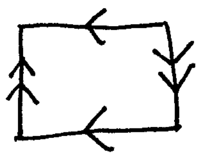
Math 444/539 Lecture 7

Goals

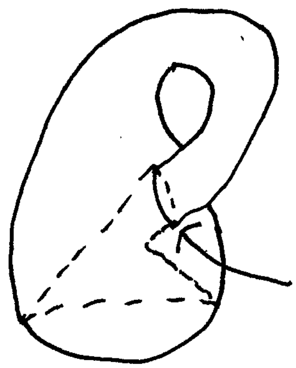
a) Prove $T^2 \# \mathbb{RP}^2 \cong \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$

b) Classify surfaces w/ bdry

Defn: The **Klein bottle** is

$K^2 =$ 

Picture: Can't draw K^2 in $\mathbb{R}^3$ w/o self-intersections. Need $\mathbb{R}^4$.

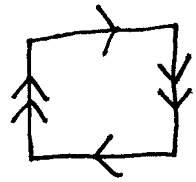


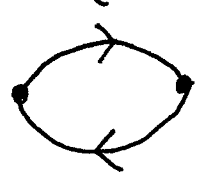
Must use 4th dim to avoid self-intersections here.

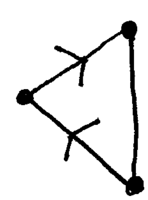
Lemma: $K^2 \cong \mathbb{RP}^2 \# \mathbb{RP}^2$

pf:

Recall that

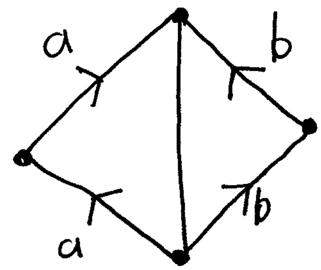
$\mathbb{RP}^2 \cong$ 

$\cong$ 

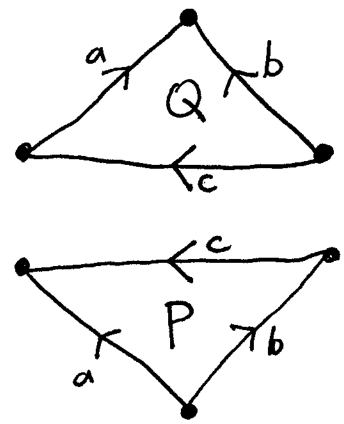
, $\mathbb{RP}^2 \setminus \text{disc} \cong$ 

Hence

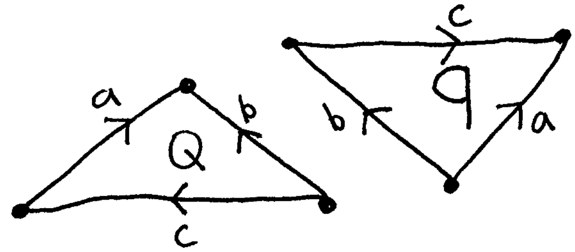
$\mathbb{RP}^2 \# \mathbb{RP}^2 \cong$



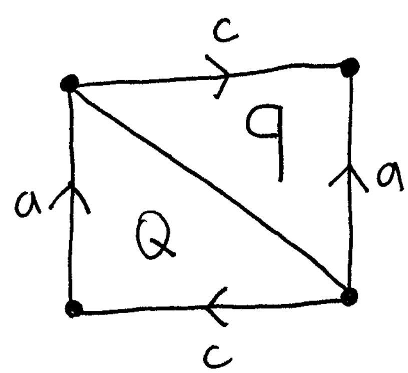
$\cong$



$\cong$



$\cong$



$\cong K^2$ ∎

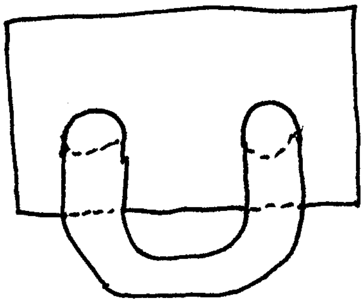
Goal a thus follows from:

Thm: $T^2 \# \mathbb{RP}^2 \cong K^2 \# \mathbb{RP}^2$

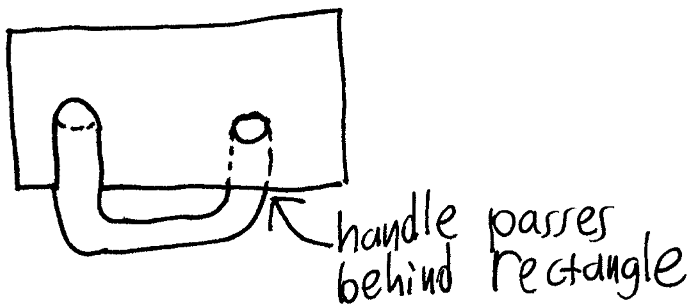
pf:

