Equivariant group presentations and the second homology group of the Torelli group

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Abstract

We develop a theory of equivariant group presentations and relate them to the second homology group of a group. Our main application says that the second homology group of the Torelli subgroup of the mapping class group is finitely generated as a $\mathbb{Z}[\mathrm{Sp}_{2g}(\mathbb{Z})]$-module.

1 Introduction

If G is a finitely presentable group, then $\mathrm{H}_2(G)$ is a finitely generated abelian group. Recent work on representation stability has focused on groups that do not satisfy finiteness conditions like finite presentability, but where the lack of finiteness is explained by the action of a larger group. In this spirit, for a group G acted upon by a group Γ, we introduce "finite Γ-equivariant presentations" for G and show that in many situations, having such a presentation implies that $\mathrm{H}_2(G)$ is finitely generated as a $\mathbb{Z}[\Gamma]$-module. As an application, we prove a conjecture of Church–Farb about the second homology group of the Torelli group.

Torelli group. Let Σ_g^b be a compact oriented genus g surface with b boundary components and let Mod_g^b be its mapping class group, i.e. the group of isotopy classes of orientation-preserving diffeomorphisms f of Σ_g^b with $f|_{\partial \Sigma_g^b} = \mathrm{id}$. For $b \in \{0, 1\}$, Poincaré duality implies that the algebraic intersection form on $\mathrm{H}_1(\Sigma_g^b; \mathbb{Z})$ is a Mod_g^b-invariant symplectic form. The Mod_g^b-action on $\mathrm{H}_1(\Sigma_g^b; \mathbb{Z})$ thus gives a representation $\mathrm{Mod}_g^b \to \mathrm{Sp}_{2g}(\mathbb{Z})$ whose kernel $\mathcal{I}_g^b$ is the *Torelli group*. This is summarized in the short exact sequence

$$1 \longrightarrow \mathcal{I}_g^b \longrightarrow \mathrm{Mod}_g^b \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1.$$

See [J1, FMa, Pu5] for surveys about the mapping class group and Torelli group.[1]

Combinatorial group theory. The group Mod_g^b has strong finiteness properties. For instance, it is finitely presentable [McCo] and all of its homology groups are finitely generated [Har]. Since $\mathcal{I}_g^b$ is an infinite-index subgroup of Mod_g^b, there is no formal reason for it to inherit any of these finiteness properties. Indeed, McCullough–Miller [McCuMi] proved that $\mathcal{I}_2^b$ is not even finitely generated. Mess [Me] strengthened this by showing that $\mathcal{I}_2$ is

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[1]Defining the Torelli group on a surface with multiple boundary components is subtle and there are several possibilities; see [Pu1]. We will thus restrict ourselves to surfaces with at most 1 boundary component.

an infinite rank free group. However, a remarkable theorem of Johnson [J2] says that $\mathcal{I}_g^b$ is finitely generated for $g \geq 3$. This has been strengthened in various ways; for instance, more efficient generating sets can be found in [Pu4] and generation results for deeper subgroups can be found in [CPu, CEPu, EHe]. However, it is not known whether $\mathcal{I}_g^b$ is finitely presentable for $g \geq 3$.

Homology. Even the easier question of whether $H_2(\mathcal{I}_g^b)$ is finitely generated is open (though large pieces of it have been calculated by Hain [Hai] and by Brendle–Farb [BrF]), so it is natural to study weaker finiteness properties. The conjugation action of Mod_g^b on $\mathcal{I}_g^b$ descends to an action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $H_k(\mathcal{I}_g^b)$. Church–Farb [CF] made a series of conjectures about this action which assert that it exhibits various forms of "representation stability". Precisely stating all of their conjectures would take us too far afield, so we will only do so for the one that we prove.

The group $H_1(\mathcal{I}_g^b)$ was calculated by Johnson [J3], and this calculation shows that all of Church–Farb's conjectures hold for it. All subsequent work has focused on $H_2(\mathcal{I}_g^b)$. Boldsen–Dollerup [BolDo] proved that $H_2(\mathcal{I}_g^1; \mathbb{Q})$ satisfies a regularity condition introduced by Church–Farb called "surjective representation stability". Miller–Patzt–Wilson [MiPaW] strengthened this by showing that $H_2(\mathcal{I}_g^1; \mathbb{Q})$ is "centrally stable" in the sense of [PuSa] (which generalizes to groups like $\mathrm{Sp}_{2g}(\mathbb{Z})$ ideas from [Pu6] concerning the symmetric group).

Church–Farb also conjectured that $H_k(\mathcal{I}_g^b)$ is finitely generated as a $\mathbb{Z}[\mathrm{Sp}_{2g}(\mathbb{Z})]$-module for $g \gg 0$. In other words, there exists a finite set $S \subset H_k(\mathcal{I}_g^b)$ such that the $\mathrm{Sp}_{2g}(\mathbb{Z})$-orbit of S spans $H_k(\mathcal{I}_g^b)$. Our first main theorem verifies this for $k = 2$.

