Math 444/539 Lecture 15

Setup: X top space w/ open cover $\{U_\alpha\}$
$p \in \bigcap_\alpha U_\alpha$
$U_\alpha \cap U_\beta$ + $U_\alpha \cap U_\beta \cap U_\gamma$ path-connected for all α, β, γ
$\varphi_{\alpha\beta} : \pi_1(U_\alpha \cap U_\beta, p) \to \pi_1(U_\beta, p)$ induced map
$\Psi : \ast_\alpha \pi_1(U_\alpha, p) \to \pi_1(X, p)$ map from univ. property of $\ast$

Thm (Seifert-van Kampen): Ψ surjective and $\mathrm{Ker}(\Psi) = R$, w/ R normal subgrp gen by
$$\{(\varphi_{\alpha\beta}(x))(\varphi_{\beta\alpha}(x))^{-1} \mid x \in \pi_1(U_\alpha \cap U_\beta, p),\ \alpha, \beta \text{ arbitrary}\}$$

pf:
Already proved Ψ surjective, must prove $\mathrm{Ker}(\Psi) = R$

Def'n: A factorization of a p-based loop γ in X is expression
$$[\gamma] = [\gamma_1] \cdots [\gamma_k]$$
w/ $\gamma_i \subseteq U_{\alpha_i}$ for some $\alpha_1, \ldots, \alpha_k$

Def'n: Let $[\gamma] = [\gamma_1] \cdots [\gamma_k]$ w/ $\gamma_i \subseteq U_{\alpha_i}$ be factorization

a) A type I move: if for some i have $\gamma_i \subseteq U_{\alpha_i'}$, then replace U_{α_i} w/ $U_{\alpha_i'}$.

b) A type II move: If $U_{\alpha_i} = U_{\alpha_{i+1}}$, then change to
$$[\gamma_1] \cdots [\gamma_{i-1}][\gamma_i \cdot \gamma_{i+1}] \cdot [\gamma_{i+2}] \cdots [\gamma_k]$$

2 factorizations of $[\gamma]$ are equivalent if differ by sequence of type I/II moves or their inverses.

Key Claim: Any 2 factorizations of a p-based loop γ are equivalent.

Key Claim $\Longrightarrow \mathrm{Ker}(\Psi) = R$:

If $[\gamma_1]\cdots[\gamma_k] \in \mathrm{Ker}(\Psi)$, then Key claim says equivalent to trivial loop. But type I moves correspond to applying rel'ns from R and type II moves correspond to applying rel'ns in $*$. Conclude: $[\gamma_1]\cdots[\gamma_k] \in R$.

<u>Pf of Key Claim</u>:

Asm

$$[\gamma] = [\gamma_1]\cdots[\gamma_k] \quad \& \quad [\gamma] = [\gamma_1']\cdots[\gamma_\ell']$$

2 factorizations. Set

$$\delta = \gamma_1 \cdots \gamma_k \quad \& \quad \delta' = \gamma_1' \cdots \delta_\ell'$$

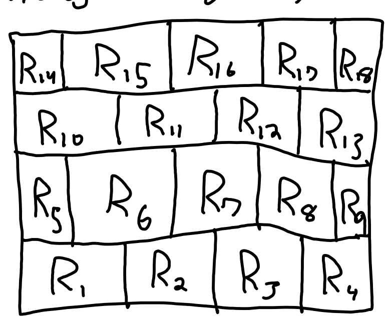
$\delta \sim \delta' \Longrightarrow \exists F : I \times I \to X$ st

$$F(s,0) = \delta(s), \quad F(s,1) = \delta'(s),$$
$$F(0,t) = F(1,t) = p$$

Can decompose $I \times I$ into rectangles $R_1, \ldots, R_N$ st $F(R_i) \subseteq U_{\beta_i}$ and every pt of $I \times I$ lies in ≤ 3 rectangles:

Set $V_\alpha = F^{-1}(U_\alpha)$ & let $\varepsilon > 0$ be Leb. # of $\{V_\alpha\}$. Then need only choose R_i w/ diam $< \varepsilon$ in following pattern:

