Math 444/539 Lecture 2

<u>Def</u> (gluing): X, Y spaces
$A \subseteq X$ subspace
$f: A \to Y$ map.

Then <u>X glued to Y using f</u> is quotient of $X \sqcup Y$ by $\mathcal{I} = \{E_a \mid a \in A\}$ w/ $E_a = \{a, f(a)\}$.

Denoted $(X \sqcup Y)/_{a \in A \sim f(a)}$

<u>Thm</u> (gluing fcns). X, Y spaces, $A \subseteq X$, $f: A \to Y$.
Asm have fcns $\varphi_1: X \to Z$ and $\varphi_2: Y \to Z$
s.t. diagram

$$\begin{tikzcd} & X \arrow[dr, "\varphi_1"] & \\ A \arrow[ur, hook, "\text{inclusion}"] \arrow[dr, "f"'] & & Z \\ & Y \arrow[ur, "\varphi_2"'] & \end{tikzcd}$$

commutes. Then $\exists! \, \Psi: (X \sqcup Y)/_{a \in A \sim f(a)} \to Z$

s.t.

$$\begin{tikzcd} & X \arrow[dl, bend right] \arrow[dr, "\varphi_1"] & \\ (X \sqcup Y)/_{a \in A \sim f(a)} \arrow[rr, "\Psi"] & & Z \\ & Y \arrow[ul, bend left] \arrow[ur, "\varphi_2"'] & \end{tikzcd}$$

commutes

pf:
Immediate from univ property of quotient maps $\square$

Ex: $X = Y = D^2$

$A \subseteq X$ the bdry S^1

$f : A \to Y$ "obvious" identification of bdry

Claim: $X \sqcup Y /_{a \in A \sim f(a)} \cong S^2$

Define

$\varphi_1 : X \to S^2$

$\varphi_1(x,y) = (x, y, \sqrt{1-x^2-y^2})$

$\varphi_2 : Y \to S^2$

$\varphi_2(x,y) = (x, y, -\sqrt{1-x^2-y^2})$

Clearly

$$\begin{tikzcd}
& X \arrow[dr, "\varphi_1"] & \\
A \arrow[ur, hook] \arrow[dr, "f"'] & & S^2 \\
& Y \arrow[ur, "\varphi_2"'] &
\end{tikzcd}$$

Commutes. Gluing fcns lemma gives fcn $\Psi : (X \sqcup Y)/_{(a \sim f(a))} \to S^2$.

Easy to check: Ψ homeo.

Defn: A <u>CW-complex</u> is a space X w/ filtration

$$X^{(0)} \subseteq X^{(1)} \subseteq \cdots \subseteq X$$

s.t.

a) $\bigcup_n X^{(n)} = X$

b) $U \subseteq X$ open $\iff U \cap X^{(n)}$ open $\forall n$.

c) $X^{(n)}$ constructed inductively

i) $X^{(0)}$ = discrete set of pts

ii) For $n \geq 1$, $\exists$ n-discs $\{D^n_\alpha\}$ and fcns $\varphi^n_\alpha : \underset{\cong S^{n-1}}{\partial D^n_\alpha} \to X^{(n-1)}$ s.t.

$$X^{(n)} = \left(X^{(n-1)} \cup \left(\bigsqcup_\alpha D^n_\alpha\right)\right) \Big/ p \in \partial D^n_\alpha \sim \varphi^n_\alpha(p)$$

Vocabulary

a) $X^{(n)}$ is <u>n-skeleton</u>

b) D^n_α is <u>n-cell</u>

c) φ^n_α is <u>attaching map</u>

d) Map $D^n_\alpha \to X^{(n)} \hookrightarrow X$ is <u>characteristic map</u>

<u>Rmk:</u> char. maps injective on $\mathrm{Int}(D^n_\alpha)$, but not necessarily injective on ∂D^n_α

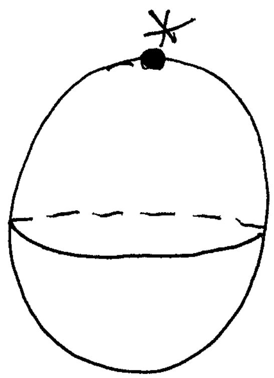
e) X is <u>n-dim'l</u> if $X^{(n)} = X$
(ie if no m-cells for $m > n$).

f) Subspace $Y \subseteq X$ is <u>subcomplex</u> if
Y union of cells

<u>Rmk</u>: Subcpx inherits CW-cpx structure

<u>Ex</u>: $X^{(n)} \subseteq X$ always a subcpx.

