Irreducible Sp-representations and subgroup distortion in the mapping class group

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Abstract

We prove that various subgroups of the mapping class group $\mathrm{Mod}(\Sigma)$ of a surface Σ are at least exponentially distorted. Examples include the Torelli group (answering a question of Hamenstädt), the "point-pushing" and surface braid subgroups, and the Lagrangian subgroup. Our techniques include a method to compute lower bounds on distortion via representation theory and an extension of Johnson theory to arbitrary subgroups of $\mathrm{H}_1(\Sigma; \mathbb{Z})$.

1 Introduction

We begin with a basic motivating question. Manifolds M are commonly presented as gluings or (possibly singular) fiberings of simpler manifolds, together with the data specifying the gluing or fibering. In low dimensions, the gluing/fibering data commonly takes the form of an element, or a finite collection of elements, of the *mapping class group* Mod_g, which is the group of homotopy classes of orientation-preserving homeomorphisms of the closed, oriented, genus g surface Σ_g. Examples include Heegaard decompositions of 3-manifolds, monodromies of surface bundles, and monodromies of Lefschetz fibrations.

The topology of the resulting manifold M can often be controlled by requiring that the gluing data lie in various special subgroups $K < \mathrm{Mod}_g$. We then have a purely group-theoretic problem: determine whether or not a given element $f \in \mathrm{Mod}_g$, given as a product of generators of Mod_g (say a generating set of Dehn twists), lies in K. The problem of finding such an algorithm is called solving the *generalized word problem* for K in Mod_g. The generalized word problem is a classical problem in combinatorial group theory; it was introduced and studied by Nielsen and Magnus.

A basic example is provided by taking K to be the *Torelli group* $\mathcal{I}_g < \mathrm{Mod}_g$, that is, the kernel of the natural symplectic representation $\pi : \mathrm{Mod}_g \to \mathrm{Sp}(2g, \mathbb{Z})$ given by the action of Mod_g on $\mathrm{H}_1(\Sigma_g; \mathbb{Z})$. In this case, the solution to the generalized word problem is easy: one simply computes the induced linear map $f_* \in \mathrm{Sp}(2g, \mathbb{Z})$ via matrix multiplication and checks whether or not $f_* = \mathrm{Id}$. With this in mind, the more refined and useful problem is to actually express a given element of $\mathcal{I}_g$, written in generators for Mod_g, in terms of generators of the subgroup $\mathcal{I}_g$. A standard quantitative measure of the (in)efficiency of doing this is the notion of *distortion*, which we now explain.

Distortion in groups. Let Γ be a finitely generated group endowed with the word metric $\|\cdot\|_\Gamma$. Any finitely generated subgroup $K < \Gamma$ comes equipped with its own word metric $\|\cdot\|_K$, and it is a basic problem in geometric group theory to understand the geometry of the embedding $K \hookrightarrow \Gamma$, that is, to compare the "intrinsic metric" $\|\cdot\|_K$ with the "extrinsic metric" on K given by the restriction of $\|\cdot\|_\Gamma$ to K. It is clear that there exist constants C, C' such that

$$\|h\|_\Gamma \leq C\|h\|_K + C' \quad \text{for all } h \in K.$$

However, there may be "shortcuts" in Γ between elements of K. This can be encoded by a function $\delta : \mathbb{N} \to \mathbb{N}$, called the *distortion* of K in Γ, which is defined to be the smallest function satisfying

$$\|h\|_K \leq \delta\left(\|h\|_\Gamma\right) \quad \text{for all } h \in K.$$

It is easy to see that δ changes by at most a linear factor if different generating sets are chosen for Γ or for K; thus the *growth type* of δ (e.g. polynomial of degree d, exponential, etc.) is independent of these choices[1]. For an introduction to distortion and its basic properties, see [Gro, Far1].

We note here that for groups Γ with solvable word problem, the distortion of K in Γ gives a quantitative measure of the efficiency of solving the generalized word problem for K in Γ. In particular, the generalized word problem for K in Γ is solvable if and only if the distortion of K in Γ is recursive (see [Far1]). This problem can be unsolvable in simple examples. For instance, Mihailova (see, e.g. [Far1]) found a finitely generated subgroup K in a product $F_m \times F_m$ of free groups which has an unsolvable generalized word problem, and hence has nonrecursive distortion.

Statement of results. In this paper we give bounds for the distortions of various subgroups of Mod_g. Some results in this direction are already known. Stillwell [Sti] used Mihailova's example to find finitely generated subgroups of Mod_g with nonrecursive distortion. In every other case which has been previously investigated, the distortion has turned out to be linear. Examples of linearly distorted subgroups of Mod_g include abelian subgroups [FLM], subgroups corresponding to mapping class groups of subsurfaces [MM] (see [Ham1] for another proof), and convex, cocompact subgroups [FM].

These results led Hamenstädt to pose the problem (see [Ham2, Problem 6]) of finding subgroups of Mod_g with (recursive) super-linear distortion. In particular, she asked if the Torelli group $\mathcal{I}_g$ is linearly distorted in Mod_g. Recall that $\mathcal{I}_g$ is finitely generated for $g > 2$ by a deep theorem of Johnson [Jo3, Theorem 2]. It was conjectured in [Far2] (Problem 3.7 and the discussion following it) that $\mathcal{I}_g$ has exponential distortion in Mod_g. Our first result confirms the lower bound of this conjecture.

To state our results in their full generality, let $\Sigma_{g,b}^p$ be an oriented genus g surface with b boundary components and p punctures (thought of as marked points), and let $\mathrm{Mod}_{g,b}^p$ be the group of homotopy classes of homeomorphisms of $\Sigma_{g,b}^p$ which fix the punctures and the boundary components pointwise (either b or p will be omitted from our notation if they equal 0). When $b \in \{0, 1\}$, the *Torelli group* $\mathcal{I}_{g,b}$ is defined as the kernel of the action of $\mathrm{Mod}_{g,b}$ on $\mathrm{H}_1(\Sigma_{g,b}; \mathbb{Z})$.

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[1]Some people use terminology differing from ours by a linear factor, so for example what we call "linearly distorted" they call "undistorted".

Theorem 1.1 (Distortion of the Torelli group). *For $g \geq 3$, the distortion of $\mathcal{I}_g$ (resp. $\mathcal{I}_{g,1}$) in Mod_g (resp. $\mathrm{Mod}_{g,1}$) is at least exponential and at most doubly exponential.*

We conjecture that the upper bound in Theorem 1.1 is, like the lower bound, exponential. The upper bound is strongly related to the isoperimetric and isodiametric inequalities in $\mathrm{Sp}(2g, \mathbb{Z})$. Thurston has conjectured that for $n \geq 4$ the group $\mathrm{SL}(n, \mathbb{Z})$ satisfies a quadratic isoperimetric inequality. The analogous conjecture for $\mathrm{Sp}(2g, \mathbb{Z})$ (together with a theorem of Papasoglu) would imply our conjectured upper bound. See §3 below, especially Remark 3.1, for a discussion.