Theorem A. $H_2(\mathcal{I}_g^b)$ *is finitely generated as a* $\mathbb{Z}[\mathrm{Sp}_{2g}(\mathbb{Z})]$*-module for* $g \geq 3$ *and* $b \in \{0, 1\}$.

Remark 1.1. Day–Putman [DaPu] proved an analogue of Theorem A for the Torelli subgroup of $\mathrm{Aut}(F_n)$. Our proof of Theorem A shares some features with [DaPu]; for instance, both start with infinite presentations for the groups in question. However, the proofs are fundamentally different. For instance, [DaPu] identifies an explicit generating set for H_2, while our proof of Theorem A is inherently non-constructive. $\qquad\square$

Equivariant presentations. For a group G, the group $H_2(G)$ is connected to the relations in a presentation for G. We derive Theorem A from a special kind of presentation for $\mathcal{I}_g^b$ that incorporates the action of Mod_g^b. For a set S, let $F(S)$ be the free group on S.

Definition 1.2. Let G and Γ be groups such that Γ acts on G. A *finite* Γ-*equivariant presentation* for G consists of a pair (S_0, R_0) as follows:
- $S_0 \subset G$ is a finite set whose orbits $S \coloneqq \Gamma \cdot S_0$ generate G.
- $R_0 \subset F(S)$ is a finite set of relations for G whose orbits $R \coloneqq \Gamma \cdot R_0$ form a complete set of relations for G. Here Γ acts on $F(S)$ via its action on S. $\qquad\square$

Example 1.3. Let
$$G = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}/2$$
and let Γ be the symmetric group on the set $\mathbb{N}$. The group Γ acts on G via its action on $\mathbb{N}$. For $n \in \mathbb{N}$, let $s_n \in G$ be the generator of the nth summand. Then G has a finite Γ-equivariant presentation (S_0, R_0) with $S_0 = \{s_1\}$ and $R_0 = \{s_1^2, [s_1, s_2]\}$. $\qquad\square$

Example 1.4. For some $n \geq 3$, let $\mathfrak{S}_{n+1}$ be the symmetric group on $(n+1)$ letters $\{1, \ldots, n+1\}$ and let $\Gamma \subset \mathfrak{S}_{n+1}$ be any subgroup acting 3-transitively on $\{1, \ldots, n\}$ and fixing $n+1$. The group Γ acts on $\mathfrak{S}_{n+1}$ by conjugation. For $1 \leq i \leq n$, let $s_i \in \mathfrak{S}_{n+1}$ be the transposition $(i, n+1)$. Then $\mathfrak{S}_{n+1}$ has a finite Γ-equivariant presentation (S_0, R_0) with $S_0 = \{s_1\}$ and $R_0 = \{s_1^2, (s_1 s_2)^3, (s_1 s_2 s_3)^4\}$. Indeed, unpacking the definition of a Γ-equivariant presentation, we see that this corresponds to the ordinary group presentation

$$\langle s_1, \ldots, s_n \mid s_i^2, (s_i s_j)^3, (s_i s_j s_k)^4 \rangle.$$

Here distinct indices represent distinct numbers. This presentation of $\mathfrak{S}_{n+1}$ is a small variant on a presentation of Burnside [Bu, p. 464] and Miller [Mi, p. 366]; see [GKasKanL, §2.2] for more details. $\qquad\square$

Remark 1.5. Finite Γ-equivariant presentations are related to but different from the L-presentations defined by Bartholdi [Ba] and used to study the Torelli subgroup of $\mathrm{Aut}(F_n)$ by Day–Putman [DaPu]. $\qquad\square$

We prove that subject to some technical conditions, a group G with a finite Γ-equivariant presentation has $\mathrm{H}_2(G)$ finitely generated as a $\mathbb{Z}[\Gamma]$-module. A group Γ is of type FP_n if the trivial $\mathbb{Z}[\Gamma]$-module $\mathbb{Z}$ has a length n partial resolution by finitely generated projective modules. This holds, for instance, if Γ has a $K(\Gamma, 1)$ whose n-skeleton is compact.

Theorem B. *Let G and Γ be groups such that Γ acts on G. Assume the following:*
- *G has a finite Γ-equivariant presentation (S_0, R_0).*
- *$\mathrm{H}_1(G)$ is finitely generated as an abelian group.*
- *Γ is of type FP_2.*
- *The Γ-stabilizers of all elements of S_0 are finitely generated.*

Then $\mathrm{H}_2(G)$ is finitely generated as a $\mathbb{Z}[\Gamma]$-module.

Remark 1.6. If Γ is of type FP_n for $n > 2$ and the Γ-stabilizers of all elements of S_0 are of type FP_{n-1}, then our proof of Theorem B almost proves that $\mathrm{H}_2(G)$ is a $\mathbb{Z}[\Gamma]$-module of type FP_n. The only thing that goes wrong is Claim 2.4 from the proof, which would require some kind of higher regularity for the relations that seems hard to verify in practice. $\qquad\square$

Back to Torelli. The group Mod_g^b acts on $\mathcal{I}_g^b$ by conjugation, and we prove the following.

