Abelian covers of surfaces and the homology of the level L mapping class group

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Abstract

We calculate the first homology group of the mapping class group with coefficients in the first rational homology group of the universal abelian $\mathbb{Z}/L$-cover of the surface. If the surface has one marked point, then the answer is $\mathbb{Q}^{\tau(L)}$, where $\tau(L)$ is the number of positive divisors of L. If the surface instead has one boundary component, then the answer is $\mathbb{Q}$. We also perform the same calculation for the level L subgroup of the mapping class group. Set $H_L = \mathrm{H}_1(\Sigma_g;\mathbb{Z}/L)$. If the surface has one marked point, then the answer is $\mathbb{Q}[H_L]$, the rational group ring of H_L. If the surface instead has one boundary component, then the answer is $\mathbb{Q}$.

1 Introduction

Let $\Sigma_{g,b}^n$ be an oriented genus g surface with b boundary components and n marked points and let $\mathrm{Mod}_{g,b}^n$ be its *mapping class group*. This is the group of homotopy classes of orientation-preserving diffeomorphisms of $\Sigma_{g,b}^n$ that act as the identity on the boundary components and marked points. We will usually omit the b and the n if they vanish. The homology groups of $\mathrm{Mod}_{g,b}^n$, which play important roles in both algebraic geometry and low-dimensional topology, have been studied intensely for the past 40 years. The culmination of much recent work is the resolution of the Mumford conjecture by Madsen and Weiss [17], which identifies $\mathrm{H}^*(\mathrm{Mod}_{g,b}^n;\mathbb{Q})$ in a stable range.

Twisted coefficient systems. For many applications, it is important to know the homology groups of $\mathrm{Mod}_{g,b}^n$ with respect to various twisted coefficient systems. For simplicity, assume that $(b,n) \in \{(0,0),(1,0),(0,1)\}$. A lot is known about coefficient systems that factor through the standard *symplectic representation* of $\mathrm{Mod}_{g,b}^n$. This is the natural representation $\mathrm{Mod}_{g,b}^n \to \mathrm{Sp}_{2g}(\mathbb{Z})$ that arises from the action of $\mathrm{Mod}_{g,b}^n$ on $\mathrm{H}_1(\Sigma_{g,b}^n;\mathbb{Z})$. Its target is the symplectic group because the action preserves the algebraic intersection form. For any rational representation V of the algebraic group Sp_{2g}, Looijenga [15] has completely determined $\mathrm{H}^*(\mathrm{Mod}_g;V)$ as a module over $\mathrm{H}^*(\mathrm{Mod}_g;\mathbb{Q})$ in a stable range. Over $\mathbb{Z}$, a bit less is known. Morita [18] has calculated $\mathrm{H}_1(\mathrm{Mod}_{g,b}^n;\mathrm{H}_1(\Sigma_{g,b}^n;\mathbb{Z}))$ for $g \geq 3$. For $b \geq 1$, this was later generalized by Kawazumi [14], who calculated $\mathrm{H}^*(\mathrm{Mod}_{g,b}^n;(\mathrm{H}^1(\Sigma_{g,b}^n;\mathbb{Z}))^{\otimes k})$ as a module over $\mathrm{H}^*(\mathrm{Mod}_{g,b}^n;\mathbb{Z})$ in a stable range.

Fix some $L \geq 2$. In this paper, we calculate the first homology group of the mapping class group with coefficients in the first rational homology group of the universal abelian $\mathbb{Z}/L$-cover of the surface (see below for the definition). We remark that this representation does *not* factor through $\mathrm{Sp}_{2g}(\mathbb{Z})$. Our techniques also give results for certain finite-index subgroups of the mapping class group. These results play an important technical role in a recent pair of papers by the author [20, 21] that study the second cohomology group and Picard group of the moduli space of curves with level L structures.

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Universal abelian $\mathbb{Z}/L$-cover. Let K_g be the kernel of the natural map $\pi_1(\Sigma_g) \to \mathrm{H}_1(\Sigma_g;\mathbb{Z}/L)$. The group K_g is the fundamental group of the universal abelian $\mathbb{Z}/L$-cover of Σ_g. Since Mod_g^1 fixes a basepoint on Σ_g, it acts on $\pi_1(\Sigma_g)$. This action preserves K_g. We thus obtain an action of Mod_g^1 on $\mathrm{H}_1(K_g;\mathbb{Q})$, the first homology group of the universal abelian $\mathbb{Z}/L$-cover of Σ_g. This representation has been previously studied by Looijenga [16], who essentially determined its image.

Remark. In [16], Looijenga more generally studied the actions of appropriate finite-index subgroups of Mod_g^1 on the first rational homology groups V of arbitrary finite abelian covers of Σ_g. Letting $\mathrm{Mod}_g^1(L)$ denote the level L subgroup of Mod_g^1 (see below), we can choose L so that $\mathrm{Mod}_g^1(L)$ acts on V. It then follows from Lemma 2.2 below that V appears a direct summand in the $\mathrm{Mod}_g^1(L)$-module $\mathrm{H}_1(K_g;\mathbb{Q})$, so one can use our results to study V as well.

Statements of theorems. Let $\tau(L)$ be the number of positive divisors of L (including 1 and L). Our first theorem is as follows.

Theorem A. *For $g \geq 4$ and $L \geq 2$, we have* $\mathrm{H}_1(\mathrm{Mod}_g^1;\mathrm{H}_1(K_g;\mathbb{Q})) \cong \mathbb{Q}^{\tau(L)}$.

In fact, our proof of Theorem A also gives a result for the *level L subgroup* of Mod_g^1, denoted $\mathrm{Mod}_g^1(L)$. This is the kernel of the action of Mod_g^1 on $\mathrm{H}_1(\Sigma_g^1;\mathbb{Z}/L)$. Far less is known about its homology. The only previous result of which the author is aware is a paper of Hain [9] that calculates $\mathrm{H}_1(\mathrm{Mod}_{g,b}^n(L);V)$ for rational representations V of the algebraic group Sp_{2g}. Our theorem is as follows.

Theorem B. *For $g \geq 4$ and $L \geq 2$, we have* $\mathrm{H}_1(\mathrm{Mod}_g^1(L);\mathrm{H}_1(K_g;\mathbb{Q})) \cong \mathbb{Q}[H_L]$*, where H_L equals* $\mathrm{H}_1(\Sigma_g;\mathbb{Z}/L)$ *and $\mathbb{Q}[H_L]$ is the rational group ring of the abelian group H_L.*

Remark. Both $\mathrm{H}_1(\mathrm{Mod}_g^1(L);\mathrm{H}_1(K_g;\mathbb{Q}))$ and $\mathbb{Q}[H_L]$ possess natural Mod_g^1-actions. The action of Mod_g^1 on $\mathrm{H}_1(\mathrm{Mod}_g^1(L);\mathrm{H}_1(K_g;\mathbb{Q}))$ comes from conjugation, and the action on $\mathbb{Q}[H_L]$ factors through the symplectic group. The isomorphism in Theorem B is equivariant with respect to these actions.

