VIC-modules over noncommutative rings

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Abstract

For a finite ring R, not necessarily commutative, we prove that the category of $\mathtt{VIC}(R)$-modules over a left Noetherian ring $\mathbf{k}$ is locally Noetherian, generalizing a theorem of the authors that dealt with commutative R. As an application, we prove a very general twisted homology stability for $\mathrm{GL}_n(R)$ with R a finite noncommutative ring.

1   Introduction

The program of representation stability was introduced by Church and Farb [3, 6]. The idea is that many of the representations that occur in nature depend on a parameter n, and it is useful to study algebraic structures that encode all of these representations simultaneously. For instance, the cohomology groups of the space $\mathrm{Conf}_n(\mathbb{R}^2)$ of configurations of n labeled points in $\mathbb{R}^2$ are representations of the symmetric group S_n, which acts by permuting the n points. Individually, these are hard to understand; however, taken together they have a lot of global structure, especially as $n \mapsto \infty$.

Representations of categories. This can be encoded in many ways. One of the most fruitful is Church–Ellenberg–Farb's [1] theory of $\mathtt{FI}$-modules. Here $\mathtt{FI}$ is the category whose objects are the finite sets $[n] = \{1, \ldots, n\}$ and whose morphisms are injections. For a category $\mathtt{C}$ like $\mathtt{FI}$ and a ring $\mathbf{k}$, a $\mathtt{C}$-*module* over $\mathbf{k}$ is a functor M from $\mathtt{C}$ to the category $\mathbf{k}$-Mod of left $\mathbf{k}$-modules. Thus M consists of a $\mathbf{k}$-module M_c for every object $c \in \mathtt{C}$ and a $\mathbf{k}$-module map $f : M_c \to M_d$ for every $\mathtt{C}$-morphism $f : c \to d$. For an $\mathtt{FI}$-module M, we will write M_n for $M_{[n]}$. The $\mathtt{FI}$-endomorphisms of $[n]$ form the symmetric group S_n. These endomorphisms act on M_n, making each M_n a representation of S_n.

Example 1.1. For a fixed p, we can define an $\mathtt{FI}$-module M over $\mathbb{Z}$ with $M_n = \mathrm{H}^p(\mathrm{Conf}_n(\mathbb{R}^2); \mathbb{Z})$. The induced $S_n = \mathrm{End}_{\mathtt{FI}}([n])$-action on M_n is precisely the S_n-action on $\mathrm{H}^p(\mathrm{Conf}_n(\mathbb{R}^2); \mathbb{Z})$ from the previous paragraph. We therefore get a single object encoding all these representations together with the various ways that they are related as $n \mapsto \infty$. □

Homological algebra. For a category $\mathtt{C}$, the collection of $\mathtt{C}$-modules over $\mathbf{k}$ forms an abelian category whose morphisms are natural transformations between functors $\mathtt{C} \to \mathbf{k}$-Mod. An important insight of Church–Ellenberg–Farb [1] is that one can do commutative and homological algebra in this category in a way that is similar to the category $\mathbf{k}$-Mod. For instance, one can construct projective resolutions, take derived functors, etc.

Local Noetherianity. Perhaps the most important technical result for this is a version of the Hilbert basis theorem. A C-module M over a ring $\mathbf{k}$ is *finitely generated* if there exist objects $c_1, \ldots, c_k \in \mathtt{C}$ and elements $x_i \in M_{c_i}$ such that the smallest C-submodule of M containing all the x_i is M. In other words, for each $c \in \mathtt{C}$ the set

$$\bigcup_{i=1}^{k} \{ f(x_i) \mid f \colon c_i \to c \text{ is a C-morphism} \} \subset M_c.$$

spans M_c. We say that the category of C-modules over $\mathbf{k}$ is *locally Noetherian* if for all finitely generated C-modules M over $\mathbf{k}$, all C-submodules of M are finitely generated. Generalizing previous work that dealt for instance with fields $\mathbf{k}$ of characteristic 0, Church–Ellenberg–Farb–Nagpal [2] proved that the category of FI-modules over a left Noetherian ring $\mathbf{k}$ is locally Noetherian.

VIC-modules. The category of FI-modules encodes representations of the symmetric groups, and there has been a huge amount of work developing analogues for other families of groups (see, e.g., [8, 15, 16, 19, 22]). One particularly important family of groups are the general linear groups $\mathrm{GL}_n(R)$ over a ring R. Here it is natural to look at categories whose objects are the finite-rank free right[1] R-modules R^n with $n \geq 0$. As for the morphisms, there are several potential choices. To help keep the notation for our morphisms straight, we will write $[R^n]$ when we mean to regard R^n as an object of one of our categories and R^n when we mean to regard it as an R-module.
- The category $\mathtt{V}(R)$, whose morphisms $[R^n] \to [R^m]$ are R-linear maps $R^n \to R^m$. Versions of this go back to work of Lannes and Schwartz and are the focus of the *Artinian conjecture* (see [11, Conjecture 3.12]), which was resolved independently by the authors [15] and by Sam–Snowden [19].
- The category $\mathtt{VI}(R)$, whose morphisms $[R^n] \to [R^m]$ are injective R-linear maps $f \colon R^n \to R^m$ that are splittable in the sense that there exists some $g \colon R^m \to R^n$ with $g \circ f = \mathrm{id}$. Equivalently, the image of f is a summand of R^m. This was introduced by Scorichenko in his thesis ([20]; see [7] for a published account).
- The category $\mathtt{VIC}(R)$, whose morphisms $[R^n] \to [R^m]$ are pairs (f_1, f_2), where $f_1 \colon R^n \to R^m$ is an injective R-linear map and $f_2 \colon R^m \to R^n$ is a splitting of f_1, so $f_2 \circ f_1 = \mathrm{id}$. This was introduced by the authors in [15].

Remark 1.2. One motivation for studying $\mathtt{VIC}(R)$ is that it is the only one of these categories where there is a functor $\mathtt{VIC}(R) \to \mathrm{Groups}$ taking $R^n \in \mathtt{VIC}(R)$ to $\mathrm{GL}_n(R)$. For a morphism $(f_1, f_2) \colon [R^n] \to [R^m]$, the induced group homomorphism $\mathrm{GL}_n(R) \to \mathrm{GL}_m(R)$ is as follows. Set $C = \ker(f_2)$, so $R^m = \mathrm{im}(f_1) \oplus C$. Our homomorphism then takes $\phi \in \mathrm{GL}_n(R)$ to the map $R^m \to R^m$ obtained from $f_1 \circ \phi \circ f_1^{-1} \colon \mathrm{im}(f_1) \to \mathrm{im}(f_1)$ by extending over C by the identity. □

Remark 1.3. Our definition of $\mathtt{VIC}(R)$ is slightly different from the one in [15], which requires that a $\mathtt{VIC}(R)$-morphism (f_1, f_2) also have $\ker(f_2)$ free. For finite (and, more generally, Artinian) rings, this added condition is superfluous: $\ker(f_2)$ is in any case stably free, and for Artinian rings finitely generated stably free modules are free (see [13, Example I.4.7.3]; rings with this property are called *Hermite rings*). □

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[1]The purpose of considering R^n as a right R-module is that $\mathrm{GL}_n(R)$ acts on R^n on the left by right R-module automorphisms.

Main theorem. Fix a left Noetherian ring **k**. In [15], it is proven that for a finite *commutative* ring R, the categories of $\mathtt{V}(R)$- and $\mathtt{VI}(R)$- and $\mathtt{VIC}(R)$-modules over **k** are all locally Noetherian (see [19] for alternate proofs for $\mathtt{V}(R)$ and $\mathtt{VI}(R)$, but not for $\mathtt{VIC}(R)$). However, in many situations (e.g., in algebraic K-theory), it is important to study $\mathrm{GL}_n(R)$ where R is a noncommutative ring. For instance, R might be the group ring $\mathbb{F}_p[G]$ of a finite group G. Our main theorem addresses this more general situation:

Theorem A. *Let R be a finite ring, not necessarily commutative, and let **k** be a left Noetherian ring. Then the categories of $\mathtt{V}(R)$- and $\mathtt{VI}(R)$- and $\mathtt{VIC}(R)$-modules over **k** are locally Noetherian.*

Remark 1.4. In Theorem A, we allow not just the finite rings R to be noncommutative, but also the base rings **k**. In fact, for R commutative the proof of Theorem A in [15] works in that level of generality. $\qquad\square$

Remark 1.5. For infinite commutative R, the authors proved in [15] that the categories of $\mathtt{V}(R)$- and $\mathtt{VI}(R)$- and $\mathtt{VIC}(R)$-modules over a ring **k** are *not* locally Noetherian. The same argument works for infinite noncommutative R. See [9] for one way to get around this for $R = \mathbb{Z}$. $\qquad\square$

Application: twisted homological stability. A basic theorem of van der Kallen [21] says that for rings R satisfying mild hypotheses (for instance, all finite rings), the groups $\mathrm{GL}_n(R)$ satisfy *homological stability*, i.e., for all p, we have

$$\mathrm{H}_p(\mathrm{GL}_n(R); \mathbb{Z}) \cong \mathrm{H}_p(\mathrm{GL}_{n+1}(R); \mathbb{Z}) \quad \text{for } n \gg p.$$

In fact, building on ideas of Dwyer [5], van der Kallen is even able to prove this for certain twisted coefficient systems (those that are "polynomial" in an appropriate sense). For example, he is able to show for all $m \geq 0$ that we have

$$\mathrm{H}_p(\mathrm{GL}_n(R); (R^n)^{\otimes m}) \cong \mathrm{H}_p(\mathrm{GL}_{n+1}(R); (R^{n+1})^{\otimes m}) \quad \text{for } n \gg p.$$

See [17] and [14] for alternate proofs of this that use the language of $\mathtt{VIC}(R)$-modules to encode the twisted coefficients.

In [15, §4], the authors showed how to deduce a much more general version of this for finite commutative rings from the local Noetherianity of $\mathtt{VIC}(R)$. Given our new Theorem A, the exact same argument gives the following result for finite noncommutative rings. For a $\mathtt{VIC}(R)$-module M, write M_n for the value of M on $[R^n] \in \mathtt{VIC}(R)$. The $\mathtt{VIC}(R)$-endomorphisms of $[R^n]$ form the group $\mathrm{GL}_n(R)$, so M_n is a representation of $\mathrm{GL}_n(R)$.

Theorem B. *Let R be a finite ring, not necessarily commutative, and let M be a finitely generated $\mathtt{VIC}(R)$-module over a left Noetherian ring **k**. Then for all $p \geq 0$, we have*

$$\mathrm{H}_p(\mathrm{GL}_n(R); M_n) \cong \mathrm{H}_p(\mathrm{GL}_{n+1}(R); M_{n+1})$$

for $n \gg p$.

Remark 1.6. The proof of Theorem B for commutative rings in [15, §4] uses the more stringent definition of $\mathtt{VIC}(R)$ discussed in Remark 1.3, which as we discussed there is equivalent to ours for finite rings. $\qquad\square$

Remark 1.7. The references [21, 17, 14] give explicit estimates of when this stability occurs. Since we apply our non-effective Noetherianity theorem, we are not able to give such an estimate. However, our theorem applies to much more general coefficient systems than the polynomial ones considered in [21, 17, 14]. □

Ideas from proof. We will derive Theorem A for $\mathtt{V}(R)$ and $\mathtt{VI}(R)$ from the case of $\mathtt{VIC}(R)$, so we will focus on that category. In [15], this is dealt with for finite commutative R by a sort of Gröbner basis argument that was introduced to the theory of representation stability in [19] (though the general theorems of [19] do not apply to $\mathtt{VIC}(R)$; also, we remark that a similar kind of argument appeared much earlier in work of Richter [18]). We do the same thing, but the details are far harder. The main issue is that finite noncommutative rings are much more complicated than finite commutative rings. Indeed, the starting point of the proof in [15] is the fact that finite commutative rings are Artinian, and thus are the product of finitely many local rings. Local rings are not that different from fields, so in the end we can mostly focus on the case of finite fields. Unfortunately, noncommutative Artinian rings are not nearly as well-behaved, which greatly complicates the proof.