Theorem C. *$\mathcal{I}_g^b$ has a finite Mod_g^b-equivariant presentation for $g \geq 3$ and $b \in \{0, 1\}$.*

Given this, we can apply Theorem B to deduce Theorem A. Indeed, the other conditions of Theorem B are satisfied:
- Johnson [J2] proved that $\mathcal{I}_g^b$ is finitely generated, so $\mathrm{H}_1(\mathcal{I}_g^b)$ is a finitely generated abelian group. Johnson later calculated $\mathrm{H}_1(\mathcal{I}_g^b)$ in [J3].
- Harer [Har] proved that Mod_g^b is of type FP_∞, so it certainly is of type FP_2.
- Let (S_0, R_0) be the finite Mod_g^b-equivariant presentation for $\mathcal{I}_g^b$ given by Theorem C. The Mod_g^b-stabilizers of elements of S_0 are same as the Mod_g^b-centralizers of these elements, and Rafi–Selinger–Yampolsky [RSeY] proved that the Mod_g^b-centralizers of all elements of Mod_g^b are finitely generated. We remark that since elements of $\mathcal{I}_g^b$ are "pure", this could be deduced for elements of S_0 using much earlier results of Ivanov–McCarthy [IMcCa].[2]

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[2]Indeed, it is not hard to work these centralizers out explicitly and verify directly that they are finitely generated. The only moderately difficult case is that of simply intersecting pair maps.
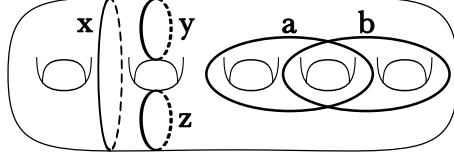
Figure 1: *A separating twist T_x, a bounding map $T_y T_z^{-1}$, and a simply intersecting pair map* $[T_a, T_b]$.

Theorem B then implies that $\mathrm{H}_2(\mathcal{I}_g^b)$ is a finitely generated $\mathbb{Z}[\mathrm{Mod}_g^b]$-module. Since the action of Mod_g^b on $\mathrm{H}_2(\mathcal{I}_g^b)$ factors through $\mathrm{Sp}_{2g}(\mathbb{Z})$, this implies that $\mathrm{H}_2(\mathcal{I}_g^b)$ is a finitely generated $\mathbb{Z}[\mathrm{Sp}_{2g}(\mathbb{Z})]$-module, as desired.

Generators. A presentation for $\mathcal{I}_g^b$ close to the one claimed by Theorem C was constructed by Putman [Pu2]. To explain what remains to be done, we discuss Putman's presentation. We begin with the following (see Figure 1):

- A *separating twist* is a Dehn twist T_x with x a separating simple closed curve.
- A *bounding pair map* is a product $T_y T_z^{-1}$, with y and z disjoint simple closed curves whose union separates Σ_g^b (note that we do not require that y and z be nonseparating).
- A *simply intersecting pair map* is a commutator $[T_a, T_b]$, where a and b are simple closed curves that intersect twice with opposite signs (again, note that we do not require that a or b be nonseparating).

These all lie in $\mathcal{I}_g^b$. Building on work of Birman [Bir], Powell [Po] proved that $\mathcal{I}_g^b$ is generated by separating twists and bounding pair maps. See [Pu1, HatMa] for modern proofs.

Putman's generating set $S(\mathcal{I})$ consists of all separating twists, all bounding pair maps, and all simply intersecting pair maps. All of these can be embedded into the surface in various ways, but the change of coordinates principle from [FMa, §1.3] shows that up to the action of Mod_g^b, there are only finitely many ways to embed each into the surface. In other words, the conjugation action of Mod_g^b on $\mathcal{I}_g^b$ restricts to a Mod_g^b-action on $S(\mathcal{I})$ with finitely many orbits. Let $S_0(\mathcal{I}) \subset S(\mathcal{I})$ contain a single representative of each of these orbits, so $S_0(\mathcal{I})$ is a finite set whose Mod_g^b-orbit is the generating set $S(\mathcal{I})$.

Relations. Let $R(\mathcal{I})$ be Putman's set of relations. The element of $R(\mathcal{I})$ fall into a number of families: there are 8 "formal relations" along with the "lantern relations", the "crossed lantern relations", the "Witt–Hall relations", and the "commutator shuffle relations". For example, the lantern and crossed lantern relations are depicted in Figure 2. Almost all of these relations correspond to a finite list of pictures that can be embedded in the surface in various ways, and again the change of coordinates principle says that up to the action of Mod_g^b there are only finitely many ways of embedding each into the surface.

