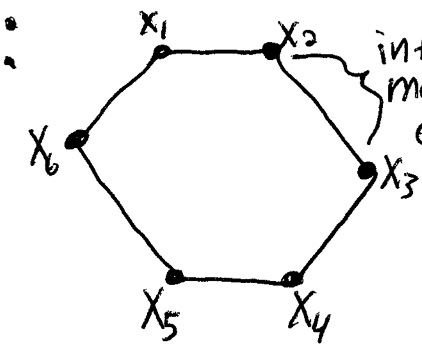
Math 444/539 Lecture 4

<u>Defn</u>: A CW cpx X is <u>regular</u> if for all cells D^k_α, the char map $\Phi^k_\alpha : D^k_\alpha \to X$ is embedding
(equiv., the attaching map is injective)

X = 2d regular CW-cpx
D^2_α = 2-cell w/ char map $\Phi^2_\alpha : D^2_\alpha \to X$
$\Longrightarrow (\Phi^2_\alpha)^{-1}(X^{(0)}) = \{x_1, \dots, x_n\} \subseteq \partial D^2_\alpha \cong S^1$

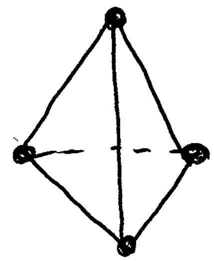
Call D^2_α an <u>n-gon</u>:

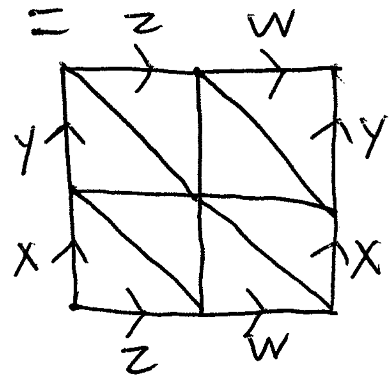


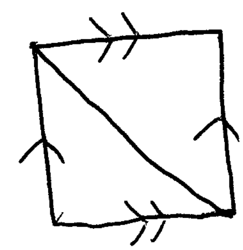
interior of edge maps homemorphically onto interior of 1-cell.

<u>Defn</u>: M^2 2-manifold w/ CW-cpx structure.
That CW-cpx structure is a <u>triangulation</u> if
a) it is regular
b) All 2-cells are triangles (ie 3-gons).

Thm (Radó): All surfaces can be triangulated.

Ex: $S^2 =$ 

Ex: $T^2 =$ 

Rmk:  is not a triangulation; 2-cells don't embed.

Recognizing 2-mflds

Defn: X = CW cpx w/ finitely many cells

c_i = # of i-cells of X $(0 \le i < \infty)$

The Euler characteristic of X is

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i c_i = c_0 - c_1 + c_2 - \cdots$$

Thm (Math 445): X, Y CW-cx's w/ finitely many cells

$$X \cong Y \Rightarrow \chi(X) = \chi(Y).$$

Ex: S^n has 2 CW-cx structures

a) 1 0-cell
1 n-cell

$$\Rightarrow \chi(S^n) = 1 + (-1)^n = \begin{cases} 0 & n \text{ odd} \\ 2 & n \text{ even} \end{cases}$$

b) 2 0-cells
2 1-cells
⋮ ⋮
2 n-cells

$$\Rightarrow \chi(S^n) = \overbrace{2 - 2 + 2 - 2 + \cdots}^{n+1 \text{ terms}}$$