Convention: left vs right modules. Throughout this paper, we emphasize that column vectors R^n are considered as right R-modules. With this convention, the group $\mathrm{GL}_n(R)$ acts on R^n on the left by right R-module homomorphisms. If we wanted to deal with left R-modules, then we would have to use row vectors and have $\mathrm{GL}_n(R)$ act on the right.

Outline. We start in §2 by reducing to proving local Noetherianity for an "ordered" version of $\mathtt{VIC}(R)$ called $\mathtt{OVIC}(R)$. The rest of the paper is devoted to this: in §3, we discuss the structure of finite noncommutative rings, in §4 we define $\mathtt{OVIC}(R)$ and give its basic properties, and finally in §5 we prove that the category of $\mathtt{OVIC}(R)$-modules is locally Noetherian.

Remark 1.8. Some parts of our argument are the same as in [15], but we tried to make this paper mostly self-contained at least for $\mathtt{VIC}(R)$. The fact that we will focus on this single category will allow us to write in a much less abstract way, so one side benefit is that we think some of the details of the proof here will be a little easier to parse. □

2 Reduction to ordered VIC

Instead of working with $\mathtt{VIC}(R)$ directly, our proof will focus on a subcategory $\mathtt{OVIC}(R)$. The "O" stands for "ordered". Its main properties are as follows:

Theorem 2.1. *Let R be a finite ring. There exists a subcategory $\mathtt{OVIC}(R)$ of $\mathtt{VIC}(R)$ with the following properties:*

  (a) The objects of $\mathtt{OVIC}(R)$ are the same as $\mathtt{VIC}(R)$: the finite-rank free R-modules R^n for $n \geq 0$.

(b) Every $\mathtt{VIC}(R)$*-morphism* $f\colon [R^d] \to [R^n]$ *can be factored as*

$$[R^d] \xrightarrow{f_1} [R^d] \xrightarrow{f_2} [R^n],$$

where $f_1\colon [R^d] \to [R^d]$ *is a* $\mathtt{VIC}(R)$*-morphism and* $f_2\colon [R^d] \to [R^n]$ *is an* $\mathtt{OVIC}(R)$*-morphism.*

(c) The category of $\mathtt{OVIC}(R)$*-modules over a left Noetherian ring* **k** *is locally Noetherian.*

The proof of Theorem 2.1 is spread throughout the rest of the paper: in §3, we discuss some ring-theoretic preliminaries, in §4 we construct $\mathtt{OVIC}(R)$ and prove part (b) of Theorem 2.1 (see Proposition 4.7), and finally in §5 we prove part (c) of Theorem 2.1 (see Proposition 5.5). Here we will show how to use Theorem 2.1 to prove Theorem A.

Proof of Theorem A, assuming Theorem 2.1. Let R be a finite ring, not necessarily commutative, and let **k** be a left Noetherian ring. In [15, §2.4], the local Noetherianity of the categories of $\mathtt{V}(R)$- and $\mathtt{VI}(R)$-modules over **k** for finite commutative rings R are derived from the local Noetherianity of the category of $\mathtt{VIC}(R)$-modules over **k**. This derivation does not make use of the commutativity of R, so we must just prove that the category of $\mathtt{VIC}(R)$-modules over **k** is locally Noetherian.

Let M be a finitely generated $\mathtt{VIC}(R)$-module over **k**. Our goal is to prove that every $\mathtt{VIC}(R)$-submodule N of M is finitely generated. Theorem 2.1 says that for the subcategory $\mathtt{OVIC}(R)$ of $\mathtt{VIC}(R)$, the category of $\mathtt{OVIC}(R)$-modules over **k** is locally Noetherian. Via restriction, we can regard M as an $\mathtt{OVIC}(R)$-module and N as an $\mathtt{OVIC}(R)$-submodule of M. We will prove that M is finitely generated as an $\mathtt{OVIC}(R)$-module. Since the category of $\mathtt{OVIC}(R)$-modules over **k** is locally Noetherian, this will imply that N is finitely generated as an $\mathtt{OVIC}(R)$-module, and hence a fortiori is also finitely generated as a $\mathtt{VIC}(R)$-module, as desired.

We will prove that M is finitely generated as an $\mathtt{OVIC}(R)$-module by studying representable $\mathtt{VIC}(R)$-modules, which function similarly to free modules. For $d \geq 0$, let $P(d)$ be the $\mathtt{VIC}(R)$-module defined as follows:

- For $n \geq 0$, the **k**-module $P(d)_n$ is the free **k**-module with basis $\mathrm{Hom}_{\mathtt{VIC}(R)}(R^d, R^n)$.
- For a $\mathtt{VIC}(R)$-module morphism $g\colon [R^n] \to [R^m]$, the induced **k**-module morphism $P(d)_n \to P(d)_m$ is the one taking a basis element $f\colon [R^d] \to [R^n]$ of $P(d)_n$ to the basis element $g \circ f\colon [R^d] \to [R^m]$ of $P(d)_m$.

By Theorem 2.1, every $\mathtt{VIC}(R)$-module morphism $f\colon [R^d] \to [R^n]$ can be factored as

$$[R^d] \xrightarrow{f_1} [R^d] \xrightarrow{f_2} [R^n],$$

where $f_1\colon [R^d] \to [R^d]$ is a $\mathtt{VIC}(R)$-morphism and $f_2\colon [R^d] \to [R^n]$ is an $\mathtt{OVIC}(R)$-morphism. This implies that as an $\mathtt{OVIC}(R)$-module, $P(d)$ is generated by the set $\mathrm{Hom}_{\mathtt{VIC}(R)}(R^d, R^d) \subset P(d)_d$, which is finite since R is a finite ring.

For all $x \in M_d$ there exists a homomorphism of $\mathtt{VIC}(R)$-modules $P(d) \to M$ taking the element $\mathrm{id}\colon [R^d] \to [R^d]$ of $P(d)_d$ to x. The image of this $\mathtt{VIC}(R)$-module homomorphism is the $\mathtt{VIC}(R)$-submodule generated by x. Since M is finitely generated as a $\mathtt{VIC}(R)$-module, for some $d_1, \ldots, d_k \geq 1$ we can find elements $x_i \in M_{d_i}$ such that $\{x_1, \ldots, x_k\}$ generates M.

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Associated to these x_i is a surjective $\mathtt{VIC}(R)$-module homomorphism

$$\bigoplus_{i=1}^{k} P(d_i) \longrightarrow M.$$

This is, a fortiori, a homomorphism of $\mathtt{OVIC}(R)$-modules, and since each $P(d_i)$ is finitely generated as an $\mathtt{OVIC}(R)$-module, we conclude that M is as well. $\qquad\square$

3   The structure of Artinian rings

To discuss $\mathtt{OVIC}(R)$, we will need some basic facts about finite rings. In fact, the results we need hold more generally for Artinian rings, so we will state them in this level of generality. A suitable textbook reference is [12]. Throughout this section, R is an Artinian ring.

Peirce decomposition, I. We begin with some generalities (see [12, §21]). Assume that $\{e_1, \ldots, e_\mu\}$ are idempotent elements of R that are orthogonal (i.e., $e_i e_j = 0$ for distinct $1 \le i, j \le \mu$) and satisfy

$$1 = e_1 + \cdots + e_\mu.$$

Each $e_i R e_j$ is an additive subgroup of R, and we have the Peirce decomposition

$$R = \bigoplus_{i,j=1}^{\mu} e_i R e_j. \tag{3.1}$$

To make this a ring isomorphism, view elements of the right hand side as $\mu \times \mu$ matrices whose (i,j)-entries lie in $e_i R e_j$. Using the fact that

$$(e_i R e_k)(e_k R e_j) \subset e_i R e_j \quad \text{and} \quad (e_i R e_k)(e_{k'} R e_j) = 0 \quad \text{if } k \ne k',$$

we can multiply these matrices as usual, turning the right hand side of (3.1) into a ring and (3.1) into a ring isomorphism. Since $e_i R e_j \subset R$, we can view (3.1) as an embedding $\Phi \colon R \hookrightarrow \mathrm{Mat}_\mu(R)$ that we will call the *Peirce embedding*.[2]

Peirce decomposition, II. Continue with the notation of the previous paragraph. A more conceptual way to think about the Peirce embedding is as follows. Each $e_i R$ is a right R-module, and letting R_R denote R considered as a right R-module we have

$$R_R = \bigoplus_{i=1}^{\mu} e_i R.$$

Via left multiplication, the ring R acts on the left[3] on R_R by right R-module endomorphisms, and in fact $R \cong \mathrm{End}(R_R)$. We thus have

$$R = \mathrm{End}(R_R) = \bigoplus_{i,j=1}^{\mu} \mathrm{Hom}(e_j R, e_i R). \tag{3.2}$$

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[2]The fact that the target of the Peirce embedding is $\mathrm{Mat}_\mu(R)$ is just a matter of convenience so we do not have to precisely define a ring of "matrices" whose entries all lie in different places. Later on we will identify the various $e_i R e_j$ with division rings $\mathbb{D}_k$ and even additive groups $\mathbb{L}_{hk}$, and we suggest to the reader that they not focus too much on how these lie inside R.

[3]This is not a typo – we are using the fact that the left and right actions of R on itself commute, i.e., that R is an (R, R)-bimodule.

For all $1 \le i, j \le \mu$, we have $\mathrm{Hom}(e_j R, e_i R) = e_i R e_j$, where $\phi \in \mathrm{Hom}(e_j R, e_i R)$ corresponds to the element[4] $\phi(e_j) \in e_i R e_j$. Making these identifications turns (3.2) into (3.1). This makes it clear that the Peirce embedding reflects the left action of R on R_R; indeed, using

$$R_R = \bigoplus_{i=1}^{\mu} e_i R,$$

we can embed R_R into the set of length-μ column vectors R^μ, which is itself a right R-module. The matrices $\mathrm{Mat}_\mu(R)$ act on the left on R^μ by right R-module endomorphisms, and we have a commutative diagram

$$
\begin{array}{ccc}
R & \xrightarrow{\ \cong\ } & \mathrm{End}(R_R) \\
{\scriptstyle\Phi}\big\downarrow & & \big\downarrow \\
\mathrm{Mat}_\mu(R) & \xrightarrow{\ \cong\ } & \mathrm{End}(R^\mu).
\end{array}
$$

Jacobson radical. Let $J(R)$ be the Jacobson radical of R, i.e., the intersection of all left ideals of R. Alternatively, $J(R)$ consists of all $y \in R$ such that for all $x, z \in R$, the element $1 - xyz$ is a unit (see [12, Lemma 4.3]). Since R is Artinian, $J(R)$ can also be characterized as the largest ideal of R that is nilpotent, i.e., such that $J(R)^k = 0$ for $k \gg 0$ (see [12, Theorem 4.12]). Let $\overline{R} = R/J(R)$. For $x \in R$, let $\overline{x} \in \overline{R}$ be its image. Also, for a matrix $M \in \mathrm{Mat}_{n,m}(R)$, let $\overline{M} \in \mathrm{Mat}_{n,m}(\overline{R})$ be its image. The following simple fact will be important for us.