Observe:

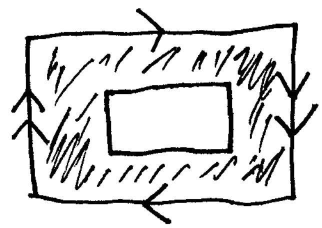
$T^2 \setminus \text{disc} \cong$



$K^2 \setminus \text{disc} \cong$

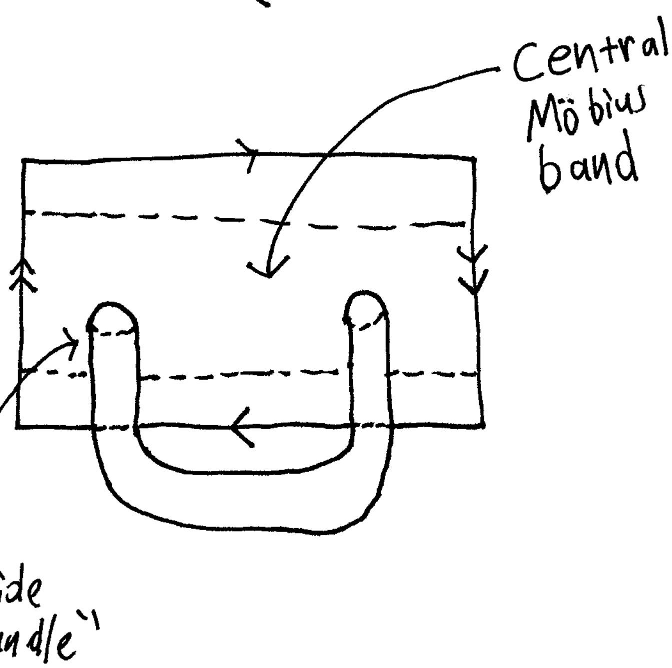


$\mathbb{RP}^2 \setminus \text{disc} \cong$

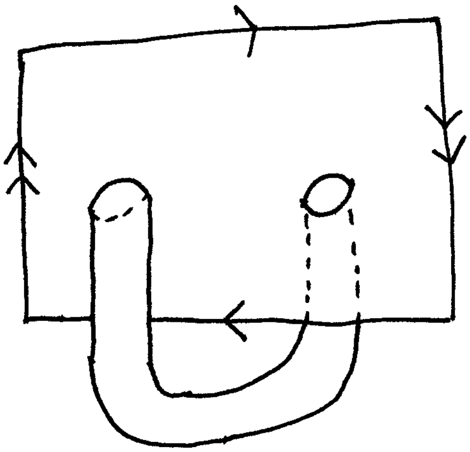


Hence

$T^2 \# \mathbb{RP}^2 \cong$



Drag left side of "handle" around central Möbius band, get that this is homeo. to



$\cong K^2 \# \mathbb{RP}^2$

Since Möbius band has "twist". $\blacksquare$

Surfaces w/ Bdry

<u>Def'n</u>: An n-<u>manifold w/ boundary</u> is a 2^{nd} countable Hausdorff space X s.t. for all $p \in X$, there exists a nbhd U of p s.t. one of the following holds:
called <u>chart</u>.

a) $U \cong \mathring{D}^n = \{\vec{x} \in \mathbb{R}^n \mid \sum x_i^2 < 1\}$

b) $U \cong \{\vec{x} \in \mathbb{R}^n \mid \sum x_i^2 < 1 \text{ and } x_n \geq 0\}$; call this latter set $\mathring{D}^n_+$.

Vocabulary: Let X be n-mnfld w/ bdry

a) $p \in X$ is interior pt if p has nbhd U w/ $U \cong \mathring{D}^n$

Define

$$\mathrm{Int}(X) = \{p \in X \mid p \text{ interior pt}\}.$$

Rmk: This is different from point-set topology def'n of interior.

b) $p \in X$ is boundary pt if p is not interior pt

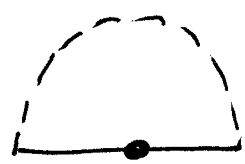
Define

$$\partial X = \{p \in X \mid p \text{ boundary pt}\}.$$

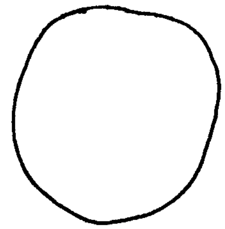
Remarks:

a) A manifold is a manifold w/ boundary X s.t. $\partial X = \emptyset$; conversely, if X is a manifold w/ boundary and $\partial X \neq \emptyset$, then X is not a manifold.

b) If $p \in \partial X$, then p has nbhd U + homeo. $\varphi : U \to \mathring{D}^n_+$ s.t. $\varphi(p) = \vec{0}$.
It's true (but annoying to prove) that conversely if such a φ exists, then p is not an interior pt. In particular, the point $\vec{0} \in \mathring{D}^n_+$ has no nbhd V w/ $V \cong \mathring{D}^n$.