Number like this

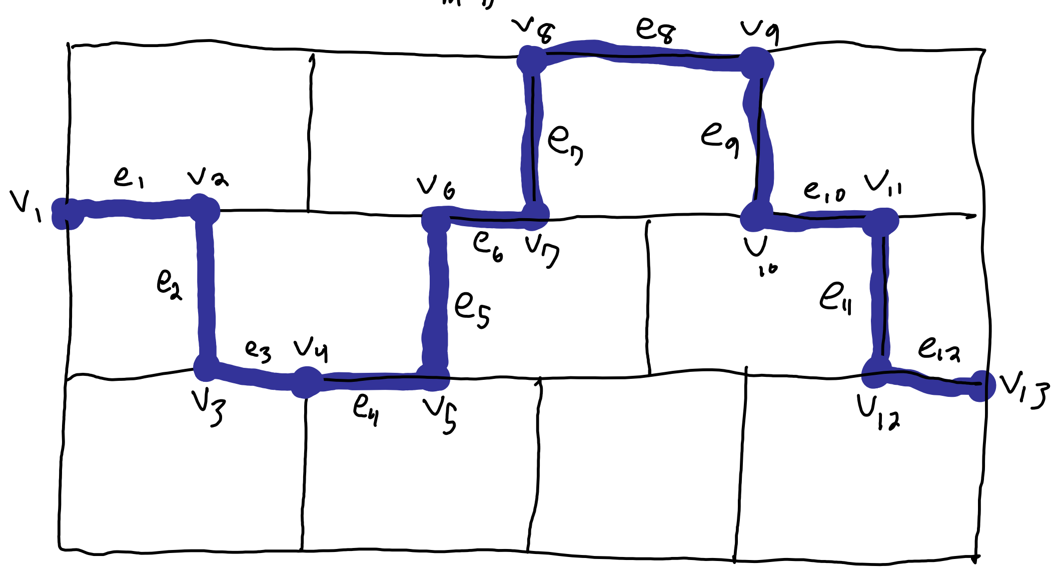


For vertex v of tiling, choose path η_v from p to $F(v)$ in $\bigcap$ of the U_α that F of the rectangles containing p lie in (at most 3 U_α's).

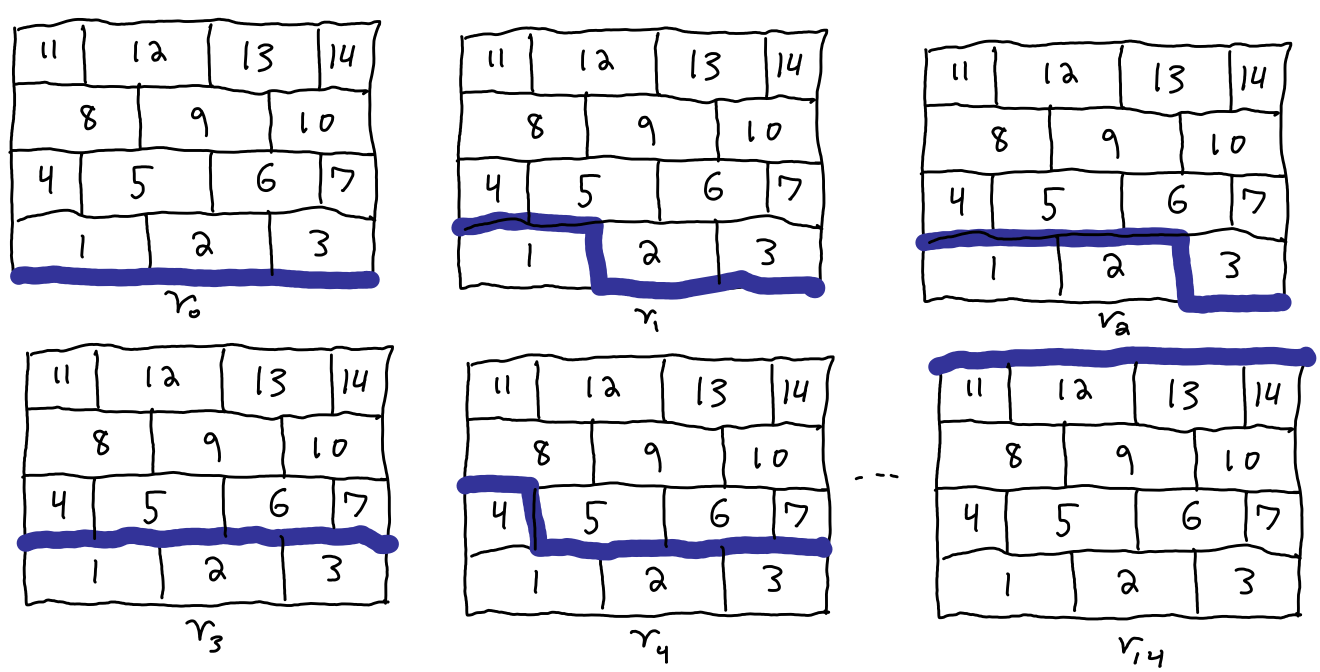
For a path γ in "grid" from LHS to RHS, get factorization $\mathcal{F}(\gamma)$ of $[\gamma]$:

Let $v_1, \dots, v_m$ be vertices of γ and let e_i be edge in grid from v_i to v_{i+1}. Then $F(v_1) = F(v_m)$ and have factorization

$$[F(e_1) \cdot \bar{\eta}_{F(v_1)}] \cdot [\eta_{F(v_2)} \cdot F(e_2) \cdot \bar{\eta}_{F(v_3)}] \cdot [\eta_{F(v_3)} \cdot F(e_3) \cdot \bar{\eta}_{F(v_4)}] \cdots [\eta_{F(v_{m-1})} \cdot F(e_{m-1})]$$



For $0 \leq i \leq N$, let γ_i be grid path separating $R_1, \dots, R_i$ from $R_{i+1}, \dots, R_N$:



Easy to check: a) $\mathcal{F}(\gamma_i)$ equivalent to $\mathcal{F}(\gamma_{i+1})$

b) $\mathcal{F}(V_0)$ equivalent to $[\gamma_1] \cdots [\gamma_k]$

c) $\mathcal{F}(V_N)$ equivalent to $[\gamma_1'] \cdots [\gamma_\ell']$

Conclude: $[\gamma_1] \cdots [\gamma_k]$ equivalent to $[\gamma_1'] \cdots [\gamma_\ell']$.

$\blacksquare$