Math 444/539 Lecture 16

Thm: A tree T is contractible to any pt $p \in V(T)$

pf:

Step 1: T finite

By induction on $|V(T)|$

$|V(T)| = 1$, trivial

$|V(T)| > 1 \Rightarrow \exists$ valence 1 vertex v_0

WLOG, $v_0 \neq p$

Let $e_0 \in E(T)$ be adjacent to v_0

Can contract T to $T' = T \setminus (\mathrm{Int}(e_0) \cup v_0)$

Induction $\Rightarrow$ Can contract T' to p

Step 2: General T

T connected $\Rightarrow \exists F' : T^{(0)} \times I \rightarrow T$ st

$F'(x,0) = x$, $F'(x,1) = p$,
$F'(p,t) = p$

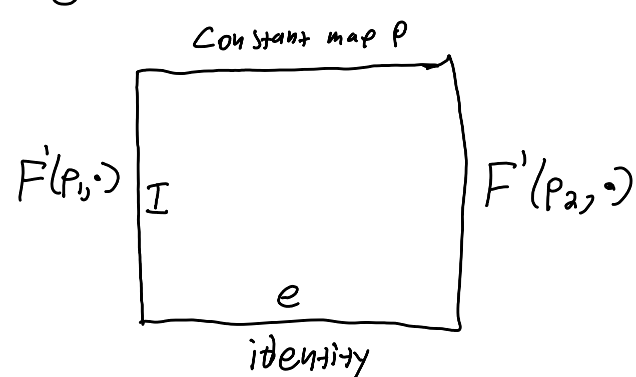
Must extend F' to $F : T \times I \rightarrow T$

Consider $e \in E(T)$ w/ endpoints $p_1, p_2 \in T^{(0)}$

$e \cong I$, so $\partial(e \times I) = $ square

Define

$F'_e : \partial(e \times I) \rightarrow T$
$F'_e(x,0) = x$, $F'_e(x,1) = p$
$F'_e(p_1,t) = F'(p_1,t)$
$F'_e(p_2,t) = F'(p_2,t)$



$F'_e(\partial(e \times I))$ compact $\Rightarrow \exists$ finite tree $T' \subseteq T$ w/ $F'_e(\partial(e \times I)) \subseteq T'$

Step 1 $\Rightarrow T'$ 1-connected

$\partial(e \times I) \cong S^1$, so 1-connectivity implies can extend

$F_e' : \partial(e \times I) \to T$ to $F_e : e \times I \to T$

Define

$$F : T \times I \to T$$

$$F|_{e \times I} = F_e$$

By construction, this is well-defined continuous fcn w/

$$F(x,0) = x, \quad F(x,1) = p, \quad F(p,t) = p \qquad \square$$

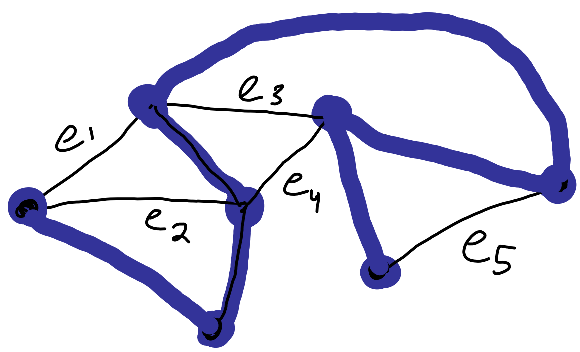
Thm: X connected graph, $p \in V(X)$

$\Rightarrow \pi_1(X,p)$ free grp

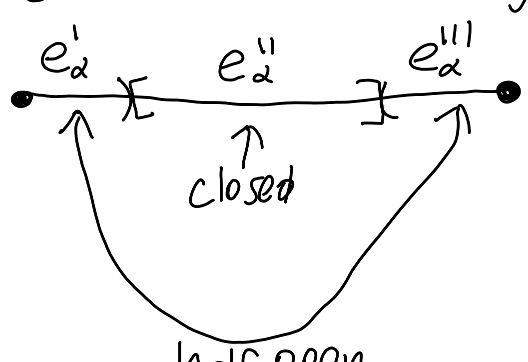
pf:

$T \subseteq X$ maximal tree

$\{e_\alpha\}$ edges of X not in T



Divide e_α into 3 segments:



Set

$$f_\alpha = e_\alpha' \cup e_\alpha'''$$

so

f_α open in e_α

Define
$$G_\alpha = T \cup e_\alpha$$
$$U_\alpha = T \cup e_\alpha \cup \Big(\bigcup_\beta f_\beta\Big)$$

<u>Facts</u>:

a) U_α open

b) U_α def. retracts onto G_α

c) $\alpha \neq \beta \Longrightarrow U_\alpha \cap U_\beta = T \cup \Big(\bigcup_\delta f_\delta\Big)$, which def retracts onto T