Somewhat surprisingly, things are quite different for surfaces with boundary. Define $\mathrm{Mod}_{g,1}(L)$ to be the kernel of the action of $\mathrm{Mod}_{g,1}$ on $\mathrm{H}_1(\Sigma_{g,1};\mathbb{Z}/L)$. Fixing a basepoint for $\pi_1(\Sigma_{g,1})$ on $\partial\Sigma_{g,1}$, the groups $\mathrm{Mod}_{g,1}$ and $\mathrm{Mod}_{g,1}(L)$ act on $\pi_1(\Sigma_{g,1})$. Define $K_{g,1}$ to be the kernel of the map $\pi_1(\Sigma_{g,1}) \to \mathrm{H}_1(\Sigma_{g,1};\mathbb{Z}/L)$. The group $K_{g,1}$ is the fundamental group of the universal abelian $\mathbb{Z}/L$-cover of $\Sigma_{g,1}$ and is preserved by the actions of $\mathrm{Mod}_{g,1}$ and $\mathrm{Mod}_{g,1}(L)$. We then have the following theorem.

Theorem C. *For $g \geq 4$ and $L \geq 2$, we have*

$$\mathrm{H}_1(\mathrm{Mod}_{g,1};\mathrm{H}_1(K_{g,1};\mathbb{Q})) \cong \mathrm{H}_1(\mathrm{Mod}_{g,1}(L);\mathrm{H}_1(K_{g,1};\mathbb{Q})) \cong \mathbb{Q}.$$

Remark. The group Mod_g does not act on the universal abelian $\mathbb{Z}/L$-cover of Σ_g. Each individual mapping class can be lifted to a diffeomorphism of the cover, but a fixed basepoint is necessary to make this lift canonical and thereby provide a representation of the entire group. The best one can achieve is as follows. There is a Birman exact sequence (see §2.3) of the form

$$1 \longrightarrow \pi_1(\Sigma_g) \longrightarrow \mathrm{Mod}_g^1 \longrightarrow \mathrm{Mod}_g \longrightarrow 1. \tag{1}$$

Since Mod_g^1 acts on $\mathrm{H}_1(K_g;\mathbb{Q})$, the group Mod_g acts on the $(\mathrm{H}_1(K_g;\mathbb{Q}))_{\pi_1(\Sigma_g)}$. However, using the transfer map (see Lemma 2.3 below) one can show that this ring of coinvariants is simply $\mathrm{H}_1(\Sigma_g;\mathbb{Q})$, so no new representation is obtained.

Comments on the proofs. The key observation underlying the proofs of our theorems is as follows. The group Mod_g^1 contains a natural copy of $\pi_1(\Sigma_g)$, known as the "point-pushing subgroup" (see §2.3 below). This fits into the Birman exact sequence (1) above. While the action of Mod_g^1 on K_g is very complicated, the action of $\pi_1(\Sigma_g)$ on $K_g \lhd \pi_1(\Sigma_g)$ is simply conjugation. Moreover, it turns out that in some vague sense the action of Mod_g^1 on $\text{H}_1(K_g;\mathbb{Q})$ is "concentrated" in the action of $\pi_1(\Sigma_g)$ on $\text{H}_1(K_g;\mathbb{Q})$. Intuitively, this happens because (as noted in the remark above) the quotient of Mod_g^1 by $\pi_1(\Sigma_g)$, namely Mod_g, does *not* act in any natural way on K_g. We prove our theorems by carefully examining all of these groups and actions.

Some related results. Some additional related results should be mentioned. First, Ivanov [12] has proven a homological stability result for the homology of $\text{Mod}_{g,1}$ with respect to very general systems of coefficients (those of "bounded degree"; the system $\text{H}_1(K_{g,1};\mathbb{Z})$ satisfies this condition). This generalizes Harer's [10] well-known untwisted homological stability theorem for the mapping class group. Ivanov's theorem has been extended to $\Sigma_{g,b}$ for $b > 1$ by Boldsen [5]. We remark that such a result is false for closed surfaces. Indeed, in [18, Corollary 5.4], Morita showed that

$$\text{H}_1(\text{Mod}_g;\text{H}_1(\Sigma_g;\mathbb{Z})) \cong \mathbb{Z}/(2g-2)\mathbb{Z}$$

for $g \geq 2$. In a somewhat different direction, a recent series of papers by Anderson and Villemoes [1, 2, 3] calculate the first homology groups of $\text{Mod}_{g,b}^n$ with coefficients in certain spaces of functions on representations varieties of $\text{Mod}_{g,b}^n$.

Outline of paper. In §2, we discuss some background results about group cohomology and the mapping class group. Next, in §3 we introduce a number of groups and group actions that will play important roles in our paper. At the end of this section, we state two key technical lemmas whose proofs are postponed until later. In §4, we prove our main theorems (assuming the truth of these two technical lemmas). In §5, we give the outline of the proof of our two key technical lemmas, reducing them to two other results, the first of which is proven in §7 (using some preliminary calculations that are first done in §6) and the second in §8.

Notation and conventions. We will denote by $i(x,y) \in \mathbb{Z}/L$ the algebraic intersection number of $x, y \in \text{H}_1(\Sigma_g;\mathbb{Z}/L)$. All surfaces we mention will contain a basepoint unless otherwise specified, and all maps between surfaces will respect this basepoint. Also, if G is a group, then we define $[g_1,g_2] = g_1g_2g_1^{-1}g_2^{-1}$ and $g_1^{g_2} = g_2g_1g_2^{-1}$ for $g_1, g_2 \in G$.

Acknowledgments. I wish to thank an anonymous referee who pointed out Lemma 3.2 below to me. This dramatically simplified my original proofs.

2 Preliminaries

2.1 Group homology

We begin by reviewing some facts about group homology and establishing some notation (see [6] for more details).

Degree zero. Let G be a group and M be a G-module. The *coinvariants* of M, denoted M_G, is the quotient M/K, where K is the submodule spanned by the set $\{g \cdot x - x \mid x \in M,\, g \in G\}$. We have $\text{H}_0(G;M) = M_G$.