For mapping class groups of surfaces with boundary components or punctures, one can construct finitely generated, normal subgroups by "filling in the punctures and boundary components"; these are the so-called *surface braid groups* (see [Bir]). For example, the *point pushing subgroup* $\pi_1(\Sigma_g) \lhd \mathrm{Mod}_g^1$ is the kernel of the surjection $\mathrm{Mod}_g^1 \twoheadrightarrow \mathrm{Mod}_g$ induced by "filling in the puncture".

Theorem 1.2 (Distortion of surface braid groups). *Let K be the kernel of the surjection*

$$\mathrm{Mod}_{g,b+n}^{p+m} \twoheadrightarrow \mathrm{Mod}_{g,b}^p,$$

where at least one of n and m is strictly greater than 0. Then K is exactly exponentially distorted in $\mathrm{Mod}_{g,b+n}^{p+m}$ for $g \geq 2$.

We prove Theorem 1.2 in §4.

In §5.2 we construct "relative Johnson homomorphisms", relative to arbitrary subgroups of $\mathrm{H}_1(\Sigma_g; \mathbb{Z})$. Applying these homomorphisms together with the general method for obtaining lower bounds on distortion given in §2.2, we obtain Theorems 1.3 and 1.5 below as well as a general result, Theorem 5.12, given in §5.2.

Theorem 1.3 (Distortion of the Lagrangian subgroup). *For $b \in \{0, 1\}$, let $L < \mathrm{Mod}_{g,b}$ be the Lagrangian subgroup, that is, the group of mapping classes which act trivially on a fixed maximal isotropic subgroup of $\mathrm{H}_1(\Sigma_{g,b}; \mathbb{Z})$. Then L is at least exponentially distorted in $\mathrm{Mod}_{g,b}$ for $g \geq 4$.*

Remark 1.4. The Lagrangian subgroup was first defined by Garoufalidis and Levine [GL] and plays an important role in the theory of finite-type invariants of 3-manifolds.

Masur-Minsky [MM] (see [Ham1] for an alternate proof) proved that for $h < g$ the natural inclusion $\mathrm{Mod}_{h,1} \hookrightarrow \mathrm{Mod}_g$ given by subsurface inclusion $\Sigma_{h,1} \hookrightarrow \Sigma_g$ is linearly distorted ("undistorted" in their terminology). We also remark that, as a consequence of [LMR], if $1 < h < g$ then $\mathrm{Sp}(2h, \mathbb{Z})$ is linearly distorted in $\mathrm{Sp}(2g, \mathbb{Z})$. One might therefore expect that the subgroup of Mod_g consisting of mapping classes "homologically supported" on $\Sigma_{h,1}$ have linear distortion. In contrast we have the following.

Theorem 1.5. *Suppose $g - h \geq 2$. Let $\Sigma_{h,1}$ be an embedded subsurface of Σ_g, and let K be the pull-back to Mod_g of the corresponding copy of $\mathrm{Sp}(2h, \mathbb{Z})$ in $\mathrm{Sp}(2g, \mathbb{Z})$. Then K is at least exponentially distorted in Mod_g.*

We would like to know what happens when $g = h + 1$, as our methods do not seem to work in this case.

A first idea. The first key idea in this paper can be illustrated by the following proof sketch of the lower bound in Theorem 1.1 in the case of $\mathcal{I}_g$ with $g \geq 3$. We begin with the standard exact sequence

$$1 \longrightarrow \mathcal{I}_g \longrightarrow \mathrm{Mod}_g \longrightarrow \mathrm{Sp}(2g, \mathbb{Z}) \longrightarrow 1$$

coming from the action of Mod_g on $H := \mathrm{H}_1(\Sigma_g; \mathbb{Z}) \cong \mathbb{Z}^{2g}$. Let $U = \wedge^3 H / H$. Johnson proved in [Jo1] that there is a surjective homomorphism

$$\tau : \mathcal{I}_g \to U$$

which is equivariant with respect to the natural actions of Mod_g. Here the Mod_g-action on U factors through the standard $\mathrm{Sp}(2g, \mathbb{Z})$ action, and the Mod_g action on $\mathcal{I}_g$ is the one induced by conjugation. This equivariance is just the formula

$$\tau(fhf^{-1}) = f_* \tau(h) \qquad \text{for all } f \in \mathrm{Mod}_g, h \in \mathcal{I}_g, \tag{1}$$

where f_* denotes the induced action of $f \in \mathrm{Mod}_g$ on $\mathrm{H}_1(\Sigma_g; \mathbb{Z})$.

We can use τ to give lower bounds for word length in $\mathcal{I}_g$ as follows. Fix a finite generating set S for $\mathcal{I}_g$, and let $\|h\|_{\mathcal{I}_g}$ denote *word length* of h with respect to S, that is, the minimal number of elements of $S^{\pm 1}$ whose product equals h. Also, fix a norm $\| \cdot \|_U$ on U. Since S is finite, there exists some $C > 0$ such that $\|\tau(s)\|_U \leq C$ for all $s \in S$. Since τ is a homomorphism, it is then clear that for every $h \in \mathcal{I}_g$ we have

$$\|h\|_{\mathcal{I}_g} \geq \frac{1}{C} \|\tau(h)\|_U. \tag{2}$$

Now it is not hard to see that we can choose a mapping class $f \in \mathrm{Mod}_g$ with the property that the linear transformation $f_* \in \mathrm{GL}(U)$ is *partially hyperbolic*, that is, there is some eigenvalue λ of f_* with $|\lambda| > 1$. Using the partial hyperbolicity of f, we will find an $h \in \mathcal{I}_g$ such that $\|f_*^n(\tau(h))\|_U$ grows exponentially in n. Then $\|f^n h f^{-n}\|_{\mathrm{Mod}_g}$ grows at most linearly with respect to n, but by (1) and (2) we have

$$\|f^n h f^{-n}\|_{\mathcal{I}_g} \geq \frac{1}{C} \|\tau(f^n h f^{-n})\|_U$$
$$= \frac{1}{C} \|f_*^n(\tau(h))\|_U,$$

which grow exponentially. See §3 for details.

Some questions. This paper is a first attempt at a systematic study of distortion in mapping class groups. While the methods here apply to many examples, there are also many examples to which they do not apply. Of the many natural questions one might ask, we would like to point out a particular one to which our methods do not seem to apply. Recall that the *handlebody group* is the subgroup of Mod_g consisting of elements which extend to a fixed handlebody. Suzuki [Suz] proved that this group is finitely generated.

Question 1.6. What is the distortion of the handlebody subgroup in the mapping class group?

