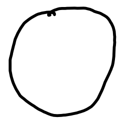
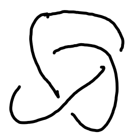
Math 444/539 Lecture 18

Def'n: A Knot is an embedding $f: S^1 \hookrightarrow \mathbb{R}^3$

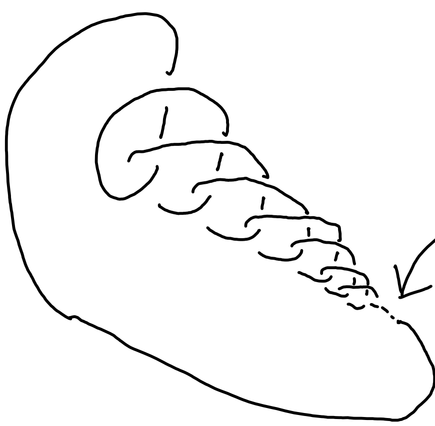
Ex: a) unknot



b) trefoil

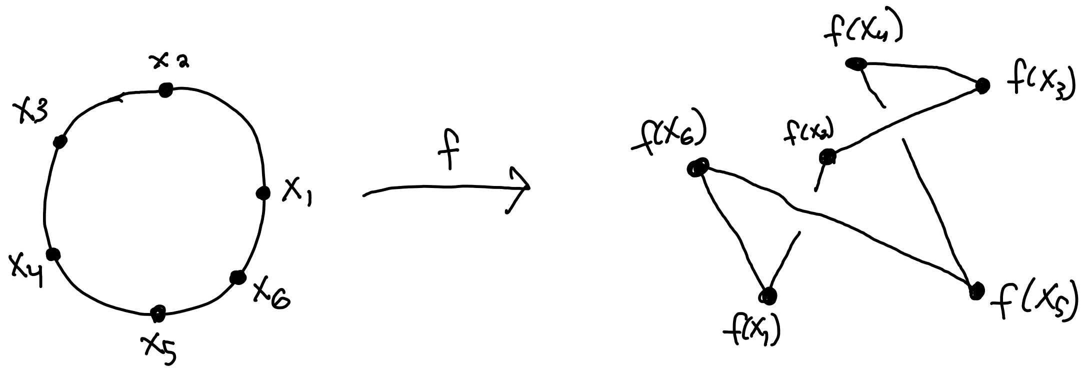


Problematic Example: wild knot



∞ many crossings getting smaller and smaller

Def'n: A tame Knot is a knot $f: S^1 \hookrightarrow \mathbb{R}^3$ st exists finitely many pts $x_1, \dots, x_n \in S^1$ st for each cpt C of $S^1 \setminus \{x_1, \dots, x_n\}$, $f(C)$ is straight line



<u>Rmk</u>: Will still draw tame knots as smooth curves. You should imagine that they are divided into so many straight segments that from a distance they appear smooth.

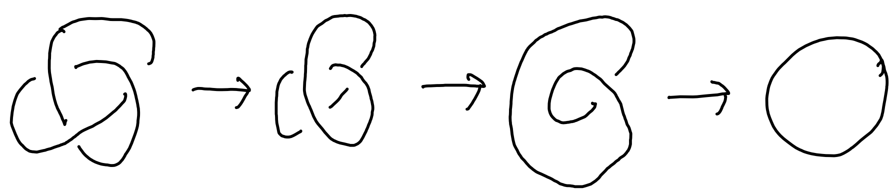
<u>Equivalence of Knots</u>

Want to say that 2 knots are equivalent if you can move one to the other in space, as if they were lengths of rope w/ ends joined.

<u>Formal Def'n</u>: Let $f, g: S^1 \hookrightarrow \mathbb{R}^3$ be knots

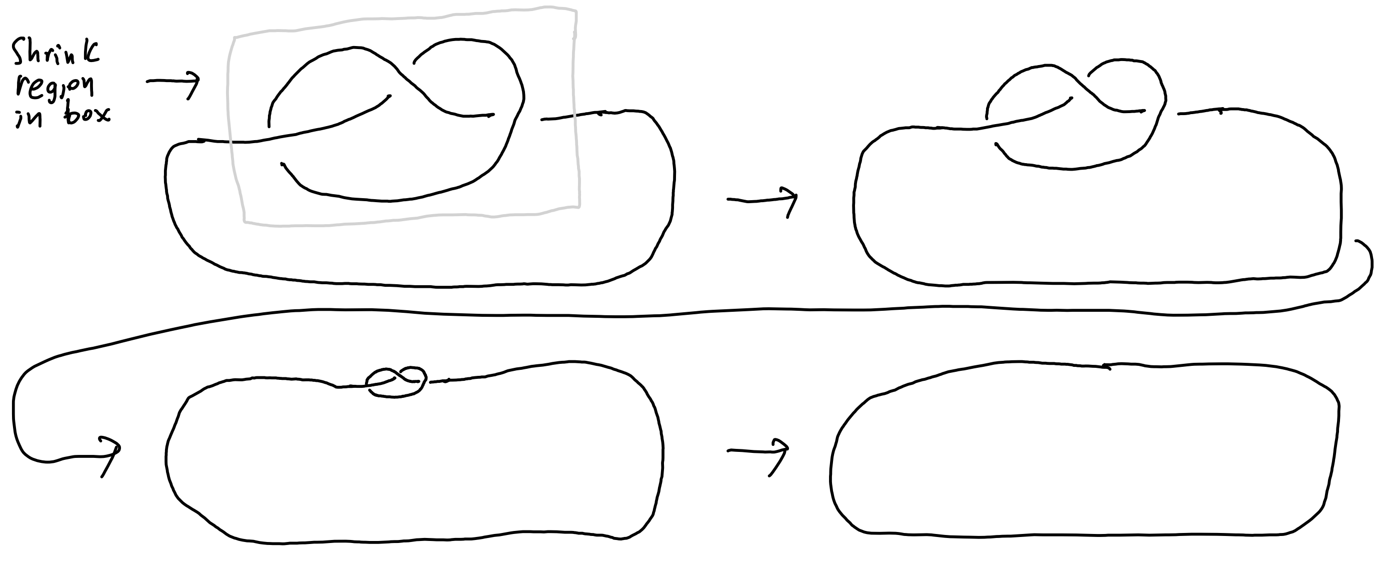
a) Attempt # 1: f + g are <u>homotopic</u> if $\exists F: S^1 \times I \longrightarrow \mathbb{R}^3$ st $F(x,0) = f(x)$ and $F(x,1) = g(x)$