The five-term exact sequence. Let

$$1 \longrightarrow K \longrightarrow G \longrightarrow Q \longrightarrow 1$$

be a short exact sequence of groups and let M be a G-module. We then have a 5-term exact sequence

$$\text{H}_2(G;M) \longrightarrow \text{H}_2(Q;M_K) \longrightarrow (\text{H}_1(K;M))_Q \longrightarrow \text{H}_1(G;M) \longrightarrow \text{H}_1(Q;M_K) \longrightarrow 0.$$

The long exact sequence. Let G be a group and let

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

be a short exact sequence of G-modules. Then there is a long exact sequence of the form

$$\cdots \longrightarrow \mathrm{H}_k(G;M_1) \longrightarrow \mathrm{H}_k(G;M_2) \longrightarrow \mathrm{H}_k(G;M_3) \longrightarrow \mathrm{H}_{k-1}(G;M_1) \longrightarrow \cdots.$$

The transfer map. If $G_2 < G_1$ are groups satisfying $[G_1 : G_2] < \infty$ and M is a G_1-module, then for all k there exists a *transfer map* of the form $t : \mathrm{H}_k(G_1;M) \to \mathrm{H}_k(G_2;M)$ (see, e.g., [6, Chapter III.9]). The key property of t (see [6, Proposition III.9.5]) is that if $i : \mathrm{H}_k(G_2;M) \to \mathrm{H}_k(G_1;M)$ is the map induced by the inclusion, then $i \circ t : \mathrm{H}_k(G_1;M) \to \mathrm{H}_k(G_1;M)$ is multiplication by $[G_1 : G_2]$. In particular, if M is a G_1-vector space over $\mathbb{Q}$, then we obtain a right inverse $\frac{1}{[G_1:G_2]}t$ to i. This yields the following standard lemma.

Lemma 2.1. *Let $G_2 < G_1$ be groups satisfying $[G_1 : G_2] < \infty$ and let M be a G_1-vector space over $\mathbb{Q}$. Then the map $\mathrm{H}_k(G_2;M) \to \mathrm{H}_k(G_1;M)$ is surjective for all $k \geq 0$.*

Assume now that $M = \mathbb{Q}$ and that Γ is a group acting on G_2 and G_1 such that the inclusion is Γ-equivariant. The induced map $\mathrm{H}_k(G_2;M) \to \mathrm{H}_k(G_1;M)$ is therefore Γ-equivariant. Moreover, the map $t : \mathrm{H}_k(G_1;M) \to \mathrm{H}_k(G_2;M)$ is also Γ-equivariant, so the surjection $\mathrm{H}_k(G_2;M) \to \mathrm{H}_k(G_1;M)$ splits in a Γ-equivariant manner. We obtain the following lemma.

Lemma 2.2. *Fix $k \geq 0$, and let $G_2 < G_1$ be groups satisfying $[G_1 : G_2] < \infty$. Let Γ be a group acting on G_1 and G_2 such that the inclusion map $G_2 \to G_1$ is Γ-equivariant. Define C to be the kernel of the surjection $\mathrm{H}_k(G_2;\mathbb{Q}) \to \mathrm{H}_k(G_1;\mathbb{Q})$. We then have a Γ-invariant splitting $\mathrm{H}_k(G_2;\mathbb{Q}) \cong \mathrm{H}_k(G_1;\mathbb{Q}) \oplus C$.*

Finally, for finite-index normal subgroups, the Hochschild-Serre spectral sequences implies the following strengthening of Lemma 2.1.

Lemma 2.3. *Let $G_2 \lhd G_1$ be groups satisfying $[G_1 : G_2] < \infty$ and let M be a G_1-vector space over $\mathbb{Q}$. Then $\mathrm{H}_k(G_1;M) \cong (\mathrm{H}_k(G_2;M))_{G_1}$ for all $k \geq 0$.*

2.2 Rational group rings

Let G be a finite group and let $\mathbb{Q}[G]$ be the rational group ring of G. We will consider $\mathbb{Q}[G]$ to be a left G-module. Let $\varepsilon : \mathbb{Q}[G] \to \mathbb{Q}$ be the *augmentation map*, i.e. the unique linear map such that $\varepsilon(g) = 1$ for all $g \in G$. The map ε is a map of G-modules, where $\mathbb{Q}$ has the trivial G-action. Its kernel is the *augmentation ideal* $I(G)$. We thus have a short exact sequence of G-modules

$$0 \longrightarrow I(G) \longrightarrow \mathbb{Q}[G] \xrightarrow{\varepsilon} \mathbb{Q} \longrightarrow 0. \tag{2}$$

Set $\theta = \sum_{g \in G} g \in \mathbb{Q}[G]$. The element θ is invariant under G, and the exact sequence (2) splits via the G-equivariant map $\psi : \mathbb{Q} \to \mathbb{Q}[G]$ defined by $\psi(1) = \frac{1}{|G|}\theta$. The associated projection $\phi : \mathbb{Q}[G] \to I(G)$ satisfies $\ker(\phi) = \langle \theta \rangle$. From these considerations, we obtain the following lemma.

Lemma 2.4. *Let G be a finite group. Then $\mathbb{Q}[G] \cong \mathbb{Q} \oplus I(G)$, where $I(G)$ is isomorphic to the quotient of $\mathbb{Q}[G]$ by $\langle \theta \rangle$.*

2.3 The mapping class group

Dehn twists. We will denote by T_γ the left Dehn twist about a simple closed curve γ on a surface.

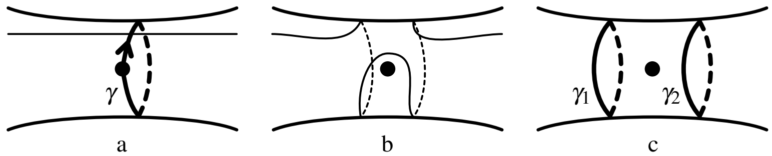


Figure 1: *a.* $\gamma \in \pi_1(\Sigma_{g,b}^p, *)$ *can be realized by a simple closed curve.* *b. The effect of the "point-pushing" map* ρ_γ. *c.* $\rho_\gamma = T_{\gamma_1} T_{\gamma_2}^{-1}$

The Birman exact sequence. For simplicity, assume that $g \geq 2$. The Birman exact sequence describes the effect on the mapping class group of deleting a marked point or gluing a disc to a boundary component. The first version, due to Birman (see [4]), is of the form

$$1 \longrightarrow \pi_1(\Sigma_{g,b}^p, *) \longrightarrow \mathrm{Mod}_{g,b}^{p+1} \longrightarrow \mathrm{Mod}_{g,b}^p \longrightarrow 1. \tag{3}$$

Here $*$ is a marked point and the map $\mathrm{Mod}_{g,b}^{p+1} \to \mathrm{Mod}_{g,b}^p$ comes from deleting $*$. For $\gamma \in \pi_1(\Sigma_{g,b}^p, *)$, the associated mapping class in the kernel of (3) "pushes" the deleted marked point around the path γ. For this reason, the kernel $\pi_1(\Sigma_{g,b}^p, *)$ is known as the "point-pushing subgroup". If $\gamma \in \pi_1(\Sigma_{g,b}^p, *)$ can be realized by a simple closed curve, then there is a nice formula for the associated "point-pushing" mapping class ρ_γ. Namely, let γ_1 and γ_2 be the boundary components of a regular neighborhood of γ (see Figure 1.c). Assume that γ_1 lies to the left of γ and γ_2 to the right. Then as is clear from Figure 1.a–b, we have $\rho_\gamma = T_{\gamma_1} T_{\gamma_2}^{-1}$.