This might lead one to think that the action of Mod_g^b on $R(\mathcal{I})$ has finitely many orbits. If this were the case, then letting $R_0(\mathcal{I})$ be a set containing a single representative of each of these orbits, the pair $(S_0(\mathcal{I}), R_0(\mathcal{I}))$ would be a finite Mod_g^b-equivariant presentation for $\mathcal{I}_g^b$.

Trouble. However, there is an issue: three of the formal relations are not of this form. In the notation of [Pu2], these are the relations (F.6), (F.7), and (F.8). They can be stated as follows. Consider $M \in S(\mathcal{I})$.
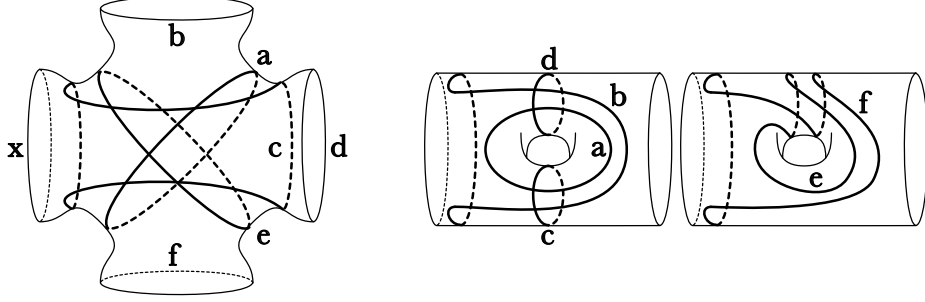
Figure 2: *On the left is the lantern relation $T_x = (T_a T_b^{-1})(T_c T_d^{-1})(T_e T_f^{-1})$, where T_x is a separating twist and the other three terms are bounding pair maps. On the right is the crossed lantern relation $(T_a T_b^{-1})(T_c T_d^{-1}) = T_e T_f^{-1}$; all three terms are bounding pair maps.*

- Relation (F.6) says that if T_x is a separating twist, then

$$MT_x M^{-1} = T_{M(x)}.$$

- Relation (F.7) says that if $T_y T_z^{-1}$ is a bounding pair map, then

$$MT_x T_y^{-1} M^{-1} = T_{M(x)} T_{M(y)}^{-1}.$$

- Relation (F.8) says that if $[T_a, T_b]$ is a simply intersecting pair map, then

$$M[T_a, T_b] M^{-1} = [T_{M(a)}, T_{M(b)}].$$

Since there is no bound on how the curves making up M intersect T_x or $T_y T_z^{-1}$ or $[T_a, T_b]$, even up to the action of Mod_g^b these relations fall into infinitely many families.

Weakly-finite equivariant presentations. In summary, what Putman constructed in [Pu2] is the following kind of equivariant presentation (a priori weaker than a finite one):

Definition 1.7. Let Γ be a group and let $G \lhd \Gamma$, so Γ acts on G by conjugation. For $\gamma \in \Gamma$ and $g \in G$, write ${}^\gamma g = \gamma g \gamma^{-1}$. A *weakly-finite Γ-equivariant presentation* for G consists of a pair (S_0, R_0) as follows:

- $S_0 \subset G$ is a finite set whose orbits

$$S := \Gamma \cdot S_0 = \{{}^\gamma s \mid \gamma \in \Gamma, \ s \in S\} \subset G$$

  generate G.
- $R_0 \subset F(S)$ is a finite set of relations for G with the following property. Let $R \subset F(S)$ be $\Gamma \cdot R_0$, and define

$$R_{\mathrm{conj}} = \{sts^{-1} ({}^s t)^{-1} \mid s, t \in S\}.$$

  See Remark 1.8 below for more explanation of the form of these relations. Then $R \cup R_{\mathrm{conj}}$ is a complete set of relations for G. □

Remark 1.8. The relations in R_{conj} might seem a bit puzzling. To unpack them, observe first that by its very definition $S \subset G$ is closed under conjugation by Γ. Since $S \subset G \subset \Gamma$, this implies that S is closed under conjugation by elements of S, i.e. for $s, t \in S$ we have ${}^s t \in S$. The elements s and t and ${}^s t$ are all elements of $S \subset G$ that a priori have no relationship in the free group $F(S)$. The relations in R_{conj} (which hold trivially in G) force $s \in F(S)$ to conjugate $t \in F(S)$ to ${}^s t \in F(S)$. □

To fix this problem, we will show that in favorable situations a weakly-finite Γ-equivariant presentation can be converted into a finite Γ-equivariant presentation:

Theorem D. *Let Γ be a group and let $G \lhd \Gamma$. Assume the following holds:*
  - *There exists a weakly-finite Γ-equivariant presentation for G.*
  - *Both Γ and G are finitely generated.*
Then there exists a finite Γ-equivariant presentation for G.

