Math 444/539 : Geometric Topology
Problem Set 13

Everyone should do all the problems.

Problems :

1. Assume that $G = G_1 * \cdots * G_n$ and that H is a subgroup of G. Let N be the subgroup of H generated by the set
$$\{xG_ix^{-1} \mid x \in G,\, 1 \leq i \leq n\} \cap H.$$
Prove that N is a normal subgroup of H and that H/N is a free group.

2. Let F be a free group on the set $\{x_1, \ldots, x_n\}$. Prove that F cannot be generated by fewer than n elements.

3. (a) Let F and G be free groups of rank n and let $\phi : F \to G$ be a surjection. Prove that ϕ is an isomorphism.
   (b) Construct groups F and G which are isomorphic and a surjection $\phi : F \to G$ which is not an isomorphism.
   (c) Let F and G be free groups of rank n. Construct an injection $\phi : F \to G$ that is not an isomorphism.
   (d) If G is a free group of rank n and $a_1, \ldots, a_n \in G$ are any n elements which generated G, then prove that $\{a_1, \ldots, a_n\}$ is a free basis for G.
   (e) If G is a free group of finite rank and N is a normal subgroup of G such that $N \neq 1$, then prove that G/N is not isomorphic to G.

4. Fix some m. Assume that $G = \cup_{n=1}^{\infty} G_n$, where each G_n is a proper subgroup of G, each G_n is a proper subgroup of G_{n+1}, and G_n is generated by at most m elements for all n. Assume that $G = H_0 * H$ with H_0 finitely generated. Prove that H_0 is generated by less than m elements and that H is not finitely generated. Hint : Prove the following in succession : 1. G is not finitely generated, 2. H is not finitely generated, 3. There exists an integer n_0 such that $H_0 \subset G_{n_0}$, 4. Apply the Kurosh subgroup theorem to obtain a free product decomposition of G_{n_0}, 5. Prove the desired result.