The second form of the Birman exact sequence, due to Johnson [13], is of the form

$$1 \longrightarrow \pi_1(U\Sigma_{g,b}^p) \longrightarrow \mathrm{Mod}_{g,b+1}^p \longrightarrow \mathrm{Mod}_{g,b}^p \longrightarrow 1. \tag{4}$$

Here $U\Sigma_{g,b}^p$ is the unit tangent bundle of $\Sigma_{g,b}^p$ and the map $\mathrm{Mod}_{g,b+1}^p \to \mathrm{Mod}_{g,b}^p$ comes from gluing a disc to a boundary component β of $\Sigma_{g,b+1}^p$ and extending mapping classes by the identity. The fiber of the kernel $\pi_1(U\Sigma_{g,b}^p)$ corresponds to the mapping class T_β.

The group $\mathrm{Mod}_{g,b+1}^p$ acts on $\mathrm{H}_1(\Sigma_{g,b+1}^p; \mathbb{Q})$. Restrict this action to $\pi_1(U\Sigma_{g,b}^p) < \mathrm{Mod}_{g,b+1}^p$. Since T_β acts trivially on $\mathrm{H}_1(\Sigma_{g,b+1}^p; \mathbb{Q})$, the action of $\pi_1(U\Sigma_{g,b}^p)$ on $\mathrm{H}_1(\Sigma_{g,b+1}^p; \mathbb{Q})$ factors through an action of $\pi_1(\Sigma_{g,b}^p, *)$. This action has the following simple description.

Lemma 2.5. *The above action of* $\pi_1(\Sigma_{g,b}^p, *)$ *on* $\mathrm{H}_1(\Sigma_{g,b+1}^p; \mathbb{Q})$ *is given by the formula*

$$\gamma(v) = v + i([\gamma], v) \cdot [\beta] \qquad (\gamma \in \pi_1(\Sigma_{g,b}^p, *),\ v \in \mathrm{H}_1(\Sigma_{g,b+1}^p; \mathbb{Q})),$$

where the boundary component β *is oriented such that the interior of the surface is to its right.*

Proof. It is enough to check this for elements $\gamma \in \pi_1(\Sigma_{g,b}^p, *)$ that can be realized by simple closed curves. For such curves, this formula is immediate from Figure 1.a–b. ☐

The level L subgroup. Now assume that $b = p = 0$ and fix some $L \geq 2$. The kernels of (3) and (4) both lie in the level L subgroup of the mapping class group. We thus have short exact sequences

$$1 \longrightarrow \pi_1(\Sigma_g) \longrightarrow \mathrm{Mod}_g^1(L) \longrightarrow \mathrm{Mod}_g(L) \longrightarrow 1$$

and

$$1 \longrightarrow \pi_1(U\Sigma_g) \longrightarrow \mathrm{Mod}_{g,1}(L) \longrightarrow \mathrm{Mod}_g(L) \longrightarrow 1.$$

We will also refer to these as Birman exact sequences.

We will need two cohomological results about the level L subgroups of the mapping class group, both of which are due to Hain. To simplify their statements, we will denote the whole mapping class group by $\mathrm{Mod}_{g,1}(1)$ and $\mathrm{Mod}_g^1(1)$.

Theorem 2.6 (Hain, [9])**.** *For $g \geq 3$ and $L \geq 1$, we have* $\mathrm{H}_1(\mathrm{Mod}_{g,1}(L);\mathbb{Q}) = 0$.

Theorem 2.7 (Hain, [9])**.** *For $g \geq 3$ and $L \geq 1$, we have*

$$\mathrm{H}_1(\mathrm{Mod}_{g,1}(L);\mathrm{H}_1(\Sigma_g;\mathbb{Q})) \cong \mathrm{H}_1(\mathrm{Mod}_g^1(L);\mathrm{H}_1(\Sigma_g;\mathbb{Q})) \cong \mathbb{Q}.$$

Remark. In fact, in [9] Hain calculated $\mathrm{H}_1(\mathrm{Mod}_{g,b}^p(L);M)$ for all rational representations M of the algebraic group Sp_{2g}.

3 The cast of characters

3.1 Homology groups of abelian covers

As in the introduction, define

$$K_g = \ker(\pi_1(\Sigma_g) \to \mathrm{H}_1(\Sigma_g;\mathbb{Z}/L)) \quad \text{and} \quad K_{g,1} = \ker(\pi_1(\Sigma_{g,1}) \to \mathrm{H}_1(\Sigma_{g,1};\mathbb{Z}/L)).$$

Here the basepoint for $\pi_1(\Sigma_{g,1})$ lies on $\partial\Sigma_{g,1}$. Also, let S_g^K (resp. $S_{g,1}^K$) be the cover of Σ_g (resp. $\Sigma_{g,1}$) corresponding to K_g (resp. $K_{g,1}$). We will identify $\mathrm{H}_1(K_g;\mathbb{Q})$ and $\mathrm{H}_1(K_{g,1};\mathbb{Q})$ with $\mathrm{H}_1(S_g^K;\mathbb{Q})$ and $\mathrm{H}_1(S_{g,1}^K;\mathbb{Q})$, respectively.

We have actions of Mod_g^1 and $\mathrm{Mod}_{g,1}$ on K_g and $K_{g,1}$, respectively. Define

$$C_g = \ker(\mathrm{H}_1(K_g;\mathbb{Q}) \to \mathrm{H}_1(\Sigma_g;\mathbb{Q})) \quad \text{and} \quad C_{g,1} = \ker(\mathrm{H}_1(K_{g,1};\mathbb{Q}) \to \mathrm{H}_1(\Sigma_{g,1};\mathbb{Q})).$$

By Lemma 2.2, we have mapping class group invariant decompositions

$$\mathrm{H}_1(K_g;\mathbb{Q}) \cong \mathrm{H}_1(\Sigma_g;\mathbb{Q}) \oplus C_g \quad \text{and} \quad \mathrm{H}_1(K_{g,1};\mathbb{Q}) \cong \mathrm{H}_1(\Sigma_{g,1};\mathbb{Q}) \oplus C_{g,1}.$$

The surjective map $\pi_1(\Sigma_{g,1}) \to \pi_1(\Sigma_g)$ induced by gluing a disc to the boundary component of $\Sigma_{g,1}$ restricts to a surjection $K_{g,1} \to K_g$. Let I_g be its kernel, so we have a short exact sequence

$$0 \longrightarrow I_g \longrightarrow K_{g,1} \longrightarrow K_g \longrightarrow 0. \tag{5}$$

We now prove the following.