A natural question analogous to the direction of this paper is to study and compute the distortions of orbits of subgroups $K < \mathrm{Mod}_g$ in the Teichmüller space $\mathcal{T}_g$ for Σ_g, say endowed with the Teichmüller metric; see [FM, KL] for related discussions. Theorem 2.1 of [FLM] (and the discussion following it) give that any Mod_g-orbit in $\mathcal{T}_g$ is exponentially distorted. Thus the problem of computing the distortion of K in Mod_g and of a K-orbit in $\mathcal{T}_g$ are *a priori* different. We believe both questions are worth pursuing.

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2 Methods for bounding distortion in groups

In this section, we give two general methods for bounding the distortion of one group inside another, one yielding lower bounds and the other upper bounds. We will apply these methods throughout this paper to subgroups of mapping class groups.

2.1 Behavior of distortion under inclusions

Before we begin, we will need to know the following basic property of distortion.

Lemma 2.1 (Behavior of distortion under inclusions)**.** *Let $K < \Gamma$ be finitely generated groups. If $\Gamma < \Gamma'$ with Γ' finitely generated, then the distortion of K in Γ' is at least the distortion of K in Γ.*

Proof. Fix a finite generating set S_Γ for Γ and let $\| \cdot \|_\Gamma$ be the associated word metric. We can choose a finite generating set $S_{\Gamma'}$ for Γ' with $S_\Gamma \subset S_{\Gamma'}$. Letting $\| \cdot \|_{\Gamma'}$ be the associated word metric, we have $\|k\|_\Gamma \geq \|k\|_{\Gamma'}$ for all $k \in K$, and the lemma follows. $\qquad\square$

2.2 Lower bounds via irreducible representations

If $K < G$, then to give a lower bound for the distortion of K in G, one must be able to give lower bounds on the word length in K. This is in general a very difficult problem, but for free abelian groups V such bounds can be easily obtained. Moreover, the resulting linear algebra is a rich source of examples of exponential growth. We say that an element of the automorphism group $\mathrm{GL}(V) := \mathrm{Aut}(V)$ is *partially hyperbolic* if the corresponding linear transformation of $V \otimes \mathbb{C}$ has some eigenvalue λ with $|\lambda| > 1$. We then have the following easy example of exponential distortion.

Example 2.2. Let Γ be the semidirect product of $V = \mathbb{Z}^n$ with any subgroup of $\mathrm{SL}(n, \mathbb{Z})$ which contains a partially hyperbolic matrix A. Then V has exponential distortion in Γ. The upper bound is easy. The exponential lower bound follows, as explained in the "First idea" on page 4 and given in detail in Proposition 2.3 below, from the exponential growth of the image of vectors in $V \otimes \mathbb{R}$ under iteration of A.

To generalize this example, we will map group/subgroup pairs to mapping tori of abelian groups like those in Example 2.2, using the abelian group as a sort of "detector" of exponential distortion.

Proposition 2.3 (Criterion for exponential distortion)*. Let K be a finitely generated normal subgroup of a finitely generated group Γ. Suppose that V is a free abelian group equipped with a Γ-action $\rho : \Gamma \to \mathrm{GL}(V)$ and that $\psi : K \to V$ is a surjective homomorphism which is Γ-equivariant, where Γ acts on K by conjugation. If $\rho(\Gamma)$ contains a partially hyperbolic matrix, then the distortion of K in Γ is at least exponential.*

Remark 2.4. When applying Proposition 2.3, we will find it useful to think of the exact sequence

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow \Gamma/K \longrightarrow 1$$

and to note that since V is abelian and $\psi : K \to V$ is surjective we have

$$\rho(k) \cdot \psi(y) = \psi(kyk^{-1}) = \psi(k) + \psi(y) - \psi(k) = \psi(y) \quad \text{for all } k, y \in K.$$

Thus the action of K on V is trivial. Hence ρ factors through a representation $\Gamma/K \to \mathrm{GL}(V)$.

Remark 2.5. By using Lemma 2.1, Proposition 2.3 can be applied beyond the context of normal subgroups. See the proofs of Theorem 1.3 and Theorem 1.5 for examples.

Proof of Proposition 2.3. The hypothesis that ψ is Γ-equivariant is precisely that

$$\psi(xyx^{-1}) = \rho(x) \cdot \psi(y) \quad \text{for all } y \in K \text{ and } x \in \Gamma. \tag{3}$$

By assumption, there exists some $x \in \Gamma$ such that

$$\rho(x) \otimes 1 : V \otimes \mathbb{C} \to V \otimes \mathbb{C}$$

has an eigenvalue λ_1 with $|\lambda_1| > 1$. Let $M = \rho(x) \otimes 1$ and let the distinct eigenvalues of M be $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$. For $1 \leq i \leq k$, let E_k be the *generalized eigenspace* for λ_i, i.e. the kernel of the map $(M - \lambda_i \mathbb{I})^{dim(V)}$, where $\mathbb{I}$ is the identity. We have a direct sum decomposition $V \otimes \mathbb{C} = E_1 \oplus \cdots \oplus E_k$ which is invariant under M. Endow $V \otimes \mathbb{C}$ with an inner product such that the E_i are orthogonal to each other and let $\| \cdot \|$ be the associated norm. It is well-known (see, e.g., [KH, §2.2]) that there exists some $C > 0$ such that if $v \in E_1$, then $\|M^n \cdot v\| \geq C|\lambda_1|^n \|v\|$ for all $n \geq 1$.

Let S be a finite generating set for K and let $S' \supseteq S$ be a finite generating set for Γ. Since S is finite, there exists some $D \geq 0$ such that

$$\|\psi(s) \otimes 1\| \leq D \quad \text{for each } s \in S. \tag{4}$$

Since ψ is surjective, its image in $V \otimes \mathbb{C}$ must span $V \otimes \mathbb{C}$. In particular, its image cannot lie in $E_2 \oplus \cdots \oplus E_k$, so we can find some $y \in K$ such that $\psi(y) \otimes 1 \notin E_2 \oplus \cdots \oplus E_k$. Let y_1 be the orthogonal projection of $\psi(y) \otimes 1$ to E_1. Using the fact that the E_i are orthogonal and invariant under M, we deduce that for all $n \geq 1$ we have

$$\begin{aligned}
\|\psi(x^n y x^{-n}) \otimes 1\| &= \|M^n \cdot (\psi(y) \otimes 1)\| \\
&\geq \|M^n \cdot y_1\| \\
&\geq C|\lambda_1|^n \|y_1\|.
\end{aligned} \tag{5}$$

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Now (4) together with (5) implies that the word length $\|x^n y x^{-n}\|_K$ with respect to the generating set S is at least $\frac{1}{D} \cdot C |\lambda_1|^n \|y_1\|$, which grows exponentially in n since $|\lambda_1| > 1$ and $y_1 \neq 0$. On the other hand, since x and y have fixed word length with respect to the generating set S' for Γ, we see that the word length in Γ of $x^n y x^{-n}$ grows at most linearly in n. This completes the proof of the proposition. $\square$

The following (purely linear-algebra) proposition gives a useful way in practice to prove that an action contains some partially hyperbolic matrix.

