Math 444/539 Lecture 11

Thm (fundamental thm of algebra):

Let

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_n \qquad (a_i \in \mathbb{C},\ n \geq 1)$$

$\Rightarrow \exists z_0 \in \mathbb{C}$ st $p(z_0) = 0$

Pf:

Asm $p(z) \neq 0$ for all $z \in \mathbb{C}$

For all $r \geq 0$, define

$$f_r : I \to S^1$$

$$f_r(x) = \frac{p(re^{2\pi i x})/p(r)}{|p(re^{2\pi i x})/p(r)|} \qquad \text{(rmk: makes sense since } p(z) \neq 0)$$

$f_r(0) = f_r(1) = 1$, so $[f_r] \in \pi_1(S^1, 1)$ for all r.

Pick

$$R > \max\{1, |a_1| + \cdots + |a_n|\},$$

and set $\gamma = [f_R] \in \pi_1(S^1, 1) \cong \mathbb{Z}$

Claim: $\gamma = 0$

$$\gamma = [f_R] = [f_0] \text{ and } f_0(x) = 1.$$

Claim: $\gamma = n > 0$, contradiction.

For $s \in I$, define

$$p_s(z) = z^n + s(a_1 z^{n-1} + \cdots + a_0)$$

If $|z| = R$, then

$$\begin{aligned}
|z|^n &= R \cdot R^{n-1} \\
&> (|a_1| + \cdots + |a_n|) R^{n-1} \\
&\geq |a_1| R^{n-1} + |a_2| R^{n-2} + \cdots + |a_n| R^0 \\
&\geq |a_1 z^{n-1} + \cdots + a_0|
\end{aligned}$$

so $p_s(z) \neq 0$.

Define

$$g_s : I \to S^1$$

$$g_s(x) = \frac{p_s(Re^{2\pi i x})/p_s(R)}{|p_s(Re^{2\pi i x})/p_s(R)|}$$

(rmk: makes sense since $p_s(z) \neq 0$ if $|z| = R$)

Then

$$g_0(x) = e^{2\pi i n x}$$

so

$$\gamma = [g_1] = [g_0] = n \quad \blacksquare$$

<u>Thm</u> (Borsuk-Ulam):

$f : S^2 \to \mathbb{R}^2$ continuous $\Rightarrow \exists x \in S^2$ s.t. $f(x) = f(-x)$

<u>Rmk</u>: True for $f : S^n \to \mathbb{R}^n$

<u>Exercise</u>: prove for $n = 1$.

<u>pf of B-U</u>:

Asm $f(x) \neq f(-x)$ for all $x \in S^2$

Define

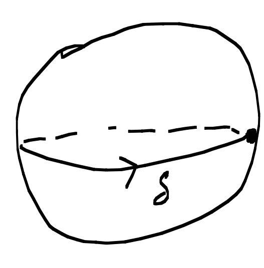
$$g : S^2 \to S^1$$

$$g(x) = \frac{f(x) - f(-x)}{\| f(x) - f(-x) \|}$$

and

$$\delta : I \to S^2$$

$$\delta(x) = (\cos(2\pi x), \sin(2\pi x), 0)$$



and $h = g \circ \delta$

Then $[h] \in \pi_1(S^1, p)$ w/ $p = g(1,0,0)$

$\cong \mathbb{Z}$

Claim: $[h]=0$

$[\gamma]\in\pi_1(S^2,(1,0,0)) \overset{\text{HW}}{\cong} 1$, so $[h]=g_*(\gamma)=0$

Claim: $[h]\neq 0$, a contradiction

$g(-x)=g(x) \implies h(s+1/2)=-h(s)$ for $s\in[0,1/2]$ $\quad (*)$

Define

$$\rho: \mathbb{R}\to S^1$$
$$\rho(x)=e^{2\pi i x}$$

and let $\tilde{p}\in\rho^{-1}(p)$. Let $\tilde{h}: I\to\mathbb{R}$ be lift w/ $\tilde{h}(0)=\tilde{p}$

$(*) \implies \tilde{h}(s+1/2)=\tilde{h}(s)+\frac{1}{2}q(s)$ w/ $q(s)$ odd integer

$q(s)$ continuous $\implies q(s)$ constant q

$\implies \tilde{h}(1)=\tilde{h}(1/2)+\frac{1}{2}q=\tilde{h}(0)+q$

$\implies [h]=q\neq 0$ since q odd $\boxed{\text{///}}$

<u>Def'n</u>: X is <u>contractible</u> if id_X is homotopic to a constant map

<u>Ex</u>: $\mathbb{R}^n$ is contractible

<u>Def'n</u>: X is <u>k-connected</u> if for all $f: S^\ell\to X$ w/ $\ell\leq k$, there exists $\bar{f}: D^{\ell+1}\to X$ st $\bar{f}|_{\partial D^{\ell+1}}=f$

<u>Facts</u>: a) For $k<-1$, all spaces are k-connected

b) X (-1)-connected $\iff X\neq\emptyset$

$S^{-1}=\emptyset$, so (-1)-connected reduces to existence of map $\underset{\{*\}}{\overset{\shortparallel}{D^0}}\to X$.

c) X 0-connected $\Leftrightarrow$ X path-connected and $X \neq \emptyset$

$S^0 = \{-1, 1\}$, so Condition reduces to:

$\forall p, q \in X$, $\exists f: D^1 \to X$ st $f(-1) = p$ and $f(1) = q$

d) X 1-connected $\Leftrightarrow$ $X \neq \emptyset$, X path-connected, and $\pi_1(X) = 1$.

HW

Lemma: X contractible $\Longrightarrow$ X k-connected for all k

Rmk: Amazingly, converse holds if X a CW-complex

Pf of Lemma:

Let $F: X \times I \to X$ be homotopy from id_X to constant map to $p \in X$

Consider $f: S^k \to X$.

Define

$$g: S^k \times I \to X$$
$$g(x,t) = F(f(x),t)$$

Then $g(x,1) = p$, so g induces map $\bar{g}: S^k \times I/\sim \to X$ w/ $(x,1) \sim (x',1)$ for all $x, x' \in S^k$

But $S^k \times I/\sim \cong D^{k+1}$, so we're done:

$S^k \times I$ $\qquad$ $S^k \times I/\sim$