Lemma 3.1. *For $g \geq 1$ and $L \geq 2$, the vector space I_g is isomorphic as a $\mathrm{Mod}_{g,1}$-module to the augmentation ideal of $\mathbb{Q}[H_L]$, where $H_L = \mathrm{H}_1(\Sigma_{g,1};\mathbb{Z}/L)$.*

Proof. Let $* \in \partial\Sigma_{g,1}$ be the basepoint and $\tilde{*} \in S_{g,1}^K$ be a lift of $*$, so $K_{g,1} = \pi_1(S_{g,1}^K, \tilde{*})$. The subgroup $I_g < \mathrm{H}_1(S_{g,1}^K;\mathbb{Q})$ is exactly the subgroup generated by the homology classes of the boundary components of $S_{g,1}^K$. The group of deck transformations H_L acts on these boundary components. Let $\gamma \in \pi_1(\Sigma_{g,1}, *)$ be the simple closed curve that goes once around the boundary component with the surface to its right. Since γ lifts to a simple closed curve $\tilde{\gamma} \in \pi_1(S_{g,1}^K, \tilde{*})$, the action of H_L on the boundary components of $\tilde{\Sigma}$ is free. For $v \in H_L$, let $[\![v]\!] \in I_g$ denote the homology class of the v-translate of $\tilde{\gamma}$. The only relation between the homology classes of the boundary components of a surface with boundary is that their sum is 0. We conclude that I_g is isomorphic to the quotient of the $\mathbb{Q}$-vector space with basis the formal symbols $\{[\![v]\!] \mid v \in H_L\}$ by the 1-dimensional subspace generated by $\sum_{v \in H_L} [\![v]\!]$. This is exactly the augmentation ideal of $\mathbb{Q}[H_L]$, and we are done. $\square$

Throughout the rest of this paper, we will denote by $[\![v]\!]$ the element of I_g corresponding to $v \in H_L$ as in the proof above.

Remark. It is clear that $I_g < C_{g,1}$, so the exact sequence (5) restricts to a short exact sequence

$$0 \longrightarrow I_g \longrightarrow C_{g,1} \longrightarrow C_g \longrightarrow 0.$$

3.2 Actions on homology groups of abelian covers

It is clear that $\mathrm{Mod}_{g,1}$ acts on $K_{g,1}$ and Mod_g^1 acts on K_g. Letting β be the boundary component of $\Sigma_{g,1}$, these two mapping class groups are related by a short exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Mod}_{g,1} \longrightarrow \mathrm{Mod}_g^1 \longrightarrow 1,$$

where $\mathbb{Z}$ is generated by T_β. Since the loop around β lies in $K_{g,1}$, the action of T_β on $\mathrm{H}_1(K_{g,1};\mathbb{Q})$ is trivial. This implies that the action of $\mathrm{Mod}_{g,1}$ on $\mathrm{H}_1(K_{g,1};\mathbb{Q})$ factors through an action of Mod_g^1.

Let $\pi_1(\Sigma_g) < \mathrm{Mod}_g^1$ be the point-pushing subgroup. The action of Mod_g^1 on $\mathrm{H}_1(K_g;\mathbb{Q})$ restricts to the action of $\pi_1(\Sigma_g)$ on $\mathrm{H}_1(K_g;\mathbb{Q})$ induced by conjugation. Restricting this action further to $K_g < \pi_1(\Sigma_g)$ thus yields the trivial action. However, if we instead restrict the action of Mod_g^1 on $\mathrm{H}_1(K_{g,1};\mathbb{Q})$ to K_g, we do *not* get a trivial action.

For $\gamma \in K_g$, denote by $\langle\!\langle \gamma \rangle\!\rangle$ the associated element of $\mathrm{H}_1(K_g;\mathbb{Q})$. Since $K_g < \mathrm{Mod}_g^1$ acts trivially on the kernel and cokernel of the short exact sequence

$$0 \longrightarrow I_g \longrightarrow \mathrm{H}_1(K_{g,1};\mathbb{Q}) \xrightarrow{\rho} \mathrm{H}_1(K_g;\mathbb{Q}) \longrightarrow 0,$$

the action of K_g on $\mathrm{H}_1(K_{g,1};\mathbb{Q})$ is of the form

$$\gamma(x) = x + \omega(\langle\!\langle \gamma \rangle\!\rangle, \rho(x)) \qquad (\gamma \in K_g \text{ and } x \in \mathrm{H}_1(K_{g,1};\mathbb{Q}))$$

for some I_g-valued bilinear form $\omega(\cdot,\cdot)$ on $\mathrm{H}_1(K_g;\mathbb{Q})$. This bilinear form has the following nice description. Let $\langle \cdot,\cdot \rangle_K$ be the algebraic intersection pairing on $\mathrm{H}_1(K_g;\mathbb{Q}) = \mathrm{H}_1(S_g^K;\mathbb{Q})$. The group $H_L = \mathrm{H}_1(\Sigma_g;\mathbb{Z}/L)$ acts on $\mathrm{H}_1(S_g^K;\mathbb{Q})$ via deck transformations. The bilinear pairing in the following lemma first appeared in work of Reidemeister [22, 23] and has since been studied by many people (see, e.g., [7, 11, 16]). We will call it the *Reidemeister pairing*.

Lemma 3.2. *For $\gamma \in K_g$ and $x \in K_g$, we have*

$$\omega(\langle\!\langle \gamma \rangle\!\rangle, x) = \sum_{v \in H_L} \langle v \cdot \langle\!\langle \gamma \rangle\!\rangle, x \rangle_K [\![v]\!].$$

Proof. An immediate consequence of Lemma 2.5. □

3.3 A key technical lemma

One of the linchpins of our proofs of our main theorems is the following lemma about the action of K_g on $C_{g,1} < K_{g,1}$. Its proof is lengthy and is given in §5.

Lemma 3.3. *For $g \geq 4$ and $L \geq 2$, the map $\mathrm{H}_1(K_g;C_{g,1}) \to \mathrm{H}_1(\mathrm{Mod}_g^1(L);C_g)$ is the zero map.*

Remark. The map $\mathrm{H}_1(K_g;C_{g,1}) \to \mathrm{H}_1(\mathrm{Mod}_g^1(L);C_g)$ factors through $\mathrm{H}_1(K_g;C_g)$, and most of our hard work is devoted to characterizing the image of $\mathrm{H}_1(K_g;C_{g,1})$ in $\mathrm{H}_1(K_g;C_g)$. It would be much easier if we could instead prove that the map $\mathrm{H}_1(K_g;C_g) \to \mathrm{H}_1(\mathrm{Mod}_g^1(L);C_g)$ was the zero map, but alas a careful examination of our proof of Theorem B shows that this is not true.