Lemma 3.1. *Let R be a ring and let $M \in \mathrm{Mat}_n(R)$ for some $n \ge 1$. Then M is invertible if and only if $\overline{M}$ is invertible.*

Proof. We have $J(\mathrm{Mat}_n(R)) = \mathrm{Mat}_n(J(R))$ (see [12, p. 61]), so $\overline{\mathrm{Mat}_n(R)} = \mathrm{Mat}_n(\overline{R})$. The result now follows from the fact that for *any* ring S (including, in particular, $S = \mathrm{Mat}_n(R)$), an element $x \in S$ is invertible if and only if $\overline{x} \in \overline{S}$ is invertible (see [12, Proposition 4.8]). $\square$

Artin–Wedderburn. The fact that R is Artinian implies that $\overline{R}$ is semisimple (see [12, Theorem 4.14]), which by the Artin–Wedderburn Theorem [12, Theorem 3.5] means that

$$\overline{R} \cong \mathrm{Mat}_{\mu_1}(\mathbb{D}_1) \times \cdots \times \mathrm{Mat}_{\mu_q}(\mathbb{D}_q) \tag{3.3}$$

for division rings $\mathbb{D}_1, \ldots, \mathbb{D}_q$. We remark that when R is finite as it is in most of this paper, Wedderburn's Little Theorem [12, Theorem 13.1] implies that the $\mathbb{D}_k$ are actually (commutative) fields. The decomposition (3.3) arises from orthogonal idempotents $\overline{e}_i^k \in \overline{R}$ for $1 \le k \le q$ and $1 \le i \le \mu_k$ satisfying

$$1 = (\overline{e}_1^1 + \cdots + \overline{e}_{\mu_1}^1) + \cdots + (\overline{e}_1^q + \cdots + \overline{e}_{\mu_q}^q) \tag{3.4}$$

as well as the following:

- For $1 \le k \le q$ and $1 \le i, j \le \mu_k$, we have $\overline{e}_i^k \overline{R} \overline{e}_j^k \cong \mathbb{D}_k$. This is an isomorphism of rings if $i = j$ (so $\overline{e}_i^k \overline{R} \overline{e}_j^k$ is closed under multiplication), and an isomorphism of additive groups if $i \ne j$.

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[4]Note that $\phi(e_j) = \phi(e_j^2) = \phi(e_j) e_j = e_i r e_j$ for some $r \in R$.

- For $1 \le k, k' \le q$ and $1 \le i, j \le \mu_k$ with $k \ne k'$, we have $\overline{e}_i^k \overline{R} \overline{e}_j^{k'} = 0$.

Setting $\mu = \mu_1 + \cdots + \mu_q$, the Peirce embedding associated to (3.4) is precisely the embedding $\overline{\Phi} \colon \overline{R} \hookrightarrow \mathrm{Mat}_\mu(\overline{R})$ taking an element of $\overline{R}$ to the matrices in (3.3), arranged as diagonal blocks in $\mathrm{Mat}_\mu(\overline{R})$. Here we are identifying the $\mathbb{D}_k$ with appropriate subrings and additive subgroups of $\overline{R}$ as above.

Lifting idempotents. Since $J(R)$ is nilpotent, idempotents in $\overline{R}$ can be lifted to R (see [12, Theorem 21.28]; we remark that a ring R such that $\overline{R}$ is semisimple and all idempotents in $\overline{R}$ can be lifted to R is called *semiperfect*). Combined with [12, Proposition 21.25] and the proof of [12, Theorem 23.6], this implies we can find orthogonal idempotents $e_i^k \in R$ for $1 \le k \le q$ and $1 \le i \le \mu_k$ lifting the $\overline{e}_i^k$ such that

$$1 = (e_1^1 + \cdots + e_{\mu_1}^1) + \cdots + (e_1^q + \cdots + e_{\mu_q}^q). \tag{3.5}$$

What is more, by [12, Proposition 21.21], we have

$$e_i^k R \cong e_{i'}^{k'} R \quad \Leftrightarrow \quad \overline{e}_i^k \overline{R} \cong \overline{e}_{i'}^{k'} \overline{R} \quad \Leftrightarrow \quad k = k'. \tag{3.6}$$

For $1 \le h, k \le q$, pick some $1 \le i \le \mu_h$ and $1 \le j \le \mu_k$ and set

$$\mathbb{L}_{hk} = e_i^h R e_j^k \cong \mathrm{Hom}(e_i^k R, e_j^h R).$$

This is a ring if $i = j$ and $k = h$, and an additive group otherwise. By (3.6), up to isomorphisms of rings or additive groups this does not depend on the choice of i and j. In particular, by taking $i = j$ we can identify $\mathbb{L}_{kk}$ with a ring. Observe the following:

- For distinct h and k the additive group $\mathbb{L}_{hk}$ projects to 0 in $\overline{R}$, so $\mathbb{L}_{hk}$ is an additive subgroup of the Jacobson radical $J(R)$.
- As in the previous paragraph, we can identify $\mathbb{D}_k$ with a corresponding subring of $\overline{R}$ such that $\mathbb{L}_{kk}$ projects to $\mathbb{D}_k$ in $\overline{R}$. By [12, Theorem 19.1], this implies that the rings $\mathbb{L}_{kk}$ are local rings, i.e., have a unique maximal left ideal (or equivalently, a unique maximal right ideal; see [12, Theorem 19.1]).

Summary. Recall that $\mu = \mu_1 + \cdots + \mu_q$. Using the isomorphisms (3.6), the Peirce embedding $\Phi \colon R \hookrightarrow \mathrm{Mat}_\mu(R)$ associated to (3.5) can be identified with a ring homomorphism that takes $x \in R$ to a $q \times q$ block matrix of the form

$$\Phi(x) = (\Phi_{hk}(x))_{h,k=1}^q \quad \text{with} \quad \Phi_{hk}(x) \in \mathrm{Mat}_{\mu_h, \mu_k}(\mathbb{L}_{hk}).$$

Here we are identifying the $\mathbb{L}_{hk}$ with appropriate subrings and additive subgroups of R. Moreover, identifying the $\mathbb{D}_k$ with appropriate subrings and additive subgroups of $\overline{R}$, we have

$$\overline{\Phi(x)} = \overline{\Phi}(\overline{x}) = \left(\overline{\Phi_{hk}(x)}\right)_{h,k=1}^q \quad \text{with} \quad \overline{\Phi_{hh}(x)} \in \mathrm{Mat}_{\mu_h}(\mathbb{D}_h) \text{ and } \overline{\Phi_{hk}(x)} = 0 \text{ for } h \ne k.$$

We will call this the *Artin–Wedderburn embedding* of R.

4 Ordered VIC: definition and basic properties

This section defines the subcategory $\mathtt{OVIC}(R)$ of $\mathtt{VIC}(R)$ and proves some basic facts about it. We do this in two steps: in §4.1, we deal with semisimple rings, and in §4.2 we deal with Artinian rings (and thus general finite rings).

4.1 Ordered `VIC` for semisimple rings

We start by introducing the notation we will use in this section. Let R be a semisimple ring, so

$$R \cong \mathrm{Mat}_{\mu_1}(\mathbb{D}_1) \times \cdots \times \mathrm{Mat}_{\mu_q}(\mathbb{D}_q) \tag{4.1}$$

for $\mu_1, \ldots, \mu_q \geq 1$ and division rings $\mathbb{D}_1, \ldots, \mathbb{D}_q$. Set $\mu = \mu_1 + \cdots + \mu_q$ and let $\Phi \colon R \hookrightarrow \mathrm{Mat}_\mu(R)$ be the Artin–Wedderburn embedding of R. For $x \in R$, the matrix $\Phi(x)$ thus consists of the matrices in (4.1), arranged as diagonal blocks in $\mathrm{Mat}_\mu(R)$.

An example. What we will do requires a lot of notation, so to help the reader follow it we start with an example. Assume that the decomposition (4.1) is of the form

$$R \cong \mathrm{Mat}_2(\mathbb{D}_1) \times \mathrm{Mat}_3(\mathbb{D}_2),$$

so the Artin–Wedderburn embedding of R is of the form $\Phi \colon R \hookrightarrow \mathrm{Mat}_5(R)$. Consider an R-linear map $h \colon R^3 \to R^2$. The map h can be represented by a 2×3 matrix with entries in R. Applying Φ to the entries of this matrix results in a block matrix $\Phi(h)$ of the form

$$\Phi(h) = \left(\begin{array}{ccc|ccc|ccc}
\star_{11} & \star_{12} & & \star_{13} & \star_{14} & & \star_{15} & \star_{16} & \\
\star_{21} & \star_{22} & & \star_{23} & \star_{24} & & \star_{25} & \star_{26} & \\
& \maltese_{11} & \maltese_{12} & \maltese_{13} & & \maltese_{14} & \maltese_{15} & \maltese_{16} & & \maltese_{17} & \maltese_{18} & \maltese_{19} \\
& \maltese_{21} & \maltese_{22} & \maltese_{23} & & \maltese_{24} & \maltese_{25} & \maltese_{26} & & \maltese_{27} & \maltese_{28} & \maltese_{29} \\
& \maltese_{31} & \maltese_{32} & \maltese_{33} & & \maltese_{34} & \maltese_{35} & \maltese_{36} & & \maltese_{37} & \maltese_{38} & \maltese_{39} \\
\hline
\star_{31} & \star_{32} & & \star_{33} & \star_{34} & & \star_{35} & \star_{36} & \\
\star_{41} & \star_{42} & & \star_{43} & \star_{44} & & \star_{45} & \star_{46} & \\
& \maltese_{41} & \maltese_{42} & \maltese_{43} & & \maltese_{44} & \maltese_{45} & \maltese_{46} & & \maltese_{47} & \maltese_{48} & \maltese_{49} \\
& \maltese_{51} & \maltese_{52} & \maltese_{53} & & \maltese_{54} & \maltese_{55} & \maltese_{56} & & \maltese_{57} & \maltese_{58} & \maltese_{59} \\
& \maltese_{61} & \maltese_{62} & \maltese_{63} & & \maltese_{64} & \maltese_{65} & \maltese_{66} & & \maltese_{67} & \maltese_{68} & \maltese_{69}
\end{array} \right),$$

where the $\star_{ij}$ are elements of $\mathbb{D}_1$, the $\maltese_{ij}$ are elements of $\mathbb{D}_2$, and the blank entries are zeros. Here we are identifying the $\mathbb{D}_k$ with appropriate subrings and additive subgroups of R as in the previous section. Let h_1 and h_2 be the submatrices of $\Phi(h)$ consisting of entries lying in $\mathbb{D}_1$ and $\mathbb{D}_2$, respectively, so

$$h_1 = \left(\begin{array}{cc|cc|cc}
\star_{11} & \star_{12} & \star_{13} & \star_{14} & \star_{15} & \star_{16} \\
\star_{21} & \star_{22} & \star_{23} & \star_{24} & \star_{25} & \star_{26} \\
\star_{31} & \star_{32} & \star_{33} & \star_{34} & \star_{35} & \star_{36} \\
\star_{41} & \star_{42} & \star_{43} & \star_{44} & \star_{45} & \star_{46}
\end{array} \right)
\qquad
h_2 = \left(\begin{array}{ccc|ccc|ccc}
\maltese_{11} & \maltese_{12} & \maltese_{13} & \maltese_{14} & \maltese_{15} & \maltese_{16} & \maltese_{17} & \maltese_{18} & \maltese_{19} \\
\maltese_{21} & \maltese_{22} & \maltese_{23} & \maltese_{24} & \maltese_{25} & \maltese_{26} & \maltese_{27} & \maltese_{28} & \maltese_{29} \\
\maltese_{31} & \maltese_{32} & \maltese_{33} & \maltese_{34} & \maltese_{35} & \maltese_{36} & \maltese_{37} & \maltese_{38} & \maltese_{39} \\
\maltese_{41} & \maltese_{42} & \maltese_{43} & \maltese_{44} & \maltese_{45} & \maltese_{46} & \maltese_{47} & \maltese_{48} & \maltese_{49} \\
\maltese_{51} & \maltese_{52} & \maltese_{53} & \maltese_{54} & \maltese_{55} & \maltese_{56} & \maltese_{57} & \maltese_{58} & \maltese_{59} \\
\maltese_{61} & \maltese_{62} & \maltese_{63} & \maltese_{64} & \maltese_{65} & \maltese_{66} & \maltese_{67} & \maltese_{68} & \maltese_{69}
\end{array} \right).$$

