^1$ and $\alpha \circ \chi_j = \psi_j$. Since $\pi_1(X^2, x)$ is a quotient of $\pi_1(X^1, x)$ it follows that $\alpha_* = \phi$.

Remarks. The condition that X has dimension 2 is essential here. However, a particularly interesting case is when Y is such that $\pi_1(Y, y) = \pi$ and for any (X, x) the map $\alpha \to \alpha_*$ gives a bijection between homotopy classes of maps $(X, x) \to (Y, y)$ and $\text{Hom}(\pi_1(X, x), \pi)$. Such a (Y, y) is called a classifying space for π (or Eilenberg-MacLane space $K(\pi, 1)$). The usual argument with universals shows its uniqueness (up to homotopy); existence is also not hard to prove, for any π. However, the existence of a Y which is e.g. a manifold, or finite complex imposes an interesting and subtle condition on π.

I now return to Propositions 1.1 and 1.2. We have obtained in some cases fairly explicit descriptions of the groups so defined. I will now give some useful generalizations of these.

The real beginning of our subject was the discovery that in the case when the maps are injective one can obtain structure theorems similar to 1.4. In fact a description of $A *_C B$ in terms of reduced words was already given by Schreier in 1927 [36], but a more thorough account and the start of the recent work of the subject is contained in Hanna Neumann's thesis [35]. The groups $A *_C$ go back to [13], though they are not actually defined in that paper. There have of course been many papers on the subject since; our account derives from those of Serre [22] and Cohen [5] and seems simpler and more natural than the original papers.

Propositions 1.1, resp. 1.2 concerned diagrams

$$\begin{array}{ccc} C & \xrightarrow{\alpha_1} & A \\ \downarrow \alpha_2 & & \downarrow \beta_2 \\ B & \xrightarrow{\beta_1} & G \end{array} \qquad \text{resp.} \qquad C \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} A \xrightarrow{\beta} G$$

Definition. If the maps α_1, α_2 are injective, the universal group G is called the free product of A and B amalgamated along C (resp. amalgamated free product of A along C), and denoted by $A *_C B$ resp. $A *_C$. If C is the trivial group, then $A *_C B$ is denoted by $A * B$ and is called the free product of A and B.

Note. There seems no reason except tradition for calling the first an amalgamated product but the second an HNN group.

We now present the traditional combinatorial arguments for analysing the structure of amalgamated products. Essentially equivalent results will be obtained below independently by geometrical reasoning.

In each of the above cases, one can give an explicit description using reduced words. The easy first half of the proof of Theorem 1.4 shows (with only slight changes) that any element of $A *_C B$ can be written as a product

$$a_1 b_1 a_2 b_2 \ldots a_n b_n \quad \text{with} \quad a_i \in \beta_1(A),\ b_i \in \beta_2(B) \quad (\text{maybe} = 1)$$

and that any element of $A *_C$ can be written as

$$a_1 t^{r_1} a_2 t^{r_2} \ldots a_n t^{r_n} \quad \text{with} \quad a_i \in \beta(A),\ r_i \in \mathbb{Z}.$$

Now we restrict our attention to $A *_C B$. Again, it is clear that some reduction is possible - e.g. for $c \in C$, $\beta_1(a)\beta_2(\alpha_2(c)\cdot b) = \beta_1(a \cdot \alpha_1(c))\beta_2(b)$. We deal with this by pushing all the c's to the right, as follows. To simplify notation, write α_1, α_2 as inclusions so $C \subset A$, $C \subset B$. Pick representatives $a_i \in A$ for the right cosets $a_i C$ of C - thus giving a section of the projection $A \to A/C$, or right transversal of C in A; then do the same for B. We impose the restriction that the identity coset C is represented by the identity element.

A reduced word is now a sequence

$$a_1 b_1 \ldots a_n b_n c$$

such that $c \in C$, a_i belongs to the chosen transversal T_A for C in A, b_i belongs to the chosen transversal T_B for C in B and

$a_i = 1 \Rightarrow i = 1, \ b_i = 1 \Rightarrow i = n.$

Any element of $A *_C B$ may be represented by a reduced word. For write the element as

$$a_1 b_1 \ldots a_n b_n$$

and use induction on n. For $n = 0$, the empty word may be represented by $1 \in C$: a reduced word. Otherwise, by inductive hypothesis, we may write

$$a_1 b_1 \ldots a_{n-1} b_{n-1} = a'_1 b'_1 \ldots a'_r b'_r c', \text{ a reduced word with } r \leq n-1.$$

If now $b'_r = 1$ resp. $a_n \in C$ then $(a'_r c' a_n) \in A$ resp. $(b'_r c' a_n b_n) \in B$ and we have a word of length $\leq r$, which may be reduced by inductive hypothesis. Otherwise write $c'a_n = a'_{r+1} c''$ with $1 \neq a'_{r+1} \in T_A$ and $c'' b_n = b'_{r+1} c$ with $b_{r+1} \in T_B$ and we have a reduced word $a'_1 b'_1 a'_2 \ldots b'_{r+1} c$.

Theorem 1.6. The maps $A \to A *_C B$, $B \to A *_C B$ are injective: every element may be represented by a unique reduced word.

Proof. Again write W for the set of reduced words.

Define an action of B on W by

$$(a_1 b_1 \ldots a_n b_n c)^b = a_1 \ldots a_n b' c' \text{ if } b_n cb = b'c' \text{ in } B \text{ with } b' \in T_B.$$

To check that this is an action, observe that the part of the word up to (and including) a_n is left fixed; for the words $a_1 \ldots a_n b_n c$ which start so, it is equivalent to the right action of B on itself.

Define an action of A on W by

$$(a_1 b_1 \ldots a_n b_n c)^a = \begin{cases} a_1 \ldots b_n a'c' & \text{if } b_n \neq 1, \ ca = a'c' \text{ with } a' \in T_A \\ a_1 \ldots b_{n-1} a''c'' & \text{if } b_n = 1, \ a_n ca = a''c'' \text{ with } a'' \in T_A \end{cases}$$

To check that this is an action, observe that the part of the word up to b_n (if $b_n \neq 1$) or b_{n-1} (if $b_n = 1$) is fixed; the rest is the standard right action. This defines maps $A \to S(W)$, $B \to S(W)$ which clearly agree on C, hence define $A *_C B \xrightarrow{\phi} S(W)$. It is now immediate by induction again that $(\ )\phi(w) = w$ for any reduced word w, so these elements of the group are distinct.

For $A *_C$ we proceed similarly. Pick right transversals T_i of $\alpha_i(C)$ in A. Now $t\alpha_2(c) = \alpha_1(c)t$ so $t^{-1}\alpha_1(c^{-1}) = \alpha_2(c^{-1})t^{-1}$ and we define a reduced word to be one of the form

$$a_1 t^{\varepsilon_1} a_2 t^{\varepsilon_2} \ldots a_n t^{\varepsilon_n} a_{n+1}$$

where $\varepsilon_i = \pm 1$, $a_i \in T_1$ if $\varepsilon_i = +1$, $a_i \in T_2$ if $\varepsilon_i = -1$ and moreover $a_i \neq 1$ if $\varepsilon_{i-1} \neq \varepsilon_i$. We let a_{n+1} be arbitrary. The above relations allow us to bring any word to a reduced form.

Theorem 1.7. The map $\beta : A \to A *_C$ is injective. Every element is represented by a unique reduced word.

Proof. Again we define an action. The element $a \in A$ acts by sending the final a_{n+1} to $a_{n+1}a$; t corresponds to the permutation τ defined by setting

$$(a_1 t^{\varepsilon_1} \ldots a_n t^{\varepsilon_n} a_{n+1})\tau =$$

$$a_1 t^{\varepsilon_1} \ldots a_{n-1} t^{\varepsilon_{n-1}} (a_n \alpha_2 \alpha_1^{-1}(a_{n+1})) \text{ if } \varepsilon_n = -1 \text{ and } a_{n+1} \in \alpha_1(C)$$

$$a_1 t^{\varepsilon_1} \ldots a_n t^{\varepsilon_n} a'_{n+1} t \alpha_2(c') \text{ otherwise, where}$$

$$a_{n+1} = a'_{n+1} \alpha_1(c') \text{ with } a'_{n+1} \in T_1 .$$

We see that this is a permutation by verifying that an inverse is given by

$$(a_1 t^{\varepsilon_1} \ldots a_n t^{\varepsilon_n} a_{n+1})\bar{\tau} =$$

$$a_1 t^{\varepsilon_1} \ldots a_{n-1} t^{\varepsilon_{n-1}} (a_n \alpha_1 \alpha_2^{-1}(a_{n+1})) \text{ if } \varepsilon_n = +1 \text{ and } a_{n+1} \in \alpha_2(C),$$

$$a_1 t^{\varepsilon_1} \ldots a_n t^{\varepsilon_n} a''_{n+1} t^{-1} \alpha_1(c'') \text{ otherwise, where}$$

$$a_{n+1} = a''_{n+1} \alpha_2(c'') \text{ with } a''_{n+1} \in T_2 .$$

The proof now concludes as before.

2. GRUŠKO'S THEOREM

Gruško's Theorem. *Let F be a finitely generated free group, $G = G_1 * G_2$ and let $\phi : F \to G$ be an epimorphism. Then there are subgroups F_1 and F_2 of F such that $F = F_1 * F_2$ and $\phi(F_i) = G_i$.*

This is a subtle result about generators of G. It says that if G can be generated by n elements, then there exists a set of n generators for G with each element in G_1 or G_2. This gives us the inequality

$$\mu(G) \geq \mu(G_1) + \mu(G_2)$$

where $\mu(G)$ denotes the minimal number of generators of a group. But the reverse inequality is obvious, so we deduce

Corollary 2.1. *If $G = G_1 * G_2$, then $\mu(G) = \mu(G_1) + \mu(G_2)$.*

As only the trivial group can have μ equal to zero, we see that $\mu(G_i) < \mu(G)$ when G_1 and G_2 are nontrivial.

Corollary 2.2. *If G is a finitely generated group, then $G = G_1 * \ldots * G_n$ for some n, where each G_i is indecomposable. (I.e. $G_i = A * B$ implies A or B is trivial.)*

We now give Stallings' proof [25] of Gruško's Theorem. See [12] for a proof using groupoids and see [3] for a proof using Bass-Serre theory (Chapter 4 of these notes).

Pick two CW-complexes with fundamental groups G_1 and G_2 and construct a CW-complex X with fundamental group $G_1 * G_2$ by joining these two complexes with an interval E. Let v denote the midpoint of E and subdivide E so that v is a vertex of E. We will take v as the basepoint of X. Let X_i denote the closure of the component of $X - \{v\}$ whose fundamental group is G_i.

Let K be a based space and let $f : K \to X$ be a based map. We will say that f *represents* ϕ if there is an isomorphism of $\pi_1(K)$ with F such that the diagram below commutes.

$$\begin{array}{ccc} \pi_1(K) & \longrightarrow & F \\ & \searrow^{f_*} & \downarrow \phi \\ & & G = \pi_1(X) \end{array}$$

We consider 2-dimensional CW-complexes K and cellular maps $f : K \to X$ which represent ϕ. Such maps certainly exist for one can take K to be the wedge of n circles where n is the rank of F. However, this particular choice of K may not be the correct one for our purposes. Our aim is to choose K and a map $f : K \to X$ representing ϕ so that $f^{-1}(v)$ is a tree. Once this is achieved, the result follows easily. For let L_i denote $f^{-1}(X_i)$ and let F_i denote $\pi_1(L_i)$. As $L_1 \cup L_2 = K$, and as $L_1 \cap L_2 = f^{-1}(v)$ which is simply connected, we see that $\pi_1(K) = F_1 * F_2$. Also $f_*(F_i) \subset G_i$ as $f(K_i) \subset X_i$. Now the results of Theorem 1.6 on reduced words in a free product, show that we must have $f_*(F_i) = G_i$, because ϕ is an epimorphism. This is now the conclusion of Gruško's Theorem.

We find an appropriate choice for K by starting with a space K_0 and a map $f_0 : K_0 \to X$ representing ϕ and then performing a sequence of modifications to K_0 and f_0. We do this so as to obtain spaces $K_1, K_2, \ldots$ and maps $f_1, f_2, \ldots$ all representing ϕ, such that $f_n^{-1}(v)$ is a forest (disjoint union of trees) with α_n components and $\alpha_{n+1} < \alpha_n$, for $n \geq 0$. After at most α_0 steps, we will obtain a space K_n and map $f_n : K_n \to X$ representing ϕ such that $f_n^{-1}(v)$ is a tree, and the result will follow.

We take K_0 to be the wedge of n circles, where n is the rank of F and choose a cellular map $f_0 : K_0 \to X$ representing ϕ so that $f_0^{-1}(v)$ is a finite number of 0-cells in K_0. In particular, $f_0^{-1}(v)$ is a forest with α_0 components. If $f_0^{-1}(v)$ is connected (and hence a single point), we already have the space K and map f which we want. Otherwise, we use the following lemma to construct the sequence of spaces already described and hence deduce the result.

Lemma 2.3. *Let K be a based CW-complex and $f : K \to X$ a map representing $\phi : F \to G$ such that $f^{-1}(v)$ is a forest with α components. If $\alpha \geq 2$, then there is a based CW-complex K' and a map $f' : K' \to X$*

representing ϕ *such that* $f'^{-1}(v)$ *is a forest with* $\alpha - 1$ *components.*

Proof. Let l be a path in K with endpoints in $f^{-1}(v)$. (By a path, we mean simply that l is a map $I \to K$.) Then $f \circ l$ is a loop in X based at v. Pick a path l joining distinct components of $f^{-1}(v)$. We construct a space K' from K by attaching a 1-cell e to ∂l and then attaching a 2-cell B to $e \cup l$. We would like to extend $f : K \to X$ to a map $f' : K' \to X$ such that $f'(e) = v$ and $f'^{-1}(v)$ does not meet the interior of the 2-cell B. If this can be done, then $f'^{-1}(v) = f^{-1}(v) \cup e$ which is a forest with $\alpha - 1$ components. Also f' represents ϕ, as f' extends f and K' deformation retracts to K. Hence f' has all the required properties. We will be able to construct such an extension f' if the loop $f \circ l$ has image in X_1 (or X_2) and is contractible in X. For then $f \circ l$ is null homotopic in X_1, and f' restricted to B is essentially this null homotopy. Our aim is to show that such a l exists.

Choose two distinct components A and B of $f^{-1}(v)$ and let L be a path in K from A to B. As $f_* : \pi_1(K) \to \pi_1(X)$ is onto, there is a loop γ in K based at $L(0)$ such that $f \circ \gamma$ is homotopic to the loop $f \circ L$. Let l be the path $\gamma^{-1}L$ in K. This is a path in K joining A to B such that $f \circ l$ is a contractible loop in X.

We can suppose that l is a cellular map $I \to K$ by subdividing I and by choice of l. Thus we can express l as a union of subpaths $l_1, \ldots, l_n$ such that the ends of l_i lie in $f^{-1}(v)$ and $f \circ l_i$ is a loop in X_1 or X_2. Further we can suppose that the maps $f \circ l_i$ alternate between X_1 and X_2. (Note that l_i may meet components of $f^{-1}(v)$ in its interior.) We say that l has length n.

Let g_i denote the homotopy class of $f \circ l_i$ in $\pi_1(X, v)$. Suppose that some l_i has the two properties that g_i is trivial and that the endpoints of l_i lie in one component of $f^{-1}(v)$. Then we can alter l to l' by removing l_i and replacing it with a path l_i' in $f^{-1}(v)$ which joins the endpoints of l_i. Clearly l' has length less than n. By repeating this process, we can arrange that l has no subarcs l_i with these two properties.

Now the equation $l = l_1 \ldots l_r$ gives rise to the equation $1 = g_1 g_2 \ldots g_r$ in $\pi_1(X)$. As $\pi_1(X) = G_1 * G_2$ and the g_i's lie alternately in G_1 and G_2, we deduce that some g_i is trivial. The corresponding l_i joins distinct components of $f^{-1}(v)$ and has $f \circ l_i$ contractible. We can now construct the required space K' and map $f' : K' \to X$ as previously described.

3. SUBGROUPS AND COVERING SPACES

We will apply the theory of covering spaces to the problem of describing subgroups of amalgamated free products. First, we consider free products.

Suppose given a group $G = G_1 * G_2$. As before we construct a space X with fundamental group G by taking CW-complexes X_1, X_2 with $\pi_1(X_i) = G_i$ and joining the basepoints of X_1 and X_2 with an interval E. We take the midpoint v of E as the basepoint of X. Now suppose that H is a subgroup of G. Then H is the fundamental group of some connected covering space $\tilde{X}$ of X, with projection map $p : \tilde{X} \to X$. Inside $\tilde{X}$, we have $p^{-1}(X_1)$ which is a covering space of X_1 and so consists of various connected covering spaces of X_1. Also $p^{-1}(X_2)$ is a union of connected covering spaces of X_2. Finally, as E is simply connected, $p^{-1}(E)$ is a union of copies of E. Thus $\tilde{X}$ looks like (and agrees up to homotopy with) a graph Γ with a covering space of X_1 or X_2 at each vertex. If Γ were a tree, then H would be the free product of the fundamental groups H_λ of all the spaces at the vertices of Γ. In general, Γ consists of a tree T with extra edges attached to T, where T is a maximal tree in Γ. Thus H will be the free product of all the groups H_λ and of a free group whose generators correspond to the edges of $\Gamma - T$.