In the course of proving Lemma 3.3, we will also prove the following.

Lemma 3.4. *For $g \geq 3$ and $L \geq 2$, we have $(C_g)_{\pi_1(\Sigma_g)} = (C_{g,1})_{\pi_1(\Sigma_g)} = 0$.*

One useful consequence of Lemma 3.4 is the following.

Lemma 3.5. *For $g \geq 3$ and $L \geq 2$, the natural map $\mathrm{H}_1(\mathrm{Mod}_{g,1}(L);C_g) \to \mathrm{H}_1(\mathrm{Mod}_g^1(L);C_g)$ is an isomorphism.*

Proof. Let β be the boundary component of $\Sigma_{g,1}$. We have a short exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Mod}_{g,1}(L) \longrightarrow \mathrm{Mod}_g^1(L) \longrightarrow 1,$$

where $\mathbb{Z} = \langle T_\beta \rangle$. The last 3 terms of the associated 5-term exact sequence with coefficients C_g are

$$(C_g)_{\mathrm{Mod}_g^1(L)} \longrightarrow \mathrm{H}_1(\mathrm{Mod}_{g,1}(L); C_g) \longrightarrow \mathrm{H}_1(\mathrm{Mod}_g^1(L); C_g) \longrightarrow 0.$$

By Lemma 3.4, we have $(C_g)_{\mathrm{Mod}_g^1(L)} = 0$, and the lemma follows. □

4 Proofs of the main theorems

We now turn to the proofs of our main theorems. In this section, we will assume the truth of Lemmas 3.3 and 3.4, which are proven in subsequent sections.

4.1 Surfaces with boundary

We begin with Theorem C, which asserts that if $g \geq 4$ and $L \geq 2$, then

$$\mathrm{H}_1(\mathrm{Mod}_{g,1}; \mathrm{H}_1(K_{g,1}; \mathbb{Q})) \cong \mathrm{H}_1(\mathrm{Mod}_{g,1}(L); \mathrm{H}_1(K_{g,1}; \mathbb{Q})) \cong \mathbb{Q}.$$

As was noted in §3, we can use Lemma 2.2 to obtain a $\mathrm{Mod}_{g,1}$-invariant decomposition

$$\mathrm{H}_1(K_{g,1}; \mathbb{Q}) \cong \mathrm{H}_1(\Sigma_{g,1}; \mathbb{Q}) \oplus C_{g,1}.$$

This implies that

$$\mathrm{H}_1(\mathrm{Mod}_{g,1}; \mathrm{H}_1(K_{g,1}; \mathbb{Q})) \cong \mathrm{H}_1(\mathrm{Mod}_{g,1}; \mathrm{H}_1(\Sigma_{g,1}; \mathbb{Q})) \oplus \mathrm{H}_1(\mathrm{Mod}_{g,1}; C_{g,1}),$$

and similarly for $\mathrm{Mod}_{g,1}(L)$. Theorem 2.7 says that

$$\mathrm{H}_1(\mathrm{Mod}_{g,1}; \mathrm{H}_1(\Sigma_{g,1}; \mathbb{Q})) \cong \mathrm{H}_1(\mathrm{Mod}_{g,1}(L); \mathrm{H}_1(\Sigma_{g,1}; \mathbb{Q})) \cong \mathbb{Q}.$$

To prove Theorem C, therefore, it is enough to prove the following theorem.

Theorem 4.1. *For $g \geq 4$ and $L \geq 2$, we have*

$$\mathrm{H}_1(\mathrm{Mod}_{g,1}; C_{g,1}) \cong \mathrm{H}_1(\mathrm{Mod}_{g,1}(L); C_{g,1}) = 0.$$

Proof. Since $\mathrm{Mod}_{g,1}(L)$ is a finite-index subgroup of $\mathrm{Mod}_{g,1}$, Lemma 2.1 implies that it is enough to prove that $\mathrm{H}_1(\mathrm{Mod}_{g,1}(L); C_{g,1}) = 0$. Associated to the Birman exact sequence

$$1 \longrightarrow \pi_1(U\Sigma_g) \longrightarrow \mathrm{Mod}_{g,1}(L) \longrightarrow \mathrm{Mod}_g(L) \longrightarrow 1$$

is a 5-term exact sequence in homology with coefficients in $C_{g,1}$. The last 3 terms of this are

$$(\mathrm{H}_1(\pi_1(U\Sigma_g); C_{g,1}))_{\mathrm{Mod}_{g,1}(L)} \xrightarrow{f} \mathrm{H}_1(\mathrm{Mod}_{g,1}(L); C_{g,1}) \longrightarrow \mathrm{H}_1(\mathrm{Mod}_g(L); (C_{g,1})_{\pi_1(U\Sigma_g)}) \longrightarrow 0.$$

Lemma 3.4 says that

$$(C_{g,1})_{\pi_1(U\Sigma_g)} = (C_{g,1})_{\pi_1(\Sigma_g)} = 0.$$

To prove the theorem, therefore, it is enough to show that $f = 0$. This is equivalent to showing that the map

$$f' : \mathrm{H}_1(\pi_1(U\Sigma_g); C_{g,1}) \to \mathrm{H}_1(\mathrm{Mod}_{g,1}(L); C_{g,1})$$

is the zero map.

Our goal is to deduce the fact that $f' = 0$ from Lemma 3.3. To do this, we perform a series of reductions. From the short exact sequence

$$0 \longrightarrow I_g \longrightarrow C_{g,1} \longrightarrow C_g \longrightarrow 0$$

of $\mathrm{Mod}_{g,1}(L)$-modules we obtain a long exact sequence in homology. This long exact sequence contains the segment

$$\mathrm{H}_1(\mathrm{Mod}_{g,1}(L); I_g) \longrightarrow \mathrm{H}_1(\mathrm{Mod}_{g,1}(L); C_{g,1}) \longrightarrow \mathrm{H}_1(\mathrm{Mod}_{g,1}(L); C_g).$$