Regard $\Phi(h)$ as an R-linear map $\Phi(h) \colon R^{15} \to R^{10}$. Let

$$V_1 = \{\vec{v}(1)_1, \ldots, \vec{v}(1)_6\} \quad \text{and} \quad V_2 = \{\vec{v}(2)_1, \ldots, \vec{v}(2)_9\}$$

be the standard basis elements of R^{15} corresponding to the columns of $\Phi(h)$ where the entries must lie in $\mathbb{D}_1$ and $\mathbb{D}_2$, respectively, ordered using the natural ordering on the standard basis elements of R^{15}. In its natural ordering, the standard basis for R^{15} is thus

$$\{\vec{v}(1)_1, \vec{v}(1)_2, \vec{v}(2)_1, \vec{v}(2)_2, \vec{v}(2)_3, \vec{v}(1)_3, \vec{v}(1)_4, \vec{v}(2)_4, \vec{v}(2)_5, \vec{v}(2)_6, \vec{v}(1)_5, \vec{v}(1)_6, \vec{v}(2)_7, \vec{v}(2)_8, \vec{v}(2)_9\}.$$

Similarly, let

$$W_1 = \{\vec{w}(1)_1, \ldots, \vec{w}(1)_4\} \quad \text{and} \quad W_2 = \{\vec{w}(2)_1, \ldots, \vec{w}(2)_6\}$$

be the standard basis elements of R^{10} corresponding to the rows of $\Phi(h)$ where the entries must lie in $\mathbb{D}_1$ and $\mathbb{D}_2$, respectively. For $k = 1, 2$, the linear map $\Phi(h)$ takes each element of V_k to a linear combination of elements of W_k, and the resulting linear map from the span of the V_k to the span of the W_k is described by the matrix h_k. We will call $V_1 \cup V_2$ and $W_1 \cup W_2$ the *distinguished bases* of R^{15} and R^{10}, respectively.

Decomposing maps and labeling rows/columns. We now consider the general case, so R is of the form

$$R \cong \mathrm{Mat}_{\mu_1}(\mathbb{D}_1) \times \cdots \times \mathrm{Mat}_{\mu_q}(\mathbb{D}_q) \tag{4.2}$$

for $\mu_1, \ldots, \mu_q \geq 1$ and division rings $\mathbb{D}_1, \ldots, \mathbb{D}_q$. Set $\mu = \mu_1 + \cdots + \mu_q$ and let $\Phi \colon R \hookrightarrow \mathrm{Mat}_\mu(R)$ be the Artin–Wedderburn embedding of R. Consider an R-linear map $h \colon R^m \to R^n$. Regard h as an $n \times m$ matrix with entries in R, and let $\Phi(h) \colon R^{\mu m} \to R^{\mu n}$ be the linear map obtained by applying Φ to each entry in this matrix. Just as above, for $1 \leq k \leq q$ we can extract submatrices $h_k \colon R^{\mu_k m} \to R^{\mu_k n}$ of $\Phi(h)$ consisting of the entries of $\Phi(h)$ that are required to lie in $\mathbb{D}_k$, which is identified with appropriate subrings and additive subgroups of R.

The *labeling* on the rows of $\Phi(h)$ is the association of the pair (k, i) with $1 \leq k \leq q$ and $i \in \{1, \ldots, \mu_k n\}$ to the row of $\Phi(h)$ corresponding to the ith row of h_k. Similarly, the *labeling* on the columns of $\Phi(h)$ is the association of the pair (k, j) with $1 \leq k \leq q$ and $j \in \{1, \ldots, \mu_k m\}$ to the column of $\Phi(h)$ corresponding to the jth row of h_k.

Distinguished bases. We will need notation for the collections of basis elements of $R^{\mu m}$ and $R^{\mu n}$ corresponding to these submatrices. The *distinguished basis of $R^{\mu m}$* is defined as follows. For each $1 \leq k \leq q$, let $\{\vec{v}(k)_1, \ldots, \vec{v}(k)_{\mu_k m}\}$ be the portion of the standard basis of $R^{\mu m}$ corresponding to the columns of $\Phi(h)$ that are labeled by (k, j) for some $j \in \{1, \ldots, \mu_k m\}$, arranged in their natural increasing order. In its natural ordering, the standard basis for $R^{\mu m}$ is thus

$$\vec{v}(1)_1, \ldots, \vec{v}(1)_{\mu_1}, \vec{v}(2)_1, \ldots, \vec{v}(2)_{\mu_2}, \ldots, \vec{v}(q)_1, \ldots, \vec{v}(q)_{\mu_q}$$

followed by

$$\vec{v}(1)_{\mu_1+1}, \ldots, \vec{v}(1)_{\mu_1+\mu_1}, \vec{v}(2)_{\mu_2+1}, \ldots, \vec{v}(2)_{\mu_2+\mu_2}, \ldots, \vec{v}(q)_{\mu_q+1}, \ldots, \vec{v}(q)_{\mu_q+\mu_q},$$

etc., finally ending with

$$\vec{v}(1)_{(m-1)\mu_1+1}, \ldots, \vec{v}(1)_{(m-1)\mu_1+\mu_1}, \ldots, \vec{v}(q)_{(m-1)\mu_q+1}, \ldots, \vec{v}(q)_{(m-1)\mu_q+\mu_q}.$$

Similarly, the *distinguished basis of $R^{\mu n}$* is defined by letting $\{\vec{w}(k)_1, \ldots, \vec{w}(k)_{\mu_k n}\}$ for $1 \leq k \leq q$ be the portion of the standard basis of $R^{\mu n}$ corresponding to the rows of $\Phi(h)$ labeled by (k, i) for some $i \in \{1, \ldots, \mu_k n\}$, arranged in their natural increasing order. For all $1 \leq k \leq q$ and $0 \leq j \leq \mu_k m$, we thus have

$$\Phi(h)(\vec{v}(k)_j) \subset \bigoplus_{i=1}^{\mu_k n} \vec{w}(k)_i \cdot \mathbb{D}_k.$$

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Surjective maps. Now assume that $h\colon R^m \to R^n$ is a surjective R-linear map. The maps $h_k\colon \mathbb{D}_k^{\mu_k m} \to \mathbb{D}_k^{\mu_k n}$ discussed above are thus also surjective. Recall that linear algebra over division rings is similar to linear algebra over fields. In particular, notions of basis, dimension, etc. make sense in this noncommutative context. Considerations of dimension show that there exists some subset $S \subset \{1, \ldots, \mu_k m\}$ such that $\{\Phi(h)(\vec{v}(k)_i) \mid i \in S\}$ is a basis for the $\mathbb{D}_k$-submodule of $R^{\mu_k n}$ spanned by $\{\vec{w}(k)_1, \ldots, \vec{w}(k)_{\mu_k n}\}$. Define $\mathfrak{S}(h,k)$ to be the smallest such $S \subset \{1, \ldots, \mu_k m\}$ in the lexicographic ordering. In this ordering, for distinct $S_1, S_2 \subset \{1, \ldots, \mu_k m\}$ we have $S_1 < S_2$ if the minimal element in the symmetric difference of the S_i lies in S_1. The following lemma gives an alternate characterization of $\mathfrak{S}(h,k)$:

Lemma 4.1. *Let $h\colon R^m \to R^n$ be a surjective R-linear map. For $1 \le k \le q$, write $\mathfrak{S}(h,k) = \{j_1 < j_2 < \cdots < j_{\mu_k n}\}$. Then the j_i are the unique elements of $\{1, \ldots, \mu_k m\}$ satisfying the following two conditions:*

- *$\{\Phi(h)(\vec{v}(k)_{j_1}), \ldots, \Phi(h)(\vec{v}(k)_{j_{\mu_k n}})\}$ is a basis for the $\mathbb{D}_k$-module*

$$\bigoplus_{i=1}^{\mu_k n} \vec{w}(k)_i \cdot \mathbb{D}_k.$$

- *Consider $1 \le j \le \mu_k m$, and let $1 \le i_0 \le \mu_k n$ be the largest index such that $j_{i_0} \le j$. Then*

$$\Phi(h)(\vec{v}(k)_j) \in \bigoplus_{i=1}^{i_0} \Phi(h)(\vec{v}(k)_{j_i}) \cdot \mathbb{D}_k.$$

Proof. Immediate. $\qquad\square$

Column-adapted maps. This allows us to make the following definition. A surjective R-linear map $h\colon R^m \to R^n$ is *column-adapted* if it satisfies the following condition for each $1 \le k \le q$. Write $\mathfrak{S}(h,k) = \{j_1 < j_2 < \cdots < j_{\mu_k n}\}$. We then require that $\Phi(h)(\vec{v}(k)_{j_i}) = \vec{w}(k)_i$ for all $1 \le i \le \mu_k n$. In other words, the matrix corresponding to the linear map $h_k\colon \mathbb{D}_k^{\mu_k m} \to \mathbb{D}_k^{\mu_k n}$ discussed above is in reduced row echelon form.

Example 4.2. Let us return to the example in the beginning of this section where

$$R \cong \mathrm{Mat}_2(\mathbb{D}_1) \times \mathrm{Mat}_3(\mathbb{D}_2).$$

An $h\colon R^3 \to R^2$ satisfying the following is column-adapted:

$$\Phi(h) = \left(\begin{array}{cc|cc|cc}
\begin{matrix} 1 & \star \\ 0 & 0 \end{matrix} & & \begin{matrix} 0 & \star \\ 1 & \star \end{matrix} & & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \\
& \begin{matrix} 0 & 1 & \star \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & & \begin{matrix} 0 & 0 & \star \\ 1 & 0 & \star \\ 0 & 1 & \star \end{matrix} & & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \hline
\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & \\
& \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & & \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}
\end{array}\right).$$

Here the $\star$ are elements of $\mathbb{D}_1$, the $\star$ are elements of $\mathbb{D}_2$, and the blank entries are zeros. The $\mathfrak{S}(h,k)$ are $\mathfrak{S}(h,1) = \{1,3,5,6\}$ and $\mathfrak{S}(h,2) = \{2,4,5,7,8,9\}$. $\qquad\square$

As is well-known, the set of reduced row echelon matrices is closed under multiplication, and similarly the class of column-adapted maps is closed under composition:

Lemma 4.3. *Let $g\colon R^m \to R^n$ and $h\colon R^n \to R^\ell$ be column-adapted maps. Then $h \circ g\colon R^m \to R^\ell$ is column-adapted.*

Proof. Let $\vec{v}(k)_i$ and $\vec{w}(k)_i$ and $\vec{u}(k)_i$ be the distinguished bases for $R^{\mu m}$ and $R^{\mu n}$ and $R^{\mu \ell}$, respectively. Fix some $1 \leq k \leq q$, and write

$$\mathfrak{S}(g,k) = \{j_1 < j_2 < \cdots < j_{\mu_k n}\},$$
$$\mathfrak{S}(h,k) = \{j_1' < j_2' < \cdots < j_{\mu_k \ell}'\}.$$

For $1 \leq i \leq \mu_k \ell$, define $j_i'' = j_{j_i'}$. We thus have

$$\{j_1'' < j_2'' < \cdots < j_{\mu_k \ell}''\} \tag{4.3}$$

and

$$h \circ g(\vec{v}(k)_{j_i''}) = h \circ g(\vec{v}(k)_{j_{j_i'}}) = h(\vec{w}(k)_{j_i'}) = \vec{u}(k)_i.$$

From this, it is easy to see that (4.3) satisfies the criterion of Lemma 4.1, so $\mathfrak{S}(h \circ g, k)$ equals (4.3) and $h \circ g$ is column-adapted. $\qquad\square$

For later use, we record a corollary of the proof of this lemma.