Let $\tilde{v}$ denote the basepoint of $\tilde{X}$ and recall that $H = p_*(\pi_1(\tilde{X}, \tilde{v}))$. Let C be a component of $p^{-1}(X_1)$ and join it to $\tilde{v}$ by a path in $\tilde{X}$. We see that $p_*(\pi_1(C, \tilde{v}))$ is a conjugate of some subgroup of G_1. Thus the above description of a typical covering space of X leads at once to the following result.

Theorem 3.1 (*Kuroš' subgroup theorem*). *If* H *is a subgroup of* $G = G_1 * G_2$, *then* H *is the free product of a free group with subgroups of conjugates of* G_1 *or* G_2.

Corollary 3.2. *If H is a subgroup of a free group, then H is free.*

Corollary 3.3. *If H is indecomposable and not infinite cyclic, and if $H \subset G_1 * G_2$, then H lies in a conjugate of G_1 or G_2.*

Examples of an application of Corollary 3.3 would be when H is finite or abelian.

Exercise. Prove that a non-trivial direct product cannot be a non-trivial free product.

Lemma 3.4. *If $G = G_1 * G_2$ and if $w^{-1}G_1w \cap G_i$ is non-trivial, then $i = 1$, $w \in G_1$ and so $w^{-1}G_1w \cap G_i = G_1$.*

Proof. Clearly G_1 must be non-trivial, and we may as well suppose that G_2 is non-trivial. Now let g be a non-trivial element of G_1 such that $w^{-1}gw \in G_i$. We can write $w = \alpha w_1$, where $\alpha \in G_1$ and w_1 is a reduced word in G beginning in G_2. Thus $w^{-1}gw = w_1^{-1}(\alpha^{-1}g\alpha)w_1 = w_1^{-1}g'w_1$ where g' is a non-trivial element of G_1. Thus $w_1^{-1}g'w_1$ is a reduced word. But this is an element of G_i and so has length 1. Hence w_1 is trivial and so w lies in G_1. Hence $w^{-1}G_1w = G_1$ and we have $G_1 \cap G_i$ non-trivial. This can only happen when $i = 1$, which completes the proof of the lemma.

Theorem 3.5. *If G is a finitely generated group, then $G = G_1 * \ldots * G_n$, where each G_i is indecomposable. If also $G = G_1 * \ldots * G_n = H_1 * \ldots * H_m$ where each G_i and H_j is non-trivial and indecomposable, then $m = n$ and, by re-ordering, we have $G_i \cong H_i$ for each i. Further, for each i with G_i not infinite cyclic we have G_i conjugate to H_i.*

Proof. The first sentence is just Corollary 2.2 stated again.

Now suppose that $G = G_1 * \ldots * G_n = H_1 * \ldots * H_m$, where each G_i, H_j is indecomposable. If each G_i is infinite cyclic, then G is free and Corollary 3.2 tells us that each H_j is free. As H_j is indecomposable, it must be infinite cyclic and it now follows easily by abelianising and

using the basis theorem for f.g. abelian groups that $m = n$. Otherwise we can re-order the G_i's so that $G_1, \ldots, G_r$ are not infinite cyclic and $G_{r+1}, \ldots, G_n$ are infinite cyclic.

Corollary 3.3 applied to $G_1 \subset H_1 * \ldots * H_m$ shows that, by re-ordering the H's, we have $u^{-1}G_1u \subset H_1$ for some u in G. Hence H_1 is not infinite cyclic and Corollary 3.3 applied to $H_1 \subset G_1 * \ldots * G_n$ shows that $v^{-1}H_1v \subset G_i$ for some i. Hence $w^{-1}G_1w \subset G_i$, where $w = uv$. Now Lemma 3.4 shows that $w \in G_1$ and $i = 1$. Thus we have

$$G_1 = w^{-1}G_1w \subset v^{-1}H_1v \subset G_1 .$$

It follows that H_1 is conjugate to G_1 in G and hence also that H_1 is isomorphic to G_1.

Repeat this process for $G_2, \ldots, G_r$ to show that G_i is conjugate to H_i for $i = 1, 2, \ldots, r$. (Note that we cannot find two different G_i's conjugate to the same H_j as different G_i's cannot be conjugate to each other, by Lemma 3.4.)

Consider $G/\langle G_1 * \ldots * G_r \rangle$, where $\langle X \rangle$ denotes the normal closure of a subset X of G. As G_i is conjugate to H_i for $i = 1, 2, \ldots, r$ we have

$$G_{r+1} * \ldots * G_n \cong G/\langle G_1 * \ldots * G_r \rangle \cong G/\langle H_1 * \ldots * H_r \rangle \cong H_{r+1} * \ldots * H_m .$$

The left hand group is free, and so each H_i, $i = r+1, \ldots, m$ must be infinite cyclic. It now follows that $m = n$ and we have completed the proof of Theorem 3.5.

One might ask whether an analogue of Theorem 3.5 holds for non-f.g. groups. At the end of this chapter, we give an example which shows that the first part of the theorem fails for non-f.g. groups. However, the uniqueness result which is the second part of Theorem 3.5 clearly applies to all groups which can be expressed as a finite free product of indecomposables.

The next step is to consider the structure of subgroups of amalgamated free products. Let us consider a group $G = A *_C B$. Let X_0 be a CW-complex with fundamental group A, let X_1 be a CW-complex with fundamental group B and let X_2 be a 2-dimensional CW-complex with fundamental group C. Lemma 1.5 tells us that there are maps

$f_0 : X_2 \to X_0$ and $f_1 : X_2 \to X_1$ such that the induced maps of fundamental groups are the inclusions of C in A and B respectively. (By subdividing, we can suppose that f_0 and f_1 are cellular.) We construct a space X with fundamental group G by taking X_0, X_1 and $X_2 \times I$ and gluing $X_2 \times \{i\}$ to X_i using f_i, for $i = 0, 1$. We identify X_2 with the subspace $X_2 \times \{\frac{1}{2}\}$ of X.

Let H be a subgroup of G and let $\tilde{X}$ be the corresponding covering space of X with projection map $p : \tilde{X} \to X$. As before $p^{-1}(X_i)$ consists of a collection of connected covering spaces of X_i, for $i = 0, 1$ or 2. Thus $\tilde{X}$ is constructed from a collection of connected covering spaces of X_0 and X_1 and a collection of connected spaces of the form $Y \times I$, where Y is a covering space of X_2, by gluing $Y \times \{i\}$ to a covering space of X_i, for $i = 0, 1$. $\tilde{X}$ looks like a graph Γ with a space at each vertex and a (space $\times$ I) along each edge. If Γ were a tree, then H would be a multiple amalgamated free product where each amalgamation is of the type $A *_C B$ and not $A *_C$. In general, Γ is a tree T, with extra edges attached, and then H is a multiple amalgamated free product together with HNN extensions. Note that the form of H one obtains depends on the choice of a maximal tree T in Γ.

If $G = A *_C$ one can obtain a similar description of subgroups of H. We take a CW-complex X_0 with fundamental group A and a 2-dimensional CW-complex X_2 with fundamental group C and construct X from X_0 and $X_2 \times I$ by gluing $X_2 \times \partial I$ to X_0 appropriately.

We now introduce the terminology, due to Serre, of a graph of groups to describe the above sort of structure in a group.

Note that the word graph means a 1-dimensional CW-complex, so that a graph Γ may contain a loop i.e. an edge with its two endpoints identified. This gives rise to difficulties with orientations of such an edge. In order to avoid these difficulties we first introduce the idea of an abstract graph. Essentially this has twice as many edges as Γ, one for each orientation of an edge of Γ.

Definition. An abstract graph Γ consists of two sets $E(\Gamma)$ and $V(\Gamma)$, called the edges and vertices of Γ, an involution on $E(\Gamma)$ which sends e to $\bar{e}$, where $\bar{e} \neq e$, and a map $\partial_0 : E(\Gamma) \to V(\Gamma)$.

We define $\partial_1 e = \partial_0 \bar{e}$ and say that e joins $\partial_0 e$ to $\partial_1 e$.

An abstract graph Γ has an obvious geometric realisation $|\Gamma|$ with vertices $V(\Gamma)$ and edges corresponding to pairs $(e, \bar{e})$. When we say that Γ is connected or has some other topological property, we shall mean that the realisation of Γ has the appropriate property. An orientation of an abstract graph is a choice of one edge out of each pair $(e, \bar{e})$.

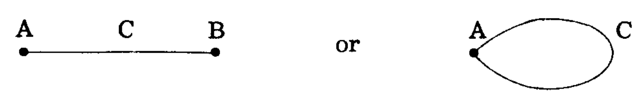
A graph of groups consists of an abstract graph Γ (which will always be tacitly assumed to be connected) together with a function $\mathcal{G}$ assigning to each vertex v of Γ a group G_v and to each edge e a group G_e, with $G_{\bar{e}} = G_e$, and an injective homomorphism $f_e : G_e \to G_{\partial_0 e}$. One may think of Γ as a partial category and $\mathcal{G}$ as a sort of functor. Similarly we may define a graph $\mathcal{X}$ of topological spaces, or of spaces with preferred basepoint: here it is not necessary for the map $X_e \to X_{\partial_0 e}$ to be injective, as we can use the mapping cylinder construction to replace the maps by inclusions, and this does not alter the total space defined below. But we will suppose for convenience that the spaces are CW complexes and maps cellular.

Given a graph $\mathcal{X}$ of spaces, we can define a total space X_Γ as the quotient of $\cup\{X_v : v \in V(\Gamma)\} \cup \cup\{X_e \times I : e \in E(\Gamma)\}$ by the identifications

$$X_e \times I \to X_{\bar{e}} \times I \quad \text{by} \quad (x, t) \to (x, 1 - t)$$

$$X_e \times 0 \to X_{\partial_0 e} \quad \text{by} \quad (x, 0) \to f_e(x).$$

If $\mathcal{X}$ is a graph of (connected) based spaces, then by taking fundamental groups we obtain a graph $\mathcal{G}$ of groups (with the same underlying abstract graph Γ). The fundamental group G_Γ of the graph $\mathcal{G}$ of groups is defined to be the fundamental group of the total space X_Γ. Observe that in the cases when Γ has just one pair $(e, \bar{e})$ of edges



we obtain the products $A *_C B$, $A *_C$ already discussed, as follows by van Kampen's theorem (1.1 and 1.2). The general case may be considered as derived by an iterated application of these constructions; however if it is treated in this way, the underlying geometry is liable to be obscured by computational complexities.

We now show that G_Γ does not depend on the choice of $\mathcal{X}$. First, for any $\mathcal{G}$ we can choose (using presentations) connected 2-dimensional CW complexes with $\pi_1(X_v, *) \cong G_v$ and $\pi_1(X_e, *) \cong G_e$. By Lemma 1.5, the homomorphisms f_e are induced by continuous maps $(X_e, *) \to (X_{\partial_0 e}, *)$. This defines a graph $\mathcal{X}$ of connected based spaces giving rise to $\mathcal{G}$. Next for any $\mathcal{X}$ we can attach cells of dimension ≥ 3 to each X_v, X_e to obtain aspherical spaces K_v, K_e still with the same fundamental group. Now the map $f_e : X_e \to X_{\partial_0 e}$ extends to a map $k_e : K_e \to K_{\partial_0 e}$ (there is no obstruction) so we have a new graph $\mathcal{K}$ of spaces, still inducing $\mathcal{G}$; its total space K_Γ is obtained from X_Γ by adding cells of dimension ≥ 3, so has the same fundamental group. But K_v is a space of type $(G_v, 1)$: its homotopy type is entirely determined by G_v (similarly K_e, G_e). Also the map k_e is determined up to homotopy by $f_e : G_e \to G_{\partial_0 e}$. Thus K_Γ is determined up to homotopy, and its fundamental group is unique up to isomorphism.

In order to relate the topology more closely to the group theory, we have insisted above on preservation of base points. However if the attaching maps $X_e \to X_{\partial_0 e}$ are altered by any homotopy (not necessarily base point preserving), the homotopy type and hence the fundamental group of X_Γ are unchanged. If the base point is pulled round a loop defining $g \in \pi_1(X_{\partial_0 e}, *)$, the homomorphism $G_e \to G_{\partial_0 e}$ is changed by conjugation by g. Thus even such a change will not alter G_Γ.

Corresponding to and generalising (1.6) and (1.7) we now have the

Proposition 3.6. (i) If $\mathcal{G}$ is a graph of groups as above, each map $G_v \to G_\Gamma$ is injective.

(ii) If $\mathcal{K}$ is a graph of aspherical spaces as above, the total space K_Γ is aspherical.

Proof. We start from the graph $\mathcal{K}$ of spaces. Observe that for each vertex v of Γ, the space

$$L_v = K_v \cup \bigcup_{\partial_0 e = v} (K_e \times I)$$

admits K_v as deformation retract, so its universal cover $\tilde{L}_v$ is contractible. Moreover, as each map $G_e \to G_v$ is injective, $\tilde{L}_v$ is obtained from $\tilde{K}_v$ by attaching copies of $\tilde{K}_e \times I$ with $\tilde{K}_e$ the universal cover of K_e, hence also contractible.

Now construct a space $Y = \cup Y_n$ by induction. Choose any vertex v_0 of Γ and set $Y_0 = \tilde{L}_{v_0}$. Now for any $n \geq 1$, in forming Y_{n-1} a number of copies of $\tilde{K}_e \times I$ will have been attached (each along $\tilde{K}_e \times 0$), for various edges e. We define Y_n to be the union of Y_{n-1} with a copy of $\tilde{L}_{\partial_1 e}$ for each such copy of $\tilde{K}_e \times I$, identified along $\tilde{K}_e \times I$. Since we are attaching contractible sets along contractible subsets, each Y_n is contractible.

Set $Y = \cup Y_n$ with the weak topology. Then Y also is contractible. There is an evident projection $Y \to K_\Gamma$; by construction K_Γ is evenly covered by Y. This proves (ii), and (i) follows since for each $K_v \subset K_\Gamma$, the induced covering of K_v contains the universal covering.

Remark. Assertion (ii) is equivalent to the exact sequences of Chiswell [30]. A normal form, in the style of Theorems 1.6 and 1.7, is given by Higgins [32].

We can now state our first result about subgroups of amalgamated free products.

Theorem 3.7. If $G = A *_C B$ or $A *_C$ and if $H \subset G$, then H is the fundamental group of a graph of groups, where the vertex groups are subgroups of conjugates of A or B and the edge groups are subgroups of conjugates of C.

Remarks. This result has an obvious generalization to the case where G is the fundamental group of a graph Γ of groups. The theorem covers the special case when the geometric realisation of Γ has a single edge.

There is a corollary of this result which is analogous to Corollary 3.3 in the case of free products. We say that a group G splits over a subgroup C if $G = A *_C$ or $G = A *_C B$ with $A \neq C \neq B$. If G splits over some subgroup, we say that G is splittable. Note that $\mathbb{Z}$ is splittable as $\mathbb{Z} = \{1\} *_{\{1\}}$.

Corollary 3.8. If $G = A *_C B$ or $A *_C$ and if H is a finitely generated non-splittable subgroup of G, then H lies in a conjugate of A or B.

Proof. We know that H is the fundamental group of a graph Γ of groups. As H is finitely generated, there is a finite subgraph Γ' whose fundamental group equals H. Now the fact that H is not splittable implies that one of the vertex groups of Γ' is equal to H. The result follows.

Remark. The finite generation of H allows one to deduce that some vertex group of Γ equals H. If we consider non-finitely generated subgroups H, we see that H need not lie in a conjugate of A or B. For example, consider $G = \mathbb{Z} *_{\mathbb{Z}}$ where the two inclusion maps are the identity and multiplication by 2. Thus G has presentation $\{a, t : t^{-1}at = a^2\}$. The subgroup H of G generated by $t^n a t^{-n}$ for all integers n is isomorphic to the dyadic rationals and is therefore non-splittable, but of course H cannot be contained in an infinite cyclic group.

We must now consider covering spaces more closely. Recall paragraph 2 on page 138 of these notes. Let (X, x_0) be a based connected space with fundamental group G. Let H be a subgroup of G and let $(\tilde{X}, \tilde{x}_0)$ be the corresponding connected covering space, with projection map $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$. Thus $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = H$.

Lemma 3.9. There is a natural bijection $\bar{\phi} : H\backslash G \to p^{-1}(x_0)$, where $H\backslash G$ denotes the quotient of G under the action of H by left multiplication.

Remark. The path lifting property of covering maps can be used to define bijections $p^{-1}(x_0) \to p^{-1}(x)$ for each $x \in X$.

Proof. First we define a map $\phi : G \to p^{-1}(x_0)$. Given $g \in G$, choose a map $(I, \partial I) \to (X, x_0)$ representing g. We will call such a map a loop. Let l be the lift of this map starting at $\tilde{x}_0$ and define $\phi(g) = l(1) \in p^{-1}(x_0)$. This definition is independent of the choice of the loop chosen to represent g.

The map ϕ is a surjection. For given $y \in p^{-1}(x_0)$, choose a path l in $\tilde{X}$ from x_0 to y. Then $p \circ l$ is a loop in X representing some element $g \in \pi_1(X, x_0)$ and $\phi(g) = y$.

