Math 444/539 Lecture 9

$f: X \to Y$

$\gamma: I \to X$ path from p to q

Define $f_*(\gamma) = f \circ \gamma$

Facts:

a) $f_*(\gamma)$ path from $f(p)$ to $f(q)$

b) $\gamma \sim \gamma' \Longrightarrow f_*(\gamma) = f_*(\gamma')$

c) γ_1 path from p to q, γ_2 path from q to r

$\Longrightarrow f_*(\gamma_1 \cdot \gamma_2) = f_*(\gamma_1) \cdot f_*(\gamma_2)$

Conclude: For $p \in X$, f induces homom.

$$f_*: \pi_1(X, p) \to \pi_1(Y, f(p))$$

Properties of f_*

a) $f: X \to Y$, $g: Y \to Z$

$\Longrightarrow (g \circ f)_* = g_* \circ f_*$

b) $i: X \to X$ identity

$\Longrightarrow i_* =$ identity

$\Bigg\}$ π_1 is a "functor"

Def'n: $f, g: X \to Y$ are homotopic if $\exists F: X \times I \to Y$ s.t.

$$F(x,0) = f(x), \quad F(x,1) = g(x)$$

Denote by $f \sim g$

Rmk: Setting $f_t(x) = F(x,t)$, think of f_t as "continuous family" of maps from $f_0 = f$ to $f_1 = g$. In other words, f can be deformed to g

$f \simeq g$ almost implies $f_* = g_*$

Problem: homotopy might move basepoint

Defn: $f, g : X \to Y$ are homotopic rel $A \subseteq X$ if $\exists F : X \times I \to Y$ st.

$F(x,0) = f(x)$, $F(x,1) = g(x)$, and

$F(a,t) = F(a,t')$ for $a \in A$, $t, t' \in I$.

Write $f \simeq g$ rel A

Rmk: $f \simeq g$ rel $A \Rightarrow f(a) = g(a)$ for $a \in A$

Lemma: $f, g : X \to Y$, $p \in X$

$f \simeq g$ rel $\{p\} \Rightarrow f_* = g_*$

pf:

Obvious ☐

Defn: $A \subseteq X$ a retract of X if $\exists r : X \to A$ st $r|_A = id$

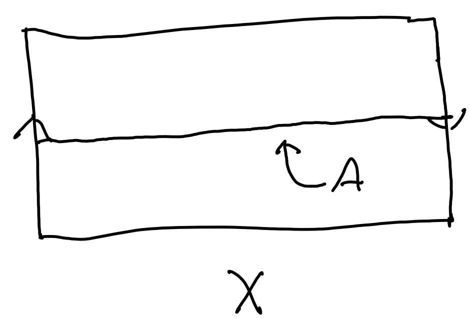
a retraction

Ex: $X =$ Möbius band

$= I \times I / \sim$

$A =$ "central circle" of X

$= I \times \{1/2\} / \sim$



Then A retract of X via retraction

$r : X \to A$

$r(s,t) = (s, 1/2)$ for $(s,t) \in I \times I / \sim$

compatible w/ $\sim$

Thm A: There does not exist a retraction from D^2 to $\partial D^2 \cong S^1$.

For pf, need following, which will be proven next time:

Thm B: $\pi_1(S^1, p) \cong \mathbb{Z}$ for all $p \in S^1$

pf of Thm A:

Asm $f: D^2 \to \partial D^2$ is retraction
Let $i: \partial D^2 \hookrightarrow D^2$ be inclusion
Pick $p \in \partial D^2$, so $i(p) = f(p) = p$
Have $Id_{\partial D^2} = f \circ i$
On π_1, get that $Id: \underbrace{\pi_1(S^1, p)}_{\cong \mathbb{Z}} \to \underbrace{\pi_1(S^1, p)}_{\cong \mathbb{Z}}$
factors as

$$\underbrace{\pi_1(S_1, p)}_{\cong \mathbb{Z}} \xrightarrow{i_*} \underbrace{\pi_1(D^2, p)}_{\cong 1} \xrightarrow{f_*} \underbrace{\pi_1(\partial D^2, p)}_{\cong \mathbb{Z}}$$

Contradiction

$\boxed{///}$

Cor (Brouwer Fixed pt thm):
$f: D^2 \to D^2$ continuous
$\Rightarrow \exists x \in D^2$ s.t. $f(x) = x$

Rmk: a) True for D^n (n arbitrary)
Can be proved w/ either π_n or homology

b) Exercise: Prove for $n=1$ using intermediate value thm

Proof of Brouwer fixed pt thm

Asm $f: D^2 \to D^2$ satisfies $f(x) \neq x \ \forall x$.

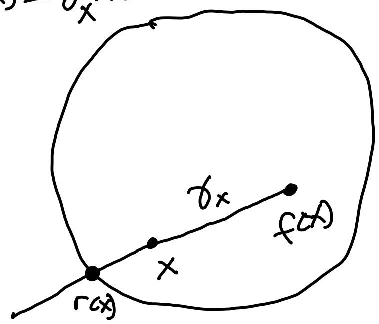
Will construct retraction $r: D^2 \to \partial D^2$, contradicting above thm

Construction of r:

$x \in D^2$

$f(x) \neq x \Rightarrow$ Can form ray γ_x starting at $f(x)$ and passing through x

Define $r(x) = \gamma_x \cap \partial D^2$

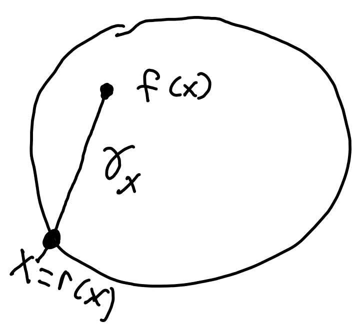


Claim: r continuous

Indeed, can easily find formula for $r(x)$ in terms of x and $f(x)$

Claim: r retraction

$x \in \partial D^2 \Rightarrow \gamma_x \cap \partial D^2 = x$, so $r(x) = x$



$\blacksquare$

Defn: $A \subseteq X$ a deformation retract if $Id_X \simeq r$ rel A for some retraction $r: X \to A$

Ex: $S^n \subseteq \mathbb{R}^{n+1}$ a deformation retract

Define

$$F: \mathbb{R}^{n+1} \times I \to \mathbb{R}^{n+1}$$
$$F(x,t) = (1-t)x + t\frac{x}{\|x\|}$$

Then

$$F(x,0) = x$$
$$F(x,1) = \frac{x}{\|x\|} \in S^n$$
$$F(x,1) = x \text{ if } x \in S^n$$