Lemma 4.4. *Let $m \geq n \geq \ell$ and $1 \leq k \leq q$. Order subsets of $\{1, \ldots, \mu_k m\}$ and $\{1, \ldots, \mu_k n\}$ lexicographically. Let $f\colon R^m \to R^n$ and $g, h\colon R^n \to R^\ell$ be column-adapted maps such that $\mathfrak{S}(g,k) \leq \mathfrak{S}(h,k)$. Then $\mathfrak{S}(g \circ f, k) \leq \mathfrak{S}(h \circ f, k)$, with equality if and only if $\mathfrak{S}(g,k) = \mathfrak{S}(h,k)$.*

Proof. Immediate from the proof of Lemma 4.3. $\qquad\square$

Ordered VIC, semisimple case. By Lemma 4.3, it makes sense to define $\mathtt{OVIC}(R)$ to be the subcategory of $\mathtt{VIC}(R)$ whose objects are all the R^n with $n \geq 1$ and whose morphisms $f\colon [R^n] \to [R^m]$ are all the $\mathtt{VIC}(R)$-morphisms $f = (f', f'')$ such that f'' is column-adapted. Since the only column-adapted maps $R^n \to R^n$ are the identity, it follows that the identity is the only $\mathtt{OVIC}(R)$-endomorphism of $[R^n]$. In the next section, we will show how to generalize all of this to the case of Artinian R, and thus in particular to all finite R.

4.2   Ordered VIC for general Artinian rings

Let R be an Artinian ring. The structure of R was discussed in §3, and we briefly recall it. The quotient ring $\overline{R} = R/J(R)$ is semisimple, so

$$\overline{R} \cong \mathrm{Mat}_{\mu_1}(\mathbb{D}_1) \times \cdots \times \mathrm{Mat}_{\mu_q}(\mathbb{D}_q)$$

for $\mu_1, \ldots, \mu_q \geq 1$ and division rings $\mathbb{D}_1, \ldots, \mathbb{D}_q$. Set $\mu = \mu_1 + \cdots + \mu_q$. Let $\Phi\colon R \hookrightarrow \mathrm{Mat}_\mu(R)$ and $\overline{\Phi}\colon \overline{R} \hookrightarrow \mathrm{Mat}_\mu(\overline{R})$ be the Artin–Wedderburn embeddings of R and $\overline{R}$, so $\overline{\Phi(x)} = \overline{\Phi}(\overline{x})$ for all $x \in R$. Also, for $1 \leq h, k \leq q$ let $\mathbb{L}_{hk}$ be as defined in §3, so the $\mathbb{L}_{kk}$ are local rings and the $\overline{\mathbb{L}}_{hk}$ for distinct h and k are additive subgroups of the Jacobson radical $J(R)$. The

Artin–Wedderburn embedding $\Phi \colon R \hookrightarrow \mathrm{Mat}_\mu(R)$ can then be decomposed into a $q \times q$ block matrix of the form

$$\Phi(x) = (\Phi_{hk}(x))_{h,k=1}^q \quad \text{with} \quad \Phi_{hk}(x) \in \mathrm{Mat}_{\mu_h,\mu_k}(\mathbb{L}_{hk}),$$

and

$$\overline{\Phi(x)} = \overline{\Phi}(\overline{x}) = \left(\overline{\Phi_{hk}(x)}\right)_{h,k=1}^q \quad \text{with} \quad \overline{\Phi_{hh}(x)} \in \mathrm{Mat}_{\mu_h}(\mathbb{D}_h) \text{ and } \overline{\Phi_{hk}(x)} = 0 \text{ for } h \neq k.$$

Here the $\mathbb{L}_{hk}$ are embedded as appropriate subrings and additive subgroups of R, and similarly the $\mathbb{D}_k$ are embedded as appropriate subrings and additive subgroups of $\overline{R}$.

Distinguished bases and labeling rows/columns. Consider an R-linear map $h \colon R^m \to R^n$. Let $\overline{h} \colon \overline{R}^m \to \overline{R}^n$ be the induced map, and let $\Phi(h) \colon R^{\mu m} \to R^{\mu n}$ and $\overline{\Phi}(\overline{h}) \colon \overline{R}^{\mu m} \to \overline{R}^{\mu n}$ be the maps obtained by applying Φ and $\overline{\Phi}$ to the entries of matrices representing h and $\overline{h}$, respectively. As we discussed in §4.1, the rows of $\overline{\Phi}(\overline{h})$ are labeled by pairs (k, i) with $1 \leq k \leq q$ and $i \in \{1, \ldots, \mu_k n\}$ and the columns of $\overline{\Phi}(\overline{h})$ are labeled by pairs (k, j) with $1 \leq k \leq q$ and $j \in \{1, \ldots, \mu_k m\}$. We will similarly label the rows and columns of $\Phi(h)$.

For $0 \leq k \leq q$, let

$$\{\overline{\vec{v}(k)_1}, \ldots, \overline{\vec{v}(k)_{\mu_k m}}\} \quad \text{and} \quad \{\overline{\vec{w}(k)_1}, \ldots, \overline{\vec{w}(k)_{\mu_k n}}\} \tag{4.4}$$

be the distinguished bases for $\overline{R}^{\mu m}$ and $\overline{R}^{\mu n}$ discussed in §4.1. These were introduced to make sense of $\overline{\Phi}(\overline{h})$. We will need the exact same bases for $R^{\mu m}$ and $R^{\mu n}$, so let

$$\{\vec{v}(k)_1, \ldots, \vec{v}(k)_{\mu_k m}\} \quad \text{and} \quad \{\vec{w}(k)_1, \ldots, \vec{w}(k)_{\mu_k n}\}$$

be the subsets of the standard bases for $R^{\mu m}$ and $R^{\mu n}$ that map to (4.4) under the maps $R^{\mu m} \to \overline{R}^{\mu m}$ and $R^{\mu n} \to \overline{R}^{\mu n}$. For all $1 \leq k \leq q$ and $1 \leq j \leq \mu_k m$, we thus have

$$\Phi(h)(\vec{v}(k)_j) \in \bigoplus_{k'=1}^q \left(\bigoplus_{i=1}^{\mu_{k'} n} \vec{w}(k')_i \cdot \mathbb{L}_{k'k} \right). \tag{4.5}$$

S-function. Given a surjective map $h \colon R^m \to R^n$, the induced map $\overline{h} \colon \overline{R}^m \to \overline{R}^n$ is also surjective. For $1 \leq k \leq q$, we define

$$\mathfrak{S}(h, k) = \mathfrak{S}(\overline{h}, k) \subset \{1, \ldots, \mu_k m\},$$

so $|\mathfrak{S}(h, k)| = \mu_k n$.

Column-adapted maps. A surjective map $h \colon R^m \to R^n$ is said to be *column-adapted* if it satisfies the following two conditions:
  (i) The map $\overline{h} \colon \overline{R}^m \to \overline{R}^n$ is column-adapted in the sense of §4.1.
  (ii) For each $1 \leq k \leq q$, write $\mathfrak{S}(h, k) = \{j_1 < j_2 < \cdots < j_{\mu_k n}\}$. We then require that $\Phi(h)(\vec{v}(k)_{j_i}) = \vec{w}(k)_i$ for all $1 \leq i \leq \mu_k n$.
This class of maps is closed under composition:

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Lemma 4.5. *Let $h_1: R^m \to R^n$ and $h_2: R^n \to R^\ell$ be column-adapted maps. Then $h_2 \circ h_1: R^m \to R^\ell$ is column-adapted.*

Proof. By Lemma 4.3, the map $\overline{h_2 \circ h_1} = \overline{h}_2 \circ \overline{h}_1$ is column-adapted, so condition (i) is satisfied for $h_2 \circ h_1$. The same argument used in the proof of Lemma 4.3 then shows that condition (ii) is satisfied for $h_2 \circ h_1$. The lemma follows. $\qquad\qquad\square$

Canonical splittings. One of the key features of column-adapted maps is the following lemma. We will call the map g constructed in it the *canonical splitting* of h; as the lemma says, it only depends on the $\mathfrak{S}(h, k)$.

Lemma 4.6. *For each $1 \le k \le q$, let $S(k) \subset \{1, \dots, \mu_k m\}$ be a $\mu_k n$-element set. There then exists an R-linear map $g: R^n \to R^m$ such that if $h: R^m \to R^n$ is a column-adapted map with $\mathfrak{S}(h, k) = S(k)$ for all $1 \le k \le q$, then $h \circ g = \mathrm{id}$.*

Proof. Let $\vec{v}(k)_i$ and $\vec{w}(k)_i$ be the distinguished bases for $R^{\mu m}$ and $R^{\mu n}$, respectively. For $1 \le k \le q$, write
$$S(k) = \{j(k)_1, \dots, j(k)_{\mu_k n}\}.$$
Define $G: R^{\mu n} \to R^{\mu m}$ via the formula
$$G(\vec{w}(k)_i) = \vec{v}(k)_{j(k)_i} \qquad (1 \le k \le q, 1 \le i \le \mu_k n).$$
Since for all $1 \le k \le q$ and $1 \le i \le \mu_k n$ we trivially have
$$G(\vec{w}(k)_i) \in \bigoplus_{k'=1}^{q} \left(\bigoplus_{j=1}^{\mu_{k'} m} \vec{v}(k')_j \cdot \mathbb{L}_{k'k} \right),$$
it follows that there exists some $g: R^n \to R^m$ with $\Phi(g) = G$. If $h: R^m \to R^n$ is a column-adapted map with $\mathfrak{S}(h, k) = S(k)$ for all $1 \le k \le q$, then for all $1 \le k \le q$ and $1 \le i \le \mu_k n$ we have
$$\Phi(h) \circ \Phi(g)(\vec{w}(k)_i) = \Phi(h)(\vec{v}_{j(k)_i}) = \vec{w}(k)_i,$$
so $\Phi(h) \circ \Phi(g) = \mathrm{id}$ and thus $h \circ g = \mathrm{id}$. $\qquad\qquad\square$

Ordered VIC, Artinian case. By Lemma 4.5, it makes sense to define $\mathtt{OVIC}(R)$ to be the subcategory of $\mathtt{VIC}(R)$ whose objects are all the R^n with $n \ge 0$ and whose morphisms $f: [R^n] \to [R^m]$ are all the $\mathtt{VIC}(R)$-morphisms $f = (f', f'')$ such that f'' is column-adapted. Since the only column-adapted maps $R^n \to R^n$ are the identity, it follows that the identity is the only $\mathtt{OVIC}(R)$-endomorphism of $[R^n]$.