Since Mod_g^b is finitely generated for all g and b (a theorem due essentially to Dehn; see [FMa]) and $\mathcal{I}_g^b$ is finitely generated for $g \geq 3$ and $b \in \{0, 1\}$ (a theorem of Johnson [J2]), we can combine Theorem D with Putman's result from [Pu2] to deduce Theorem C, which asserts that $\mathcal{I}_g^b$ has a finite Mod_g^b-equivariant presentation for $g \geq 3$ and $b \in \{0, 1\}$. As we discussed above, this implies Theorem A.

Remark 1.9. Putman's construction of a weakly-finite Mod_g^b-equivariant presentation for $\mathcal{I}_g^b$ used the main result of [Pu3], which explains how to find presentations for groups acting on simplicial complexes without identifying a fundamental domain for the action. This machine inherently gives weakly-finite equivariant presentations; indeed, the relations R_{conj} in the above definition are precisely the "conjugation relations" from [Pu3]. Because of this, we expect that Theorem D will prove useful in other applications of combinatorial group theory to representation stability. $\qquad\square$

Outline. The two results we must prove are Theorems B and D. The proof of Theorem B is in §2 and the proof of Theorem D is in §3. Here are brief descriptions of how those proofs go:
  - For Theorem B, the main idea is to embed $\mathrm{H}_2(G)$ into the five-term exact sequence in group homology and study the finiteness properties of the various terms of this sequence as $\mathbb{Z}[\Gamma]$-modules.
  - For Theorem D, the main idea is to show that all the conjugation relations can be deduced from those that show how to express the conjugate of a G-generator by a Γ-generator in terms of the finite generating set for G.

2   The second homology group of normal subgroups

This section contains the proof of Theorem B. We begin with some preliminary results in §2.1 and then give the proof in §2.2.

2.1   Preliminaries on finiteness conditions

All our rings have a unit, and unless otherwise specified all modules are left modules. We start with the following definition.

Definition 2.1. Let R be a ring and let M be an R-module. We say that M is of type FP_n if there exists a length n partial resolution

$$P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

of M by finitely generated projective R-modules. $\qquad\square$

Remark 2.2. Being of type FP_0 is equivalent to being finitely generated and being of type FP_1 is equivalent to being finitely presentable. $\qquad\square$

The following lemma implies among other things that if M has type FP_n for all n, then M has an infinite length resolution by finitely generated projective R-modules.

Lemma 2.3 ([Bie]). *Let R be a ring and let M be an R-module. Then M is of type FP_n if and only if M is finitely generated and for all partial resolutions*

$$P_{n'} \longrightarrow P_{n'-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

of M by finitely generated projective R-modules of length $n' < n$, the kernel of the map $P_{n'} \to P_{n'-1}$ is finitely generated.

The following lemma describes how our finiteness conditions behave under extensions.

Lemma 2.4 ([Bie]). *Let R be a ring and let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be a short exact sequence of R-modules. The following hold.
  1. *If M is of type FP_n and M'' is of type FP_m for some $n \geq 0$ and $m \geq 1$, then M' is of type FP_r for $r = \min(n, m-1)$.*
  2. *If M' is of type FP_n and M'' is of type FP_m for some $n, m \geq 0$, then M is of type FP_r for $r = \min(n, m)$.*
  3. *If M' is of type FP_n and M is of type FP_m for some $n, m \geq 0$, then M'' is of type FP_r for $r = \min(n+1, m)$.*

We now turn to groups.

Definition 2.5. A group G is of type FP_n if the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$ is of type FP_n. $\quad\square$

For a group G of type FP_n, the following lemma give a large supply of $\mathbb{Z}[G]$-modules of type FP_n. We expect that this lemma is known to the experts, but we do not know a reference.

Lemma 2.6. *Let G be a group of type FP_n and let M be a $\mathbb{Z}[G]$-module that is finitely generated as an abelian group. Then M is of type FP_n.*

Proof. Let M_{tor} be the torsion subgroup of M, so M/M_{tor} is free abelian and

$$0 \longrightarrow M_{\mathrm{tor}} \longrightarrow M \longrightarrow M/M_{\mathrm{tor}} \longrightarrow 0$$

is a short exact sequence of $\mathbb{Z}[G]$-modules. Lemma 2.4 says that it is enough to prove that M_{tor} and M/M_{tor} are of type FP_n, so we are reduced to proving the following two special cases of the lemma.

Case. *M is a finite abelian group.*

In this case, the kernel G' of the map $G \to \mathrm{Aut}(M)$ is a finite-index subgroup of G. Since a partial resolution of $\mathbb{Z}$ by finitely generated projective $\mathbb{Z}[G]$-modules restricts to a partial resolution of $\mathbb{Z}$ by finitely generated projective $\mathbb{Z}[G']$-modules, the group G' is of type FP_n. We claim that M is of type FP_n as a $\mathbb{Z}[G']$-module. Indeed, there exists a finitely generated free abelian group $\widetilde{M}$ and a surjection $f \colon \widetilde{M} \to M$. Endow $\widetilde{M}$ with the trivial $\mathbb{Z}[G']$-module structure. We then have a short exact sequence

$$0 \longrightarrow \ker(f) \longrightarrow \widetilde{M} \xrightarrow{\ f\ } M \longrightarrow 0$$

of $\mathbb{Z}[G']$-modules (all trivial!). Both $\widetilde{M}$ and $\ker(f)$ are direct sums of the trivial $\mathbb{Z}[G']$-module $\mathbb{Z}$. Since G' is of type FP_n, so are the $\mathbb{Z}[G']$-modules $\widetilde{M}$ and $\ker(f)$. Lemma 2.4 thus implies that M is also of type FP_n as a $\mathbb{Z}[G']$-module, as claimed.