Proposition 2.6 (Representation-theoretic criterion for hyperbolicity). *Let Γ be a group and let V be a free abelian group equipped with a Γ-action. Suppose that V contains a nontrivial Γ-submodule W satisfying the following two properties:*

1. *W is irreducible in the sense that if $W' < W$ is a nontrivial Γ-submodule, then $W' \otimes \mathbb{R} = W \otimes \mathbb{R}$.*

2. *The image of the map $\nu : \Gamma \to \mathrm{GL}(W)$ is infinite.*

Then $\nu(\Gamma)$ contains a partially hyperbolic matrix.

Remark 2.7. In all the examples we consider in this paper, one could prove that the relevant representations contain partially hyperbolic matrices by hand; however, we believe that Proposition 2.6 gives a conceptual reason for the ubiquity of representations with this property.

Proof of Proposition 2.6. Choosing a basis, we can identify $\mathrm{GL}(W)$ with $\mathrm{GL}(m, \mathbb{Z})$ for some $m \geq 1$. If a matrix $A \in \mathrm{GL}(W)$ is not partially hyperbolic, then every eigenvalue λ of A satisfies $|\lambda| = 1$. Since A has integer entries, each of its eigenvalues is an algebraic integer. An old theorem of Kronecker (see, e.g., [Gr]) states that if λ is any algebraic integer with $|\lambda| = 1$ and with $|\lambda'| = 1$ for every Galois conjugate λ' of λ, then λ is a root of unity.

Let $\mathrm{GL}(n, \mathbb{Z})[L]$ denote the *level L congruence subgroup* of $\mathrm{GL}(n, \mathbb{Z})$, which consists of those $A \in \mathrm{GL}(n, \mathbb{Z})$ for which, when the entries of A are taken mod L, the result is the identity matrix. Theorem 5.61 of [Mo] states that there exists some $L > 1$ such that no eigenvalue of any element of $\mathrm{GL}(n, \mathbb{Z})[L]$ is a nontrivial root of unity. Pulling back $\mathrm{GL}(n, \mathbb{Z})[L]$ via ν then gives us a finite index subgroup Γ' of Γ with the property that no eigenvalue of any element of $\nu(\Gamma')$ is a nontrivial root of unity. Since we are assuming (assumption (2)) that $\nu(\Gamma)$ is infinite, we know $\nu(\Gamma')$ is nontrivial (indeed infinite).

Suppose $\nu(\Gamma')$ contains no partially hyperbolic matrix. Then by the above two paragraphs, every $A \in \nu(\Gamma')$ must be *unipotent*, that is, A has 1 as its only eigenvalue. But any subgroup of $\mathrm{GL}(n, \mathbb{R})$ consisting of unipotent matrices must be nilpotent, and indeed conjugate into the upper triangular group with 1's on the diagonal (see, e.g., Corollary 17.5 of [Hu]). Setting

$$U = \{w \in W \mid \Gamma' \cdot w = w\},$$

this implies that $U \neq 0$. Since $\nu(\Gamma)$ is infinite (assumption (2)), we have $U \neq W$, for otherwise $\nu(\Gamma')$ would be trivial and so $\nu(\Gamma)$ would be finite. Also, since Γ' is a normal subgroup of Γ, we have that $\Gamma \cdot U \subset U$; indeed, for $h \in \Gamma$ and $u \in U$, we have for all $g \in \Gamma'$ that

$$g \cdot (h \cdot u) = h \cdot ((h^{-1} g h) \cdot u) = h \cdot u,$$

so $h \cdot u \in U$. We conclude that $\nu(\Gamma)$ is reducible, contradicting assumption (1). Thus $\nu(\Gamma') \subset \nu(\Gamma)$ must contain some partially hyperbolic matrix, and we are done. $\square$

To apply Proposition 2.6, we will need a method for establishing the irreducibility of the action of the discrete group Γ. In the cases which arise in this paper, this action will factor through various arithmetic subgroups (such as $\mathrm{Sp}(2g, \mathbb{Z})$) of semisimple Lie groups, and hence the following theorem of Borel can be applied.

Theorem 2.8 ([Bor1, Proposition 3.2])**.** *Let G be a connected, semisimple, real algebraic group which is defined over $\mathbb{Q}$ and which has no compact factors (for example, $G = \mathrm{Sp}(2g, \mathbb{R})$ for $g \geq 1$ or $G = \mathrm{SL}(n, \mathbb{R})$ for $n \geq 2$). Let V be a finite dimensional vector space over $\mathbb{R}$ which is an irreducible G-module. Then V is also an irreducible $G_{\mathbb{Z}}$-module, where $G_{\mathbb{Z}}$ is the group of integer points of G.*

Remark 2.9. The conditions in [Bor1, Proposition 3.2] do not mention integer points, but instead refer to subgroups satisfying a certain "property (S)". However, it is easy to see that lattices satisfy this property, and a well-known theorem of Borel-Harish-Chandra says that arithmetic subgroups are lattices. We also remark that [Bor1, Proposition 3.2] is one of the steps in the proof of an early version of the Borel Density Theorem, but later proofs do not make use of this result, and Theorem 2.8 can be easily deduced from the Borel Density Theorem.

2.3   Isoperimetric inequalities and upper bounds on distortion

The goal of this subsection is to prove that an upper bound for the distortion of a normal subgroup $K \lhd \Gamma$ can be obtained from isoperimetric and isodiametric functions on the quotient group Γ/K. We first recall some definitions. While the notion of isoperimetric function and isodiametric function for a group are usually defined as independent quantities (see, e.g., [Ger]), we will be interested in their simultaneous realization, as discussed in [GR] and in [Ril, Definition 2.2.2]. We refer the reader to these references for background on this topic.

Definition 2.10 (Simultaneous (isoperimetric,isodiametric) pairs)**.** Let $\Gamma \cong \langle S | R \rangle$ be a finitely-presented group. Denote the free group on S by $F(S)$. Also, denote the word length of an element w in $F(S)$ by $\ell(w)$. We say that a pair of functions (μ_P, μ_D) is a *simultaneous (isoperimetric,isodiametric) pair for* Γ if for all $w \in F(S)$ representing the identity in Γ, we can write w as a word

$$w = \prod_{i=1}^{N} x_i r_i x_i^{-1}, \quad x_i \in F(S), r_i \in R$$

such that

$$N \leq \mu_P(\ell(w)) \quad \text{and} \quad \ell(x_i) \leq \mu_D(\ell(w)) \quad \text{for each } 1 \leq i \leq N.$$

Remark 2.11. Though the precise functions in an (isoperimetric,isodiametric) pair for a group depend on the choice of finite generating set, changing the generators preserves the equivalence classes of these functions up to linear substitution. Thus as with distortion we will consider (isoperimetric,isodiametric) pairs only up to their well-defined equivalence classes.

The main result of this subsection is the following, which is a small generalization of a theorem of Arzhantseva and Osin [AO, Lemma 3.6].