Factoring VIC-morphisms. The following proposition verifies part (b) of Theorem 2.1:

Proposition 4.7. *Let R be an Artinian ring. Every $\mathtt{VIC}(R)$-morphism $f: [R^d] \to [R^n]$ can be factored as*
$$[R^d] \xrightarrow{f_1} [R^d] \xrightarrow{f_2} [R^n],$$
where $f_1: [R^d] \to [R^d]$ is a $\mathtt{VIC}(R)$-morphism and $f_2: [R^d] \to [R^n]$ is an $\mathtt{OVIC}(R)$-morphism.

Proof. Write $f = (f', f'')$, where $f': R^d \to R^n$ is an injection and $f'': R^n \to R^d$ is a splitting of f', so $f'' \circ f' = \mathrm{id}$.

Let $\vec{v}(k)_i$ and $\vec{w}(k)_i$ be the distinguished bases of $R^{\mu n}$ and $R^{\mu d}$, respectively. Also, write

$$\mathfrak{S}(f'', k) = \{j(k)_1 < \cdots < j(k)_{\mu_k d}\} \subset \{1, \ldots, \mu_k n\}.$$

Define $G: R^{\mu d} \to R^{\mu d}$ via the formula

$$G(\vec{w}(k)_i) = \Phi(f'')(\vec{v}(k)_{j(k)_i}) \qquad (1 \le k \le q, 1 \le i \le \mu_k d).$$

Using (4.5) for $h = f''$, for $1 \le k \le q$ and $1 \le i \le \mu_k d$ we have

$$G(\vec{w}(k)_i) \in \bigoplus_{k'=1}^{q} \left(\bigoplus_{j=1}^{\mu_{k'} d} \vec{w}(k')_j \cdot \mathbb{L}_{k'k} \right).$$

From this, we see that there exists some $g: R^d \to R^d$ such that $G = \Phi(g)$.

Since the columns of $\Phi(\overline{g})$ are a basis for $\overline{R}^{\mu d}$, it follows that $\overline{g}$ is an isomorphism, so by Lemma 3.1 it follows that g is an isomorphism. By construction, the map $g^{-1} \circ f''$ is column-adapted, so $f_2 = (f' \circ g, g^{-1} \circ f'')$ is an $\mathtt{OVIC}(R)$-morphism. Setting $f_1 = (g^{-1}, g)$, the map f_1 is a $\mathtt{VIC}(R)$-morphism and $f = f_2 \circ f_1$, as desired. $\qquad\square$

Free and dependent rows. Consider an $\mathtt{OVIC}(R)$-morphism $f: [R^n] \to [R^m]$ with $f = (f', f'')$. The condition that f'' is column-adapted is a condition on the columns of $\Phi(f'') \in \mathrm{Mat}_{\mu n, \mu m}(R)$. We now discuss the rows of $\Phi(f') \in \mathrm{Mat}_{\mu m, \mu n}(R)$. We will call the rows of $\Phi(f')$ that are labeled (k, i) for some $1 \le k \le q$ and $i \in \mathfrak{S}(f'', k) \subset \{1, \ldots, \mu_k m\}$ the *dependent rows*, and all the other rows will be called the *free rows*. The reason for this terminology is the following lemma:

Lemma 4.8. *Let R be an Artinian ring. Consider $\mathtt{OVIC}(R)$-morphisms $f_1, f_2: [R^n] \to [R^m]$ with $f_i = (f_i', f_i'')$. Assume that $f_1'' = f_2''$ and that the free rows of $\Phi(f_1')$ and $\Phi(f_2')$ are equal. Then $f_1 = f_2$.*

Proof. What this lemma is saying is that the dependent rows of $\Phi(f_i')$ are determined by the free rows together with the fact that $f_i'' \circ f_i' = \mathrm{id}$. This is immediate from Lemma 4.9 below. $\qquad\square$

Lemma 4.9. *Let R be any ring. For some $a \le b$, let $X \in \mathrm{Mat}_{a,b}(R)$ and $Y, Y' \in \mathrm{Mat}_{b,a}(R)$ be matrices such that $XY = \mathrm{id}$ and $XY' = \mathrm{id}$. Also, let $I \subset [b]$ be a set with $|I| = a$ such that the submatrix of X consisting of the columns of X lying in I is a permutation matrix, i.e., can be transformed into the identity matrix by reordering its columns. Assume that the submatrices of Y and Y' consisting of the rows lying in $[b] \setminus I$ are equal. Then $Y = Y'$.*

Proof. This is a simple fact about matrix multiplication that is easier to grasp from an example rather than a formal proof: if for instance we have

$$X = \begin{pmatrix} * & 0 & 0 & * & 1 & * \\ * & 1 & 0 & * & 0 & * \\ * & 0 & 1 & * & 0 & * \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} * & * & * \\ \diamond & \diamond & \diamond \\ \diamond & \diamond & \diamond \\ * & * & * \\ \diamond & \diamond & \diamond \\ * & * & * \end{pmatrix},$$

then the $\diamond$-entries of Y are determined by the $*$-entries of X and Y along with the fact that $XY = \mathrm{id}$. $\qquad\square$

5   Ordered `VIC`: local Noetherianity

The goal of this section is to prove that the category of $\mathtt{OVIC}(R)$-modules over $\mathbf{k}$ is locally Noetherian for a finite ring R and a left Noetherian ring $\mathbf{k}$. This is proved in §5.3, which is preceded by two preliminary sections: §5.1 discusses well partial orders and §5.2 constructs a specific ordering that is needed for the proof.

5.1   Well partial orders

Let $(\mathfrak{P}, \leq)$ be a poset. We say that $\mathfrak{P}$ is *well partially ordered* if for any infinite sequence

$$p_1, p_2, p_3, \ldots \qquad (p_i \in \mathfrak{P}),$$

we can find indices $i_1 < i_2 < i_3 < \cdots$ such that

$$p_{i_1} \leq p_{i_2} \leq p_{i_3} \leq \cdots. \tag{5.1}$$

In fact, it is enough to just prove that

$$\text{there exist indices } i < j \text{ with } p_i \leq p_j. \tag{5.2}$$

Here is a quick proof of this. Letting $I = \{i \mid \text{there does not exist } j > i \text{ with } p_j \geq p_i\}$, if I is infinite then it provides a sequence of elements of $\mathfrak{P}$ violating (5.2), so I must be finite and we can find the sequence (5.1) starting with any index larger than all the indices in I.

We will need the following specific well partial ordering. Fix a finite set Σ, and let Σ^* be the set of words $s_1 \cdots s_p$ whose letters s_i are in Σ. Define a partial ordering on Σ^* by saying that $s_1 \cdots s_p \leq t_1 \cdots t_q$ if there exists a strictly increasing function $\lambda \colon \{1, \ldots, p\} \to \{1, \ldots, q\}$ with the following two properties:

- $s_i = t_{\lambda(i)}$ for $1 \leq i \leq p$, and
- for all $1 \leq j \leq q$, there exists some $1 \leq i \leq p$ such that $\lambda(i) \leq j$ and $t_{\lambda(i)} = t_j$.

We then have the following theorem, which is a variant on Higman's Lemma [10].

Lemma 5.1 ([19, Proposition 8.2.1])**.** *For all finite sets Σ, the ordering $(\Sigma^*, \leq)$ is a well partial ordering.*

Remark 5.2. An alternate proof of Lemma 5.1 can be found in [4, Proof of Prop. 7.5]. $\qquad\square$

5.2   An ordering of the generators

The key to our proof that the category of $\mathtt{OVIC}(R)$-modules over $\mathbf{k}$ is locally Noetherian for a finite ring R and a left Noetherian ring $\mathbf{k}$ is the following lemma.

Lemma 5.3. *Let R be a finite ring and let $d \geq 0$. Define*

$$\mathfrak{P}(d) = \bigsqcup_{n=0}^{\infty} \mathrm{Hom}_{\mathtt{OVIC}(R)}(R^d, R^n).$$

There then exists a well partial ordering $\preceq$ on $\mathfrak{P}(d)$ along with an extension $\leq$ of $\preceq$ to a total ordering such that the following holds. Consider $\mathtt{OVIC}(R)$-morphisms $f: [R^d] \to [R^n]$ and $g: [R^d] \to [R^m]$ with $f \preceq g$. There then exists an $\mathtt{OVIC}(R)$-morphism $\phi: [R^n] \to [R^m]$ with the following two properties:

(i) $g = \phi \circ f$, and
(ii) if $h: [R^d] \to [R^n]$ is an $\mathtt{OVIC}(R)$-morphism such that $h < f$, then

$$\phi \circ h < \phi \circ f = g.$$

Remark 5.4. A total ordering $\leq$ that extends a well partial ordering $\preceq$ (as in Lemma 5.3) is a well-ordering. □

Proof of Lemma 5.3. The notation will be as in §4.2. Our finite ring R is Artinian, so $\overline{R} = R/J(R)$ is semisimple and

$$\overline{R} \cong \mathrm{Mat}_{\mu_1}(\mathbb{D}_1) \times \cdots \times \mathrm{Mat}_{\mu_q}(\mathbb{D}_q)$$

for $\mu_1, \ldots, \mu_q \geq 1$ and division rings $\mathbb{D}_1, \ldots, \mathbb{D}_q$. Set $\mu = \mu_1 + \cdots + \mu_q$. Let $\Phi: R \hookrightarrow \mathrm{Mat}_\mu(R)$ and $\overline{\Phi}: \overline{R} \hookrightarrow \mathrm{Mat}_\mu(\overline{R})$ be the Artin–Wedderburn embeddings of R and $\overline{R}$, so $\overline{\Phi(x)} = \overline{\Phi}(\overline{x})$ for all $x \in R$.

Step 1. *We construct the total order $\leq$ on $\mathfrak{P}(d)$.*

Fix an arbitrary total order on $R^{\mu d}$. Consider $\mathtt{OVIC}(R)$-morphisms $f: [R^d] \to [R^n]$ and $g: [R^d] \to [R^m]$ in $\mathfrak{P}(d)$. Write $f = (f', f'')$ and $g = (g', g'')$. We determine if $f < g$ via the following procedure:

- If $n < m$, then $f < g$.
- Otherwise, assume that $n = m$. For each $1 \leq k \leq q$, we have the $\mu_k d$-element subsets $\mathfrak{S}(f'', k)$ and $\mathfrak{S}(g'', k)$ of $\{1, \ldots, \mu_k m\}$. Order subsets of $\{1, \ldots, \mu_k m\}$ lexicographically, and then further order tuples $(I_1, \ldots, I_q)$ with $I_k \subset \{1, \ldots, \mu_k m\}$ lexicographically. If

$$(\mathfrak{S}(f'', 1), \ldots, \mathfrak{S}(f'', q)) < (\mathfrak{S}(g'', 1), \ldots, \mathfrak{S}(g'', q))$$

  using this order, then $f < g$.
- Otherwise, assume that $n = m$ and that $\mathfrak{S}(f'', k) = \mathfrak{S}(g'', k)$ for all $1 \leq k \leq q$. Compare the columns of matrices in $\mathrm{Mat}_{\mu d, \mu n}(R)$ using our fixed total order on $R^{\mu d}$ and the lexicographic order. If under this ordering the columns of $\Phi(f'')$ are less than the columns of $\Phi(g'')$, then $f < g$.
- Otherwise, assume that $n = m$ and that $f'' = g''$. Compare the free rows of $\Phi(f') \in \mathrm{Mat}_{\mu n, \mu d}(R)$ and $\Phi(g') \in \mathrm{Mat}_{\mu n, \mu d}(R)$ using our fixed total order on $R^{\mu d}$ and the lexicographic order. If under this ordering the free rows of $\Phi(f')$ are less than the rows of $\Phi(g')$, then $f < g$.