If $\phi(g_1) = \phi(g_2)$, then $l_1 l_2^{-1}$ is a loop in $\tilde{X}$ based at $\tilde{x}_0$, where l_i is a lift of a loop in X representing g_i. Thus $p \circ (l_1 l_2^{-1})$ represents an element h of H and we have the equation $g_1 g_2^{-1} = h$. Conversely if $g_1 g_2^{-1} = h \in H$, then $g_1 = hg_2$ and it is clear that $\phi(g_1) = \phi(g_2)$. Thus $\phi(g_1) = \phi(g_2)$ if and only if $g_1 g_2^{-1} \in H$ and so ϕ induces a bijection $\bar{\phi} : H\backslash G \to p^{-1}(x_0)$.

Lemma 3.10. If H is a normal subgroup of G, then G/H acts on $\tilde{X}$ by covering homeomorphisms with quotient X.

Proof. Let $g \in G$ and let $y \in p^{-1}(x_0)$ be the point determined by g. Then $p_*(\pi_1(\tilde{X}, y)) = g^{-1}Hg$ which equals H as H is normal in G. The uniqueness of covering spaces corresponding to H shows that there is a unique covering homeomorphism $\psi_g : (\tilde{X}, \tilde{x}_0) \to (\tilde{X}, y)$. I claim that this process defines a homomorphism of G to the group of covering homeomorphisms of $\tilde{X}$. One need only show that $\psi_{g_1 g_2}(\tilde{x}_0) = \psi_{g_1} \circ \psi_{g_2}(\tilde{x}_0)$, as two covering homeomorphisms which agree on $\tilde{x}_0$ must be equal by the uniqueness result again. Let l_1 and l_2 be paths in $\tilde{X}$ starting at $\tilde{x}_0$ which are lifts of loops in X representing g_1 and g_2 respectively. Then $\psi_{g_1} \circ l_2$ is a path in $\tilde{X}$ starting at $\psi_{g_1}(\tilde{x}_0)$ and still lifting a loop in X representing g_2. Thus $\psi_{g_1} \circ l_2$ begins where l_1 ends and we deduce that $\psi_{g_1 g_2}(\tilde{x}_0) = \psi_{g_1} \circ l_2(1) = \psi_{g_1} \circ \psi_{g_2}(\tilde{x}_0)$ as required. It is clear that the kernel of this homomorphism is H, so that we do have an action of G/H on $\tilde{X}$.

The quotient of $\tilde{X}$ by the action of G/H has a natural projection π to X and π is a covering map. Also, for each $x \in X$, $\pi^{-1}(x)$ is a single point as $\pi^{-1}(x_0)$ is a single point. Hence π is a homeomorphism and this completes the proof of the lemma.

Before going further, we give an application of this result.

Theorem 3.11. If $G = A * B$ where A, B are non-trivial and H is a finitely generated, normal subgroup of G, then H is trivial or has finite index in G.

Proof. We suppose that H has infinite index in G and will prove that H must be trivial. We know that H is the fundamental group of a graph Γ of groups, where the edge groups are trivial and the vertex groups H_λ are subgroups of conjugates of A or B. If T is a maximal tree in Γ, then $H = F * (* H_\lambda)$ where F is a free group whose generators correspond to the edges of $\Gamma - T$. The fact that H is normal in G tells us that G/H acts on Γ with quotient an interval.

As H has infinite index in G, we deduce that Γ has infinitely many edges. As H is finitely generated, we deduce that $\Gamma - T$ is finite and only finitely many of the groups H_λ are non-trivial. Thus H is the fundamental group of some connected finite subgraph Γ' of Γ. Let E be an edge of $\Gamma - \Gamma'$. Then removing E from Γ gives two subgraphs Γ_1 and Γ_2 one of which has trivial fundamental group. As G/H acts transitively on the edges of Γ, we deduce that every edge of Γ has these properties. Thus Γ must be a tree and at most one vertex group can be non-trivial. Thus H is contained in a conjugate of A or B. As H is normal in G, it must lie in the intersection of all conjugates of A (or of B). But $A \cap b^{-1}Ab$ is trivial for any non-trivial element $b \in B$. Hence H is trivial.

Q. E. D.

Exercise. Is there an analogous result when $G = A *_C B$ or $A *_C$?

We now return to covering spaces. The aim of our next result is to give a more precise structure theorem for subgroups of amalgamated free products.

Let H be a subgroup of $G = \pi_1(X, x_0)$, and let $\tilde{X}$, $\tilde{x}_0$ be as before. Let Y be a subspace of X which contains x_0, such that inclusion of Y in X induces an injective map $\pi_1(Y, x_0) \to \pi_1(X, x_0)$. We denote the image group by A and identify $\pi_1(Y)$ with this subgroup of G.

Lemma 3.12. There is a natural correspondence $\bar{\theta}$ between the double cosets HgA and the components of $p^{-1}(Y)$ in $\tilde{X}$.

Proof. Let i denote the inclusion of $p^{-1}(x_0)$ in $p^{-1}(Y)$ and recall the map $\phi : G \to p^{-1}(x_0)$. We define $\theta : G \to p^{-1}(Y)$ by $\theta = i \circ \phi$.

Suppose that $\theta(g_1)$ and $\theta(g_2)$ lie in the same component of $p^{-1}(Y)$. Then they can be joined by a path l in $p^{-1}(Y)$. Let $\alpha \in A$ be the element of $\pi_1(X, x_0)$ represented by the loop $p \circ l$. Then, by projecting into X, we see that $g_1 \alpha g_2^{-1} \in H$, so that $g_2 = h g_1 \alpha$ for some $h \in H$. Conversely, if $g_2 = h g_1 \alpha$ for some elements $\alpha \in A$ and $h \in H$, then lifting to $\tilde{X}$ tells us that $\theta(g_1)$ and $\theta(g_2)$ can be joined by a path l in $p^{-1}(Y)$, where l is a lift of α.

Hence θ induces the required bijection $\bar{\theta}$.

Lemma 3.13. Let $g \in G$, $y = \phi(g) \in p^{-1}(x_0)$ and let C be the component of $p^{-1}(Y)$ which contains y. Let λ be a loop in X representing g and let l be the lift of λ which goes from $\tilde{x}_0$ to y. Then $p_*(\pi_1(C, \tilde{x}_0)) = H \cap gAg^{-1}$, where we define $\pi_1(C, \tilde{x}_0)$ by using the path l.

Proof. We know that $p_*(\pi_1(C, y)) \subset A$, and so $p_*(\pi_1(C, \tilde{x}_0)) \subset gAg^{-1}$. As $p_*(\pi_1(C, \tilde{x}_0)) \subset H$, we have $p_*(\pi_1(C, \tilde{x}_0)) \subset H \cap gAg^{-1}$.

Consider an element $\beta = g\alpha g^{-1}$ of $H \cap gAg^{-1}$, where $\alpha \in A$. Let μ be a loop in Y representing α. Then $\lambda\mu\lambda^{-1}$ is a loop in X representing β. We know that $\lambda\mu\lambda^{-1}$ lifts to a loop in $\tilde{X}$, because $\beta \in H$. Thus λ lifts to l and λ^{-1} lifts to l^{-1} and so $\lambda\mu\lambda^{-1}$ lifts to $l m l^{-1}$ where m is some loop in $p^{-1}(Y)$ based at y. Therefore β lies in $p_*(\pi_1(C, \tilde{x}_0))$ and we have shown that $p_*(\pi_1(C, \tilde{x}_0)) = H \cap gAg^{-1}$ as required.

We can now state a more precise version of the subgroup theorem. Similar statements can be found in [4], [15], [22], [28]. For convenience in the case $G = A *_C$, where we have two injections i_1 and i_2 of C into A, we identify C with $i_1(C)$. Then $i_2(C) = t^{-1}Ct$, and the subgroup $C = i_1(C)$ of A is also a subgroup of tAt^{-1}.

Subgroup Theorem 3.14. If H is a subgroup of $G = A *_C$ (or $A *_C B$), then H is the fundamental group of a graph Γ of groups. The vertices of Γ correspond to the double cosets HgA (and HgB), and the corresponding groups are $H \cap gAg^{-1}$ (and $H \cap gBg^{-1}$). The edges of Γ correspond to the double cosets HgC and the corresponding groups are $H \cap gCg^{-1}$.

If $G = A *_C B$, the two ends of the edge HgC are the vertices HgA and HgB and the injections of the associated groups are simply the inclusion mappings.

If $G = A *_C$, the two ends of the edge HgC are the vertices HgA and $HgtA$. The injections of the associated groups are simply the inclusion mappings.

A corresponding theorem for subgroups H of G, where G is the fundamental group of any graph of groups, follows from the same lemmas. We leave the precise formulation to the reader.

As applications of the subgroup theorem, we prove the following results.

Lemma 3.15. If $G = G_1 *_C G_2$, then either $gG_1g^{-1} \cap G_i$ is a subgroup of a conjugate of C, or $i = 1$ and $g \in G_1$, so that $gG_1g^{-1} \cap G_i = G_1$.

Proof. Let $H = gG_1g^{-1} \cap G_i$. Then H is the fundamental group of a graph Γ of groups. The vertices v_1, v_2 of Γ corresponding to the double cosets HgG_1 and HG_i have associated groups $H \cap gG_1g^{-1} = H$ and $H \cap G_i = H$. If v_1 and v_2 are distinct vertices of Γ, choose a path in Γ joining them. As the inclusion of each of the groups associated to v_1, v_2 is an isomorphism with H, we see that each vertex and edge of this path has associated group H. Thus H is the group associated to some edge and so lies in some conjugate of C. The only other possibility is that $v_1 = v_2$. This implies that $i = 1$ and $HgG_1 = HG_1$. Thus $g \in G_1$, and the result is proved.

Next we give the promised example of a non-f.g. group G which fails to satisfy the conclusion of Theorem 3.5. This example is due to Kuroš [17].

Example. The group $G = \{a_0, a_1, a_2, \ldots, b_1, b_2, \ldots \mid a_{n-1} = [a_n, b_n], \forall n \geq 1\}$ cannot be expressed as a free product of indecomposable subgroups.

First observe that $G = \{a_1, b_1\} *_{C_1} \{a_2, b_2\} *_{C_2} \ldots$, where each factor group is free of rank 2, each C_i is infinite cyclic, the inclusion map $C_i \to \{a_i, b_i\}$ sends a generator to a_i and the inclusion map $C_i \to \{a_{i+1}, b_{i+1}\}$ sends the same generator to $[a_{i+1}, b_{i+1}]$.

Next observe that

$$G = \{b_1\} * \{a_1, a_2, \ldots, b_2, b_3, \ldots \mid a_{n-1} = [a_n, b_n], \forall n \geq 2\} \cong \mathbb{Z} * G.$$

Hence G can be expressed as a free product involving any given (finite) number of factors. Hence G cannot be a free product of n indecomposables, for any integer n, as the proof of the uniqueness result of Theorem 3.5 would apply to show that any factorisation of G has no more than n factors - a contradiction. Hence if G can be expressed as a free product of indecomposables, then G must have an infinite number of factors. The last step is to show that this also is impossible.

Suppose that $G = G_1 * G_2 * \ldots$, where each G_i is indecomposable and non-trivial. Consider the element a_0 of G. For some n, a_0 must lie in $G_1 * \ldots * G_n$, which we denote by A. Thus $G = A * B$, with A and B non-trivial and $a_0 \in A$. Our decomposition of G as an infinite amalgamated free product shows that each a_i is a non-trivial element of G. The fact that a_0 lies in A shows that a_1, b_1 lie in A, by Lemma 3.16 below. As $a_1 \in A$, we see a_2, $b_2 \in A$ by the same lemma. By repeating this argument, we see that $G \subset A$, contradicting the hypothesis that B is non-trivial. This contradiction proves the required result.

Lemma 3.16. If a group $G = A * B$, $A \neq \{1\} \neq B$, and if $g = [g_1, g_2]$ is a non-trivial element of A, then g_1 and g_2 lie in A.

Proof. Let H be the subgroup of G generated by g_1, g_2. By the Subgroup Theorem, $H = (H \cap A) * C$, for some subgroup C of H. We must prove that $H \subset A$. If this is not the case, then each of $H \cap A$ and

C is non-trivial and so Corollary 2.1 of Grusko's Theorem shows that each group is cyclic. Hence the abelianisation homomorphism $H \to H/H'$ injects $H \cap A$ and C. This contradicts the fact that g is a non-trivial commutator in $H \cap A$. Hence H must be contained in A.

We finish this section by proving the famous embedding theorem of Higman and Neumann [13] which states that any countable group can be embedded in a 2-generator group. A nice example to consider is the subgroup K of F_2, the free group on a and b, which is the kernel of the homomorphism $F_2 \to \mathbb{Z}$, which sends b to a generator and a to the identity. Let X denote the wedge of 2 circles, so that $\pi_1(X) \cong F_2$. The covering space $\tilde{X}$ of X corresponding to K consists of a copy of $\mathbb{R}$ together with a circle attached at each integer point. Thus K has basis $\{b^{-n}ab^n : n \in \mathbb{Z}\}$. If one started with a countably generated free group $K = \{x_n : n \in \mathbb{Z}\}$, one would embed it in a finitely generated group by adding an element b to K which makes all the x_i's conjugate. More precisely, one has the shift automorphism of K sending x_n to x_{n+1}, for each n, and one takes the extension of K by $\mathbb{Z}$ determined by this automorphism. The new group generated by x_0 and b is, of course, isomorphic to F_2. The idea of the proof of the embedding result is to do the same sort of thing in general, i.e. make lots of generators conjugate.

Theorem 3.17. If G is a countable group, then G can be embedded in a 2-generator group.

Proof. Let $x_1, x_2, \ldots$ be a generating set for G. We embed G in $G_1 = G * \mathbb{Z}$. Let t be a generator of $\mathbb{Z}$, and write $y_i = x_i t$, $y_0 = t$. Then G_1 is generated by $y_0, y_1, \ldots$ and each y_i has infinite order. Now let $G_2 = \{G_1, t_0, t_1, \ldots : t_i^{-1} y_i t = y_{i+1}\}$. Then G_2 is obtained from G_1 by an infinite sequence of HNN extensions and so $G_1 \subset G_2$. A set of generators for G_2 is $y_0, t_0, t_1, \ldots$. The subgroup K of G_2 generated by the t_i's is free and has the t_i's as a basis. To see this observe that by killing G_1 one obtains a homomorphism of G_2 to a free group F which maps the t_i's to a basis for F. Thus we can embed K in F_2 as in the discussion preceding the theorem and we let

$G_3 = G_2 *_K F_2$. Again $G_2 \subset G_3$ and G_3 is generated by y_0, a and b where a denotes one of the t_i's. Finally we observe that the subgroup H of G_3 generated by y_0 and b is free of rank two. This can be seen by applying the Subgroup Theorem, as $H \cap G_2$ and $H \cap F_2$ are infinite cyclic and $H \cap K$ is trivial. Thus we can construct $G_4 = G_3 *_{F_2}$ where our two inclusions of F_2 in G_3 have images H and F_2. We choose $G_4 = \{G_3, s : s^{-1}as = b, s^{-1}bs = y_0\}$. Then G_4 is generated by a and s which completes the proof of the embedding theorem.

4. GROUPS ACTING ON GRAPHS

The subject matter of this chapter is a reworking of the Bass-Serre theory [22]. We consider a (continuous) action of a group G on a (topological) graph Γ: clearly this corresponds also to an action on the corresponding abstract graph. We say that G acts without inversions if whenever an element g of G fixes an edge e of Γ, it fixes each point of e. In the abstract setting, this means that $g.e = \bar{e}$ is forbidden. Given any action, the process of subdividing each edge once by an extra vertex in the middle gives us an action without inversions.

The following simple example will be of use in Section 6. Let G be a group, $S \subset G$ a subset (not subgroup), $\Gamma = \Gamma(S, G)$ the (geometric) graph with vertex set G and, for each $(g, s) \in G \times S$ a single edge $e(g, s)$ joining g to gs. There is an obvious action of G on Γ, where $h \in G$ takes the vertex g to the vertex hg; the edge $e(g, s)$ to the edge $e(hg, hgs)$. Only the identity element of G can leave a vertex or edge fixed, so G acts freely (without inversions). Note that even if $s^2 = 1$ we do not identify the edges $e(g, s)$ and $e(gs, s)$ with the same endpoints.

Proposition 4.1. (i) Γ contains no loop $\iff 1 \notin S$.

(ii) Γ is a simplicial complex $\iff S \cap S^{-1} = \emptyset$.

(iii) Γ is connected $\iff$ S generates G.

(iv) Γ is a tree $\iff$ S freely generates G.

Proof. (i) and (ii) are trivial.

If H denotes the subgroup generated by S, and Γ' the full subgraph on H, then Γ' is open (there are no edges with just one end in Γ') and connected (any word - e.g. $s_1 s_2 s_3^{-1} s_4$ - in $S \cup S^{-1}$ defines, in an obvious way, a path in Γ' joining 1 to the element in H it represents). Now (iii) follows.

Finally if Γ is a tree, it is contractible; as G acts freely on Γ, we have an isomorphism of G on $\pi_1(G\backslash\Gamma, *)$. But $G\backslash\Gamma$ is the graph with one vertex and edges labelled by S, so its fundamental group is the free group $F(S)$. Moreover the composed isomorphism $F(S) \to G$ takes $s \in F(S)$ to the class of the loop 's'; lifting this, we get the edge of Γ from 1 to s, which corresponds to $s \in G$. This argument is reversible: if S freely generates G, then $G \cong F(S) \cong \pi_1(G\backslash\Gamma, *)$, so Γ is the universal cover of $G\backslash\Gamma$, hence contractible - i.e. a tree.