<u>Ex</u>: CW-cpx structure on S^n



$X^{(0)} =$ one pt $*$.

one n-cell D_1^n w/ attaching map
$\varphi_1^n : \partial D_1^n \to X^{(0)}$
$\varphi_1^n(x) = *$

no m-cells for $m \neq 0, n$.

<u>Rmk</u>: W/ this CW-structure, $S^m \subseteq S^n$ for $m < n$ <u>not</u> a subcpX.

<u>Ex</u>: CW-cpx structure on S^n II. (cf example on p. 2)

Will define CW-cpx X s.t.

a) $X^{(k)} \cong S^k$ for $0 \leq k \leq n$.

b) X has 2 k-cells for $0 \leq k \leq n$

Set
$X^{(0)} = 2$ pts
$\cong S^0$

For $1 \leq k < n$, assm $X^{(k)} \cong S^k$ constructed.

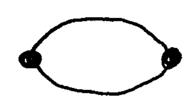
Let $X^{(k+1)} = X^{(k)} + 2$ $(k+1)$ cells D_1^k & D_2^k w/ attaching maps obvious homeomorphisms

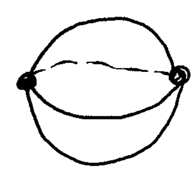
$$\varphi_i^k : \underset{\cong S^{k-1}}{\partial D_i^k} \longrightarrow \underset{\cong S^{k-1}}{X^{(k-1)}}$$

(D_1^k "upper hemisphere", D_2^k "lower hemisphere")

Like in ex on p. 2, can then prove that $X^{(k+1)} \cong S^{k+1}$.

$k=0$: 

$k=1$: 

$k=2$: 

Ex: S^∞ = CW-cpx w/ $(S^\infty)^{(n)}$ = CW-structure on S^n from previous example.

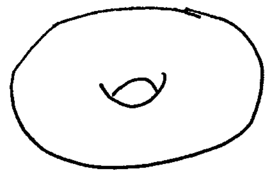
$\Longrightarrow$ As space,

$$S^\infty = \bigcup_{n=0}^{\infty} S^n$$

w/ $S^n \subseteq S^{n+1}$ "equator" and topology

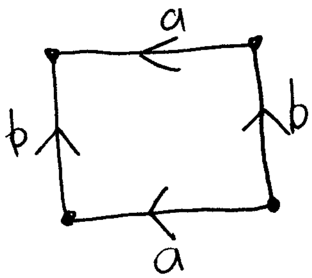
$$U \subseteq S^\infty \text{ open} \iff U \cap S^n \text{ open } \forall n.$$

Ex: T^2 = torus

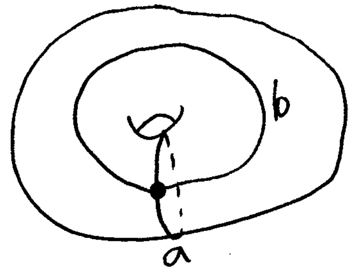


$= S^1 \times S^1$

= square w/ sides glued together:



All vertices of square identified:



Above gives CW-cpx structure w/

1 0-cell: •

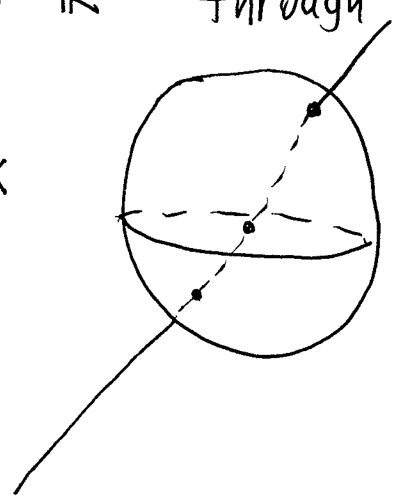
2 1-cells: a & b

1 2-cell: square.

Ex: $\mathbb{R}P^n$ = "lines in $\mathbb{R}^{n+1}$ through 0"

$= S^n/_{x \sim -x}$

line determined by 2 antipodal pts on S^n



$$= D^n \Big/ x \in \partial D^n \cong S^{n-1} \sim -x$$

$$= \mathbb{R}P^{n-1} \cup D^n$$

every line determined by isection w/ upper hemisphere

<u>Conclusion</u>: $\mathbb{R}P^n$ has CW-structure w/ 1 cell in each dim & $(\mathbb{R}P^n)^{(k)} \cong \mathbb{R}P^k$ for $0 \le k \le n$.

<u>Ex</u>: Like for spheres, can define

$$\mathbb{R}P^\infty = \bigcup_{n=0}^{\infty} \mathbb{R}P^n$$