By Lemma 4.8, this determines a total order $\leq$ on $\mathfrak{P}(d)$.

Step 2. *We define the notion of a stabilization of an* $\texttt{OVIC}(R)$*-morphism.*

Let $f\colon[R^d]\to[R^n]$ be an $\texttt{OVIC}(R)$-morphism and let $1\le a\le b\le n$. An (a,b)*-stabilization* of f (or simply a *stabilization* if we do not want to specify a and b) is an $\texttt{OVIC}(R)$-morphism $g\colon[R^d]\to[R^{n+1}]$ related to f as follows. Write $f=(f',f'')$ and $g=(g',g'')$. Regard $f'\colon R^d\to R^n$ and $f''\colon R^n\to R^d$ and $g'\colon R^d\to R^{n+1}$ and $g''\colon R^{n+1}\to R^d$ as matrices. We then require the following three conditions to hold:

- g'' is obtained from f'' by inserting a copy of the ath column of f'' after the bth column.
- Let $\widehat{g}'\colon R^d\to R^{n+1}$ be the matrix obtained from f' by inserting a copy of the ath row of f' after the bth row. We then require all the free rows[5] of $\Phi(g')$ to equal the corresponding rows of $\Phi(\widehat{g}')$.
- None of the dependent rows of $\Phi(f')$ are contained in rows coming from the ath row of f'.

Note that these three conditions imply that $\mathfrak{S}(f'',k)=\mathfrak{S}(g'',k)$ for all $1\le k\le q$. We remark that Lemma 4.8 implies that an (a,b)-stabilization is unique if it exists. Technically, our proof does not require proving that it exists, but we remark that its existence can be derived from the argument of Step 5 below, which constructs an $\texttt{OVIC}(R)$-morphism $\phi\colon[R^n]\to[R^{n+1}]$ such that $g=\phi\circ f$ is the (a,b)-stabilization of f.

Step 3. *We construct the partial order* $\preceq$ *such that* $\le$ *is a refinement of* $\preceq$.

Consider $\texttt{OVIC}(R)$-morphisms $f\colon[R^d]\to[R^n]$ and $g\colon[R^d]\to[R^m]$ in $\mathfrak{P}(d)$. We then say that $f\prec g$ if for some $r\ge 1$ there exists a sequence of $\texttt{OVIC}(R)$-morphisms

$$f=h_0,h_1,\ldots,h_r=g,$$

where for $0\le i<r$ the $\texttt{OVIC}(R)$-morphism h_{i+1} is a stabilization of the $\texttt{OVIC}(R)$-morphism h_i. Since a stabilization increases the rank of the codomain by 1, this implies that $m=n+r$, and in particular that $n<m$. This clearly defines a partial ordering $\preceq$ on $\mathfrak{P}(d)$, and since $f\prec g$ required $n<m$ our total ordering $\le$ from Step 1 refines $\preceq$.

Step 4. *We prove that* $\preceq$ *is a well partial order.*

We will embed $(\mathfrak{P}(d),\preceq)$ into a poset $(\Sigma^*,\preceq)$ of words, where Σ is a finite set of letters and $\preceq$ is as[6] in Lemma 5.1. That lemma says that $(\Sigma^*,\preceq)$ is a well partial ordering, so this will imply that $(\mathfrak{P}(d),\preceq)$ is as well.

First, define

$$\widehat{R}=R\sqcup\{\clubsuit\},$$

where $\clubsuit$ is a formal symbol. Though $\widehat{R}$ is not a ring, it still makes sense to speak about matrices with entries in $\widehat{R}$. Define

$$\Sigma=\{(M_1,M_2)\mid M_1\in\operatorname{Mat}_{\mu,\mu d}(\widehat{R})\text{ and }M_2\in\operatorname{Mat}_{d,1}(R)\}.$$

---

[5]We cannot require $g'=\widehat{g}'$ since we need $g''\circ g'=\operatorname{id}$, which requires changing the dependent rows.

[6]Recall that for words $s_1\cdots s_p$ and $t_1\cdots t_q$ in Σ^*, we have $s_1\cdots s_p\preceq t_1\cdots t_q$ if there exists a strictly increasing function $\lambda\colon\{1,\ldots,p\}\to\{1,\ldots,q\}$ with the following two properties:

- $s_i=t_{\lambda(i)}$ for $1\le i\le p$, and
- for all $1\le j\le q$, there exists some $1\le i\le p$ such that $\lambda(i)\le j$ and $t_{\lambda(i)}=t_j$.

We then define a map $\iota\colon\mathfrak{P}(d) \to \Sigma^*$ in the following way.

Consider an element $f\colon[R^d] \to [R^n]$ of $\mathfrak{P}(d)$. Write $f = (f', f'')$. Let $r_1, \ldots, r_n \in \mathrm{Mod}_{1,d}(R)$ be the rows of the matrix representing $f'\colon R^d \to R^n$ and let $c_1, \ldots, c_n \in \mathrm{Mod}_{d,1}(R)$ be the columns of the matrix representing $f''\colon R^n \to R^d$. We have

$$\Phi(r_1), \ldots, \Phi(r_n) \in \mathrm{Mod}_{\mu,\mu d}(R),$$

and the word

$$(\Phi(r_1), c_1)\cdots(\Phi(R_n), c_n) \in \Sigma^*$$

contains all the information about f. However, this encoding is redundant since it contains not just the data of the free rows of f', but also the data of the dependent rows.

For each $1 \le i \le n$, we modify $\Phi(r_i) \in \mathrm{Mat}_{\mu,\mu d}(R)$ to form $\widehat{r_i} \in \mathrm{Mat}_{\mu,\mu d}(\widehat{R})$ in the following way. Each row of $\Phi(r_i)$ is a row of $\Phi(f')$. If that row is a free row, then do not change it. Otherwise, if it is a dependent row, then replace each entry in it with the formal symbol $\clubsuit$. The result is a matrix $\widehat{r_i} \in \mathrm{Mat}_{\mu,\mu d}(\widehat{R})$ some of whose entire rows consist of repeated $\clubsuit$'s.

We now define

$$\iota(f) = (\widehat{r_1}, c_1)(\widehat{r_2}, c_2)\cdots(\widehat{r_n}, c_n) \in \Sigma^*.$$

This is an injection since knowing $\iota(f)$, we can reconstruct f'' and all the free rows of $\Phi(f')$, and this determines f' and hence $f = (f', f'')$ by Lemma 4.8. That ι is order-preserving is immediate from the definitions.

Step 5. *We construct the ϕ satisfying (i).*

Consider $\mathtt{OVIC}(R)$-morphisms $f\colon[R^d] \to [R^n]$ and $g\colon[R^d] \to [R^m]$ with $f \le g$. Our goal is to construct an $\mathtt{OVIC}(R)$-morphism $\phi\colon[R^n] \to [R^m]$ such that $g = \phi \circ f$. Examining the definition of the partial ordering $\le$ in Step 3, we see that it is enough to deal with the case where g is an (a, b)-stabilization of f for some $1 \le a \le b \le n$. The general case can be dealt with by iterating this $m - n$ times.

Write $f = (f', f'')$ and $g = (g', g'')$. By definition, the following three things hold:

- g'' is obtained from f'' by inserting a copy of the ath column of f'' after the bth column.
- Let $\widehat{g}'\colon R^d \to R^{n+1}$ be the matrix obtained from f' by inserting a copy of the ath row of f' after the bth row. Then all the free rows of $\Phi(g')$ equal the corresponding rows of $\Phi(\widehat{g}')$.
- None of the dependent rows of $\Phi(f')$ are contained in rows coming from the ath row of f'.

Let $\psi\colon R^d \to R^n$ be the canonical splitting of f'' (see Lemma 4.6). Let $\mathfrak{c} \in R^d$ be the ath column of the matrix representing f'', and set $\widehat{\mathfrak{c}} = \psi(\mathfrak{c}) \in R^n$. We then define $\phi = (\phi', \phi'')$ in the following way:

- $\phi''\colon R^{n+1} \to R^n$ is represented by the matrix obtained by inserting $\widehat{\mathfrak{c}}$ after the bth column of $\mathrm{id}\colon R^n \to R^n$.
- $\phi'\colon R^n \to R^{n+1}$ is represented by the matrix obtained by first subtracting $\widehat{\mathfrak{c}}$ from the ath column of $\mathrm{id}\colon R^n \to R^n$, and then inserting the row $(0, \ldots, 0, 1, 0, \ldots, 0)$ with a 1 in position a after the bth row.

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For example, for $n = 7$ and $a = 3$ and $b = 4$ we would have

$$\phi'' = \begin{pmatrix} 1 & 0 & 0 & 0 & \widehat{\mathfrak{c}}_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \widehat{\mathfrak{c}}_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \widehat{\mathfrak{c}}_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \widehat{\mathfrak{c}}_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \widehat{\mathfrak{c}}_5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \widehat{\mathfrak{c}}_6 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \widehat{\mathfrak{c}}_7 & 0 & 0 & 1 \end{pmatrix} \qquad \phi' = \begin{pmatrix} 1 & 0 & -\widehat{\mathfrak{c}}_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\widehat{\mathfrak{c}}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\widehat{\mathfrak{c}}_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\widehat{\mathfrak{c}}_4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\widehat{\mathfrak{c}}_5 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\widehat{\mathfrak{c}}_6 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\widehat{\mathfrak{c}}_7 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is clear that $\phi'' \circ \phi' = \text{id}$ and that the matrix representing $f'' \circ \phi'' \colon R^{n+1} \to R^d$ is obtained by inserting $f''(\widehat{\mathfrak{c}}) = \mathfrak{c}$ after the bth column of the matrix representing f''. Moreover, examining the construction of the canonical splitting in Lemma 4.6, we see that the entries of $\Phi(\widehat{\mathfrak{c}}) \in R^{\mu n}$ lying in the free rows of $\Phi(f')$ are all 0, so the matrix corresponding to $\phi' \circ f'$ is obtained by first inserting a copy of the ath row of the matrix representing f' after the bth row of that matrix, and then possibly modifying the dependent rows.

Step 6. *We prove that the ϕ we constructed satisfy (ii).*

Just like in the previous step, it is enough to deal with the case $g \colon [R^d] \to [R^{n+1}]$ is a stabilization of $f \colon [R^d] \to [R^n]$. Consider some $\texttt{OVIC}(R)$-morphism $h \colon [R^d] \to [R^n]$ such that $h < f$. Our goal is to prove that $\phi \circ h < \phi \circ f$. Write $f = (f', f'')$ and $h = (h', h'')$ and $\phi = (\phi', \phi'')$.

Examining the construction of the total ordering $\leq$ in Step 1, we see that there are three cases we have to deal with. The first is where

$$(\mathfrak{S}(h'', 1), \ldots, \mathfrak{S}(h'', q)) < (\mathfrak{S}(f'', 1), \ldots, \mathfrak{S}(f'', q)),$$

where subsets of $\{1, \ldots, \mu_k n\}$ are ordered using the lexicographic ordering and these tuples are further ordered using the lexicographic ordering. Lemma 4.4 then implies that

$$(\mathfrak{S}(h'' \circ \phi'', 1), \ldots, \mathfrak{S}(h'' \circ \phi'', q)) < (\mathfrak{S}(f'' \circ \phi'', 1), \ldots, \mathfrak{S}(f'' \circ \phi'', q)),$$

so $\phi \circ h < \phi \circ f$.