Problem: Can pass strands through each other, so all knots are homotopic



b) Attempt #2: f + g are <u>isotopic</u> if $\exists F: S^1 \times I \longrightarrow \mathbb{R}^3$ st $F(x,0) = f(x)$, $F(x,1) = g(x)$, and maps $f_t: S^1 \longrightarrow \mathbb{R}^3$ w/ $f_t(x) = F(x,t)$ are embeddings for all t

Problem: Can "shrink" knotted region through embeddings until it disappears, so all tame knots are isotopic



c) Attempt #3: f, g are <u>ambient isotopic</u> if $\exists H: \mathbb{R}^3 \times I \longrightarrow \mathbb{R}^3$ st maps $h_t: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ w/ $h_t(x) = H(x,t)$ are homeomorphisms for all t, $h_0 = id$, and $g = h_1 \circ f$

"Move space along w/ knot"

<u>Rmk</u>: The maps $f_t = h_t \circ f : S^1 \rightarrow \mathbb{R}^3$ are embeddings for all t, so ambient isotopic knots are isotopic

Will say knots $f, g: S^1 \hookrightarrow \mathbb{R}^3$ are <u>equivalent</u> if they are ambient isotopic

<u>Rmk</u>: Above wild knot <u>not</u> equivalent to any tame knot

<u>Main Problem of Knot Theory</u>: Find invariants to distinguish non-equivalent knots.

Will confuse knot $f: S^1 \hookrightarrow \mathbb{R}^3$ w/ its image $f(S^1)$

<u>Observation</u>: If knots K_1 & K_2 are equivalent, then $\mathbb{R}^3 \setminus K_1 \cong \mathbb{R}^3 \setminus K_2$

$\mathbb{R}^3 \setminus K$ knows a lot about K:

<u>Thm</u> (Gordon-Luecke): $K_1, K_2 \subseteq \mathbb{R}^3$ tame knots

K_1 equivalent to K_2 or $\overline{K_2}$ $\iff \mathbb{R}^3 \setminus K_1 \cong \mathbb{R}^3 \setminus K_2$

$\overline{K_2}$: K_2 w/ orientation reversed.

<u>Def'n</u>: The <u>group</u> of a knot K is $\pi_1(\mathbb{R}^3 \setminus K)$

Often useful to view knot $K \subseteq \mathbb{R}^3$ as living in $S^3 = \mathbb{R}^3 \cup \{\infty\}$.

<u>Lemma</u>: $K \subseteq \mathbb{R}^3$ knot $\Rightarrow \pi_1(\mathbb{R}^3 \setminus K) \cong \pi_1(S^3 \setminus K)$.

pf: $U_1 = \mathbb{R}^3 \setminus K$, $U_2 \subseteq S^3 \setminus K$ small open ball around ∞

Then

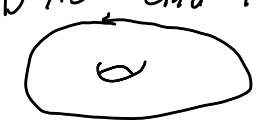
$$S^3 \setminus K = U_1 \cup U_2$$

$$U_1 \cap U_2 = U_2 \setminus \{\infty\} \text{ path-connected}$$

$$\pi_1(U_2) = \pi_1(U_1 \cap U_2) = 1$$

$$SvK \Rightarrow \pi_1(S^3 \setminus K) \cong \pi_1(U_1) \quad \boxtimes$$

Recall from HW 1: $S^3 = X_1 \cup X_2$ w/ $X_i \cong D^2 \times S^1$ and $X_1 \cap X_2 \cong T^2$ embedded in std way:



<u>Thm</u>: K unknot. Then $\pi_1(S^3 \setminus K) \cong \mathbb{Z}$.

pf:

Let $S^3 = X_1 \cup X_2$ be as above. Then $S^3 \setminus K$ def retracts to X_2, so $\pi_1(S^3 \setminus K) \cong \pi_1(X_2) \cong \mathbb{Z}$ ∎