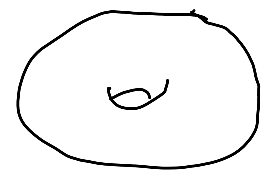
Math 444/539 Lecture 20

Recall from last time:

$GCD(m,n)=1 \Longrightarrow$ elt $(m,n) \in \mathbb{Z}^2 \cong \pi_1(T^2)$ can be realized by simple closed curve

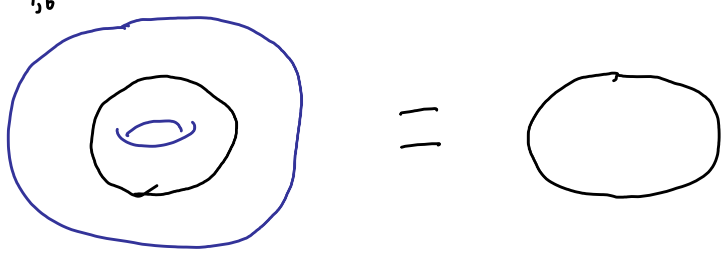
Let $\gamma_{m,n}: S^1 \to T^2$ be embedding of (m,n)-curve

Let $i: T^2 \hookrightarrow \mathbb{R}^3$ be std embedding:

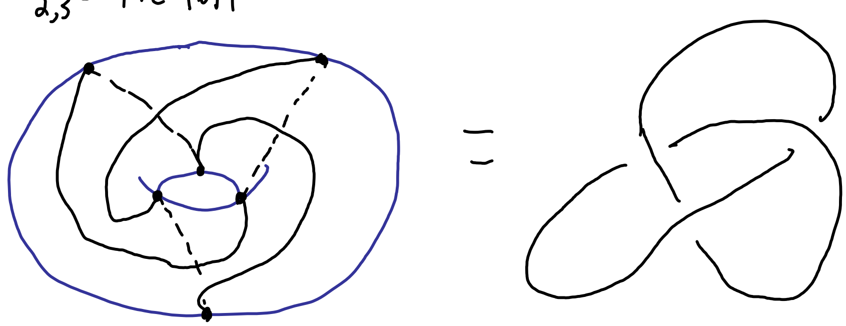


The <u>(m,n)-torus knot</u> $T_{m,n}$ is knot $i \circ \gamma_{m,n}$.

<u>Ex</u>: a) $T_{1,0}$ = unknot



b) $T_{2,3}$ = trefoil



Goal today:

<u>Thm</u>: $T_{m,n}$ equiv to $T_{m',n'} \Longrightarrow$ either $(m,n)=(m',n')$ or $(m,n)=(n',m')$

<u>Cor</u>: The trefoil is <u>not</u> equiv. to the unknot

For integers m, n, define
$$\Gamma_{m,n} = \langle \alpha, \beta \mid \alpha^m = \beta^n \rangle$$

<u>Prop</u>: $\pi_1(S^3 \setminus T_{m,n}) \cong \Gamma_{m,n}$

pf:
Proof is clearer if we ignore requirement in SvK for $\{U_\alpha\}$ to be open — to make it rigorous, just "thicken" everything to open sets that def. retract onto sets we define.

Write $S^3 = X_1 \cup X_2$, where
- $X_i \cong D^2 \times S^1$
- $X_1 \cap X_2 = T^2 \subseteq S^3$ embedded in std way, so $T_{m,n} \subseteq X_1 \cap X_2$

Set $Y_i = X_i \setminus T_{m,n}$. Then
- $S^3 \setminus T_{m,n} = Y_1 \cup Y_2$
- Y_i def retracts onto $\{0\} \times S^1$, so $\pi_1(Y_i) \cong \mathbb{Z}$
- $Y_1 \cap Y_2 = T^2 \setminus T_{m,n}$ path connected

SvK $\Rightarrow \pi_1(S^3 \setminus T_{m,n}) \cong \pi_1(Y_1) * \pi_1(Y_2) / R$

$T_{m,n}$ does not separate T^2, so Euler char
$\Rightarrow T^2 \setminus T_{m,n} \cong S^1 \times (0,1)$, and hence
$\pi_1(T^2 \setminus T_{m,n}) \cong \mathbb{Z}$.

Let $r \in \pi_1(T^2 \setminus T_{m,n})$ & $\alpha \in \pi_1(Y_1)$ & $\beta \in \pi_1(Y_2)$ be generators

Let $\varphi_i : \pi_1(T^2 \setminus T_{m,n}) \to \pi_1(Y_i)$ be induced maps, and let

$\varphi_1(r) = \alpha^k$ and $\varphi_2(r) = \beta^l$

SvK then says that

$\pi_1(S^3 \setminus T_{m,n}) \cong \langle \alpha, \beta \mid \alpha^k = \beta^l \rangle$

Thus must prove:

Claim: $k = m$ and $l = n$

$r \in \pi_1(T^2 \setminus T_{m,n})$ is loop parallel to $T_{m,n}$, so it wraps m times around T^2 in one direction and n times in other. Claim follows. $\square$

Thus to prove goal thm from page 1, enough to prove

Prop: $\Gamma_{m,n} \cong \Gamma_{m',n'} \Rightarrow$ either $(m,n) = (m',n')$ or $(m,n) = (n',m')$

pf:

Recall that if G a grp, then

$Z(G) =$ center of G

$= \{x \in G \mid xy = yx \ \forall y \in G\}$

$\Gamma_{m,n} \cong \Gamma_{m',n'} \Rightarrow \Gamma_{m,n}/Z(\Gamma_{m,n}) \cong \Gamma_{m',n'}/Z(\Gamma_{m',n'})$

Claim: $\Gamma_{m,n}/Z(\Gamma_{m,n}) \cong \mathbb{Z}/m * \mathbb{Z}/n$

α^m commutes w/ α + β (since $\alpha^m = \beta^n$), so $\alpha^m \in Z(\Gamma_{m,n})$

$$\Gamma_{m,n}/\langle \alpha^m \rangle \cong \langle \alpha, \beta \mid \alpha^m = \beta^n, \alpha^m \rangle$$
$$\cong \langle \alpha, \beta \mid \alpha^m = 1, \beta^n = 1 \rangle$$
$$\cong \mathbb{Z}/m * \mathbb{Z}/n$$

From HW: $Z(\mathbb{Z}/m * \mathbb{Z}/n) = 1$

Hence if $x \in Z(\Gamma_{m,n})$, then x projects to 1 in $\Gamma_{m,n}/\langle \alpha^m \rangle$; ie $x \in \langle \alpha^m \rangle$.

Claim: $\mathbb{Z}/m * \mathbb{Z}/n \cong \mathbb{Z}/m' * \mathbb{Z}/n' \Longrightarrow$ either $(m,n) = (m',n')$ or $(m,n) = (n',m')$

Define
$$\mathcal{T}_{m,n} = \{ k \mid \text{exists cyclic subgrp of } \mathbb{Z}/m * \mathbb{Z}/n \text{ of order } k \}$$
Have $\mathcal{T}_{m,n} = \mathcal{T}_{m',n'}$

From HW, any torsion elt of $\mathbb{Z}/m * \mathbb{Z}/n$ is conjugate into $\mathbb{Z}/m$ or $\mathbb{Z}/n$.
$\Longrightarrow \mathcal{T}_{m,n} = \{1, m, n\}$ ▨

Rmk: Can show that torus knots only knots whose groups have nontrivial centers.