Since $\mathrm{Mod}_{g,1}(L)$ acts trivially on I_g, Theorem 2.6 implies that $\mathrm{H}_1(\mathrm{Mod}_{g,1}(L); I_g) = 0$. It follows that the map $\mathrm{H}_1(\mathrm{Mod}_{g,1}(L); C_{g,1}) \to \mathrm{H}_1(\mathrm{Mod}_{g,1}(L); C_g)$ is injective. Lemma 3.5 says that $\mathrm{H}_1(\mathrm{Mod}_{g,1}(L); C_g) \cong \mathrm{H}_1(\mathrm{Mod}_g^1(L); C_g)$, so we deduce that to prove that $f' = 0$, it is enough to prove that the map

$$\mathrm{H}_1(\pi_1(U\Sigma_g); C_{g,1}) \longrightarrow \mathrm{H}_1(\mathrm{Mod}_g^1(L); C_g)$$

is the zero map. This map factors through the map

$$\mathrm{H}_1(\pi_1(\Sigma_g); C_{g,1}) \longrightarrow \mathrm{H}_1(\mathrm{Mod}_g^1(L); C_g). \tag{6}$$

Since K_g is a finite-index subgroup of $\pi_1(\Sigma_g)$, Lemma 2.1 says that the map $\mathrm{H}_1(K_g; C_{g,1}) \longrightarrow \mathrm{H}_1(\pi_1(\Sigma_g); C_{g,1})$ is surjective. Thus to show that the map in (6) vanishes, it is enough to show that the map

$$\mathrm{H}_1(K_g; C_{g,1}) \to \mathrm{H}_1(\mathrm{Mod}_g^1(L); C_g)$$

vanishes, which is exactly the content of Lemma 3.3. □

4.2 Closed surfaces

We now turn to Theorems A and B, which assert that if $g \geq 4$ and $L \geq 2$, then

$$\mathrm{H}_1(\mathrm{Mod}_g^1; \mathrm{H}_1(K_g; \mathbb{Q})) \cong \mathbb{Q}^{\tau(L)} \quad \text{and} \quad \mathrm{H}_1(\mathrm{Mod}_g^1(L); \mathrm{H}_1(K_g; \mathbb{Q})) \cong \mathbb{Q}[H_L].$$

Here $\tau(L)$ is the number of positive divisors of L and $H_L \cong \mathrm{H}_1(\Sigma_g; \mathbb{Z}/L)$. Also, the second isomorphism should be equivariant with respect to Mod_g^1 actions on $\mathrm{H}_1(\mathrm{Mod}_g^1(L); \mathrm{H}_1(K_g; \mathbb{Q}))$ and $\mathbb{Q}[H_L]$. We begin by deriving Theorem A from Theorem B.

Proof of Theorem A, assuming Theorem B. Lemma 2.3 implies that

$$\mathrm{H}_1(\mathrm{Mod}_g^1; \mathrm{H}_1(K_g; \mathbb{Q})) \cong (\mathrm{H}_1(\mathrm{Mod}_g^1(L); \mathrm{H}_1(K_g; \mathbb{Q})))_{\mathrm{Mod}_g^1}.$$

Applying Theorem B, we must show that $(\mathbb{Q}[H_L])_{\mathrm{Mod}_g^1} \cong \mathbb{Q}^{\tau(L)}$. The vector space $\mathbb{Q}[H_L]$ has a basis that is permuted by the action of Mod_g^1, namely the elements of H_L. It is enough, therefore, to show that there are $\tau(L)$ orbits of the action of Mod_g^1 on H_L. This action factors through the surjection $\mathrm{Mod}_g^1 \to \mathrm{Sp}_{2g}(\mathbb{Z}/L)$. Let $v \in H_L$ be a fixed primitive vector, and set

$$X = \{cv \mid c \text{ is a positive divisor of } L\} \subset H_L.$$

The set X has cardinality $\tau(L)$, and clearly no two elements of X are in the same $\mathrm{Sp}_{2g}(\mathbb{Z}/L)$-orbit. Also, if $w \in H_L$, then there is a primitive vector w' and a positive divisor c of L such that $w = cw'$. Since $\mathrm{Sp}_{2g}(\mathbb{Z}/L)$ acts transitively on the set of primitive vectors, there is some $\phi \in \mathrm{Sp}_{2g}(\mathbb{Z}/L)$ such that $v = \phi(w')$. Thus w is in the same $\mathrm{Sp}_{2g}(\mathbb{Z}/L)$-orbit as $cv \in X$. We conclude that X contains a unique representative from every $\mathrm{Sp}_{2g}(\mathbb{Z}/L)$-orbit, and we are done. □

Finally, we discuss Theorem B. Lemma 2.2 implies that there is a Mod_g^1-invariant decomposition

$$\mathrm{H}_1(K_g;\mathbb{Q}) \cong \mathrm{H}_1(\Sigma_g;\mathbb{Q}) \oplus C_g,$$

so we have a Mod_g^1-invariant decomposition

$$\mathrm{H}_1(\mathrm{Mod}_g^1(L);\mathrm{H}_1(K_g;\mathbb{Q})) \cong \mathrm{H}_1(\mathrm{Mod}_g^1(L);\mathrm{H}_1(\Sigma_g;\mathbb{Q})) \oplus \mathrm{H}_1(\mathrm{Mod}_g^1(L);C_g).$$

Also, Theorem 2.7 says that

$$\mathrm{H}_1(\mathrm{Mod}_g^1(L);\mathrm{H}_1(\Sigma_g;\mathbb{Q})) \cong \mathbb{Q},$$

so we obtain a Mod_g^1-invariant decomposition

$$\mathrm{H}_1(\mathrm{Mod}_g^1(L);\mathrm{H}_1(K_g;\mathbb{Q})) \cong \mathbb{Q} \oplus \mathrm{H}_1(\mathrm{Mod}_g^1(L);C_g).$$

By Lemma 2.4, we have $\mathbb{Q}[H_L] \cong \mathbb{Q} \oplus I$, where I is the augmentation ideal of $\mathbb{Q}[H_L]$. Lemma 3.1 says that I_g is isomorphic to the augmentation ideal of $\mathbb{Q}[H_L]$, so we conclude that it is enough to prove the following theorem.

Theorem 4.2. *For $g \geq 4$ and $L \geq 2$, we have a Mod_g^1-equivariant isomorphism $\mathrm{H}_1(\mathrm{Mod}_g^1(L);C_g) \cong I_g$.*

Proof. Lemma 3.5 says that there is an isomorphism

$$\mathrm{H}_1(\mathrm{Mod}_{g,1}(L);C_g) \cong \mathrm{H}_1(\mathrm{Mod}_g^1(L);C_g). \tag{7}$$