Proposition 2.12 (Upper bound on distortion). *Let $K \lhd \Gamma$ be a finitely generated, normal subgroup of a finitely generated group Γ. Suppose that Γ/K is finitely presented and that (μ_P, μ_D) is a simultaneous (isoperimetric,isodiametric) pair for Γ/K. Then there exists some $C > 0$ such that the distortion of K in Γ is at most $\mu_P C^{\mu_D}$.*

Proof. Let S_Γ and S_K be finite generating sets for Γ and K respectively, and let $F(S_\Gamma)$ be the free group on S_Γ. Let $\|\cdot\|_\Gamma : \Gamma \to \mathbb{N}$ be the S_Γ word metric and $\|\cdot\|_K : K \to \mathbb{N}$ be the S_K word metric. Let $\ell : F(S_\Gamma) \to \mathbb{N}$ give the word length in $F(S_\Gamma)$ and $\pi : F(S_\Gamma) \to \Gamma$ be the natural projection. Finally, choose $R \subset F(S_\Gamma)$ such that $\Gamma/K \cong \langle S_\Gamma | R \rangle$ is a finite presentation.

We begin by claiming that there is some $C > 0$ such that if $r \in R$ and $x \in F(S_\Gamma)$, then

$$\|\pi(xrx^{-1})\|_K < C^{\ell(x)}.$$

Indeed, an easy induction shows that if

$$C_1 = \max\{\|\pi(r)\|_K \mid r \in R\},$$

and

$$C_2 = \max\{\|\pi(s_\Gamma s_K s_\Gamma^{-1})\|_K \mid s_\Gamma \in S_\Gamma \text{ and } s_K \in S_K\},$$

then $C = C_1 C_2$ satisfies the claim.

Now given $k \in K$ choose an efficient word $w \in F(S_\Gamma)$ such that $\pi(w) = k$ and $\ell(w) = \|k\|_\Gamma$. By Definition 2.10, we can write

$$w = \prod_{i=1}^{N} x_i r_i x_i^{-1}, \quad x_i \in F(S_\Gamma), \quad r_i \in R$$

with

$$N \le \mu_P(\ell(w)) \quad \text{and} \quad \ell(x_i) \le \mu_D(\ell(w)) \quad \text{for each } 1 \le i \le N.$$

We then calculate:

$$
\begin{aligned}
\|k\|_K &= \|\pi(w)\|_K \\
&\le \sum_{i=1}^{N} \|\pi(w_i r_i w_i^{-1})\|_K \\
&\le \sum_{i=1}^{N} C^{\mu_D(\ell(w))} \\
&\le \mu_P(\ell(w)) C^{\mu_D(\ell(w))} \\
&= \mu_P(\|k\|_\Gamma) C^{\mu_D(\|k\|_\Gamma)},
\end{aligned}
$$

as desired. $\qquad\square$

3 Distortion of the Torelli group

In this section, we apply Propositions 2.3 and 2.12 to give lower and upper bounds on the distortion of $\mathcal{I}_{g,b}$ in $\mathrm{Mod}_{g,b}$ for $b \in \{0,1\}$ and $g \geq 3$.

Proof of Theorem 1.1. Consider the standard exact sequence

$$1 \longrightarrow \mathcal{I}_{g,b} \longrightarrow \mathrm{Mod}_{g,b} \longrightarrow \mathrm{Sp}(2g,\mathbb{Z}) \longrightarrow 1. \tag{6}$$

We begin with the lower bound. Set $H = \mathrm{H}_1(\Sigma_{g,b};\mathbb{Z})$. In [Jo1], Johnson constructed the *Johnson homomorphisms*, which are for $g \geq 3$ surjective $\mathrm{Mod}_{g,b}$-equivariant homomorphisms

$$\tau : \mathcal{I}_{g,1} \rightarrow \wedge^3 H$$

and

$$\tau : \mathcal{I}_g \rightarrow (\wedge^3 H)/H$$

Here H is embedded in $\wedge^3 H$ as $H \wedge \omega$, where $\omega = a_1 \wedge b_1 + \cdots + a_g \wedge b_g$ for any symplectic basis $\{a_1, b_1, \ldots, a_g, b_g\}$ of H. The action of $\mathrm{Mod}_{g,b}$ on $(\wedge^3 H)/H$ or $\wedge^3 H$ factors surjectively through the infinite group $\mathrm{Sp}(2g,\mathbb{Z})$.

Now $H \otimes \mathbb{R}$ is an irreducible representation of $\mathrm{Sp}(2g,\mathbb{R})$. By Theorem 2.8 it is also an irreducible $\mathrm{Sp}(2g,\mathbb{Z})$-representation. We can apply Proposition 2.6 to the inclusion $H \subset \wedge^3 H$ to show that some element of $\mathrm{Mod}_{g,1}$ acts partially hyperbolically on $\wedge^3 H$. Similarly, $((\wedge^3 H)/H) \otimes \mathbb{R}$ is an irreducible representation of $\mathrm{Sp}(2g,\mathbb{R})$. By Theorem 2.8 it is also an irreducible $\mathrm{Sp}(2g,\mathbb{Z})$-representation. We can apply Proposition 2.6 with $W = V = (\wedge^3 H)/H$ to show that some element of $\mathrm{Mod}_{g,0}$ acts partially hyperbolically on $(\wedge^3 H)/H$.

Now apply Proposition 2.3 to $\mathcal{I}_{g,b} \lhd \mathrm{Mod}_{g,b}$ with the homomorphism ψ equal to the relevant Johnson homomorphism to conclude that $\mathcal{I}_{g,b}$ is at least exponentially distorted in $\mathrm{Mod}_{g,b}$ for $g \geq 3$ and $b \in \{0,1\}$, as desired.

We now establish the upper bound. As explained in §5 of [Leu] (see in particular Corollary 5.4), the nonpositively curved symmetric space X for $\mathrm{Sp}(2g,\mathbb{R})$ admits an $\mathrm{Sp}(2g,\mathbb{Z})$-equivariant retraction $r : X \rightarrow \Omega$ onto a submanifold with boundary Ω on which $\mathrm{Sp}(2g,\mathbb{Z})$ acts cocompactly and properly by isometries (in the path metric). Now X is a nonpositively curved Riemannian manifold, and so it has a simultaneous (isoperimetric,isodiametric) function which is (quadratic, linear). The retraction r distorts lengths and hence volumes by at most an exponential factor. It follows that Ω, hence $\mathrm{Sp}(2g,\mathbb{Z})$, has at worst a simultaneous (exponential,exponential) (isoperimetric,isodiametric) pair. While [Leu] explains this only for isoperimetric functions, the argument for the (isoperimetric,isodiametric) pair follows exactly his argument.