$\Longrightarrow U_\alpha \cap U_\beta$ path-connected and $\pi_1(U_\alpha \cap U_\beta, p) = 1$ $(*)$

d) α, β, γ distinct $\Longrightarrow U_\alpha \cap U_\beta \cap U_\gamma = T \cup \Big(\bigcup_\delta f_\delta\Big)$

$\Longrightarrow U_\alpha \cap U_\beta \cap U_\gamma$ path-connected

$\therefore$ Can apply Seifert-van Kampen to $\{U_\alpha\}$, and by $(*)$ the "relations" R are trivial

$$\Longrightarrow \pi_1(X, p) \cong \mathop{\ast}_\alpha \pi_1(U_\alpha, p)$$
$$\cong \mathop{\ast}_\alpha \pi_1(G_\alpha, p)$$

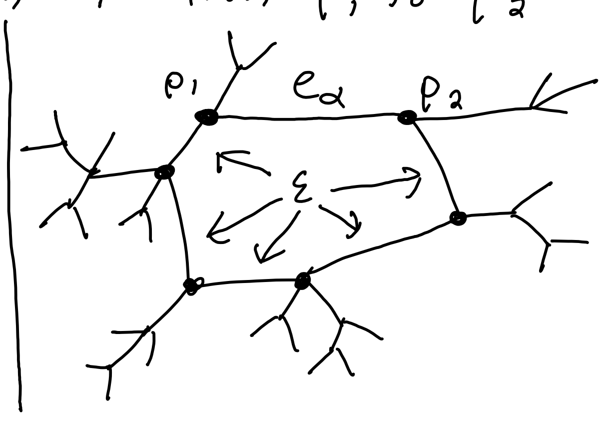
Hence enough to prove:

<u>Claim</u>: $\pi_1(G_\alpha, p) \cong \mathbb{Z}$

p_1, p_2 endpts of e_α

ε = injective path in T from p_1 to p_2

<u>Clear</u>: each cpt of $G_\alpha \setminus (\varepsilon \cup e_\alpha)$ is tree, so G_α def. retracts to $\varepsilon \cup e_\alpha \cong S^1$



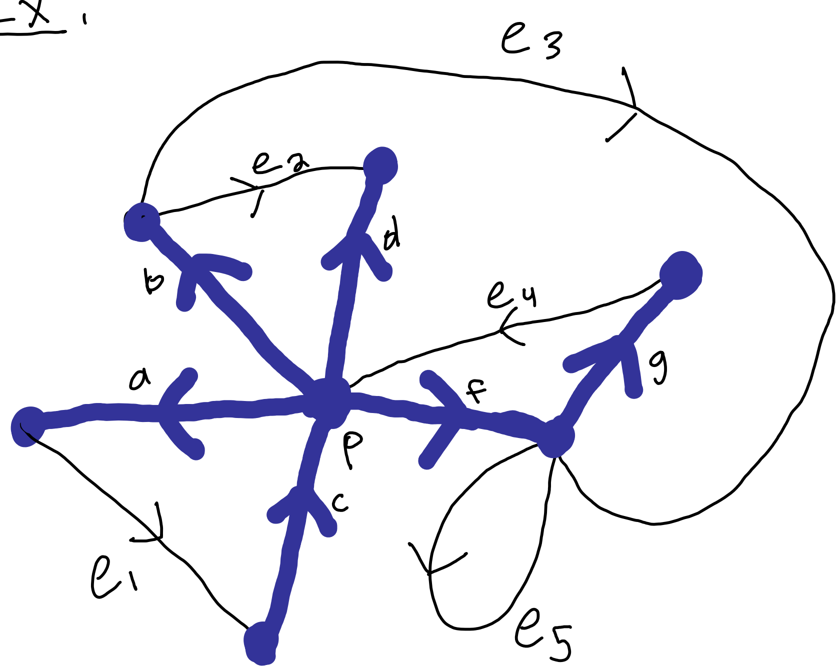
$\square$

Algorithm for finding $\pi_1(X,p)$ for graph X

1) Find max tree $T \subseteq X$
2) Let $\{e_\alpha\}$ be edges not in T
3) Orient e_α and let i_α and t_α be its initial and terminal vertices
4) For $v \in V(X)$, let δ_v be unique injective path in T from p to v

Conclusion: $\pi_1(X,p) \cong$ free grp w/ generators $\{x_\alpha\}$, w/ $x_\alpha = \underbrace{\delta_{i_\alpha} \cdot e_\alpha \cdot \delta_{t_\alpha}}_{\text{loop based at } p}$

Ex:



$\pi_1(X,p) \cong$ free grp on $x_1, \ldots, x_5$, where:

$x_1 = a e_1 c$
$x_2 = b e_2 \overline{d}$
$x_3 = b e_3 \overline{f}$
$x_4 = f g e_4$
$x_5 = f e_5 \overline{f}$

Rmk: Basis for $\pi_1(X,p)$ not canonical 'cause it depends on T. However, # of generators indep. of T