The second case is where $\mathfrak{S}(h'', k) = \mathfrak{S}(f'', k)$ for all k, but the columns of $\Phi(h'')$ are less than the columns of $\Phi(f'')$ in the lexicographic ordering (using our fixed total ordering on $R^{\mu d}$). In this case, it follows from our construction of ϕ that the matrix representing $h'' \circ \phi''$ is obtained from the matrix representing h'' by inserting a copy of the ath column of the matrix representing f'' after the bth column, and similarly for $f'' \circ \phi''$. This implies that the columns of $\Phi(h'' \circ \phi'')$ remain less than the columns of $\Phi(f'' \circ \phi'')$, so $\phi \circ h < \phi \circ f$.

The final case is where $h'' = f''$, but the free rows of $\Phi(h')$ are less than the free rows of $\Phi(f')$ in the lexicographic ordering. In this case, $\Phi(\phi' \circ h')$ is obtained from $\Phi(h')$ by taking a bunch of free rows and duplicating them lower in the matrix, and similarly for $\Phi(\phi' \circ f')$ (with the same rows). It follows that the free rows of $\Phi(\phi' \circ h')$ remain less than the free rows of $\Phi(\phi' \circ f')$, so $\phi \circ h < \phi \circ f$. $\qquad\square$

5.3 Local Noetherianity

We now prove the following, which verifies part (c) of Theorem 2.1:

Proposition 5.5. *Let R be a finite ring and let $\mathbf{k}$ be a left Noetherian ring. Then the category of $\mathtt{OVIC}(R)$-modules over $\mathbf{k}$ is locally Noetherian.*

Proof. Just like in the proof of Theorem A in §2, we will prove this by studying representable modules. For $d \geq 0$, let $P(d)$ be the $\mathtt{OVIC}(R)$-module defined via the formula

$$P(d)_n = \mathbf{k}[\mathrm{Hom}_{\mathtt{OVIC}(R)}(R^d, R^n)] \qquad (n \geq 0).$$

As we discussed in the proof of Theorem A, every finitely generated $\mathtt{OVIC}(R)$-module over $\mathbf{k}$ is the surjective image of a direct sum of finitely many $P(d)$ (for differing choices of d). To prove that every submodule of such a finitely generated module is finitely generated, it is thus enough to prove this for $P(d)$.

We start with some preliminaries. Let $\preceq$ and $\leq$ be the orderings on

$$\mathfrak{P}(d) = \bigsqcup_{n=0}^{\infty} \mathrm{Hom}_{\mathtt{OVIC}(R)}(R^d, R^n)$$

provided by Lemma 5.3. For a nonzero $x \in P(d)_n$, define the *initial term* of x, denoted $\mathrm{init}(x)$, as follows. Write

$$x = \alpha_1 f_1 + \cdots + \alpha_k f_k \quad \text{with } \alpha_1, \ldots, \alpha_k \in \mathbf{k} \smallsetminus \{0\}$$
$$\text{and } f_1, \ldots, f_k \in \mathrm{Hom}_{\mathtt{OVIC}(R)}(R^d, R^n) \text{ pairwise distinct.}$$

Order these terms such that $f_1 < f_2 < \cdots < f_k$. Then $\mathrm{init}(x) = \alpha_k f_k$.

Next, for an $\mathtt{OVIC}(R)$-submodule M of $P(d)$, define the *initial module* $\mathcal{I}(M)_{\bullet}$ of M to be the ordered sequence of $\mathbf{k}$-modules defined via the formula

$$\mathcal{I}(M)_n = \mathbf{k}\{\mathrm{init}(x) \mid x \in M_n\} \qquad (n \geq 1).$$

Be warned that this need not be an $\mathtt{OVIC}(R)$-submodule of $P(d)$. However, we do have the following.

Claim. *If N and M are $\mathtt{OVIC}(R)$-submodules of $P(d)$ with $N \subset M$ and $\mathcal{I}(N)_{\bullet} = \mathcal{I}(M)_{\bullet}$, then $N = M$.*

Proof of claim. Assume otherwise, and let $n \geq 0$ be such that $N_n \subsetneq M_n$. Recall that $\leq$ is a well-order (c.f. Remark 5.4). Consider the nonempty set

$$\{f \mid \text{there exists } x \in M_n \smallsetminus N_n \text{ and } \alpha \in \mathbf{k} \smallsetminus \{0\} \text{ such that } \mathrm{init}(x) = \alpha f\}.$$

Since $\leq$ is a well total ordering, this set has a $\leq$-minimal element $f\colon [R^d] \to [R^n]$; indeed, if it did not we could find an infinite strictly decreasing sequence of elements in it. Let $x \in M_n \smallsetminus N_n$ satisfy $\mathrm{init}(x) = \alpha f$ with $\alpha \in \mathbf{k} \smallsetminus \{0\}$. By assumption, there exists some $y \in N_n$ such that $\mathrm{init}(y) = \alpha f$. The αf terms cancel in $x - y$, so $\mathrm{init}(x - y) = \beta g$ with $\beta \in \mathbf{k} \smallsetminus \{0\}$ and $g < f$. Since $x \in M_n \smallsetminus N_n$ and $y \in N_n$, we have $x - y \in M_n \smallsetminus N_n$, so this contradicts the minimality of f. $\qquad\square$

We now commence with the proof that every $\mathtt{OVIC}(R)$-submodule of $P(d)$ is finitely generated. Assume otherwise, so there exists a strictly increasing chain

$$M^0 \subsetneq M^1 \subsetneq M^2 \subsetneq \cdots$$

of $\mathtt{OVIC}(R)$-submodules of $P(d)$. By the above claim, the initial modules $\mathcal{I}(M^i)_\bullet$ must all be distinct, so for all $i \geq 1$ we can find some $n_i \geq 0$ such that there exists some

$$\alpha_i f_i \in \mathcal{I}(M^i)_{n_i} \smallsetminus \mathcal{I}(M^{i-1})_{n_i} \quad \text{with } \alpha_i \in \mathbf{k} \smallsetminus \{0\} \text{ and } f_i \colon [R^d] \to [R^{n_i}].$$

Let $x_i \in M^i_{n_i}$ be an element with $\mathrm{init}(x_i) = \alpha_i f_i$.

Since $\preceq$ is a well partial ordering, there exists some increasing sequence $i_1 < i_2 < i_3 < \cdots$ of indices such that

$$f_{i_1} \preceq f_{i_2} \preceq f_{i_3} \preceq \cdots.$$

Since $\mathbf{k}$ is a left Noetherian ring, there exists some $m \geq 1$ such that α_{m+1} is in the left $\mathbf{k}$-ideal generated by $\alpha_{i_1}, \ldots, \alpha_{i_m}$, i.e., we can write

$$\alpha_{m+1} = c_1 \alpha_{i_1} + \cdots + c_m \alpha_{i_m} \quad \text{with } c_1, \ldots, c_m \in \mathbf{k}.$$

For $1 \leq j \leq m$, the fact that $f_{i_j} \preceq f_{i_{m+1}}$ implies by part (i) of Lemma 5.3 that there exists some $\mathtt{OVIC}(R)$-morphism $\phi_j \colon [R^{n_{i_j}}] \to [R^{n_{i_{m+1}}}]$ such that $f_{i_{m+1}} = \phi_j \circ f_{i_j}$. Conclusion (ii) of Lemma 5.3 implies that $\mathrm{init}(\phi_j \circ x) = \alpha_j f_{i_{m+1}}$. Setting

$$y = \sum_{j=1}^{m} c_j (\phi_j \circ x_{i_j}) \in M^{i_m}_{n_{i_{m+1}}},$$

we thus see that

$$\mathrm{init}(y) = \sum_{j=1}^{m} c_j \alpha_j f_{i_{m+1}} = \alpha_{m+1} f_{i_{m+1}} = \mathrm{init}(x_{i_{m+1}}).$$

This contradicts the fact that

$$\mathrm{init}(x_{i_{m+1}}) \in \mathcal{I}(M^{i_{m+1}})_{n_{i_{m+1}}} \smallsetminus \mathcal{I}(M^{i_m})_{n_{i_{m+1}}}.$$

The proposition follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

References

[1] T. Church, J. S. Ellenberg and B. Farb, FI-modules and stability for representations of symmetric groups, Duke Math. J. 164 (2015), no. 9, 1833–1910, `arXiv:1204.4533v4`.

[2] T. Church, J. S. Ellenberg, B. Farb and R. Nagpal, FI-modules over Noetherian rings, Geom. Topol. 18 (2014), no. 5, 2951–2984, `arXiv:1210.1854v2`.

[3] T. Church and B. Farb, Representation theory and homological stability, Adv. Math. 245 (2013), 250–314, `arXiv:1008.1368v3`.

[4] J. Draisma and J. Kuttler, Bounded-rank tensors are defined in bounded degree, Duke Math. J. 163 (2014), no. 1, 35–63, `arXiv:1103.5336v2`.

[5] W. G. Dwyer, Twisted homological stability for general linear groups, Ann. of Math. (2) 111 (1980), no. 2, 239–251.

[6] B. Farb, Representation stability, Proceedings of the 2014 International Congress of Mathematicians. Volume II, 1173–1196, `arXiv:1404.4065v1`.

[7] V. Franjou and A. Touzé (eds.), *Lectures on functor homology*, Progress in Mathematics, 311, Birkhäuser/Springer, Cham, 2015.

[8] W. L. Gan and L. Li, Noetherian property of infinite EI categories, New York J. Math. **21** (2015), 369–382, `arXiv:1407.8235v3`.

[9] N. Harman, Effective and Infinite-Rank Superrigidity in the Context of Representation Stability, preprint 2019, `arXiv:1902.05603v1`.

[10] G. Higman, Ordering by divisibility in abstract algebras, Proc. London Math. Soc. (3) 2 (1952), 326–336.

[11] N. J. Kuhn, Generic representations of the finite general linear groups and the Steenrod algebra. II, *K*-Theory 8 (1994), no. 4, 395–428.

[12] T. Y. Lam, *A first course in noncommutative rings*, second edition, Graduate Texts in Mathematics, 131, Springer-Verlag, New York, 2001.

[13] T. Y. Lam, *Serre's problem on projective modules*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2006.

[14] A. Putman, A new approach to twisted homological stability, with applications to congruence subgroups, preprint 2021, `arXiv:2109.14015`

[15] A. Putman and S. V. Sam, Representation stability and finite linear groups, Duke Math. J. **166** (2017), no. 13, 2521–2598, `arXiv:1408.3694v3`.

[16] A. Putman, S. V. Sam and A. Snowden, Stability in the homology of unipotent groups, Algebra Number Theory **14** (2020), no. 1, 119–154, `arXiv:1711.11080v4`.

[17] O. Randal-Williams and N. Wahl, Homological stability for automorphism groups, Adv. Math. **318** (2017), 534–626, `arXiv:1409.3541`.

[18] G. Richter, Noetherian semigroup rings with several objects, in *Group and semigroup rings (Johannesburg, 1985)*, 231–246, North-Holland Math. Stud., 126, Notas Mat., 111, North-Holland, Amsterdam.

[19] S. V Sam and A. Snowden, Gröbner methods for representations of combinatorial categories, J. Amer. Math. Soc. 30 (2017), 159–203, `arXiv:1409.1670v3`.

[20] A. Scorichenko, Stable K-theory and functor homology over a ring, thesis 2000.

[21] W. van der Kallen, Homology stability for linear groups, Invent. Math. 60 (1980), no. 3, 269–295.

[22] J. C. H. Wilson, FI_W-modules and stability criteria for representations of classical Weyl groups, J. Algebra **420** (2014), 269–332, `arXiv:1309.3817v2`.

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