Proposition 2.12 applied to exact sequence (6) therefore implies that for $g \geq 3$ and $b \in \{0,1\}$, the group $\mathcal{I}_{g,b}$ is at most doubly exponentially distorted in $\mathrm{Mod}_{g,b}$, as desired. $\square$

Remark 3.1. The isoperimetric/isodiametric bounds for $\mathrm{Sp}(2g,\mathbb{Z})$ used in the proof of Theorem 1.1 are probably not sharp. In fact, Thurston has conjectured that $\mathrm{SL}(n,\mathbb{Z})$ satisfies a quadratic isoperimetric inequality for $n \geq 4$, and one would expect the same to hold for $\mathrm{Sp}(2g,\mathbb{Z})$ for $g \geq 3$. If $\mathrm{Sp}(2g,\mathbb{Z})$ satisfied a quadratic isoperimetric inequality, then a theorem of Papasoglu [Pap] (see [GR, Theorem 2] for a generalization and alternate proof) would imply that it has a

(quadratic,linear) (isoperimetric,isodiametric) pair. This would imply that our exponential lower bound on the distortion of $\mathcal{I}_g$ in Mod_g (and of $\mathcal{I}_{g,1}$ in $\mathrm{Mod}_{g,1}$) is sharp.

In fact one could get by with much less than Thurston's conjecture. Sharpness of the exponential lower bound on the distortion of the Torelli group would follow if $\mathrm{Sp}(2g, \mathbb{Z})$ satisfied an (exponential,linear) (isoperimetric,isodiametric) pair.

4   Distortion of surface braid groups

Proof of Theorem 1.2. Consider the exact sequence

$$1 \longrightarrow K \longrightarrow \mathrm{Mod}^{p+m}_{g,b+n} \longrightarrow \mathrm{Mod}^p_{g,b} \longrightarrow 1. \tag{7}$$

We first prove that K is at least exponentially distorted in $\mathrm{Mod}^{p+m}_{g,b+n}$. Setting $H = \mathrm{H}_1(\Sigma_g; \mathbb{Z})$, we will construct a $\mathrm{Mod}^{p+m}_{g,b+n}$-equivariant surjection $\psi : K \to H^{n+m}$. Here the $\mathrm{Mod}^{p+m}_{g,b+n}$-action on H^{n+m} factors through the natural projection

$$\mathrm{Mod}^{p+m}_{g,b+n} \longrightarrow \mathrm{Mod}_g \longrightarrow \mathrm{Sp}(2g, \mathbb{Z}).$$

Since H^{n+m} contains the nontrivial irreducible (by Theorem 2.8) $\mathrm{Sp}(2g, \mathbb{Z})$-module H, we can apply Propositions 2.6 and 2.3 to $K \lhd \mathrm{Mod}^{p+m}_{g,b+n}$ with the homomorphism ψ to conclude that K is at least exponentially distorted in $\mathrm{Mod}^{p+m}_{g,b+n}$, as desired.

The construction of ψ goes as follows. Let K' be the kernel of the surjection $\mathrm{Mod}^{p+m+n}_{g,b} \to \mathrm{Mod}^p_{g,b}$. The map $\mathrm{Mod}^{p+m}_{g,b+n} \to \mathrm{Mod}^{p+m+n}_{g,b}$ induces a surjection $K \to K'$. A basic result of Birman [Bir] shows that K' is the fundamental group of the *configuration space of $n+m$ points on Σ^p_{g+b}*, that is, that $K' \cong \pi_1((\Sigma^p_{g,b})^{n+m} \setminus \Delta)$ where

$$\Delta = \{(x_1, \ldots, x_{n+m}) \in (\Sigma^p_{g,b})^{n+m} \mid x_i = x_j \text{ for some } i \neq j\}.$$

The map ψ is then the composition

$$K \longrightarrow K' = \pi_1\big((\Sigma^p_{g,b})^{n+m} \setminus \Delta\big) \longrightarrow \pi_1\big((\Sigma^p_{g,b})^{n+m}\big) \longrightarrow \pi_1\big((\Sigma_g)^{n+m}\big) \longrightarrow H^{n+m}.$$

We now prove that K is at most exponentially distorted in $\mathrm{Mod}^{p+m}_{g,b+n}$. In [Mos], Mosher proved that $\mathrm{Mod}^p_{g,b}$ is automatic. This implies [ECHLPT] that $\mathrm{Mod}^p_{g,b}$ satisfies a simultaneous (quadratic,linear) (isoperimetric,isodiametric) pair. We can thus deduce the desired upper bound by applying Proposition 2.12 to exact sequence (7). $\qquad\square$

5   Relative Johnson homomorphisms and the distortion of homologically specified subgroups

In this section, we generalize the Johnson homomorphism to other subgroups of $\mathrm{Mod}_{g,b}$ defined by a variety of homological conditions. We then apply these homomorphisms to give lower bounds on distortion.

5.1 Relative Johnson homomorphisms

Fix $g \geq 3$ and $b \in \{0,1\}$, and set $H = \mathrm{H}_1(\Sigma_{g,b}; \mathbb{Z})$. We will consider the following subgroups of the mapping class group $\mathrm{Mod}_{g,b}$.

Definition 5.1. Let W be a subgroup of H. Define

$$\mathrm{Mod}_{g,b}^W := \{f \in \mathrm{Mod}_{g,b} \mid f_*(W) \subset W\}.$$

Observe that $\mathrm{Mod}_{g,b}^W$ acts on H/W. The kernel of this action will be denoted by $\mathcal{I}_{g,b}^W$.

Example 5.2. The classical Torelli group $\mathcal{I}_{g,b}$ corresponds to $\mathcal{I}_{g,b}^W$ with $W = 0$.

Example 5.3. Fix $m \in \mathbb{N}$, and let W be the kernel of the map $H \to \mathrm{H}_1(\Sigma_{g,b}; \mathbb{Z}/m\mathbb{Z})$. Then $\mathrm{Mod}_{g,b}^W = \mathrm{Mod}_{g,b}$ and $\mathcal{I}_{g,b}^W$ is the *level m subgroup of* $\mathrm{Mod}_{g,b}$, that is, the kernel of natural map $\mathrm{Mod}_{g,b} \to \mathrm{Sp}(2g, \mathbb{Z}/m\mathbb{Z})$.

Example 5.4. Let $\Sigma' \subset \Sigma_{g,b}$ be a subsurface, and let W be the image of $\mathrm{H}_1(\Sigma'; \mathbb{Z})$ in H. Then $\mathcal{I}_{g,b}^W$ is generated by $\mathcal{I}_{g,b}$ together with the set mapping classes supported on Σ'. One may think of this as the subgroup of mapping classes which are "homologically supported" on Σ'.

Example 5.5. Let $W \subset \mathrm{H}_1(\Sigma_{g,b}; \mathbb{Z})$ be a Lagrangian, that is, a maximal isotropic subgroup. Then $\mathcal{I}_{g,b}^W$ is the subgroup of mapping classes which preserve W and act trivially on H/W. It is easy to see that $\mathcal{I}_{g,b}^W$ must also act trivially on W. In other words, these are the Lagrangian subgroups of Theorem 1.3.

