Math 444/539, Lecture 1

Arbitrary top space = 

"Good" top spaces

(a) $\mathbb{R}^n$

(b) $D^n = \{\vec{x} \in \mathbb{R}^n \mid \sum x_i^2 \leq 1\}$

(c) $S^{n-1} = \{\vec{x} \in \mathbb{R}^n \mid \sum x_i^2 = 1\}$

Goal: Use $\mathbb{R}^n$, D^n, + S^n to study other "good" (ie geometric) spaces

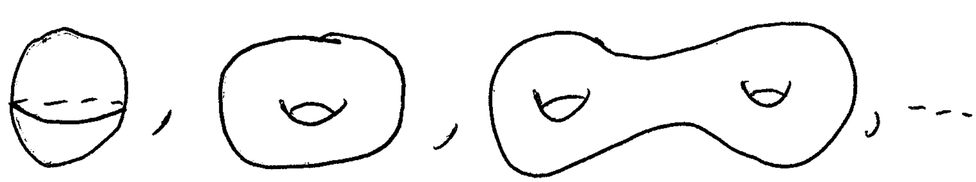
Math 444 : $n=1$ ← nonabelian

Math 445 : $n>1$ ← abelian

Outline

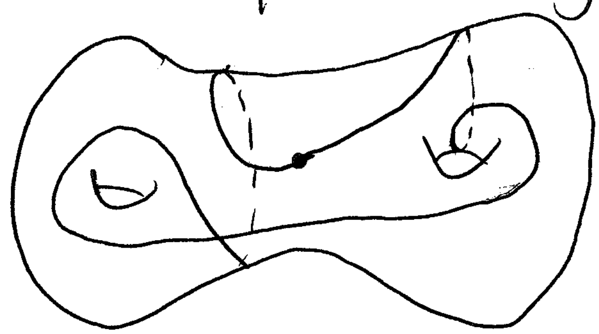
I. Formalize "good spaces": CW cpx's, mnflds

Highlight = classification of surfaces

, ...

II. Fund grp

algebraic "probe" using S^1



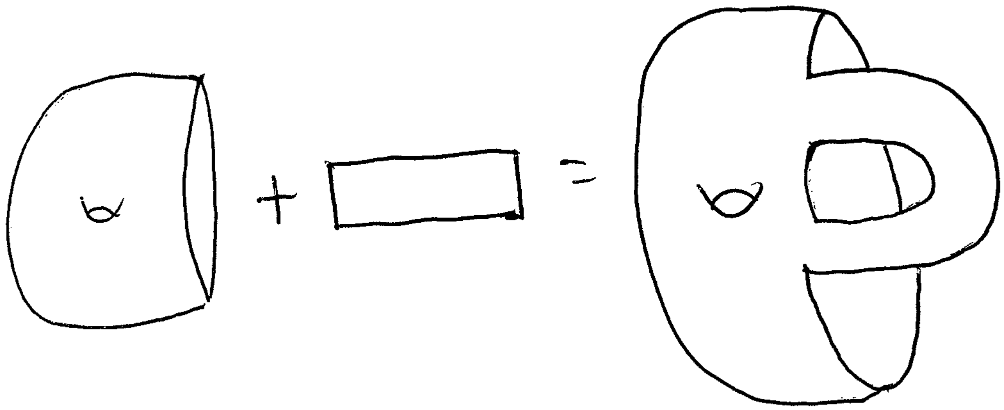
III. Covering spaces

"unwrap" fund. grp

"good spaces" = simple spaces glued together (in complicated ways)

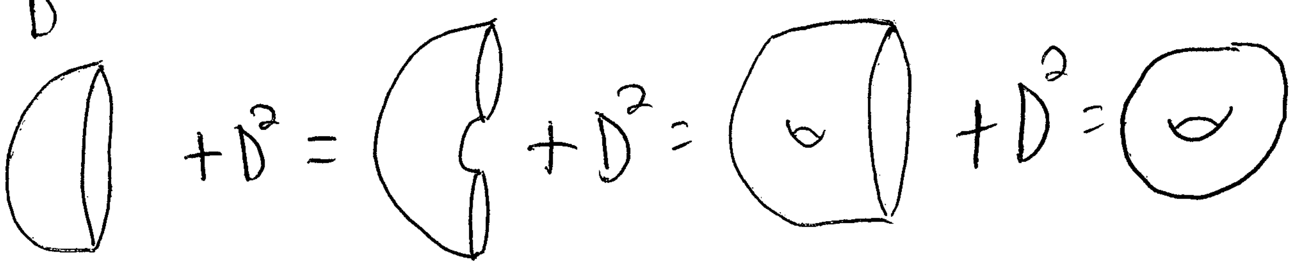
<u>1st step</u>: formalize gluing (quotient top)

