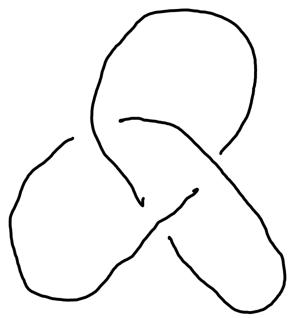
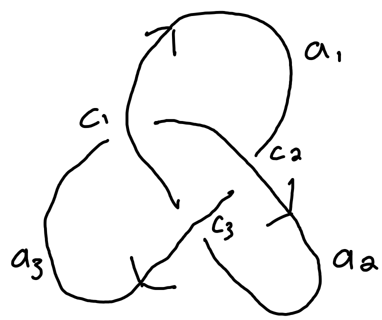
Math 444/539 Lecture 19

Goal: Construct Wirtinger presentation for knot group

Consider a knot $K \subseteq S^3$ w/ a knot diagram

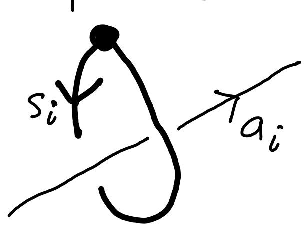


Orient knot and label arcs in diagram $\{a_1, \dots, a_k\}$ and crossings $c_1, \dots, c_k$

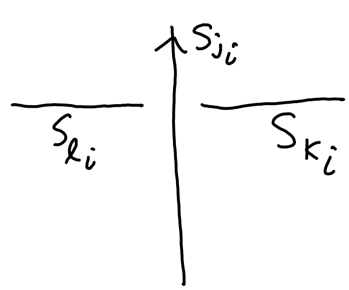


Thm: $\pi_1(S^3 \setminus K) \cong \langle s_1, \dots, s_k \mid r_1, \dots, r_k \rangle$ w/

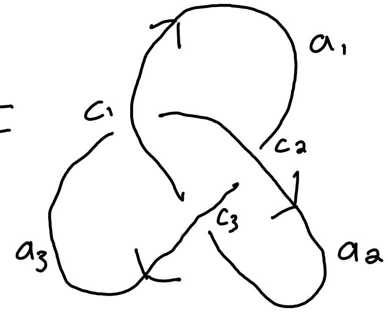
(a) s_i loop encircling a_i:



(b) $r_i = s_{j_i} s_{k_i} s_{j_i}^{-1} s_{l_i}^{-1}$, where c_i looks like



Eg: From

$K =$ 

get $\pi_1(S^3 \setminus K) = \langle s_1, s_2, s_3 \mid s_1 s_2 s_1^{-1} s_3^{-1},\ s_2 s_3 s_2^{-1} s_1^{-1},\ s_3 s_1 s_3^{-1} s_2 \rangle$

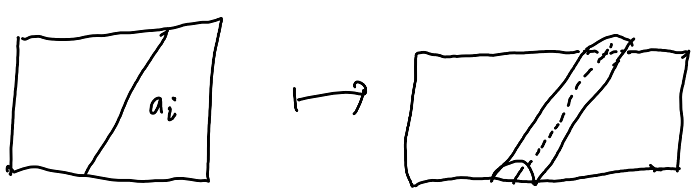
Pf of thm:

K' = change overcrossings in K to undercrossings

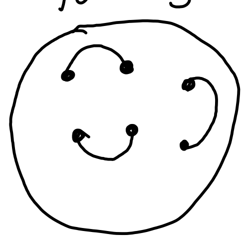
Observe: $S^3 \setminus K \cong S^3 \setminus K'$

Will construct $S^3 \setminus K'$ w/ indicated π_1

Draw K' on $S^2 \subseteq S^3$
Attach disjoint strips (tunnels) above arcs to get space X:

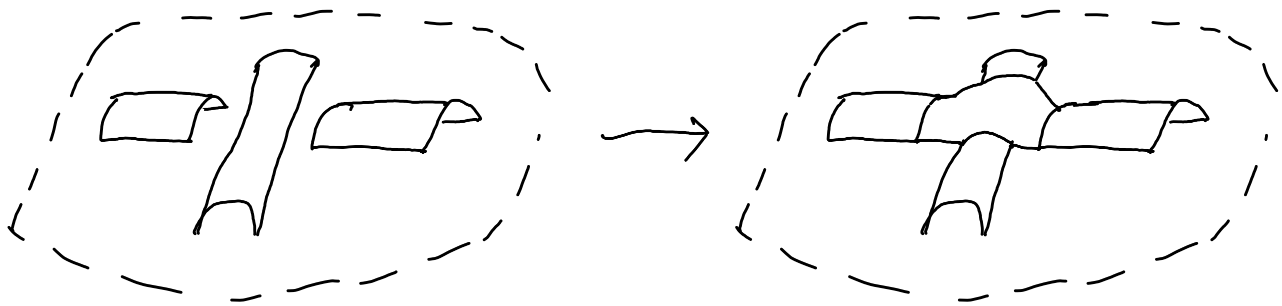


X def. retracts to S^2 w/ k 1-cells attached:



$\Rightarrow \pi_1(X) = \langle s_1, \dots, s_k \mid \rangle$

For each crossing c_i, attach 2-cell "overpass" to join tunnels to get space Y



relation from 2-cell at c_i : $s_{j_i} s_{k_i} s_{j_i}^{-1} s_{l_i}^{-1}$

$\Rightarrow \pi_1(Y) = \langle s_1, \dots, s_k \mid r_1, \dots, r_k \rangle$

Attach 3-cells to "inside" and "outside" of S^2 + tunnels get $Z \cong S^3 \setminus \text{nbhd}(K)$

$S^3 \setminus K$ def retracts to Z, so

$$\pi_1(S^3 \setminus K) \cong \pi_1(Z) \cong \langle s_1, \dots, s_k \mid r_1, \dots, r_k \rangle \quad \blacksquare$$

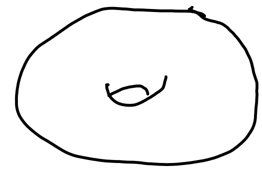
<u>Next Topic</u>: Torus knots

Recall from HW:

$GCD(m,n) = 1 \Rightarrow$ elt $(m,n) \in \mathbb{Z}^2 \cong \pi_1(T^2)$ can be realized by simple closed curve

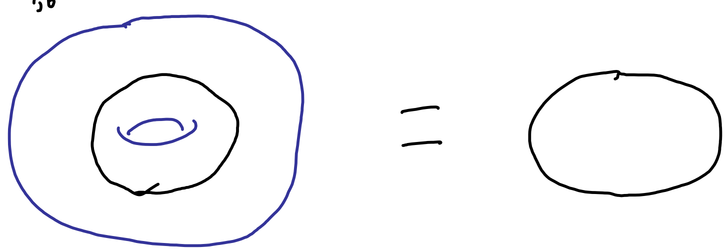
Let $\gamma_{m,n} : S^1 \longrightarrow T^2$ be embedding of (m,n)-curve

Let $i : T^2 \hookrightarrow \mathbb{R}^3$ be std embedding:



The *(m,n)-torus knot* $T_{m,n}$ is knot $i \circ \gamma_{m,n}$.

Ex: a) $T_{1,0}$ = unknot



b) $T_{2,3}$ = trefoil