Now suppose given a graph $\mathcal{G}$ of groups, with vertices v and edges e corresponding to groups G_v, G_e with $G_{\bar{e}} = G_e$, and injections $\alpha(e) : G_e \to G_{\partial_0 e}$. As before, we choose a corresponding graph of connected spaces X_v, X_e with total space X_Γ, and fundamental group G_Γ. Since by (3.6) the natural maps $G_v \to G_\Gamma$, $G_e \to G_\Gamma$ are injective, the universal cover $\tilde{X}_\Gamma$ is a union of copies of the universal covers $\tilde{X}_v$, $\tilde{X}_e \times I$.

In $\tilde{X}_\Gamma$, identify each copy of $\tilde{X}_v$ to a point, and each copy of $\tilde{X}_e \times I$ to a copy of I, giving a quotient space Z with projection $\pi : \tilde{X}_\Gamma \to Z$. Clearly Z is a graph. We define $j : Z \to \tilde{X}_\Gamma$ by first choosing for each vertex (edge) of Z a point $V(E)$ in the corresponding copy of $\tilde{X}_v(\tilde{X}_e)$. Then divide each edge of Z into three parts: j maps the middle part to $E \times I$, and the end parts to paths in the connected space $\tilde{X}_v$ joining the corresponding points E, V. Clearly $\pi \circ j$ is homotopic to the identity, so Z is connected and simply-connected, and hence is a tree.

The map π is compatible with the natural action of G_Γ on $\tilde{X}_\Gamma$, so we inherit an action of G_Γ on Z. This action has no inversions. The isotropy group of each vertex (obtained from collapsing a copy of $\tilde{X}_v$) is a conjugate of G_v, and of an edge (collapsed from $\tilde{X}_e \times I$) is a conjugate of G_e. As G_Γ acts without inversions, $G_\Gamma\backslash Z$ is also a graph and in fact coincides with the geometric realization $|\Gamma|$ of the original graph Γ: there is an obvious map onto $|\Gamma|$ and each vertex (edge) of $|\Gamma|$ determines $X_v(X_e \times I)$ in X_Γ, hence a collection of copies of $\tilde{X}_v(\tilde{X}_e \times I)$ in $\tilde{X}_\Gamma$, transitively permuted by G_Γ, hence a single vertex (edge) of $G_\Gamma\backslash Z$. Thus we recover the original graph $\mathcal{G}$ of groups from the action of G_Γ on Z. There is a slight problem here: how to choose the subgroups within their conjugacy classes to obtain the desired inclusions. This will be dealt with below.

We now start from an action of a group G on a tree Y, having no inversions, and show how to construct a graph of groups with fundamental group G. Choose a connected CW complex U with fundamental group G: then $\tilde{U}$ is simply connected and G acts freely on it, hence also (diagonally) on $\tilde{U} \times Y$. Consider the quotient X and the projection

$$X = G\backslash(\tilde{U} \times Y) \to G\backslash Y = \Gamma, \text{ say.}$$

Since G acts without inversions, Γ is a graph and each vertex (edge) of Y projects isomorphically onto one of Γ. For a vertex v (edge e) with isotropy group G_v (G_e) in G, we see that $G\backslash(\tilde{U} \times v)$ $(G\backslash(\tilde{U} \times e))$ has fundamental group G_v (G_e). Thus X has the structure of a graph $\mathcal{X}$ of connected spaces realising a graph $\mathcal{G}$ of groups. Since G acts freely on the 1-connected space $(\tilde{U} \times Y)$, we have $G = \pi_1(X)$ the fundamental group of $\mathcal{G}$. Observe that U plays no essential role in the construction of $\mathcal{G}$, which could be expressed purely algebraically except for a certain vagueness about conjugates which will be considered below.

In order to deal with the points at the end of the two preceding paragraphs, and also to obtain a more precise formulation of the result which can be used for explicit calculation with words, we must now consider base points. The usual procedure with a CW-complex K is to choose a maximal tree T in the 1-skeleton $K^{(1)}$ (a graph). This is contractible and contains all the vertices. Hence $K \to K/T$ is a homotopy equivalence, and K/T a complex with only one vertex, which we take as base point. Thus the edges of K not in T give generators of $\pi_1(K)$.

Now let G act without inversions on a (connected) graph Y, so that $X = G\backslash Y$ is also a graph; let T be a tree in X containing the vertex v, and let $\tilde{v} \in Y$ lie over v.

Proposition 4.2. <u>There is a lifting</u> $j : T \to Y$ <u>of the inclusion of</u> T <u>in</u> X; <u>moreover, we can take</u> $j(v) = \tilde{v}$.

Proof. Applying Zorn's Lemma, we see that there is a maximal pair (T', j'): T' a subtree of T (containing v), $j' : T' \to Y$ over the inclusion (with $j'(v) = \tilde{v}$). If $T' \neq T$, let w be a vertex of $T - T'$; since T is connected, we can join w to v by an edge path, and at least one edge in the path, say e, has one vertex v_0 which is in T' and one which is not. Now e is the image of an edge $\tilde{e}$ of Y, one of whose vertices $\tilde{v}_0$ lies over v_0. As $\tilde{v}_0$ and $j'(v_0)$ lie over v_0, they are equivalent under G. If $g.\tilde{v}_0 = j'(v_0)$, we can extend j' over e by setting $j(e) = g.\tilde{e}$. This contradicts the supposed maximality and proves the result.

Returning now to the action of G on the tree Y, we choose a maximal tree T in $\Gamma = G\backslash Y$, a lifting $j : T \to Y$ with $j(T) = \tilde{T}$, and use these as 'extended base points'. Over each vertex v of Γ there is just one vertex $\tilde{v}$ of $\tilde{T}$: we define G_v as the stabiliser of $\tilde{v}$. If e is an edge of T, we have the edge $\tilde{e} = j(e)$, and call its stabiliser G_e. For each other edge e of Γ we choose an edge $\tilde{e}$ of Y over e with $\partial_0\tilde{e} = (\partial_0 e)^\sim$, and an element $g_e \in G$ such that $\partial_1\tilde{e} = g_e.(\partial_1 e)^\sim$, and define G_e to be the stabiliser of $\tilde{e}$. The map $\alpha_0(e)$ is the inclusion map; $\alpha_1(e)$ is induced by conjugation by g_e. Note that we have implicitly chosen e from the pair $(e, \bar{e})$ (one could set $\tilde{\bar{e}} = (g_0^{-1}\tilde{e})^-$, $g_{\bar{e}} = g_e^{-1}$, but then would have $G_{\bar{e}} \neq G_e$).

We have considered two constructions above. Given a graph $\mathcal{G}$ of groups, realised by a graph $\mathcal{X}$ of spaces, we defined a quotient Z of $\tilde{X}_\Gamma$, proved it a tree, and obtained an action of G_Γ on it. Conversely, given an action (all actions supposed without inversions) of a group G on a tree Y we defined (following (4.2)), using certain choices (maximal tree T, liftings $\tilde{T}$, $\tilde{e}$, elements g_e) a graph of groups over $\Gamma = G\backslash Y$. The key result of the theory is

Theorem 4.3. <u>These two constructions are mutually inverse up to isomorphism and (for graphs of groups) replacing the</u> $\alpha_i(e)$ <u>by conjugate homomorphisms.</u>

Proof. Most of the proof was given above. Starting from the action of G on Y, the graph of groups over $\Gamma = G\backslash Y$ is realised by a graph of spaces with total space $G\backslash(\tilde{U} \times Y)$, U a connected CW complex with fundamental group G. The collapsing process on the universal cover $(\tilde{U} \times Y)$ gives back the original tree Y and action of G on it.

Now suppose given a graph $\mathcal{G}$ of groups and construct (as above) an action of the fundamental group G_Γ on a tree Z. We have already observed that the isotropy groups of vertices and edges agree up to conjugacy with the images in G_Γ of the given groups G_v, G_e. It remains to identify the injections $\alpha_i(e)$, where more care is necessary.

Take $\mathcal{X}$ as a graph of based spaces, so we can identify Γ with a subset of X, and choose a maximal tree T in Γ. Since T is contractible, we can lift it to $\tilde{T} \subset \tilde{X}_\Gamma$. For each edge e of $\Gamma - T$ there is a unique lift $\tilde{e} \subset \tilde{X}_\Gamma$ with $\partial_0\tilde{e} \in \tilde{T}$, and a unique $g_e \in G_\Gamma$ with $(g_e^{-1}.\partial_0\tilde{e}) \in \tilde{T}$. Now $\mathcal{G}$ is isomorphic to a graph of groups in which each $\alpha_0(e)$, and those $\alpha_1(e)$ with $e \in T$, are inclusions; so we can identify each G_e, G_v with a subgroup of G_Γ.

Now $\pi : \tilde{X}_\Gamma \to Z$ maps $\tilde{T}$ isomorphically to a tree $\bar{\tilde{T}}$ over T. Using the action of G_Γ on Z to define a graph of groups as above, we obtain the same subgroups G_e and G_v; each $\alpha_0(e)$ and those $\alpha_1(e)$ with $e \in T$ are inclusions. For $e \notin T$, $\alpha_1(e)$ is induced by conjugation by g_e. In the given graph of groups $\alpha_1(e)$ was induced by the map $f_e^1 : (X_e, *) \to (X_{\partial_1 e}, *)$. There is a unique path p in T joining $\partial_1 e$ to $\partial_0 e$; as we have identified G_e (via $X_e \to X_{\partial_0 e}$) with a subgroup of $G_{\partial_0 e}$, the map f_e^1 cannot be regarded as preserving base points, which have to be translated along the path p. Thus $\alpha_1(e)$ is induced by conjugation by the element g' of G_Γ represented by the closed path $p.e$. Now p lifts to the path in $\tilde{T}$ joining $(\partial_1 e)^\sim$ to $(\partial_0 e)^\sim = \partial_0\tilde{e}$, so the lift of $p.e.$ joins it to $\partial_1\tilde{e}$. Thus $g'.(\partial_1 e)^\sim = (\partial_1 e)^\sim$, which identifies g' with g_e and hence concludes the proof.

Remark. The reader may already have observed that our two inverse constructions can be formulated in purely algebraic terms. We feel however that the above proof of the key theorem is more intuitive than any involving cancellation arguments.

We now note some special cases of the theorem.

Corollary 4.4. Let $G\backslash Y$ consist of a single edge $\overset{v}{\bullet}\overset{e}{\rule{3em}{0.4pt}}\overset{v'}{\bullet}$. Then $G \cong G_v *_{H_e} G_{v'}$.

Corollary 4.5. Let $G\backslash Y$ consist of a single loop. Then Y contains the edge $\overset{\tilde{v}}{\bullet}\overset{\tilde{e}}{\rule{3em}{0.4pt}}\overset{g.\tilde{v}}{\bullet}$ and $G \cong G_v *_{H_e}$.

A somewhat different application arises from considering free actions. Of course, if G acts freely on a tree, it is free. The theorem allows us to write down a set of free generators. Suppose in particular that F is the free group on a set S, and G a subgroup of F. Write down the graph Γ_S for F: by (4.1) it is a tree, and the action of F on it induces a free action of G.

Identify the vertices of Γ_S with the corresponding elements of F. Then the path in the tree joining 1 to the element with reduced word $t_1 \dots t_n$ (where each $t_i \in S \cup S^{-1}$) goes through the successive vertices $t_1, t_1t_2, t_1t_2t_3, \dots$. Thus if $\tilde{T} \subset \Gamma_S$ contains 1, it contains the initial segments of the vertices of $\tilde{T}$.

Since the action of G is free, the lift of an edge of $G\backslash\Gamma_S = X$ is determined by its initial vertex. Thus the preferred lifts of the edges of $X - T$ are just those edges of Γ_S whose initial vertex is in T but terminal vertex is not. Applying the theorem we have

Proposition 4.6. The left cosets of G in F are canonically represented by the set R of vertices of $\tilde{T}$: if a reduced word $w = uv$ belongs to R, so does u. If $W = \{(t, s) \in R \times S : ts \notin R\}$ and for each $w = (t, s) \in W$ we write $ts = g_w u_w$ with $g_w \in G$, $u_w \in R$ then $\{g_w : w \in W\}$ is a free basis of G.

We conclude this section with brief mentions of two alternative approaches. The first follows a paper of Serre [23]. We say that G has property (FA) if for any action of G on a tree Y, there is a fixed point of G in Y.

Theorem 4.7. G has (FA) if and only if G is (i) unsplittable, and (ii) not a union of an increasing sequence of subgroups.

Note that for countable G, (ii) is equivalent to being finitely generated.

Proof. (FA) $\Rightarrow$ (i) by Corollaries 4.4 and 4.5: any decomposition induces an action without a fixed point. As to (ii), if $G = \cup G_n$ with $G_n \subset G_{n+1}$ we form a graph with vertices $\cup(G/G_n)$ and for each vertex gG_n an edge joining it to gG_{n+1}. It is immediate that this is a tree, and that the natural action of G on it has no fixed point.

Conversely if (i) and (ii) hold and G acts on Y, G is the universal group of the graph of groups $G\backslash Y$. By (ii), G is also the universal group of a finite subgraph. If this subgraph is not a tree, G is splittable; if the subgraph is a tree, we still have a splitting unless G coincides with one of the vertex groups - i.e. has a fixed point in Y.

Some interesting examples of the above are given in Serre's paper [23] and several more in his monograph [22, Chapter 6].

We conclude this paragraph by mentioning length functions. These were introduced by Lyndon [34] to permit inductive arguments: they constitute an axiomatic generalisation of the length of a reduced word as in 1.4, 1.6 or 1.7 above. It was shown by Chiswell [3] however that every function satisfying the axioms defines an action on a tree and hence comes from a decomposition of G as fundamental group of a graph of groups. Thus here we have a further equivalent concept.

5. ENDS

The definition of ends, and construction of the end point compactification (for a peripherally compact space) was achieved by Freudenthal in 1931 [9]; and the application to group theory initiated by himself [10], [11], Hopf [14] and Specker [24]. We present a somewhat simplified version, adapted to the present applications.

Let X be a locally finite simplicial complex. For each finite subcomplex K, the number of connected components of $X - K$ is finite; denote by $n(K)$ the number of infinite ones (equivalently, having noncompact closure in X). Now define the number of ends $e(X) = \sup n(K)$. Clearly $e(X) = 0 \iff X$ is finite; otherwise $e(X)$ is a positive integer or $+\infty$.

If $X = \mathbf{R}$ and K is a point, clearly $n(K) = 2$. On the other hand, any compact K is contained in a closed interval $J : \mathbf{R} - J$ has only two components, and a component of $\mathbf{R} - K$ meeting neither is contained in J, hence finite. Thus $e(\mathbf{R}) = 2$. Similarly, since the complement of a (large) disc is connected, $e(\mathbf{R}^n) = 1$ for any $n \geq 2$.

As X is locally finite, for any finite K the (open) subcomplex $st(K)$ consisting of all simplices with a vertex in K is finite, and clearly $n(\overline{st\, K}) \geq n(st\, K) \geq n(K)$. Now any point of $X - (st\, K)$ can be joined by a path avoiding $st\, K$ to a vertex not in K, and if two such vertices can be joined by a path avoiding $st\, K$, as none of the vertices of the simplices met by the path are in K, we can find a path along edges not in $st\, K$. It follows that in computing $e(X)$ we may ignore all simplices of dimension > 1, and work in the 1-skeleton. This can now be formalized. The cochain complex $C^*(X)$ of X (coefficients $\mathbf{Z}_2$ = integers mod 2 understood) contains a subcomplex $C_f^*(X)$ of cochains with finite support. Note: the fact that $C_f^*(X)$ is closed under the coboundary follows from local finiteness of X. Write $C_e^*(X)$ for the quotient complex, and $H_e^*(X)$, $H_f^*(X)$ for the cohomology groups of $C_e^*(X)$, $C_f^*(X)$. Then the short exact sequence $0 \to C_f^*(X) \to C^*(X) \to C_e^*(X) \to 0$ induces a long exact sequence of cohomology groups.

Our interest in these comes from

Proposition 5.1. $e(X)$ *is the dimension of* $H_e^0(X)$ *over* $\mathbf{Z}_2$.

Proof. Observe that $H_e^0(X) = \delta^{-1}(C_f^1(X))/C_f^0(X)$ is the quotient of 0-cochains with finite coboundary by finite 0-cochains. Also, by the above, we may suppose X 1-dimensional.

Now if the 0-cochains $c_1, \dots, c_n$ define linearly independent elements of $H_e^0(X)$, as each δc_i is finite we can choose a finite subcomplex K containing the supports of all δc_i. But then for each edge e not in K, each c_i takes the same value at both ends of e. Thus for each connected component A of $X - K$, each c_i takes a constant value $c_i(A)$ on the vertices of A. If there were only $r < n$ infinite components A, there would be a nontrivial linear relation $\Sigma \lambda_i c_i(A) = 0$ holding for all such A. But then $\Sigma \lambda_i c_i$ would be a finite cochain, contradicting our

choice. Hence $n \leq \dim H_e^0(X)$ implies $e(X) \geq n$.