We now wish to generalize the Johnson homomorphisms to the groups $\mathcal{I}_{g,b}^W$. We begin by discussing the appropriate target for these Johnson homomorphisms. We will need the following well-known lemma.

Lemma 5.6 ([Bro, Theorem V.6.4]). *If A is an abelian group, then there is an injection $i : \wedge^3 A \to \mathrm{H}_3(A; \mathbb{Z})$. If A is torsion-free, then i is an isomorphism.*

Now, using Lemma 5.6, the classical Johnson homomorphism on a surface with boundary is of the form

$$\tau : \mathcal{I}_{g,1} \longrightarrow \wedge^3 H \cong \mathrm{H}_3(H; \mathbb{Z}).$$

The relative Johnson homomorphism on a surface with boundary will be of the form

$$\tau^W : \mathcal{I}_{g,1}^W \longrightarrow \mathrm{H}_3(H/W; \mathbb{Z}).$$

On a closed surface, the classical Johnson homomorphism is of the form

$$\tau : \mathcal{I}_g \longrightarrow (\wedge^3 H)/H \cong \mathrm{H}_3(H; \mathbb{Z})/H.$$

Here H is embedded in $\wedge^3 H$ as $H \wedge \omega$, where $\omega = a_1 \wedge b_1 + \cdots + a_g \wedge b_g$ for any symplectic basis $\{a_1, b_1, \ldots, a_g, b_g\}$ of H. By Lemma 5.6, there is a copy of $\wedge^3(H/W)$ in $\mathrm{H}_3(H/W; \mathbb{Z})$. For any subgroup W of H, the aforementioned embedding of H into $\wedge^3 H$ therefore projects to a homomorphism

$$H \hookrightarrow \wedge^3 H \to \wedge^3(H/W) \hookrightarrow \mathrm{H}_3(H/W; \mathbb{Z})$$

whose image we will denote by H^W. The relative Johnson homomorphism on a closed surface will be of the form
$$\tau^W : \mathcal{I}_g^W \longrightarrow \mathrm{H}_3(H/W; \mathbb{Z})/H^W.$$

Remark 5.7. The map $H \to H^W$ need not be injective. Indeed, if W is a maximal isotropic subspace of H, then the image of ω in $\wedge^2(H/W)$ is 0. Hence in this case $H^W = 0$.

For simplicity, we will define $H^W = 0$ if $b = 1$. We can now state our theorem.

Theorem 5.8 (Relative Johnson homomorphisms). *For $g \geq 3$ and $b \in \{0, 1\}$, there exist homomorphisms*
$$\tau^W : \mathcal{I}_{g,b}^W \longrightarrow \mathrm{H}_3(H/W; \mathbb{Z})/H^W,$$
which we will call the relative Johnson homomorphisms, *satisfying the following properties:*

1. *τ^0 is the classical Johnson homomorphism.*

2. *If $W_1 \subset W_2$, then $\tau^{W_2}|_{\mathcal{I}_{g,b}^{W_1}}$ equals τ^{W_1} followed by the natural map*
$$\mathrm{H}_3(H/W_1; \mathbb{Z})/H^{W_1} \to \mathrm{H}_3(H/W_2; \mathbb{Z})/H^{W_2}.$$

3. *For $f \in \mathrm{Mod}_{g,b}^W$ and $h \in \mathcal{I}_{g,b}$, we have $\tau(fhf^{-1}) = f_* \circ \tau(h)$, where f_* is the induced map on $\mathrm{H}_3(H/W; \mathbb{Z})/H^W$.*

Proof. We begin with the case $b = 1$ (where $H^W = 0$). As described for example in [Jo2], the classical Johnson homomorphism τ can be defined in various ways. We imitate the construction based on mapping tori. For a detailed discussion of this definition of the classical Johnson homomorphism and a proof of its equivalence to the more standard definition, see §3-4 of [Hai]. For $h \in \mathcal{I}_{g,1}^W$, choose a homeomorphism h' of $\Sigma_{g,1}$ representing h and construct a homeomorphism $\overline{h}$ of Σ_g by gluing a disc D to the boundary component of $\Sigma_{g,1}$ to get a copy of Σ_g and defining $\overline{h}$ to equal 1 on D and h' on $\Sigma_{g,1} = \Sigma_g \setminus D$. Next, let $M_{\overline{h}}$ be the *mapping torus* of $\overline{h}$, that is, the quotient of $\Sigma_g \times [0, 1]$ by the equivalence relation $(x, 1) \sim (\overline{h}(x), 0)$. Fixing a basepoint v on $D \times 0 \subset \Sigma_g \times 0$, we get a canonical loop $\ell \in \pi_1(M_{\overline{h}}, v)$, via $\ell(t) = v \times t$. Observe that
$$\mathrm{H}_1(M_{\overline{h}}; \mathbb{Z})/\langle \ell \rangle \cong H/\langle \{\alpha - h_* \alpha \mid \alpha \in H\} \rangle.$$

Since $h \in \mathcal{I}_{g,1}^W$, this has a natural projection to H/W. We conclude that there is a natural map
$$\pi_1(M_{\overline{h}}, v) \longrightarrow H/W,$$
and since $M_{\overline{h}}$ is an Eilenberg-MacLane space, we have an induced map
$$\phi : M_{\overline{h}} \longrightarrow K(H/W, 1).$$

We define
$$\tau^W(h) = \phi_*([M_{\overline{h}}]) \in \mathrm{H}_3(H/W; \mathbb{Z}).$$

It is easy to see that this definition is independent of the choices involved in its construction, and τ^W clearly satisfies Condition 2 of the theorem. Moreover, if $W = 0$, then τ^W reduces to Johnson's

definition of the classical Johnson homomorphism via mapping tori, so Condition 1 follows. The proof (see [Jo2]) that the classical Johnson homomorphism is a homomorphism which satisfies Condition 3 generalizes verbatim to our situation, so we are done.

We now deal with the case $b = 0$. We have an exact sequence

$$1 \longrightarrow \pi_1(T^1\Sigma_g) \longrightarrow \mathcal{I}_{g,1}^W \longrightarrow \mathcal{I}_g^W \longrightarrow 1,$$

where $T^1\Sigma_g$ is the unit tangent bundle of Σ_g; see [Jo3]. The classical Johnson homomorphism restricted to $\pi_1(T^1\Sigma_g)$ lands in $H \subset \wedge^3 H$. It follows that τ^W restricted to $\pi_1(T^1\Sigma_g)$ lands in $H^W \subset \wedge^3 H/W$, and thus we have an induced map

$$\mathcal{I}_g^W \longrightarrow (\mathrm{H}_3(H/W;\mathbb{Z}))/H^W.$$

These maps clearly satisfy the conditions of the theorem. $\qquad\square$

Remark 5.9. In the special case of the Lagrangian subgroup, Levine [Lev] has given a different construction of τ^W. His construction imitates Johnson's original definition of τ [Jo2, First Definition], while our construction is inspired by Johnson's definition in terms of mapping tori [Jo2, Second Definition].