<u>Ex</u>:

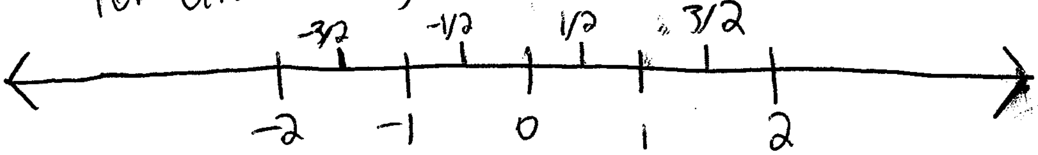


<u>Ex</u>:  = 4 D^2's glued together

D^2



Ex: $\mathbb{R}$ w/ x & $x+n$ identified for all $x \in \mathbb{R}$, $n \in \mathbb{Z}$



$= I = [0,1]$ w/ 0 & 1 identified

$= S^1$

$f: \mathbb{R} \to \mathbb{R}$ s.t. $f(x+n) = f(x)$ $\forall x \in \mathbb{R}, n \in \mathbb{Z}$ $\longleftrightarrow$ $g: S^1 \to \mathbb{R}$, $S^1 \subseteq \mathbb{R}^2$

$f(x) = \cos(x/2\pi)$ $\longleftrightarrow$ $g(x,y) = x$

Thm: X space, $\mathcal{I} \subseteq P(X)$
pwr set

$\Longrightarrow \exists!$ space Y, $\pi : X \to Y$ s.t. $\pi|_I = \text{constant}$ for $I \in \mathcal{I}$ and

(*) if $\varphi : X \to Z$ satisfies $\varphi|_I = \text{constant}$ for $I \in \mathcal{I}$, then $\exists! \varphi' : Y \to Z$ s.t. $\varphi = \varphi' \circ \pi$

Rmk: (a) (*) a "universal mapping property"

(b) $\varphi = \varphi' \circ \pi$ can be visualized w/ a $\underbrace{\text{commutative diagram}}_{\text{all ways of following arrows around diagram give same answer}}$

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\pi"'] & Z \\ Y \arrow[ur, "\varphi'"'] & \end{tikzcd}$$

(c) In above example w/ $\mathbb{R}$ + S^1, have

$X = \mathbb{R}$,

$\mathcal{I} = \{E_x \mid x \in \mathbb{R}\}$ w/ $E_x = \{x+n \mid n \in \mathbb{Z}\} \subseteq \mathbb{R}$

$Y = S^1$

pf of thm:

Uniqueness of Y: HW

Existence

For $p \in X$, set

$$E_p = \{p' \in X \mid \exists q_1, \ldots, q_n \in X \ \& \ I_1, \ldots, I_{n-1} \in \mathcal{I} \text{ st } p = q_1,\ p' = q_n, \text{ and } \{q_i, q_{i+1}\} \subseteq I_i \text{ for } 1 \leq i < n\}.$$

HW: For $p, p' \in X$, have either $E_p = E_{p'}$ or $E_p \cap E_{p'} = \emptyset$.

Set

$$Y = \{E_p \mid p \in X\}$$

w/ top

$$\mathcal{U} = \{U \subseteq Y \mid \bigcup_{E \in U} E \subseteq X \text{ open}\}$$

HW: $\mathcal{U}$ a topology

Define

$$\pi : X \to Y$$
$$\pi(p) = E_p$$

Check: π continuous

$$U \subseteq Y \text{ open} \Rightarrow \pi^{-1}(U) = \bigcup_{E \in U} E \subseteq X \text{ open by defn.}$$

<u>Check</u>: $\pi : X \to Z$ satisfies $(*)$

Consider $\varphi : X \to Z$ as in $(*)$

$E \in Y \Rightarrow \varphi(p) = \varphi(p') \ \forall p, p' \in E$ (by $(*)$)

Define

$\varphi' : Y \to Z$

$\varphi'(E) = \varphi(p) \quad (p \in E)$

<u>Clear</u>:

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\pi"'] & Z \\ Y \arrow[ur, "\varphi'"'] & \end{tikzcd}$$

commutes + no other $Y \to Z$ makes it commute

Must show φ' continuous

$U \subseteq Z$ open $\Rightarrow \varphi^{-1}(U) \subseteq X$ open

$(\varphi')^{-1}(U) = \{E_p \mid p \in X, \varphi(p) \in U\}$

$\Rightarrow \displaystyle\bigcup_{E \in (\varphi')^{-1}(U)} E = \varphi^{-1}(U)$ open

$\Rightarrow (\varphi')^{-1}(U)$ open

$\boxed{//}$