Now consider some $n' < n$ and a length n' partial resolution

$$P_{n'} \longrightarrow P_{n'-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0 \tag{2.1}$$

of M by finitely generated projective $\mathbb{Z}[G]$-modules. By Lemma 2.3, to prove that M is of type FP_n as a $\mathbb{Z}[G]$-module, it is enough to prove that the kernel of the map $P_{n'} \to P_{n'-1}$ is a finitely generated $\mathbb{Z}[G]$-module. Since G' is a finite-index subgroup of G, the restriction of (2.1) to $\mathbb{Z}[G']$ is a length n' partial resolution of M by finitely generated projective $\mathbb{Z}[G']$-modules. Since M is of type FP_n as a $\mathbb{Z}[G']$-module, Lemma 2.3 implies that the kernel of the map $P_{n'} \to P_{n'-1}$ is a finitely generated $\mathbb{Z}[G']$-module. This clearly implies that it is also a finitely generated $\mathbb{Z}[G]$-module, as desired.

Case. *M is a finitely generated free abelian group.*

We learned the argument in this case from Steinberg [St]. Recall that if R is a ring, then a free R-bimodule is a direct sum of copies of the R-bimodule $R \otimes_{\mathbb{Z}} R$. Since G is of type FP_n, a theorem of Pride [Pr, Theorem 2] shows that there exists a length n partial resolution

$$Q_n \longrightarrow Q_{n-1} \longrightarrow \cdots \longrightarrow Q_0 \longrightarrow \mathbb{Z}[G] \longrightarrow 0$$

of $\mathbb{Z}[G]$ by finitely generated free $\mathbb{Z}[G]$-bimodules. We claim that

$$Q_n \otimes_{\mathbb{Z}[G]} M \longrightarrow \cdots \longrightarrow Q_0 \otimes_{\mathbb{Z}[G]} M \longrightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} M = M \longrightarrow 0 \tag{2.2}$$

is a length n partial resolution of M by finitely generated free $\mathbb{Z}[G]$-modules. This requires checking two things:

- The chain complex (2.2) is exact. To see this, observe that each free $\mathbb{Z}[G]$-bimodule Q_k is a free right $\mathbb{Z}[G]$-module (not necessarily finitely generated). This means that the chain complex (2.2) computes $\mathrm{Tor}^{\mathbb{Z}[G]}(\mathbb{Z}[G], M)$, which vanishes since $\mathbb{Z}[G]$ is free.
- Each $Q_k \otimes_{\mathbb{Z}[G]} M$ is a finitely generated free $\mathbb{Z}[G]$-module. This follows from the fact that Q_k is a finite direct sum of copies of $\mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G]$ together with the observation that

$$(\mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G]) \otimes_{\mathbb{Z}[G]} M \cong \mathbb{Z}[G] \otimes_{\mathbb{Z}} M \cong (\mathbb{Z}[G])^{\oplus \mathrm{rank}(M)}. \qquad \square$$

2.2 From equivariant presentations to finiteness

We now prove Theorem B.

Proof of Theorem B. We first recall the setup. Let G and Γ be groups such that Γ acts on G. Assume that the following hold:
- G has a finite Γ-equivariant presentation (S_0, R_0).
- $H_1(G)$ is finitely generated as an abelian group.
- Γ is of type FP_2.
- The Γ-stabilizers of all elements of S_0 are finitely generated.

We must prove that $H_2(G)$ is finitely generated as a $\mathbb{Z}[\Gamma]$-module, i.e. that $H_2(G)$ is a $\mathbb{Z}[\Gamma]$-module of type FP_0.

Let $S = \Gamma \cdot S_0$, let $R = \Gamma \cdot R_0$, and let $\langle\!\langle R \rangle\!\rangle$ be the normal closure of R in $F(S)$. We thus have a short exact sequence

$$1 \longrightarrow \langle\!\langle R \rangle\!\rangle \longrightarrow F(S) \longrightarrow G \longrightarrow 1.$$

Since $H_2(F(S)) = 0$, the five-term exact sequence in group cohomology associated to this short exact sequence is of the form

$$0 \longrightarrow H_2(G) \longrightarrow (H_1(\langle\!\langle R \rangle\!\rangle))_G \xrightarrow{\phi} H_1(F(S)) \xrightarrow{\psi} H_1(G) \longrightarrow 0.$$

By construction, this is an exact sequence of $\mathbb{Z}[\Gamma]$-modules. The following five claims elucidate the finiteness properties of various terms of this exact sequence. The theorem itself is the fifth one.