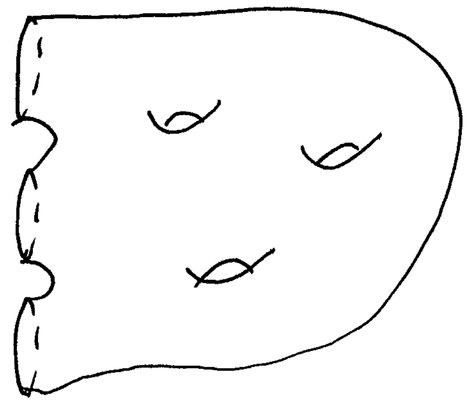


Ex: a) D^n is n-mnfld w/ boundary.
$\mathrm{Int}(D^n) = \mathring{D}^n$
$\partial D^n \cong S^{n-1}$



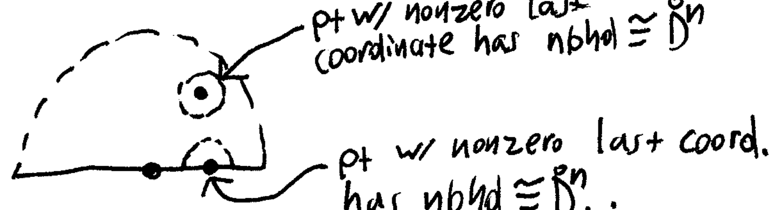
b) Σ cpt surface, $B \subseteq X$ subspace w/ $B \cong D^2$.
Then $\Sigma \setminus \mathrm{Int}(B)$ is mnfld w/ boundary

c) Can also remove multiple discs



Lemma: X mnfld w/ boundary $\Rightarrow \partial X$ is $(n-1)$-mnfld.

pf:
$p \in \partial X$. Let $\varphi : U \to \mathring{D}^n_+$ be chart w/ $\varphi(p) = 0$.
Then a pt $x \in \mathring{D}^n_+$ is image of bdry pt iff $x_n = 0$.



$\Rightarrow \varphi$ maps $U \cap \partial X$ homeo. onto
$\{\vec{x} \in \mathring{D}^n_+ \mid x_n = 0\} \cong B^{n-1}$

$\Rightarrow U \cap \partial X \cong \mathring{D}^{n-1}$ is chart for $p \in \partial X$ $\blacksquare$

<u>Cor</u>: Σ cpt surface w/ boundary
$\Rightarrow \partial\Sigma$ disjoint union of finite # of S^1's.

pf:
∂X compact 1-manifold $\blacksquare$

<u>Thm</u>: Σ_1, Σ_2 cpt surfaces w/ bdry

$\Sigma_1 \cong \Sigma_2 \iff$
a) both orientable or not orientable
b) $\chi(\Sigma_1) = \chi(\Sigma_2)$
c) both have same # of bdry cpts.

pf:
$\Rightarrow$: trivial

$\Leftarrow$: Let $\hat{\Sigma}_i$ be Σ_i w/ discs glued to all bdry cpts.

$\hat{\Sigma}_i$ cpt surface (without bdry)

<u>Claim</u>: $\chi(\hat{\Sigma}_i) = \chi(\Sigma_i) + n$, where n is # of boundary cpts.

Triangulate Σ_i

Then $\hat{\Sigma}_i$ obtained by adding 2-cell glued to each bdry cpt, so $\chi(\hat{\Sigma}_i) = \chi(\Sigma_i) + n$.

<u>Conclude</u>: $\chi(\hat{\Sigma}_1) = \chi(\hat{\Sigma}_2)$.

Since $\hat{\Sigma}_1$ + $\hat{\Sigma}_2$ either both orientable or both not orientable, classification of cpt surfaces $\Longrightarrow$ $\exists$ homeo. $\varphi: \hat{\Sigma}_1 \to \hat{\Sigma}_2$

Need following annoying lemma, whose proof is omitted:

<u>Lemma</u>: S cpt surface

$B_1, \ldots, B_n \subseteq S$ disjoint subsets w/ $B_i \cong D^2$

$B_1', \ldots, B_n' \subseteq S$ disjoint subsets w/ $B_i' \cong D^2$

$\Longrightarrow$ $\exists$ homeo $\psi: S \to S$ w/

$\psi(B_i) = B_i'$ for $1 \leq i \leq n$.

Lemma $\Longrightarrow$ we can assume that φ takes discs glued to bdry cpts of Σ_1 to discs glued to bdry cpts of Σ_2

Hence $\varphi|_{\Sigma_1}$ is homeo from Σ_1 to Σ_2

$\blacksquare$