$$= \begin{cases} 0 & n \text{ odd} \\ 2 & n \text{ even} \end{cases}$$

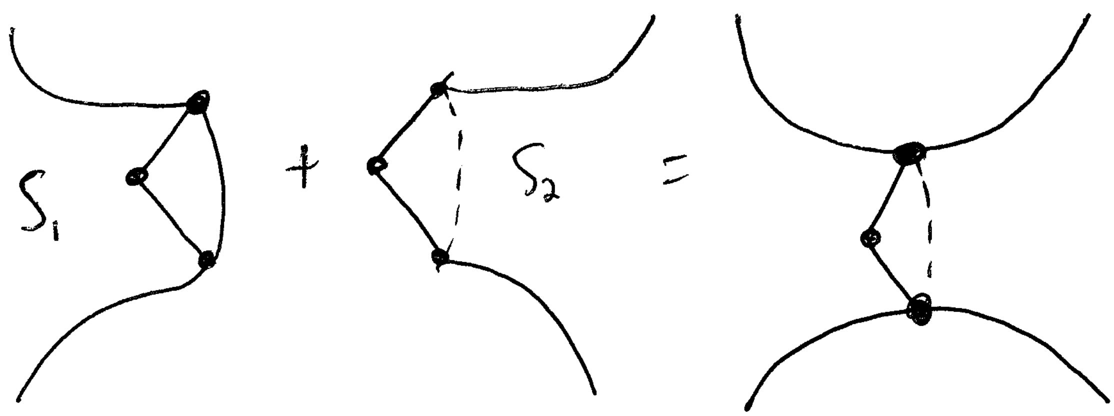
<u>Lemma</u>: S_1, S_2 cpt surfaces

$$\Longrightarrow \chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$$

pf:

Choose triangulation for S_i w/ V_i 0-cells, e_i 1-cells, f_i 2-cells.

Delete a 2-cell from S_1 & S_2 and glue boundaries of deleted triangles together to get $S_1 \# S_2$:



$$\chi(S_1 \# S_2) = (v_1 + v_2 - 3) - (e_1 + e_2 - 3) + (f_1 + f_2 - 2)$$

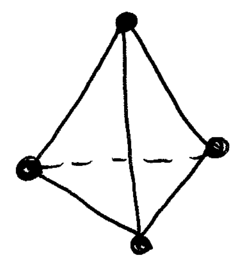
$$= (V_1 - e_1 + f_1) + (V_2 - e_2 + f_2) - 2$$

$$= \chi(S_1) + \chi(S_2) - 2$$

$\boxed{}$

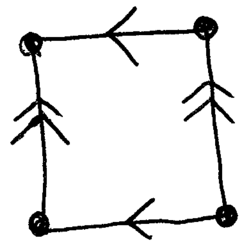
Calculations

a) $\chi(S^2) = 2$



$4 - 6 + 4 = 2$

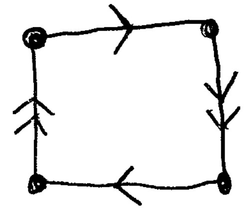
b) $\chi(T^2) = 0$



$1 - 2 + 1 = 0$

c) $\chi(\underbrace{T^2 \# \cdots \# T^2}_{g\ T^2\text{'s}}) = 2 - 2g$

d) $\chi(\mathbb{R}P^2) = 1$



$2 - 2 + 1 = 1$

e) $\chi(\underbrace{\mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2}_{g\ \mathbb{R}P^2\text{'s}}) = 2 - g$

Observation: Euler char can distinguish the $T^2 \# \cdots \# T^2$'s. It can also distinguish the $\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$'s. However, it can't always distinguish the $T^2 \# \cdots \# T^2$'s from the $\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$'s.

Orientability in 2-mnflds

Informal: A 2-mnfld is orientable if it has consistent notion of left vs right.

Locally, 2-mnfld is $\mathbb{R}^2$, so left vs right makes sense in small regions.
However, $\exists$ global obstructions.

Ex: Möbius strip <u>not</u> orientable

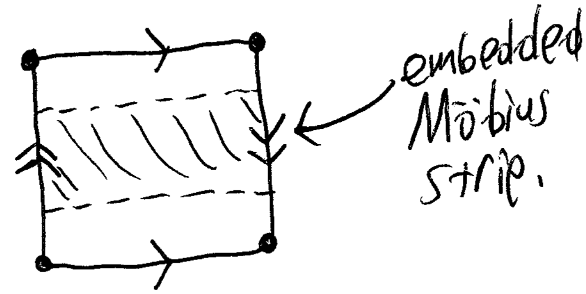
If you walk around strip, your notions of left vs right get reversed!

Turns out this is only problem, so

<u>Def'n</u>: A 2-mnfld is <u>orientable</u> if it has no embedded Möbius strip.

<u>Facts</u>: a) T^2 orientable

b) $\mathbb{RP}^2$ not orientable



c) $S_1, \dots, S_n$ orientable
$\Longrightarrow S_1 \# S_2 \# \cdots \# S_n$ orientable

d) S arbitrary, T not orientable
$\Longrightarrow S \# T$ not orientable.

Proofs of a, c, + d omitted. (not hard, just long; also, will have more enlightening proofs available after Math 445).

From facts:

$T^2 \# \cdots \# T^2$ orientable

$\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$ not orientable

Conclude:

Thm: S_1, S_2 cpt surfaces

$S_1 \cong S_2 \iff \chi(S_1) = \chi(S_2)$ and either both S_1 & S_2 are orientable or S_1 & S_2 are not orientable.