Claim 2.1. $H_1(G)$ *is a* $\mathbb{Z}[\Gamma]$-*module of type* FP_2.

This follows from Lemma 2.6, which we can apply since Γ is a group of type FP_2 and $H_1(G)$ is a finitely generated abelian group.

Claim 2.2. $H_1(F(S))$ *is a* $\mathbb{Z}[\Gamma]$-*module of type* FP_1.

By construction, we have
$$H_1(F(S)) \cong \bigoplus_{s \in S_0} \mathbb{Z}[\Gamma/\Gamma_s],$$

where Γ_s denotes the Γ-stabilizer of $s \in S_0$. Fixing some $s \in S_0$, it is thus enough to prove that $\mathbb{Z}[\Gamma/\Gamma_s]$ is a $\mathbb{Z}[\Gamma]$-module of type FP_1. By assumption, Γ_s is finitely generated; let X be a finite generating set for it. We then have a finite presentation

$$\bigoplus_{x \in X} \mathbb{Z}[\Gamma] \xrightarrow{\iota} \mathbb{Z}[\Gamma] \longrightarrow \mathbb{Z}[\Gamma/\Gamma_s] \longrightarrow 0,$$

where the map ι is of the form $\iota = \oplus_{x \in X} \iota_x$ with $\iota_x \colon \mathbb{Z}[\Gamma] \to \mathbb{Z}[\Gamma]$ the map taking $\omega \in \mathbb{Z}[\Gamma]$ to $\omega(x-1) \in \mathbb{Z}[\Gamma]$. The claim follows.

Claim 2.3. $\ker(\psi) = \operatorname{Im}(\phi)$ *is a* $\mathbb{Z}[\Gamma]$-*module of type* FP_1.

This follows from Claims 2.1 and 2.2 together with Lemma 2.4.

Claim 2.4. $(H_1(\langle\!\langle R \rangle\!\rangle))_G$ *is a* $\mathbb{Z}[\Gamma]$-*module of type* FP_0.

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Since R_0 is finite, it is enough to show that the evident map

$$\bigoplus_{r \in R_0} \mathbb{Z}[\Gamma] \longrightarrow (\mathrm{H}_1 \left(\langle\!\langle R \rangle\!\rangle \right))_G$$

is a surjection. The image of this map equals the image of the map

$$\mathrm{H}_1(R) \longrightarrow (\mathrm{H}_1 \left(\langle\!\langle R \rangle\!\rangle \right))_G.$$

But this map is surjective since we are taking G-coinvariants, which causes all G-conjugates of an element $r \in R$ to collapse to a single element of $(\mathrm{H}_1 \left(\langle\!\langle R \rangle\!\rangle \right))_G$.

Claim 2.5. $\ker(\phi) = \mathrm{H}_2(G)$ *is a* $\mathbb{Z}[\Gamma]$-*module of type* FP_0.

This follows from Claims 2.3 and 2.4 together with Lemma 2.4. $\qquad\square$

3 Upgrading weakly-finite equivariant presentations

We conclude the paper by proving Theorem D.

Proof of Theorem D. We begin by recalling the setup. Let Γ be a group and let G be a normal subgroup of Γ. Assume that both G and Γ are finitely generated, and let (S_0, R_0) be a weakly-finite Γ-equivariant presentation for G. Our goal is to construct a finite Γ-equivariant presentation for G.

For $\gamma \in \Gamma$ and $g \in G$, write $^\gamma g$ for the image of g under the action of γ. As in the definition of a weakly-finite Γ-equivariant presentation, let $S = \Gamma \cdot S_0$, let $R = \Gamma \cdot R_0$, and let

$$R_{\mathrm{conj}} = \{sts^{-1} \left({}^s t \right)^{-1} \mid s, t \in S\}.$$

By definition, $R \cup R_{\mathrm{conj}}$ is a complete set of relations for G. To prove the theorem, it is enough to construct a further finite set $R_0' \subset F(S)$ with the following property:

(†) Let $R' = \Gamma \cdot R_0'$. Then each relation in R_{conj} is a consequence of the relations in R'.

Since G is finitely generated and S generates G, there exists a finite subset X of S that generates G. Define R_0'' to be the following finite subset of R_{conj}:

$$R_0'' = \{sxs^{-1} \left({}^s x \right)^{-1} \mid s \in S_0, \, x \in X\}.$$

Now let Y be a finite generating set for Γ that is symmetric in the sense that if $y \in Y$, then $y^{-1} \in Y$. For $y \in Y$ and $x \in X$, we can find some $w_{y,x} \in F(X) \subset F(S)$ that maps to $^y x \in S \subset G$ under the composition $F(X) \hookrightarrow F(S) \to G$. Define

$$R_0''' = \{ \left({}^y x \right) \left(w_{y,x} \right)^{-1} \mid y \in Y, \, x \in X \} \subset F(S)$$

and $R_0' = R_0'' \cup R_0'''$.

