Math 444/539 : Geometric Topology
Problem Set 3

Everyone has to do all the problems. Here is some vocabulary that will be used in the problems.

- A *simple closed curve* on a surface is the image of an embedding of S^1 into the surface.
- $\Sigma_{g,n}$ will denote an orientable genus g surface with n boundary components. The n will be omitted if it equals 0.
- A simple closed curve γ on S *bounds a disc* if there exists an embedding $i : \Sigma_{0,1} \hookrightarrow S$ that takes the boundary of $\Sigma_{0,1}$ homeomorphically onto γ.
- Two simple closed curves γ_1 and γ_2 are *parallel* if there exists an embedding $\Sigma_{0,2} \hookrightarrow S$ that takes the boundary components of $\Sigma_{0,2}$ homeomorphically onto γ_1 and γ_2. Observe that parallel simple closed curves must be disjoint.

Also, in these problems you will frequently invoke the classification of surfaces with boundary to construct homeomorphisms. When doing so, you can assume that the resulting homeomorphisms have any reasonable conditions on the boundary components that you like.

Problems :

1. Let X be a compact triangulated surface and let T be a subcomplex of $X^{(1)}$ which is a tree. Prove that T has a neighborhood which is homeomorphic to D^2. Hint : You may use the following standard result:

   Lemma : Let X be a topological space and let $A, B \subset X$ be subspaces such that $X = A \cup B$. Assume that both A and B are homeomorphic to D^2 and that $A \cap B$ is homeomorphic to $[0,1]$. Then X is homeomorphic to D^2.

2. Let γ_1 and γ_2 be simple closed curves on Σ_g. Assume that $\Sigma_g \setminus \gamma_i$ is connected for $1 \leq i \leq 2$. Prove that there exists some homeomorphism $f : \Sigma_g \to \Sigma_g$ such that $f(\gamma_1) = \gamma_2$. Hint : What surface results from cutting Σ_g along γ_i?

3. Let $\gamma_1, \ldots, \gamma_k$ be disjoint simple closed curves on Σ_g. Assume that $\Sigma_g \setminus (\gamma_1 \cup \cdots \cup \gamma_k)$ is connected. Prove that $k \leq g$. Hint : What surface results from cutting Σ_g along $\gamma_1 \cup \cdots \cup \gamma_k$?

4. Let γ be a simple closed curve on $\Sigma_{0,3}$. Prove that γ either bounds a disc or is parallel to a boundary component. Hint : What surface results from cutting $\Sigma_{0,3}$ along γ?

5. In this problem, assume that $g > 1$. Let $\gamma_1, \ldots, \gamma_k$ be a collection of disjoint simple closed curves on Σ_g none of which bound a disc. Assume that γ_i is not parallel to γ_j for all $1 \leq i, j \leq k$ such that $i \neq j$. Also, assume that there does not exist any simple closed curve γ_{k+1} on Σ_g such that $\gamma_1, \ldots, \gamma_{k+1}$ has the above properties. Prove that $k = 3g - 3$. Hint : Prove first that all the components of Σ_g cut along $\gamma_1 \cup \cdots \cup \gamma_k$ must be homeomorphic to $\Sigma_{0,3}$; the previous problem might be helpful here.