Remark 5.10. Another approach sufficient for the proofs of Theorems 1.3 and 1.5 would be to use Morita's [Mor1] extension of the classical Johnson homomorphism to a crossed homomorphism $\mathrm{Mod}_{g,1} \to \frac{1}{2} \wedge^3 H$.

5.2   Applications to subgroup distortion

We now apply the relative Johnson homomorphisms τ^W together with Propositions 2.6 and 2.3 to give lower bounds on the distortions of some of the groups $\mathcal{I}_{g,b}^W$ inside $\mathrm{Mod}_{g,b}$. For these distortions to make sense, however, we must first prove the following.

Proposition 5.11. *For $g \geq 3$ and $b \in \{0,1\}$, let W be any subgroup of $\mathrm{H}_1(\Sigma_{g,b};\mathbb{Z})$. Then the groups $\mathcal{I}_{g,b}^W$ and $\mathrm{Mod}_{g,b}^W$ are finitely generated.*

Proof. Let Γ be either $\mathcal{I}_{g,b}^W$ or $\mathrm{Mod}_{g,n}^W$. We have an exact sequence

$$1 \longrightarrow \mathcal{I}_{g,b} \longrightarrow \Gamma \longrightarrow B \longrightarrow 1,$$

where B is the image of $\Gamma < \mathrm{Mod}_{g,b}$ in $\mathrm{Sp}(2g,\mathbb{Z})$. As has already been mentioned, Johnson [Jo3] showed that $\mathcal{I}_{g,b}$ is finitely generated. It is enough, therefore, to show that B is finitely generated. To see this, first note that B is the set of $\mathbb{Z}$-points of the $\mathbb{Q}$-algebraic group consisting of matrices in $\mathrm{SL}(2g,\mathbb{Q})$ which preserve the (integral) subgroup $W < \mathbb{Q}^{2g}$ and the symplectic form and (for $\Gamma = \mathcal{I}_{g,b}^W$) which act trivially on H/W. Hence B is arithmetic, and as such it is finitely generated (see, e.g. [Mo], Theorem 5.57). $\qquad\square$

We now prove Theorems 1.3 and 1.5 from the introduction.

Proof of Theorem 1.3. Let $H = \mathrm{H}_1(\Sigma_{g,b}; \mathbb{Z})$ with symplectic basis $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ and let $W = \langle a_1, \ldots a_g \rangle$ be the standard Lagrangian. Then the Lagrangian subgroup L is $\mathcal{I}_{g,b}^W$. By Lemma 2.1 and Proposition 5.11, to prove that $\mathcal{I}_{g,b}^W$ is at least exponentially distorted in $\mathrm{Mod}_{g,b}$, it is enough to prove that $\mathcal{I}_{g,b}^W$ is at least exponentially distorted in $\mathrm{Mod}_{g,b}^W$. The proof is based on the exact sequence

$$1 \longrightarrow \mathcal{I}_{g,b}^W \longrightarrow \mathrm{Mod}_{g,b}^W \longrightarrow A \longrightarrow 1, \tag{8}$$

where $A < \mathrm{Sp}(2g, \mathbb{Z})$ consists of symplectic matrices with a $g \times g$ block of zeros in the lower left-hand corner. Now, by Remark 5.7, we have $H^W = 0$. By Lemma 5.6, the relative Johnson homomorphism τ^W given by Theorem 5.8 is of the form

$$\tau^W : \mathcal{I}_{g,b}^W \longrightarrow \mathrm{H}_3(H/W; \mathbb{Z}) \cong \wedge^3(H/W) \cong \wedge^3 \langle b_1, \ldots, b_g \rangle.$$

Moreover, it is easy to see that A acts on $\mathrm{H}_3(H/W; \mathbb{Z})$ via the surjection $A \to \mathrm{SL}(g, \mathbb{Z})$ given by projection to the lower right hand $g \times g$ block. For $g \geq 4$, Theorem 2.8 says that the A-module $\wedge^3 \mathbb{Z}^g$ is nontrivial and irreducible, so we can apply Propositions 2.6 and 2.3 to $\mathcal{I}_{g,b}^W \lhd \mathrm{Mod}_{g,b}^W$, setting $\psi = \tau^W$, to conclude that $\mathcal{I}_{g,b}^W$ is at least exponentially distorted in $\mathrm{Mod}_{g,b}^W$, as desired. $\quad\square$

Proof of Theorem 1.5. First note that $K = \mathcal{I}_g^W$, where $W = \mathrm{H}_1(\Sigma_{h,1}; \mathbb{Z})$. Again, by Lemma 2.1 and Proposition 5.11, to prove that $\mathcal{I}_g^W$ is at least exponentially distorted in Mod_g, it is enough to prove that $\mathcal{I}_g^W$ is at least exponentially distorted in Mod_g^W. To prove this we consider the exact sequence

$$1 \longrightarrow \mathcal{I}_g^W \longrightarrow \mathrm{Mod}_g^W \longrightarrow A \longrightarrow 1, \tag{9}$$

where $A < \mathrm{Sp}(2g, \mathbb{Z})$ is isomorphic to $\mathrm{Sp}(2(g-h), \mathbb{Z})$. Now, by Lemma 5.6, the homomorphism τ^W given by Theorem 5.8 is of the form

$$\tau^W : \mathcal{I}_g^W \longrightarrow \mathrm{H}_3(H/W; \mathbb{Z})/H^W \cong (\wedge^3 \mathrm{H}_1(\Sigma_{g-h,1}; \mathbb{Z}))/\mathrm{H}_1(\Sigma_{g-h,1}; \mathbb{Z}).$$

Since $g - h \geq 2$, Theorem 2.8 says that the $\mathrm{Sp}(2(g-h), \mathbb{Z})$-module

$$(\wedge^3 \mathrm{H}_1(\Sigma_{g-h,1}; \mathbb{Z}))/\mathrm{H}_1(\Sigma_{g-h,1}, \mathbb{Z})$$

is nontrivial and irreducible. Applying Propositions 2.6 and 2.3 to $\mathcal{I}_g^W \lhd \mathrm{Mod}_g^W$ with $\psi = \tau^W$ then gives that $\mathcal{I}_g^W$ is at least exponentially distorted in Mod_g^W, as desired. $\quad\square$

The following theorem gives lower bounds for the distortion of $\mathcal{I}_{g,b}^W$ in $\mathrm{Mod}_{g,b}$ for arbitrary W. As the proof is similar to the proofs of Theorems 1.3 and 1.5, we omit it.

Theorem 5.12. *Let $g \geq 3$, $b \in \{0, 1\}$, and $W \subset H$. Assume that some element of $\mathrm{Mod}_{g,b}^W$ acts partially hyperbolically on $\mathrm{H}_3(H/W; \mathbb{Z})/H^W$. Then $\mathcal{I}_{g,b}^W$ is at least exponentially distorted in $\mathrm{Mod}_{g,b}$.*

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