Conversely, if $e(X) \geq n$ we choose K finite with $n(K) \geq n$, and let $A_1, \dots, A_n$ be distinct infinite components of $X - K$. Define the cochain c_i to take the value 1 on vertices of A_i, 0 on other vertices of X. Then if $\delta c_i(e) = 1$, e has one end in A_i, the other not (and hence in K), so e is one of the finitely many edges of $st\, K$. So each δc_i is finite and, by construction, the c_i are independent modulo finite cochains. Hence $\dim H_e^0(X) \geq n$.

We next construct another theory, analogous to the above. For any group G, let PG be the power set of all subsets. Under Boolean addition ('symmetric difference') this is an additive group of exponent 2. Write FG for the additive subgroup of finite subsets. Now define

$$QG = \{A \subset G : \forall g \in G,\ A + Ag \text{ is finite}\}.$$

We refer to two sets A and B whose difference lies in FG as <u>almost equal</u>, and write $A \overset{a}{=} B$. This amounts to equality in the quotient group PG/FG. Moreover G acts by right translation on these groups, and QG/FG is the subgroup of elements invariant under this action. Elements of QG are said to be <u>almost invariant</u>. We define the number of ends of G to be

$$e(G) = \dim_{\mathbf{Z}_2} (QG/FG).$$

If G is finite, all subsets are finite and clearly $e(G) = 0$. Otherwise, G is an infinite set which is invariant (not merely 'almost'), so $e(G) \geq 1$.

For finitely generated groups G we can identify these two definitions as follows. Choose a finite set S of generators, and form the Cayley graph $\Gamma_S = \Gamma(S, G)$. Clearly this is locally finite.

Proposition 5.2. $e(G) = e(\Gamma_S)$.

Corollary 5.3. $e(\mathbf{Z}) = 2$. *For* $\{1\}$ *generates* $\mathbf{Z}$, *and the corresponding graph is homeomorphic to* $\mathbf{R}$.

Proof. We can identify the vertices of Γ_S with elements of G, and hence $C^0(\Gamma_S)$ with PG and $C_f^0(\Gamma_S)$ with FG. What we have to show, then, is that if the 0-cochain c corresponds to the subset A, then

$$\delta c \text{ is finite} \iff A \in QG.$$

Now δc is supported by the set of edges (g, gs) $(g \in G,\ s \in S)$ with just one end in A. For fixed s, this means that g belongs to just one of A, As^{-1}; i.e. $g \in A + As^{-1}$. If A is almost invariant, for each s we have finitely many g, hence a finite number of edges in total. Conversely if δc is finite, the class of A in PG/FG is invariant under each s^{-1} $(s \in S)$, hence under the group, G, which they generate.

This connection can now be extended.

Theorem 5.4. Let G act freely on the connected complex X, with finite quotient K (equivalently, $X \to K$ is a connected regular covering, with group G). Then $e(G) = e(X)$.

Proof. As before, we may suppose X a graph by ignoring cells of dimension > 1. Let T be a maximal tree in K, $\tilde{T}$ a lift to X. The trees $g\tilde{T}$ $(g \in G)$ are all disjoint; if we identify each to a point (obtaining Y, say) $H_e^0(X)$ is unaltered. For if c has finite coboundary, it is constant on all but finitely many $g\tilde{T}$, hence almost equal to a c' which is constant on each. The natural map $C^0Y \to C^0X$ preserves the subgroups of finite cochains and of cochains with finite coboundaries, hence induces $H_e^0(Y) \to H_e^0(X)$. This is clearly injective, and the above observation proves it surjective.

We may thus suppose K/T has only one vertex. But now Y can be identified with a suitable graph Γ_S, and the result follows from (5.2).

Corollary 5.5. If G acts freely on $\mathbb{R}^n$ with compact quotient, e.g. if $G = \mathbb{Z}^n$, we have $e(G) = 1$.

The connection with topology is valid only for finitely generated G. However, an interpretation in terms of group cohomology can always be given. For any G, $H^n(G; PG) \cong \mathbb{Z}_2$ $(n = 0)$, 0 $(n \neq 0)$. Moreover FG can be identified with the group ring $\mathbb{Z}_2G$. The invariant subgroup $QG/FG = H^0(G; PG/\mathbb{Z}_2G)$, and for G infinite, since $H^0(G; \mathbb{Z}_2G) = 0$, we deduce that

$$e(G) = 1 + \dim H^1(G; \mathbb{Z}_2G).$$

We now begin work on calculating the number of ends of various groups. We start with lemmas noting the invariance of $e(G)$ under commensurability and isogeny.

Lemma 5.6. If H is a subgroup of finite index in G, $e(G) = e(H)$.

Proof. If G is finitely generated, we may use (5.2) and observe that H acts freely on Γ_S with finite quotient (a covering of $G\backslash\Gamma_S$, of degree $|G : H|$).

In general, if $A \subset G$ is almost invariant in G so is $A \cap H$ in H. For $(A \cap H) + (A \cap H)h = (A + Ah) \cap H$ is finite, for any $h \in H$. Thus intersection induces a homomorphism $QG/FG \xrightarrow{\phi} QH/FH$. Choose a left transversal T for H in G.

Now ϕ is injective, for if $A \cap H$ is finite so is each $Ag \cap H$, hence $A \cap Hg^{-1}$. Letting g run through the finitely many elements of T^{-1}, we deduce A is finite.

And ϕ is surjective, for if $B \subset H$ is almost invariant consider $A = BT$. Certainly $A \cap H = B$. For any $g \in G$, $t \in T$, write $tg = h_t s$ $(s \in T)$. Then $A + Ag = \Sigma(Bt + Btg) = \Sigma(Bt + Bh_t s)$. But Bh_t is almost equal to B, and s runs through T as t does. Hence $A + Ag$ is finite.

Lemma 5.7. If K is a finite normal subgroup of G,

$$e(G) = e(G/K).$$

Proof. Write $p : G \to G/K$ for the natural map and $p_t : PG \to P(G/K)$, $p^{-1} : P(G/K) \to PG$ for the direct and inverse image maps induced by p. Then $p_t p^{-1} B = B$ for any $B \subset G/K$, while for $A \subset G$, $p^{-1} p_t A = AK$. Trivially B is almost invariant $\iff p^{-1}B$ is, and if A is almost invariant it is almost equal to AK, so $p_t(A)$ is almost invariant. Hence p_t, p^{-1} preserve the subgroups Q and F and the induced maps between QG/FG and $Q(G/K)/F(G/K)$ are two sided inverses, hence isomorphisms.

We now come to the main result of this section.

Theorem 5.8. *Suppose G finitely generated, $A \in QG$ such that both A and A^* ($= G - A$) are infinite, and that $H = \{h \in G : hA \overset{a}{=} A\}$ is infinite. Then G has an infinite cyclic subgroup of finite index.*

Since the left translations on PG commute with the right, there is an induced action of G from the left on QG/FG. As H is the stabilizer of A for this action, it is a subgroup.

Corollary 5.9. *If G is finitely generated, $e(G) = 0, 1, 2$ or ∞.*

For suppose $e(G) \neq 0, 1$ or ∞. Then G is infinite ($e(G) \neq 0$) and acts on the finite ($e(G) \neq \infty$) group QG/FG. As $e(G) \neq 1$, we can find an A as above; the isotropy group H has finite index in G, so is infinite. Then by (5.8) there is a subgroup Z of finite index which is infinite cyclic, by (5.6) $e(G) = e(Z)$ and by (5.3), $e(Z) = 2$.

We also have a characterization of groups with 0 ends (finite) or 2 ends (finite extensions of $\mathbb{Z}$). Our next main objective will be a study of groups with ∞ ends. The restriction to finitely generated groups is not essential: the result of Corollary 5.9 is proved in Cohen's book [5] for groups which are not locally finite; he also shows that a countable locally finite group has ∞ ends, and (see Goalby [31]) an uncountable one also has 1 or ∞.

We begin the proof of (5.8) with a lemma, which (together with the corollary) will also be repeatedly used in chapter 6.

Lemma 5.10. *Let $A_0, A_1 \in QG$. For almost all $g \in A_0$, either $gA_1 \subseteq A_0$ or $gA_1^* \subseteq A_0$.*

Proof. Choose a finite set S of generators of G, and use (as in (5.2)) the action of G on Γ_S. Pick connected finite subgraphs C_i of Γ_S containing δA_i.

For each vertex c of C_1, $gc \in A_0$ for almost all $g \in A_0$. As C_i is finite, $gC_1 \cap C_0 = \phi$ for almost all $g \in G$. Hence for almost all $g \in A_0$, we have $gC_1 \cap C_0 = \phi$, and $gc \in A_0$ for each vertex c of C_1.

For any collection A of vertices of Γ, let $\overline{A}$ denote the maximal subgraph of Γ with vertex set equal to A. Each component E of $\overline{A_1}$ or $\overline{A_1^*}$ contains a vertex of C_1, so gE meets A_0: if it also meets A_0^*, it meets C_0. But C_0 is connected and disjoint from gC_1, so lies in a single component gE. Thus A_0^* cannot meet both gA_1 and gA_1^*.

Corollary 5.11. *If $A_0, A_1 \in QG$, then for almost all $g \in G$ one (at least) of $gA_1 \subset A_0$, $gA_1^* \subset A_0$, $gA_1 \subset A_0^*$, $gA_1^* \subset A_0^*$ holds.*

Proof of 5.8. Interchanging A, A^* if necessary, we may assume $H \cap A$ infinite. We may also adjoin 1 to A. By the lemma, for almost all $g \in A$ either $gA \subseteq A - \{1\}$ or $gA^* \subseteq A - \{1\}$. Hence we can choose $c \in H \cap A$ satisfying one of these: necessarily $cA \subseteq A - \{1\}$. We will show that c generates the required subgroup.

If $n > 0$, $c^nA \subseteq cA \subset A$. Thus $c^n \neq 1$, so c has infinite order. As $1 \in A$, $c^n \in A$ for $n > 0$, and as $c^nA \subseteq A - \{1\}$ for $n > 0$, we have $c^{-n} \in A^*$ for $n > 0$.

If $d \in \cap\{c^nA : n > 0\}$, then $c^{-n} \in Ad^{-1}$ for $n > 0$, contradicting the fact that $Ad^{-1} + A$ is finite, and all the c^n distinct. Hence $\cap\{c^nA : n > 0\} = \phi$. So

$$A = \cup\{c^nA - c^{n+1}A : n \geq 0\}$$
$$= \cup\{c^n(A - cA) : n \geq 0\}$$

is contained in the union of finitely many (right) cosets of $\langle c \rangle$ in G: recall that $c \in H$, so $A - cA$ is finite. The same holds for A^* (replacing c by c^{-1}). Hence the infinite cyclic subgroup $\langle c \rangle$ has finite index in G.

Alternate proof of 5.8. As before, we may assume that $H \cap A$ is infinite. Lemma 5.10 and its proof tells us that for almost all $g \in A$, $g(\delta A) \cap \delta A$ is empty and either $gA \subset A$ or $gA^* \subset A$. Hence there is an element c of $H \cap A$, such that $c(\delta A) \cap \delta A$ is empty and either $cA \subset A$ or $cA^* \subset A$. As cA is almost equal to A, we must have $cA \subset A$ and the inclusion must be strict, as $c(\delta A) \cap \delta A = \phi$. Let $B = A + cA$. Then B is non-empty, finite and $B \subset A$, $B \cap cA = \phi$. Further for any two integers r, s, with $r > s$, we have $c^rB \cap c^sB = \phi$. For $c^rB \cap c^sB = c^s(c^{r-s}B \cap B)$, and $c^{r-s}B \subset c^{r-s}A \subset cA$. Thus $c^{r-s}B \cap B \subset cA \cap B = \phi$.

Now consider $\sum_{n\in\mathbf{Z}} c^n B$, which equals $\bigcup_{n\in\mathbf{Z}} c^n B$ by the above. As δ is additive, we have $\delta\left(\sum_{n\in\mathbf{Z}} c^n B\right) = \sum_{n\in\mathbf{Z}}\left(c^n\delta A + c^{n+1}\delta A\right) = 0$. (Note that these infinite sums make sense.) Thus $\bigcup_{n\in\mathbf{Z}} c^n B$ must equal G, and so the cyclic subgroup of G generated by c has index equal to the order of B. As G is infinite, c must have infinite order and the result follows.

One can also give a more direct proof of the result (5.9) that a finitely generated group must have 0, 1, 2 or ∞ ends. Let G be a finitely generated, infinite group and suppose that e(G) is a positive integer n. Choose a finite generating set for G and let Γ be the corresponding graph. G acts on Γ on the left with finite quotient. Let L be a finite connected subgraph of Γ such that $\Gamma - L$ consists of n infinite components $V_1, \ldots, V_n$. As G is infinite, there exists $g \in G$ with $gL \cap L = \phi$. Thus gL lies in one of the V's, V_1 say. Exactly one of the components of $V_1 - gL$ is infinite, for $\Gamma - (L \cup gL)$ has only n infinite components. Now $L \cup V_2 \cup \ldots \cup V_n$ is connected, so that $\Gamma - gL$ has at most two infinite components. As g is a homeomorphism, $\Gamma - L$ must have at most two infinite components and this proves the required result.

We finish this section by giving some more information about groups with two ends.

Theorem 5.12. The following conditions on a finitely generated group G are equivalent:

(i) $e(G) = 2$,

(ii) G has an infinite cyclic subgroup of finite index,

(iii) G has a finite normal subgroup with quotient $\mathbf{Z}$ or $\mathbf{Z}_2 * \mathbf{Z}_2$,

(iv) $G = F *_F$ with F finite, or $G = A *_F B$ with F finite and $|A : F| = |B : F| = 2$.

Proof. (i) ⇒ (ii) by Theorem 5.8, for H will have index at most 2 in G.

(ii) ⇒ (i) by Lemma 5.6 and the fact that $e(\mathbf{Z}) = 2$.

(iii) ⇒ (iv) If F is a finite normal subgroup of G with quotient

$\mathbf{Z}$, then $G = F *_F$. If the quotient is $\mathbf{Z}_2 * \mathbf{Z}_2$, then $G = A *_F B$, where A and B are the inverse images of the $\mathbf{Z}_2$-factors. Thus $|A : F| = |B : F| = 2$ as required.

(iv) ⇒ (iii) If $G = F *_F$ with F finite, then both inclusions of F in F must be isomorphisms and so F is normal in G with quotient $\mathbf{Z}$. If $G = A *_F B$, with F finite and $|A : F| = |B : F| = 2$, then F is normal in A and B. Hence F is normal in G and $G/F \cong (A/F) * (B/F) \cong \mathbf{Z}_2 * \mathbf{Z}_2$.

(iii) ⇒ (i) by Lemma 5.7. Note that $\mathbf{Z}_2 * \mathbf{Z}_2$ is isomorphic to $D(\infty)$, the infinite dihedral group, and so has two ends.

Finally, we prove (ii) ⇒ (iii), to complete the theorem.

First, G must contain an infinite cyclic subgroup K of finite index which is also normal in G. One takes for K the intersection of all the conjugates of the original infinite cyclic subgroup. Let H denote the centralizer of K in G. Thus $|G : H| \leq 2$. H is finitely generated and its centre is a subgroup of finite index. A theorem of Schur (see e.g. W. R. Scott, Group Theory, Prentice-Hall, 1964, §15.1.13) tells us that H', the commutator subgroup of G, is finite. Now H/H' must have rank 1, and so there is an epimorphism $\phi : H \to \mathbf{Z}$ with finite kernel L. If $G = H$, our result is proved. Otherwise observe that H is normal in G and L is characteristic in H, as L is the torsion subgroup of H. Thus L is normal in G and we have the exact sequence

$$1 \to H/L \to G/L \to \mathbf{Z}_2 \to 1.$$

We know that G/L must be non-abelian, and therefore G/L is isomorphic to $\mathbf{Z}_2 * \mathbf{Z}_2$. This completes the proof of Theorem 5.12.

6. THE STRUCTURE THEOREM FOR GROUPS WITH INFINITELY MANY ENDS

The aim of this section is to describe which finitely generated groups have infinitely many ends. The neatest formulation of the result includes the case of two ends.

Theorem 6.1. *If G is a finitely generated group, then $e(G) \geq 2$ if and only if G splits over a finite subgroup.*

Remark. Theorem 5.12 tells us that $e(G) = 2$ if and only if either $G = F *_F$ with F finite or $G = A *_F B$ with F finite and $|A : F| = |B : F| = 2$.

This remarkable result is due to Stallings [37], [26], but our treatment of the proof is an amalgam of results of Cohen [5], Dunwoody [7] and Stallings [26].

There is a close connection between Theorem 6.1 and the Sphere Theorem. In fact, Stallings discovered the result by considering the proof of the Sphere Theorem due to Papakyriakopoulos [19] and Whitehead [29].

Sphere Theorem. *If M is an orientable 3-manifold with $\pi_2(M) \neq 0$, there is an embedded 2-sphere S in M which represents a non trivial element of $\pi_2(M)$.*

Let M be a closed orientable 3 manifold with fundamental group G. One can show easily (see below) that the hypothesis that $\pi_2(M)$ is non-zero is equivalent to asserting that $e(G) \geq 2$. Also the conclusion of the Sphere Theorem implies that G splits over the trivial subgroup. Thus the Sphere Theorem is extremely like Theorem 6.1, when M is a closed manifold. Further, it is possible [26] to give a proof of the Sphere Theorem which uses Theorem 6.1.