We claim that R_0' satisfies (†). To see this, let $R' = \Gamma \cdot R_0'$. The claim (†) is the third of the following claims. For words $h, k \in F(S)$, write $h \equiv k$ if h equals k modulo R'.

Claim 3.1. *For all $u \in S$, there is some word $w \in F(X)$ such that $u \equiv w$.*

There exists some $u_0 \in S_0$ and $\gamma \in \Gamma$ such that

$$u = {}^\gamma u_0.$$

The proof of the claim will be by induction on the length of the shortest word in the generating set Y for Γ needed to write γ. The base case where that word has length 0 is trivial, so assume that it has positive length. Using the fact that Y is symmetric, we can write $\gamma = y\gamma'$, where $y \in Y$ and γ' can be written as a shorter word than γ. Our inductive hypothesis say that there exists some $w' \in F(X)$ such that

$$ {}^{\gamma'} u_0 \equiv w'.$$

Since the relations in R' are closed under the action of Γ, this implies that

$$u = {}^{y\gamma'} u_0 \equiv {}^y w'.$$

Now write $w = x_1^{e_1} \cdots x_n^{e_n}$ with $x_i \in X$ and $e_i \in \{\pm 1\}$. We then have

$$ {}^y w' = ({}^y x_1)^{e_1} ({}^y x_2)^{e_2} \cdots ({}^y x_n)^{e_n}.$$

Using the relations in $R_0''' \subset R'$, we see that each term on the right hand side of the previous equation is equivalent modulo the relations in R' to a word in $F(X)$. The claim follows.

Claim 3.2. *For all $s_0 \in S_0$ and $u \in S$, we have*

$$s_0 u s_0^{-1} \equiv {}^{s_0} u. \tag{3.1}$$

Using Claim 3.1, we can find a word $w \in F(X)$ such that

$$u \equiv w. \tag{3.2}$$

Since the relations in R' are closed under the action of Γ, this implies that

$$ {}^{s_0} u \equiv {}^{s_0} w. \tag{3.3}$$

Write $w = x_1^{e_1} \cdots x_n^{e_n}$ with $x_i \in X$ and $e_i \in \{\pm 1\}$. The equation (3.1) can then be deduced as follows:

$$\begin{aligned}
s_0 u s_0^{-1} &\equiv s_0 \left(x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n} \right) s_0^{-1} \\
&= \left(s_0 x_1 s_0^{-1} \right)^{e_1} \left(s_0 x_2 s_0^{-1} \right)^{e_2} \cdots \left(s_0 x_n s_0^{-1} \right)^{e_n} \\
&\equiv ({}^{s_0} x_1)^{e_1} ({}^{s_0} x_2)^{e_2} \cdots ({}^{s_0} x_n)^{e_n} \\
&= {}^{s_0} \left(x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n} \right) \\
&= {}^{s_0} w \\
&\equiv {}^{s_0} u.
\end{aligned}$$

Here the first $\equiv$ is (3.2), the second $\equiv$ uses the relations in $R_0'' \subset R$, and the third $\equiv$ is (3.3).

Claim 3.3. *For all $s, t \in S$, we have*

$$sts^{-1} \equiv {}^s t. \tag{3.4}$$

We can write $s = {}^{\gamma}s_0$ for some $\gamma \in \Gamma$ and $s_0 \in S_0$. Define $u = {}^{\gamma^{-1}}t$. Claim 3.2 implies that

$$s_0 u s_0^{-1} \equiv {}^{s_0}u \tag{3.5}$$

The relations in R' are closed under the action of Γ, so we can apply γ to both sides of (3.5) and deduce that

$$^{\gamma}\left(s_0 u s_0^{-1}\right) \equiv {}^{\gamma s_0}u. \tag{3.6}$$

The left hand side of (3.6) is

$$^{\gamma}\left(s_0 u s_0^{-1}\right) = ({}^{\gamma}s_0)\left({}^{\gamma\gamma^{-1}}t\right)({}^{\gamma}s_0)^{-1} = sts^{-1}, \tag{3.7}$$

while the right hand side of (3.6) is

$$^{\gamma s_0}u = {}^{\gamma s_0 \gamma^{-1}}t = {}^{s}t; \tag{3.8}$$

here we are using the fact that Γ acts on G by conjugation, so

$$\gamma s_0 \gamma^{-1} = {}^{\gamma}s_0 = s$$

in G. Since (3.7) and (3.8) are also the left and right hand sides of (3.4), we conclude that (3.4) holds, as desired.

Claim 3.3 was precisely (†) above, which as we noted implies that $(S_0, R_0 \cup R_0')$ is a finite Γ-equivariant presentation of G. The theorem follows. $\qquad\square$

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