The reason why $\pi_2(M) \neq 0$ if and only if $e(G) \geq 2$ is as follows. Let $\tilde{M}$ denote the universal covering space of M. Then Theorem 5.4 tells us that $e(G) = e(\tilde{M})$. For these purposes it will be convenient to use coefficients $\mathbb{Z}$, not $\mathbb{Z}_2$, when defining the groups $H^n(\tilde{M})$, $H^n_f(\tilde{M})$, $H^n_e(\tilde{M})$. The natural analogue of Proposition 5.1 is that $e(\tilde{M})$ equals the rank of $H^0_e(\tilde{M})$, where the rank of an abelian group is defined to be the maximal rank of all finitely generated free abelian subgroups or ∞ if this maximum does not exist. Now consider the long exact sequence connecting the groups $H^n(\tilde{M})$, $H^n_f(\tilde{M})$, $H^n_e(\tilde{M})$. This begins

$$H^0_f(\tilde{M}) \to H^0(\tilde{M}) \to H^0_e(\tilde{M}) \to H^1_f(\tilde{M}) \to H^1(\tilde{M}) \to \ldots .$$

As $H^1(\tilde{M}) = 0$, we see that $e(\tilde{M}) \geq 2$ if and only if $H^1_f(\tilde{M})$ is non-zero. Now Poincaré duality for $\tilde{M}$ gives an isomorphism between $H^1_f(\tilde{M})$ and $H_2(\tilde{M})$, and we have $H_2(\tilde{M}) \cong \pi_2(\tilde{M}) \cong \pi_2(M)$. Thus $\pi_2(M) \neq 0$ if and only if $e(G) \geq 2$.

The conclusion of the Sphere Theorem implies that $G \cong A *_{\pi_1(S)} B$ or $A *_{\pi_1(S)}$ according to whether S separates M or not. As $\pi_1(S)$ is trivial, this implies that G splits over the trivial subgroup unless S separates M into two components one of which is simply connected. We show that this is impossible. If this did happen, we would have a compact, simply connected 3-manifold X with boundary a 2-sphere. Hence Poincaré duality tells us that $H_2(X, \partial X) \cong H^1(X) = 0$. Now the exact homology sequence of the pair $(X, \partial X)$

$$0 \to H_3(X, \partial X) \to H_2(\partial X) \to H_2(X) \to H_2(X, \partial X) \to \ldots$$

shows that $H_2(X) = 0$. The Hurewicz Theorem then implies that $\pi_2(X) = 0$, so that S is null-homotopic in X, contradicting our assumption on S.

A purely group theoretic result, which follows easily from Theorem 6.1 is the following.

Theorem 6.2. *If G is a finitely generated, torsion free group with a free subgroup of finite index, then G is free.*

Remark. Swan [27] has extended this result by removing the restriction that G be finitely generated.

Proof. Let $\mu(G)$ denote the minimal number of generators of G. If $\mu(G) = 0$, then G is trivial, so the theorem holds. If $\mu(G) > 0$, then G is non-trivial. As G cannot be finite, G has a non-trivial free subgroup F of finite index and so $e(G) \geq 2$. Thus Theorem 6.1 tells us that G splits over a finite subgroup. Now the only finite subgroup of G is the trivial subgroup, so that either $G \cong \mathbb{Z}$ or $G \cong G_1 * G_2$, where each G_i is non-trivial. In the first case, our result is proved. In the second case, we observe $\mu(G_i) < \mu(G)$ by Corollary 2.1 and that $F \cap G_i$ is a free subgroup of G_i which is of finite index. Thus the required result follows by induction on $\mu(G)$. Note that we have used the fact that subgroups of free groups are free.

We now come to the proof of Theorem 6.1 and we start with the easy half.

Lemma 6.3. If G splits over a finite subgroup, then $e(G) \geq 2$.

Remark. For this result, G need not be finitely generated.

Proof. It suffices to produce an almost invariant subset E of G such that E and E^* are infinite.

First suppose that $G = A *_C B$, where C is finite, and recall the canonical form for elements of G given by Theorem 1.6. One chooses based transversals T_A and T_B for C in A and B and obtains the form $a_1 b_1 \ldots a_n b_n c$ for any element of G, where $c \in C$, $a_i \in T_A$, $b_i \in T_B$ and $a_i = 1 \Rightarrow i = 1$, $b_i = 1 \Rightarrow i = n$. Let E be the subset of G consisting of elements for which a_1 is non-trivial. Clearly E and E^* are infinite. (This uses the fact that $A \neq C \neq B$.) If $b \in B$, then $Eb = E$. If $a \in A$, then $Ea \subset E \cup C$ and so also $Ea^{-1} \subset E \cup C$. Hence

$$E \subset Ea \cup Ca \subset E \cup C \cup Ca$$

so that $E \stackrel{a}{=} Ea$. As A and B together generate G, we have $Eg \stackrel{a}{=} E$ for all g in G.

Secondly suppose that $G = A *_C$, where C is finite, and recall the canonical form for elements of G given by Theorem 1.7. One chooses based transversals T_i of $\alpha_i(C)$ in A and obtains the form $a_1 t^{\varepsilon_1} a_2 t^{\varepsilon_2} \ldots a_n t^{\varepsilon_n} a_{n+1}$, where $a_{n+1} \in A$, $a_i \in T_1$ if $\varepsilon_i = 1$, $a_i \in T_2$ if $\varepsilon_i = -1$, and moreover $a_i \neq 1$ if $\varepsilon_{i-1} \neq \varepsilon_i$. Let E be the subset of elements of G for which a_1 is trivial and $\varepsilon_1 = 1$. If $a \in A$, then $Ea = E$. Also $Et \subset E$ and $Et^{-1} \subset E \cup \alpha_1(C)$. Therefore, as before, E is almost invariant in G. This completes the proof of Lemma 6.3.

The hard part of Theorem 6.1 is the result that a finitely generated group G with infinitely many ends must split over a finite subgroup. Our aim, following Dunwoody [7], is to produce a tree T on which G acts with quotient a single edge. We start by considering graphs and trees in more detail than before. Let Γ be an abstract graph.

Definition. An edge path in Γ is a sequence $e_1, \ldots, e_n$ of edges such that $\partial_1 e_i = \partial_0 e_{i+1}$ and $e_i \neq \bar{e}_{i+1}$, for $i = 1, 2, \ldots, n-1$. If e, f are edges of Γ, we will write $e \leq f$ if there is an edge path with $e_1 = e$ and $e_n = f$.

The relation $\leq$ has the following properties.

(A) For any graph Γ, the relation $\leq$ is reflexive and transitive.

(B) For any graph Γ and any edges e, f of Γ, if $e \leq f$ then $\bar{f} \leq \bar{e}$.

(C) The graph Γ is connected if and only if for any pair e, f of edges of Γ, at least one of $e \leq f$, $e \leq \bar{f}$, $\bar{e} \leq f$, $\bar{e} \leq \bar{f}$ holds.

(D) The graph Γ has no circuits if and only if whenever $e \leq f$ and $f \leq e$, then $e = f$.

(E) If Γ has no circuits, then for no pair e, f of edges can we have $e \leq f$ and $e \leq \bar{f}$.

(F) If Γ has no circuits, then for any pair e, f of edges there are only finitely many edges g with $e \leq g \leq f$.

If a relation satisfies the conditions in (A) and (D), we shall call it a partial order. Thus if Γ is a tree, the relation $\leq$ on $E(\Gamma)$ is a partial order and the following conditions hold.

(1) If $e \leq f$, then $\bar{f} \leq \bar{e}$.

(2) If $e, f \in E(\Gamma)$, there are only finitely many $g \in E(\Gamma)$ such that $e \leq g \leq f$.

(3) If $e, f \in E(\Gamma)$, at least one of $e \leq f$, $e \leq \bar{f}$, $\bar{e} \leq f$, $\bar{e} \leq \bar{f}$ holds.

(4) If $e, f \in E(\Gamma)$, we cannot have $e \leq f$ and $e \leq \bar{f}$.

Remark. If two of the inequalities in (3) hold, then $e = f$ or $e = \bar{f}$.

The next step is to show that if we start with a partially ordered set E satisfying all the above conditions, then we can construct a tree out of E.

Let E be a partially ordered set with an involution $e \to \bar{e}$, where $e \neq \bar{e}$, and suppose that conditions (1)-(4) hold. Write $e < f$ if $e \leq f$ and $e \neq f$. Write $e << f$ if $e < f$, and $e \leq g \leq f$ implies $g = e$ or $g = f$. We need the following technical result.

Lemma 6.4. The relation $\sim$ on E, defined by $e \sim f$ if and only if $e = f$ or $e << \bar{f}$, is an equivalence relation.

Proof. The relation is obviously reflexive and is symmetric because $e << \bar{f}$ implies $f << \bar{e}$.

We suppose $e \sim f$ and $e \sim h$ and will show $f \sim h$. The first step is to show that $f = h$ or $f < \bar{h}$. Condition (3) implies that one of $f \leq h$, $\bar{f} \leq \bar{h}$, $\bar{f} \leq h$, $f < \bar{h}$ holds. If $f \leq h$, then $\bar{h} \leq \bar{f}$ and we would have $e < \bar{h} \leq \bar{f}$ which implies $h = f$ as $e << \bar{f}$. If $\bar{f} \leq \bar{h}$, then $e < \bar{f} \leq \bar{h}$ implies $h = f$ as $e << \bar{h}$. If $\bar{f} \leq h$, we would have $e < \bar{f} \leq h$. As $e \leq \bar{h}$, this contradicts (4). The only remaining possibility is $f < \bar{h}$. Hence either $f = h$ or $f < \bar{h}$ as required.

The last step is to suppose $f < \bar{h}$ and $f \leq g \leq \bar{h}$ and prove $g = f$ or $g = \bar{h}$. One of the inequalities $\bar{e} \leq g$, $\bar{e} \leq \bar{g}$, $e < g$, $e < \bar{g}$ must hold. If $\bar{e} \leq g$, then $\bar{e} \leq \bar{h}$. As $e \leq \bar{h}$, this contradicts (4). If $\bar{e} \leq \bar{g}$, then $\bar{e} \leq \bar{f}$ which again contradicts (4). If $e < g$, then $e < g \leq \bar{h}$ shows that $g = \bar{h}$ as $e << \bar{h}$. If $e < \bar{g}$, then $e < \bar{g} \leq \bar{f}$ shows that $g = f$ as $e << \bar{f}$. Hence $g = f$ or $g = \bar{h}$ as required. This completes the proof of Lemma 6.4.

We construct a graph Γ out of E as follows. Let $t(e)$ denote the equivalence class of e in E under the relation $\sim$. Let $V = \{t(e): e \in E\}$ and let $\partial_0 e = t(\bar{e})$. Then the sets E, V and the map ∂_0 form an abstract graph Γ.

Theorem 6.5. Γ <u>is a tree and the order relation which</u> Γ <u>induces on</u> E <u>is the same as the original relation.</u>

Proof. We will prove the second part of the theorem. The fact that Γ is a tree will then follow from properties (C) and (D). Lemma 6.4 tells us that for distinct elements e, f of E we have $t(f) = t(e)$ if and only if $e << \bar{f}$. Thus $\partial_1 e = \partial_0 f$ if and only if $e << f$. It follows that if e and f are joined by an edge path in Γ, then $e \leq f$, and condition (2) shows that if $e \leq f$, then e and f can be joined by an edge path in Γ. Hence Γ does induce the original order relation on E, as required.

Consider the following examples of Theorem 6.5 in action.

Let G be the free group of rank 2 with generators a, b and let Γ be the corresponding graph for G. If e is an edge of Γ it separates Γ into two components. Thus $G = A \cup A^*$ where $\delta A = \delta A^* = e$. Let E be the set of all subsets A of G with δA equal to a single edge of Γ. We partially order E by inclusion. Then E has an involution $A \to A^*$ and satisfies conditions (1)-(4). The tree constructed in Theorem 6.5 is the graph Γ again.

One can obtain an action of G on a different tree T as follows. Let A be the subset of G which contains a and whose coboundary is the edge (e, a) of Γ. Let F be the set of all translates gA, gA^* of A and A^*, partially ordered by inclusion. Again F satisfies conditions (1)-(4). Hence we can construct a tree T from Theorem 6.5. The natural left action of G on F induces an action of G on T which has quotient a single edge. Let g be an element of G such that $gA = A$ or A^*. Then $g(\delta A) = \delta A$ so that the edges (g, ga) and (e, a) are equal. This is only possible if $g = e$ so that G acts on T without inversions and the stabilizer of any edge of T is trivial.

Recall that $t(A)$ consists of A together with every element B of F such that $A << B^*$. We will call an edge of Γ of the form (g, ga) an a-<u>edge</u>, and an edge of the form (g, gb) a b-<u>edge.</u> If $A << B^*$, then δB is an a-edge of Γ which has no vertices in A and such that the path from δB to δA consists only of b-edges. It is now easy to see that

$$t(A) = \{b^n A : n \in \mathbb{Z}\} \cup \{b^n a^{-1} A^* : n \in \mathbb{Z}\}.$$

Hence the stabilizer of $t(A)$ is the infinite cyclic subgroup H of G generated by b. One can also see that $\partial_0 A = \partial_1 A^* = a\partial_1 A$. Hence G acts transitively on the vertices of T, so that $G\backslash T$ is a loop. This action of G on T corresponds to expressing G as $H *_{\{1\}}$. The graph T is obtained from Γ by identifying each b-edge of I to a point.

We can now sketch the proof of the main part of Theorem 6.1, i.e. if G is a finitely generated group and $e(G) = \infty$, then G splits over a finite subgroup. The idea is to proceed, as in the example above, to construct a tree T on which G acts so that the stabilizer of any edge is finite and the quotient $G\backslash T$ is a single edge. However, we need to partially order our almost invariant sets by almost inclusion and not by strict inclusion in order to be able to prove that condition (3) holds.

For any almost invariant set B of G, write $[B]$ for the set of all almost invariant sets of G which are almost equal to B. Define $[B] \leq [C]$

if and only if $B \overset{a}{\subset} C$. Fix a proper almost invariant set A of G i.e. A and A^* are infinite, and let E be the set of $[gA]$ and $[gA^*]$ for all g in G, partially ordered by $\leq$. We have the involution $[A] \to [A^*]$ on E. Our aim is to choose A so that E satisfies conditions (1)-(4). Assuming that we can do this, we can construct the required tree T. The stabilizer of the edge $[A]$ will be finite because $\{g \in G : gA \overset{a}{=} A\}$ is finite by Theorem 5.8. G may invert edges but if so we simply subdivide T. Hence, by Theorem 4.3, G is an amalgamated free product over a finite subgroup F. Hence G splits over F so long as no vertex of T has stabilizer equal to G. We show that this is impossible.

Suppose that G fixes a vertex v of T. Then every edge of T has v as a vertex. In particular, v must be one of the original vertices of T, and was not introduced by subdivision. Now we consider the original T. If e and f are edges of T with $e < f$, we have $e << f$. Now Lemma 5.10 tells us that for almost all elements x of A, $xA \subset A$ or $xA^* \subset A$. As $\{g \in G : gA \overset{a}{=} A\}$ is finite, by Theorem 5.8, we deduce that there is an element x of A such that xA is not almost equal to A or A^* and either $xA \subset A$ or $xA^* \subset A$. Similarly, there is an element y of A^* such that yA^* is not almost equal to A or A^* and either $yA \subset A^*$ or $yA^* \subset A^*$. If $xA \subset A$, then we have $x^2A \subset xA \subset A$ and this contradicts the fact that if e, f are edges of T with $e < f$, then $e << f$. Similarly, if $yA^* \subset A^*$, we obtain a contradiction. If $xA^* \subset A$ and $yA \subset A^*$, then $xyA \subset xA^* \subset A$ and again we have a contradiction. Therefore G cannot fix a vertex of T.

In order to complete the proof of Theorem 6.1, we must show how to find a proper almost invariant set A in G such that the partially ordered set E satisfies conditions (1)-(4). Conditions (1) and (4) hold automatically for any choice of A. Our next result says that condition (2) also holds for any choice of A.

Lemma 6.6 Let G be a finitely generated group with infinitely many ends. If B, C, D are proper almost invariant subsets of G, then $\{g \in G : B \overset{a}{\subset} gC \overset{a}{\subset} D\}$ is finite.

Proof. The result is trivial if B is not almost contained in D. If B is strictly contained in D, we can add an element of D^* to B without altering the problem. Hence we suppose that $B \overset{a}{\subset} D$ but $B \not\subset D$.

If $B \overset{a}{\subset} gC \overset{a}{\subset} D$, then either $gC \not\subset D$ or $B \not\subset gC$. We will show that $\{g \in G : gC \overset{a}{\subset} D,\ gC \not\subset D\}$ and $\{g \in G : B \overset{a}{\subset} gC,\ B \not\subset gC\}$ are both finite. This will prove the required result. Now Corollary 5.11 states that for almost all elements g in G one of $gC \subset D$, $gC \subset D^*$, $gC^* \subset D$, $gC^* \subset D^*$ holds. If $gC \overset{a}{\subset} D$ and $gC \not\subset D$, none of these four inclusions can hold except for $gC^* \subset D^*$. If $gC \overset{a}{\subset} D$ and $gC^* \subset D^*$, then $gC \overset{a}{=} D$. As $\{g \in G : gC \overset{a}{=} C\}$ is finite, by Theorem 5.8, we deduce that $\{g \in G : gC \overset{a}{\subset} D,\ gC \not\subset D\}$ is finite. Similarly $\{g \in G : B \overset{a}{\subset} gC,\ B \not\subset gC\}$ is finite.

Finally, we show that it is possible to choose A so that E satisfies condition (3). Note that for almost all g in G, we know that one of $gA \subset A$, $gA \subset A^*$, $gA^* \subset A$, $gA \subset A^*$ holds. We must arrange that this holds for every element of G, when we replace strict inclusion by almost inclusion.

We fix a finite generating set S for G and let $\Gamma = \Gamma(S, G)$ be the corresponding graph. If A is an almost invariant set in G, we denote the number of edges in δA by $|\delta A|$. Let k be the smallest value taken by $|\delta A|$ as A ranges over proper almost invariant sets in G. We say that a set A in G is narrow if $|\delta A| = k$.

Lemma 6.7. Let $A_1 \supset A_2 \supset \ldots$ be a sequence of narrow sets in G. If $B = \bigcap_{n \geq 1} A_n$ is non-empty, then the sequence stabilizes, i.e. there is an integer K such that $A_n = B$, when $n \geq K$.

Proof. Let e be an edge of δB. Then e has one vertex in every A_n and the other vertex is outside every A_n for which n exceeds some integer N. Therefore e is an edge of δA_n, when $n > N$. If δB contains $k+1$ edges, the above argument shows that δA_n would also contain $k+1$ edges for a suitably large value of n. It follows that $|\delta B| \leq k$ and that $\delta B \subset \delta A_n$ for all suitably large n. In particular, B is almost invariant in G. We have the equations $A_n = (A_n + B) + B$ and $\delta A_n = \delta(A_n + B) + \delta B$. As $\delta B \subset \delta A_n$ we see that $\delta(A_n + B) \cap \delta B$ is empty. As A_n is infinite, one of B and $(A_n + B)$ must be infinite.

The infinite one, X, must have $|\delta X| = k$, as X is a proper almost invariant subset of G. The other one must then have empty coboundary and so be empty. As B is non-empty, we deduce $B = A_n$, which completes the proof of the lemma.

Let g be any element of G, and let A be a narrow set in G. Then A^* is also narrow so that g must lie in a narrow set in G. Lemma 6. 7 tells us that the set of all narrow subsets of G which contain g has minimal elements, where we partially order narrow sets by inclusion.

Lemma 6. 8. Let A be a narrow set, minimal with respect to containing some element g of G. Then for any narrow set A_1, one of $A \overset{a}{\subset} A_1$, $A \overset{a}{\subset} A_1^*$, $A^* \overset{a}{\subset} A_1$, $A^* \overset{a}{\subset} A_1^*$ holds.

Proof. The required result is equivalent to proving that one of the sets $A \cap A_1$, $A \cap A_1^*$, $A^* \cap A_1$, $A^* \cap A_1^*$ is finite. For convenience we call these sets X_1, X_2, X_3, X_4. For each i, $\delta X_i \subset \delta A \cup \delta A_1$. As the X_i's are disjoint, any edge in $\delta A \cup \delta A_1$ has its ends in exactly two of the X_i's. Hence each edge in $\delta A \cup \delta A_1$ lies in the coboundary of exactly two of the X_i's. Hence

$$|\delta X_1| + |\delta X_2| + |\delta X_3| + |\delta X_4| = 2|\delta A \cup \delta A_1| \leq 4k,$$

where $|\delta A| = |\delta A_1| = k$.

If each X_i is infinite, then we must have $|\delta X_i| \geq k$ for each i, because each X_i^* is infinite. Hence $|\delta X_i| = k$ for each i. But one of $A \cap A_1$, $A \cap A_1^*$ (say $A \cap A_1$) is then a narrow subset of G which contains g. Hence $A \cap A_1 = A$ by the minimality of A, and so $A \cap A_1^*$ is empty - a contradiction. Therefore some X_i must be finite which completes the proof of Lemma 6. 8.

In order to carry out the proof of Theorem 6. 1 as sketched after Theorem 6. 5, we simply need to choose a narrow set A in G which is minimal with respect to containing some element of G.

7. APPLICATIONS AND EXAMPLES

Many of the most important applications of Stallings' structure theorem for groups with infinitely many ends have to do with the cohomology of groups. We will only consider more simple minded examples. We start by discussing the problem of accessibility first posed by Wall [28].

Think of a splitting of a group over a finite subgroup as a kind of factorization. Stallings' theorem tells us that if G is finitely generated and $e(G) \geq 2$, then G has such a factorization. The first natural question to ask is whether one can go on factorizing G for ever, or whether the process of factorization must stop.

We will say that a f. g. group with at most one end is 0-accessible, and that a group G is n-accessible if G splits over a finite subgroup with each of the factor groups (n-1)-accessible. We will call a group accessible if it is n-accessible for some n.

Conjecture. Any finitely generated group is accessible.

Bamford and Dunwoody [1] have shown that accessibility is equivalent to a certain condition on the cohomology of the group, but, in general, one has no proof that their condition is satisfied. However, it is easy to see that any f. g. torsion free group G is accessible. Corollary 2. 2, which follows from Gruško's Theorem, tells us that G is a free product of indecomposables. Each factor in this decomposition has at most one end or is infinite cyclic and so G is accessible.

The following result seems to clarify the concept of accessibility.

Lemma 7. 1. Let G be a finitely generated group. Then G is accessible if and only if G is the fundamental group of a finite graph Γ of groups, where each edge group is finite and each vertex group has at most one end.

Proof. If Γ exists, G is obviously accessible. We prove the converse by induction on n, where G is n-accessible. If $n = 0$, we can take Γ to be a single vertex.

If $G = G_1 *_C G_2$ where G_1 and G_2 are already the fundamental groups of graphs Γ_1 and Γ_2 of groups, and C is finite, we construct Γ as follows. By Corollary 3.8, there are vertex groups H_1, H_2 of Γ_1, Γ_2 and elements g_1, g_2 of G_1, G_2 such that $C \subset g_i^{-1}H_i g_i$, for $i = 1, 2$. By replacing every vertex and edge group H of Γ_1 by $g_1 H g_1^{-1}$ we can suppose that $C \subset H_1$, and similarly for Γ_2. Now Γ consists of Γ_1, Γ_2 and an edge e joining the vertices underlying H_1 and H_2, where e has associated group C.

If $G = A *_C$ where A is already the fundamental group of a graph Γ_1 of groups, and C is finite, we proceed as follows. We have two inclusions α_1, α_2 of C in A and each $\alpha_i(C)$ must lie in a conjugate of some vertex group H_i of Γ_1. As above, we can suppose that $\alpha_1(C) \subset H_1$, and $\alpha_2(C) \subset sH_2s^{-1}$. (Note that possibly $H_1 = H_2$.) Now G has presentation $\{A, t : t^{-1}\alpha_1(c)t = \alpha_2(c), \forall c \in C\}$. Write $u = ts$. Then G also has presentation $\{A, u : u^{-1}\alpha_1(c)u = s^{-1}\alpha_2(c)s, \forall c \in C\}$. We replace α_2 by β_2 where $\beta_2(c) = s^{-1}\alpha_2(c)s$. As $\beta_2(C) \subset H_2$, we can take Γ to be Γ_1 together with an edge e joining the vertices of Γ_1 which underlie H_1 and H_2, where e has associated group C.

We can now re-define accessibility to allow for infinite factorization. A group is <u>accessible</u> if and only if it is the fundamental group of a graph of groups in which every edge group is finite and every vertex group has at most one end. For f. g. groups, this is equivalent to the old definition. One can ask if all groups are accessible, but the Kuroš example in Section 3 shows that the answer is negative. For Kuroš's group is an infinite amalgamated free product of free groups and hence is torsion free. Thus his group is accessible if and only if it can be expressed as a free product of indecomposable subgroups.

There is one other class of groups known to be accessible. That is groups with a free subgroup of finite index. We have already shown (Theorem 6.2) that if such a group is torsion free it must be free. We now state a general structure theorem for such groups. This was proved by Karrass, Pietrowski and Solitar in the f. g. case [16], Cohen in the countable case [6], and Cohen [6] and Scott [20] in the general case. See also Dunwoody [7] for a more recent proof.

Theorem 7.3. <u>A group</u> G <u>has a free subgroup of finite index if and only if</u> G <u>is the fundamental group of a graph</u> Γ <u>of groups in which every vertex group of</u> Γ <u>is finite and the orders of all the vertex groups are bounded.</u>

Remark. We will prove this theorem only in the case when G is f.g. We can then assume that Γ is finite, so that the boundedness condition is redundant.

Proof. Suppose that G is f. g. and has a free subgroup of finite index. We will show that Γ exists by induction on $r(G)$, where $r(G)$ is the minimal rank of free subgroups of G of finite index. If $r(G) = 0$, then G is finite and the result follows. If $r(G) > 0$, then $e(G) \geq 2$ so G splits over a finite subgroup. If $G = A *_C B$, or $G = A *_C$, I claim that $r(A)$ and $r(B)$ are each less than $r(G)$, so that the result will follow by induction as in the proof of Lemma 7.1. Let F be a free subgroup of G of finite index and of minimal rank. As C is finite, F meets any conjugate of C trivially. Hence the Subgroup Theorem applied to $F \subset A *_C B$ or $A *_C$ tells us that $F = (F \cap A) * (F \cap B) * K$ or $F = (F \cap A) * K$, for some subgroup K of F. Hence the ranks of $F \cap A$ and $F \cap B$ are each less than that of F unless one of them equals F. But then we would have F contained in A or B which is impossible as F has finite index in G, but A and B have infinite index in G.

Now suppose that G is the fundamental group of a finite graph Γ of finite groups. We use induction on the number n of edges of Γ. If $n = 0$, then G is finite and the result is obvious.

If $n \geq 1$, we pick an edge e of Γ with associated group C. Then $G = A *_C B$ or $A *_C$, according to whether e separates Γ or not, where A and B are the fundamental groups of the subgraphs of Γ obtained by removing e. Thus, by our induction hypothesis, each of A and B has a free subgroup of finite index and hence a normal free subgroup of finite index. Let A_1, B_1 denote the quotients of A and B by their normal free subgroups of finite index. As C is finite, the natural maps from A and B to A_1 and B_1 both inject C. Hence we have a natural map $A *_C B \to A_1 *_C B_1$ or $A *_C \to A_1 *_C$, which injects any finite subgroup of G. Lemma 7.4 below tells us that there are maps of $A_1 *_C B_1$ or $A_1 *_C$ to a finite group which inject A_1 and B_1. By composing these maps we obtain a homomorphism from G to a finite group which injects every finite subgroup of G. The kernel of this homomorphism must be a free group, by the Subgroup Theorem, which completes the proof of Theorem 7.3.

Lemma 7.4. If $G = A *_C B$ or $A *_C$, with A, B and C finite, then G has a free subgroup of finite index.

Proof. We construct a homomorphism from G to a finite group which injects A and B. The kernel must be free, by the Subgroup Theorem.

Case $G = A *_C B$

Let $X = A/C \times C \times B/C$, where A/C denotes the set of all cosets aC of C in A. We will represent A and B faithfully as permutation groups of the finite set X, in such a way that C acts on X in the same way for each action. There will then be a homomorphism $G \to S(X)$, the group of permutations of X, which injects A and B.

Choose a transversal $t : A/C \to A$. We have a bijection $A/C \times C \to A$ sending (α, c) to $t(\alpha)c$. The action of A on itself by right multiplication gives an action of A on $A/C \times C$, by using this bijection. We let A act on X by defining $(\alpha, c, \beta)a = ((\alpha, c)a, \beta)$. If $c' \in C$, then $(\alpha, c, \beta)c' = (\alpha, cc', \beta)$. Similarly we use a transversal of C in B to define an action of B on X. For this action also, we have $(\alpha, c, \beta)c' = (\alpha, cc', \beta)$ for all $c' \in C$.

Case $G = A *_C$

We have two injections of C into A. We use one of them to identify C with a subgroup of A. Thus we have a subgroup C of A and an injective map $\phi : C \to A$, whose image we denote by C_1.

Let $X = A$, and let A act on X by right multiplication. The two induced actions of C are each multiples of the right regular representation so are equivalent. We can write down an equivalence as follows. Choose transversals $T : A/C \to A$, $T_1 : A/C_1 \to A$ and a bijection $\psi : A/C \to A/C_1$.

Then T, T_1 induce bijections $U : A/C \times C \to A$, $U_1 : A/C_1 \times C_1 \to A$ (as above), and we define θ to be the composite

$$A \xrightarrow{U^{-1}} A/C \times C \xrightarrow{\psi \times \phi} A/C_1 \times C_1 \xrightarrow{U_1} A.$$

Then $\theta(ac) = \theta(a)\phi(c)$. Now we can define a homomorphism $r : G \to S(X)$ by letting $r|A$ be the right regular representation and $r(t) = \theta$.

Remark. These constructions - which do not depend on finiteness (except to suppose the existence of a bijection ψ) - give an alternative proof of the assertion (1.6, 1.7) that if $C \to A$, $C \to B$ are injective, so are $A \to A *_C B$, $A \to A *_C$.

Having discussed the accessibility of groups i.e. the existence of a factorization, the next question to consider is that of uniqueness of the factorization. One would like some analogue of Theorem 3.5 for free products. The first point is that given any graph Γ of groups with fundamental group G, one can construct a larger graph Γ', also with fundamental group G by adding an edge e to Γ with only one vertex of e in Γ and an isomorphism at the other end of e. This corresponds to expressing G as $G *_C C$ for some subgroup C.

We will say that an edge e in a graph of groups Γ is trivial if the two ends of e are distinct vertices of Γ and e has an isomorphism at one end. If Γ has such an edge, we can replace Γ by a new graph Γ' obtained from Γ by identifying e to a point, such that Γ' has the same fundamental group as Γ. Hence if we start with a finite graph Γ, we can eliminate all the trivial edges. However, this is false for infinite graphs. For example, let Γ be the graph with vertices 1, 2, ... and edges e_i joining i to i + 1. We associate an infinite cyclic group A_i to the vertex i and an infinite cyclic group B_i to the edge e_i. The map $B_i \to A_i$ is an isomorphism and the map $B_i \to A_{i+1}$ is multiplication by two. The fundamental group of Γ is the dyadic rationals, but every edge of Γ is trivial.

We will say that a graph of groups with no trivial edges is minimal. Then any finitely generated accessible group G is the fundamental group of some minimal graph Γ, where each edge group is finite and each vertex group has at most one end. Even with minimal graphs, one still cannot expect that the graph Γ is unique. For example, if $G = G_1 * \ldots * G_n$, then one can take for Γ any tree with n vertices, and associate $G_1, \ldots, G_n$ to the vertices and the trivial group to each edge. The same problem arises for amalgamated free products e.g. when $G = G_1 *_C \cdots *_C G_n$. Also if $G = A *_C B *_D E$ with $D \subset C$, then $G = E *_D A *_C B$ giving two possible graphs for G.

We need the following result.

Lemma 7.5. Let G be the fundamental group of a finite graph Γ of groups, such that each edge group is finite and each vertex group has at most one end.

(i) If A is a subgroup of G with at most one end, then A lies in a conjugate of a vertex group of Γ.

(ii) Let v_1, v_2 be vertices of Γ with associated groups G_1, G_2. If $A = G_1 \cap G_2^g$, then either there is an edge path in Γ from v_1 to v_2 such that each of the associated edge groups contains A or $G_1 = G_2$ and $g \in G_1$.

Proof. (i) The Subgroup Theorem tells us that A is the fundamental group of a graph Γ' of groups where the associated groups are conjugates of subgroups of the groups associated to Γ. Our aim is to show that A must be a vertex group of Γ'.

The fact that $e(A) \leq 1$ tells us that each edge of Γ' is trivial and that Γ' is a tree. Thus the vertex groups of Γ' are partially ordered by inclusion. Suppose that $A_1 \subset A_2 \subset \ldots$ is an infinite ascending chain of vertex groups of Γ'. If all the inclusions are strict, then each A_i equals an edge group of Γ'. But, as Γ is a finite graph, there is an upper bound on the orders of the edge groups of Γ'. Hence, one cannot have an infinite strictly increasing chain of vertex groups of Γ'. Hence there is a maximal vertex group. This vertex group must equal A, which completes the proof of (i).

(ii) The proof of this is the same as the proof of Lemma 3.15.

Now we consider a finitely generated group G and two minimal graphs Γ and Γ' each with fundamental group G, such that each edge group is finite and each vertex group has at most one end. Note that Γ and Γ' must be finite.

Lemma 7.6. (i) There is a bijection between the vertices of Γ and Γ' such that corresponding vertex groups are conjugate in G.

(ii) Γ and Γ' have the same number of edges.

(iii) If Γ does not have distinct edges e, f with G_e lying in a conjugate of G_f, then Γ and Γ' are isomorphic as graphs and corresponding vertex or edge groups are conjugate in G.

Remarks. In (iii), the hypothesis implies that no edge group of Γ is trivial, unless Γ has only one edge.

It seems reasonable to suppose that the analogue of (i) for the edges of Γ and Γ' always holds, but we cannot prove it.

Proof. (i) Let A be a vertex group of Γ. Then $e(A) \leq 1$. Lemma 7.5 (i) tells us that A lies in a conjugate of a vertex group B of Γ'. The same lemma shows that B lies in a conjugate of a vertex group A_1 of Γ. Hence A lies in a conjugate A_1^g of A_1 for some $g \in G$. Lemma 7.5(ii) tells us that either $A = A_1$ and $g \in A$ or there is a path from A to A_1 in Γ for which each edge group contains a conjugate of A. As Γ is minimal, the second case can only occur when $A = A_1$ and the path consists of a single loop. Therefore $A = A_1^g$ and A is conjugate to B. As the groups associated to distinct vertices of Γ cannot be conjugate (because Γ is minimal), assertion (i) follows.

(ii) Let $\overline{G}$ denote the quotient of G obtained by killing all the vertex groups of Γ. This quotient is a free group of rank $E - V + 1$, where E and V are the number of edges and vertices of Γ. Part (i) tells us we obtain a group isomorphic to $\overline{G}$ by killing all the vertex groups of Γ'. Hence $E - V + 1 = E' - V' + 1$. As $V = V'$, by (i), we have $E = E'$ as required.

(iii) Let e be an edge of Γ with vertices v_1 and v_2 which may be equal. Let G_1 and G_2 be the groups associated to v_1 and v_2 and let A be the group associated to e. Then $A = G_1 \cap G_2^g$ where either $G_1 \neq G_2$ or $G_1 = G_2$ and $g \notin G_1$. It follows from Lemma 7.5 (ii), and from part (i) of this lemma, that A lies in a conjugate of an edge group B of Γ'. Similarly, B lies in a conjugate of an edge group A_1 of Γ. Our hypothesis on Γ, in (iii), implies that $A = A_1$ so that A is conjugate to B. Thus for each edge group of Γ, there is an edge group of Γ' conjugate to it in G and distinct edges of Γ correspond to distinct edges of Γ'. As Γ and Γ' have the same number of edges, by part (ii), we have a bijection between the edges of Γ and Γ'.

Suppose that A is an edge group of Γ and that A is contained in a conjugate of a vertex group H of Γ. Our condition that A is not contained in a conjugate of any other edge group of Γ implies that H is one of the vertex groups at the end of the edge to which A is associated. The same holds for Γ', so that the bijection between the edges of Γ and Γ' must actually induce an isomorphism of the graphs Γ and Γ'.

We turn now to another embedding result proved in [20]. The result and its proof are similar to those of Theorem 3.17, which told us that any countable group could be embedded in a 2-generator group. The result is an essential part of the proof of Theorem 7.3 for arbitrary cardinality of the groups involved.

Theorem 7.7. If G is the fundamental group of a countable graph Γ of finite groups, where the vertex groups have bounded order, then G can be embedded in a group H_* which is the fundamental group of a finite graph of finite groups.

Remarks. The natural homomorphism $A *_C \to \mathbb{Z}$, obtained by killing A, has kernel K equal to $\ldots *_C A *_C A *_C \ldots$. This can be seen most simply by constructing a space X whose fundamental group is K and observing that $\mathbb{Z}$ acts freely on X with quotient a space with fundamental group $A *_C$. The graph Γ corresponding to K is a copy of the real line with integer points as vertices, and all the vertex groups are copies of A, all the edge groups are copies of C. Thus $\mathbb{Z}$ acts on Γ, as a graph of groups. This is an example of how to embed a group which is the fundamental group of an infinite graph of groups into a group which is the fundamental group of a finite graph of groups. One needs a fairly uniform sort of graph Γ so that Γ admits a group action.

Proof. The aim of our proof is to work in steps so as to make Γ uniform. Since the vertex groups have bounded order we can choose a group H (for example, a symmetric group) in which they all embed.

Let G_1 be obtained from G by replacing each vertex group of Γ by a copy of H. Note that $G \subset G_1$.

Let $H_1, \ldots, H_n$ be groups, one from each isomorphism class of subgroups of H. Let $f_1, \ldots, f_N$ be the distinct embeddings of $H_1, \ldots, H_n$ in H. Then each edge of Γ_1 has a pair of f_1's associated to it. We will say that two edges of Γ_1 are of the same type if they have the same unordered pair associated.

We enlarge the graph of groups Γ_1 by adding countably many edges of each type joining each distinct pair of vertices of Γ_1. This new graph Γ_2 is still countable, and so is its fundamental group G_2. We have $G_1 \subset G_2$. Choose a maximal tree T in Γ_2 consisting of edges with the identity map of H at each end. Let Γ_3 be the graph of groups obtained from Γ_2 by identifying T to a point. Thus Γ_3 has one vertex labelled H and countably many loops of each type. Its fundamental group is still G_2.

We now have a graph which clearly admits a group action. Suppose that Γ_3 has m types of loop. We label the edges of Γ_3 by a_{ij}, where $1 \le i \le m$, and for fixed i, the suffix j runs through all the integers, thus enumerating all the loops of one given type.

G_2 has a presentation of the form

$$\{H, \{a_{ij}\} \mid a_{ij}^{-1} b_{ik} a_{ij} = c_{ik}, \text{ where } k \text{ runs through some set } K_i \text{ and } b_{ik}, c_{ik} \in H\}.$$

We define an isomorphism $\phi : G_2 \to G_2$ by $\phi(h) = h$, for $h \in H$ and $\phi(a_{ij}) = a_{i,j+1}$. This determines an extension of G_2 by $\mathbb{Z}$ which we call G_3. G_3 can be presented as

$$\{H, \{a_{i0}\}, t \mid t^{-1}ht = h \text{ for } h \in H, a_{i0}^{-1} b_{ik} a_{i0} = c_{ik} \text{ for } k \in K_i\}.$$

Hence G_3 is the fundamental group of a graph of groups which has one vertex labelled H, one loop of each type and one extra loop which has associated to it the identity map of H at each end. Hence G_3 is the required group H_*.

8. ENDS OF PAIRS OF GROUPS

The concept of the number of ends of a pair of groups (G, C), where C is a subgroup of G, is a generalization of the number of ends of a group. Recall (Section 6) the close relationship between the theory of ends of groups and the Sphere Theorem. One is also interested in conditions which will guarantee the existence of other surfaces in a 3-manifold - particularly when the fundamental group of the surface injects into the fundamental group of the 3-manifold. Thus one is interested in groups which split over infinite subgroups. The starting point of my work on ends of pairs of groups was the idea that there should be a generalization of Stallings' structure theorem to this situation. Thus one is looking for a natural definition of a number $e(G, C)$, and one hopes to prove that $e(G, C) \geq 2$ if and only if G splits over some subgroup closely related to C.

The correct definition of $e(G, C)$ is due to Houghton [33]. Recall the definition of $e(G)$. One lets PG be the power set of G, FG be the collection of finite subsets of G, each with Boolean addition, and defines $EG = PG/FG$. The right action of G on itself induces a right action of G on EG. Let $(EG)^G$ denote the subset of elements left fixed by this action (this is the same as QG/FG, where QG is as in §5). Then $e(G)$ is the dimension, as $\mathbb{Z}_2$-vector space, of $(EG)^G$.

Let C be a subgroup of G and let $H = C\backslash G$. Then we define $e(G, C)$ to be the dimension of $(EH)^G$. Clearly if C is trivial, then $e(G, C) = e(G)$. The following result justifies the claim that this is the correct definition of $e(G, C)$.

Lemma 8.1. Let X be a finite CW-complex with a connected regular covering space $\tilde{X}$ whose covering group is G. If C is a subgroup of G, then $e(G, C) = e(C\backslash\tilde{X})$.

Remark. The hypothesis that X is finite implies that G is f.g.

One can summarize the basic properties of $e(G, C)$ as follows.

Lemma 8.2. (i) $e(G, C) = 0$ if and only if $|G : C|$ is finite.

(ii) If $G \supset G_1 \supset C$, with $|G : G_1|$ finite, then $e(G, C) = e(G_1, C)$.

(iii) If K is a normal subgroup of G with quotient G_1 and $|K : K \cap C|$ is finite, then $e(G, C) = e(G_1, pC)$, where $p : G \to G_1$ is the natural projection.

(iv) If $C_1 \subset C \subset G$ and $|C : C_1| = n$, then $e(G, C) \leq e(G, C_1) \leq n. e(G, C)$.

These results are all the analogues of results about $e(G)$, except for (iv). One can give examples showing that either equality can be achieved in this part. The final basic property of $e(G, C)$ is

Lemma 8.3. If G splits over C, then $e(G, C) \geq 2$.

Proof. The proof is very similar to that of Lemma 6.3. We consider only the case when $G = A *_C B$. Recall the set E which was the subset of G consisting of elements whose canonical form starts in A. If $b \in B$, then $Eb = E$ and if $a \in A$, then $E \subset Ea \cup Ca \subset E \cup C \cup Ca$. The subset pE of $C\backslash G$ is left almost invariant by every element of A or B, and so is almost invariant in $C\backslash G$. Clearly pE and pE^* are infinite.

This leads us to the first large difference between $e(G)$ and $e(G, C)$. We know that $e(G)$ can only take the values $0, 1, 2$ or ∞, but $e(G, C)$ can take any positive integer value. This is shown by the following example. Note that both G and C are f.g. in this example.

Example. Let F be a closed surface and let X be a compact subsurface so that no component of $F - X$ has closure homeomorphic to a 2-disc. Then the natural map $\pi_1(X) \to \pi_1(F)$ is injective, and we call the groups G and C. Now $e(G, C)$ equals the number of ends of F_C, the covering space of F with fundamental group C. But one knows that X lifts to F_C and can prove easily that F_C consists of X together with half open annuli $S^1 \times [0, \infty)$ attached to each boundary component of X. Thus $e(G, C)$ equals the number of boundary components of X. By choosing F to be of appropriately high genus, one can find pairs (G, C) for which $e(G, C)$ takes any specified value.

Originally I hoped to prove that $e(G, C) \geq 2$ if and only if G splits over some subgroup closely related to C. The following example shows that no such result can hold.

Example. Let M be a closed, orientable irreducible 3-manifold. It can be shown that M is sufficiently large if and only if $\pi_1(M)$ splits over some subgroup. There exists such a 3-manifold which is not sufficiently large, but has a finite covering space which is sufficiently large. See [8] for a discussion of such examples. Thus we have an unsplittable group G with a subgroup G_1 of finite index which splits over some subgroup C. Hence $e(G, C) = e(G_1, C) \geq 2$, but G is unsplittable. (One can find a subgroup C which is finitely generated, so there is nothing pathological about this example.)

This example suggests that one must be content to prove that $e(G, C) \geq 2$ if and only if G has a subgroup G_1 of finite index such that G_1 splits over some subgroup closely related to C. It then seems reasonable that one will need a residual finiteness condition on G.

We say that a group G is residually finite if given $g \in G$, there is $G_1 \subset G$, such that $|G : G_1|$ is finite and $g \notin G_1$. If C is a subgroup of G, we say that G is C-residually finite if given $g \in G - C$ there is $G_1 \subset G$ such that $|G : G_1|$ is finite, $G_1 \supset C$ and $g \notin G_1$. The natural result seems to be the following, which is proved in [21].

Theorem 8.4. If G and C are f.g. groups and G is C-residually finite, then $e(G, C) \geq 2$ if and only if G has a subgroup G_1 of finite index such that G_1 contains C and splits over C.

The residual finiteness condition cannot be omitted.

Example. Let $G = A * C$, where A and C are infinite, simple, f.g. groups. Thus G has no subgroups of finite index and C has no subgroups or supergroups of finite index. Now for any non-trivial free product $A * C$ except for $\mathbb{Z}_2 * \mathbb{Z}_2$, it is easy to show that $e(G, C) = \infty$. But if G had a subgroup G_1 of finite index which split over some subgroup C_1 closely related to C one would be forced to have $G = G_1$ and $C_1 = C$. The example is completed by showing that G cannot split over C.

Lemma 8.5. Let $G = A * C$, where A is indecomposable and not infinite cyclic. Then G cannot split over C.

Proof. Suppose $G = X *_C Y$ or $X *_C$. As no conjugate of A meets C, we see from the Subgroup Theorem that A lies in a conjugate of X or Y. We can suppose X is involved. Use $\langle A \rangle$ to denote the normal closure of A in a group containing A. We know that $G/\langle A \rangle \cong C$. Hence we have the equations $C = X/\langle A \rangle *_C Y$ or $C = X/\langle A \rangle *_C$. The second equation is impossible, and the first equation can only hold when $X/\langle A \rangle = C = Y$. But the equation $C = Y$ contradicts the hypothesis that G splits over C.

REFERENCES

1. C. Bamford and M. J. Dunwoody. On accessible groups, J. of Pure and Applied Algebra 7 (1976), 333-46.
2. R. Brown. Elements of modern topology, McGraw-Hill, New York (1968).
3. I. M. Chiswell. Abstract length functions in groups, Math. Proc. Camb. Phil. Soc. 80 (1976), 451-63.
4. D. E. Cohen. Subgroups of HNN groups, J. Austral. Math. Soc. 17 (1974), 394-405.
5. D. E. Cohen. Groups of cohomological dimension one, Springer Lecture Notes 245 (1972).
6. D. E. Cohen. Groups with free subgroups of finite index, Conference on Group Theory, Springer Lecture Notes 319 (1972), 26-44.
7. M. J. Dunwoody. Accessibility and groups of cohomological dimension one, Proc. London Math. Soc., (3) 38 (1979), 193-215.
8. B. Evans and W. Jaco. Varieties of groups and 3-manifolds, Topology 12 (1973), 83-97.
9. H. Freudenthal. Über die Enden topologischer Raüme und Gruppen, Math. Zeit. 33 (1931), 692-713.
10. H. Freudenthal. Neuaufbau der Endentheorie, Ann. of Math. 43 (1942), 261-79.

11. H. Freudenthal. Über die Enden diskreter Raüme und Gruppen, Comm. Math. Helv. 17 (1944), 1-38.

12. P. J. Higgins. Gruško's theorem, J. Algebra 4 (1966), 365-72.

13. G. Higman, B. H. Neumann and H. Neumann. Embedding theorems for groups, J. London Math. Soc. 24 (1949), 247-54.

14. H. Hopf. Enden offener Raüme und unendliche diskontinuierliche Gruppen, Comm. Math. Helv. 16 (1943), 81-100.

15. A. Karrass, A. Pietrowski and D. Solitar. An improved subgroup theorem for HNN groups with some applications, Canad. J. Math. 26 (1974), 214-24.

16. A. Karrass, A. Pietrowski and D. Solitar. Finitely generated groups with a free subgroup of finite index, J. Austral. Math. Soc. 16 (1973), 458-66.

17. A. G. Kuroš. Zum Zerlegungsproblem der Theorie der freien Produkte, Mat. Sb. 44 (1937), 995-1001.

18. W. S. Massey. Algebraic Topology: An Introduction, Harcourt and Brace, New York (1967).

19. C. D. Papakyriakopoulos. On Dehn's lemma and the asphericity of knots, Ann. of Math. 66 (1957), 1-26.

20. G. P. Scott. An embedding theorem for groups with a free subgroup of finite index, Bull. London Math. Soc. 6 (1974), 304-6.

21. G. P. Scott. Ends of pairs of groups, J. of Pure and Applied Algebra 11 (1977), 179-98.

22. J.-P. Serre. Arbres, amalgames, SL_2, Astérisque 46 (1977).

23. J.-P. Serre. Amalgames et points fixes, Canberra Conference on Group Theory, Springer Lecture Notes in Math. 372 (1974), 633-40.

24. E. Specker. Endenverbände von Raüme und Gruppen, Math. Annalen 122 (1950), 167-74.

25. J. R. Stallings. A topological proof of Gruško's theorem on free products, Math. Zeit. 90 (1965), 1-8.

26. J. R. Stallings. Group theory and three dimensional manifolds, Yale Univ. Press, New Haven and London (1971).

27. R. G. Swan. Groups of cohomological dimension one, J. Algebra 12 (1969), 585-610.

28. C. T. C. Wall. Pairs of relative cohomological dimension one, J. Pure Appl. Algebra 1 (1971), 141-54.

29. J. H. C. Whitehead. On 2-spheres in 3-manifolds, Bull. Amer. Math. Soc. 64 (1958), 161-6.

30. I. M. Chiswell. Exact sequences associated with a graph of groups, J. Pure Applied Alg. 8 (1976), 63-74.

31. A. J. Goalby. Ends, M. Sc. dissertation, Department of Pure Mathematics, University of Liverpool (1975).

32. P. J. Higgins. The fundamental groupoid of a graph of groups, J. London Math. Soc. 13 (1976), 145-9.

33. C. H. Houghton. Ends of locally compact groups and their quotient spaces, J. Aust. Math. Soc. 17 (1974), 274-84.

34. R. C. Lyndon. Length functions in groups, Math. Scand. 12 (1963), 209-34.

35. H. Neumann. Generalized free products with amalgamated subgroups I, II, Amer. J. Math. 70 (1948), 590-625; 71 (1949), 491-540.

36. O. Schreier. Die Untergruppen der freien Gruppen, Abh. Math. Sem. Hamburg 5 (1927), 161-83.

37. J. R. Stallings. On torsion-free groups with infinitely many ends, Ann. of Math. 88 (1968), 312-34.