The action of $\mathrm{Mod}_{g,1}$ on $\mathrm{H}_1(\mathrm{Mod}_{g,1}(L);C_g)$ factors through Mod_g^1, and it is easy to see that the isomorphism in (7) is Mod_g^1-equivariant. We deduce that it is enough to construct a $\mathrm{Mod}_{g,1}$-equivariant isomorphism $\mathrm{H}_1(\mathrm{Mod}_{g,1}(L);C_g) \cong I_g$. The long exact sequence in $\mathrm{Mod}_{g,1}(L)$-homology associated to the short exact sequence

$$0 \longrightarrow I_g \longrightarrow C_{g,1} \longrightarrow C_g \longrightarrow 0$$

of $\mathrm{Mod}_{g,1}(L)$-modules contains the segment

$$\mathrm{H}_1(\mathrm{Mod}_{g,1}(L);C_{g,1}) \longrightarrow \mathrm{H}_1(\mathrm{Mod}_{g,1}(L);C_g) \longrightarrow I_g \longrightarrow (C_{g,1})_{\mathrm{Mod}_{g,1}(L)}.$$

Here we are using the fact that $\mathrm{Mod}_{g,1}(L)$ acts trivially on I_g, so

$$\mathrm{H}_0(\mathrm{Mod}_{g,1}(L);I_g) = (I_g)_{\mathrm{Mod}_{g,1}(L)} = I_g.$$

Theorem 4.1 says that $\mathrm{H}_1(\mathrm{Mod}_{g,1}(L);C_{g,1}) = 0$, and Lemma 3.4 implies that $(C_{g,1})_{\mathrm{Mod}_{g,1}(L)} = 0$. We obtain an isomorphism $\mathrm{H}_1(\mathrm{Mod}_{g,1}(L);C_g) \cong I_g$, which is easily verified to be $\mathrm{Mod}_{g,1}$-equivariant. The theorem follows. □

5 Skeleton of the proof of Lemma 3.3

This section sets the stage for the remainder of the paper by reducing the proof of Lemma 3.3 to two further lemmas which are proven in the remaining sections. It also proves Lemma 3.4. This is all done in §5.2. Before that, §5.1 sets up some notation which will be used in the remainder of the paper.

5.1 Notation for K_g

In this section, we will introduce notation for elements of $\mathrm{H}_1(K_g;\mathbb{Q})$. Let $\rho : K_g \to \mathrm{H}_1(K_g;\mathbb{Q})$ be the abelianization map. Recall that the notation $g_1^{g_2}$ stands for $g_2 g_1 g_2^{-1}$. Observe that if $a \in K_g$ and $x,y \in \pi_1(\Sigma_g)$ are such that $xy^{-1} \in K_g$, then

$$\rho(a^x) = \rho(a^{(xy^{-1})y}) = \rho((xy^{-1})(a^y)(xy^{-1})^{-1}) = \rho(xy^{-1}) + \rho(a^y) - \rho(xy^{-1}) = \rho(a^y).$$

This implies that for $a \in K_g$ and $x,y \in \pi_1(\Sigma_g)$ and $v \in H_L$, we may unambiguously define

$$\langle\!\langle a \rangle\!\rangle^v = \rho(a^{\tilde{v}}) \qquad \text{and} \qquad \langle\!\langle x,y \rangle\!\rangle^v = \rho([x,y]^{\tilde{v}}),$$

where $\tilde{v} \in \pi_1(\Sigma_g)$ is any lift of v under the map $\pi_1(\Sigma_g) \to H_L$.

As another bit of notation, for $x \in \pi_1(\Sigma_g)$ denote by $\bar{x}$ the associated element of $H_L = \mathrm{H}_1(\Sigma_g;\mathbb{Z}/L)$.

With this notation, we have the following identities.

Lemma 5.1.

1. *For $a \in K_g$ and $x \in \pi_1(\Sigma_g)$, we have $\langle\!\langle a,x \rangle\!\rangle = \langle\!\langle a \rangle\!\rangle - \langle\!\langle a \rangle\!\rangle^{\bar{x}}$.*
2. *For $x,y \in \pi_1(\Sigma_g)$, we have $\langle\!\langle x,y \rangle\!\rangle = -\langle\!\langle y,x \rangle\!\rangle$.*
3. *For $x,y,z \in \pi_1(\Sigma_g)$, we have $\langle\!\langle xy,z \rangle\!\rangle = \langle\!\langle x,z \rangle\!\rangle + \langle\!\langle y,z \rangle\!\rangle^{\bar{x}}$.*
4. *For $x,y \in \pi_1(\Sigma_g)$, we have $\langle\!\langle x^{-1},y \rangle\!\rangle = -\langle\!\langle x,y \rangle\!\rangle^{-\bar{x}}$.*

Proof. Items 1 and 2 are obvious, item 3 follows from the commutator identity $[xy,z] = [x,z][y,z]^x$, and item 4 follows from item 3. $\square$

5.2 Skeleton of the proof of Lemma 3.3

Recall that we want to prove that the map

$$\mathrm{H}_1(K_g;C_{g,1}) \to \mathrm{H}_1(\mathrm{Mod}_g^1(L);C_g)$$

is the zero map. This map factors through $\mathrm{H}_1(K_g;C_g)$. Since K_g acts trivially on C_g, this latter group is isomorphic to $\mathrm{H}_1(K_g;\mathbb{Z}) \otimes C_g$.

Our first order of business is to characterize the image of $\mathrm{H}_1(K_g;C_{g,1})$ in $\mathrm{H}_1(K_g;C_g)$. The long exact sequence in K_g-homology associated to the short exact sequence

$$0 \longrightarrow I_g \longrightarrow C_{g,1} \longrightarrow C_g \longrightarrow 0$$

of K_g-modules contains the segment

$$\mathrm{H}_1(K_g;C_{g,1}) \longrightarrow \mathrm{H}_1(K_g;\mathbb{Z}) \otimes C_g \xrightarrow{\partial} I_g. \tag{8}$$

Here we have used the fact that $\mathrm{H}_0(K_g;I_g) = (I_g)_{K_g} = I_g$. The image of $\mathrm{H}_1(K_g;C_{g,1})$ in $\mathrm{H}_1(K_g;C_g)$ is the kernel of ∂. It turns out that ∂ is closely related to the Reidemeister pairing $\omega(\cdot,\cdot)$ from Lemma 3.2.

Lemma 5.2. *The map $\partial : \mathrm{H}_1(K_g;\mathbb{Z}) \otimes C_g \to I_g$ is the restriction of the linear map $\mathrm{H}_1(K_g;\mathbb{Q}) \otimes \mathrm{H}_1(K_g;\mathbb{Q}) \to I_g$ induced by the bilinear Reidemeister pairing $\omega(\cdot,\cdot)$.*

Proof. Consider $f \otimes y \in \mathrm{H}_1(K_g;\mathbb{Z}) \otimes C_g$. Tracing through the construction of the long exact sequence (see, e.g., [6, §III.7]), we can calculate $\partial(f \otimes y)$ as follows. Lift y to $\tilde{y} \in C_{g,1}$. Then

$$\partial(f \otimes y) = f(\tilde{y}) - \tilde{y} \in I_g.$$

The lemma then follows from Lemma 3.2. $\square$

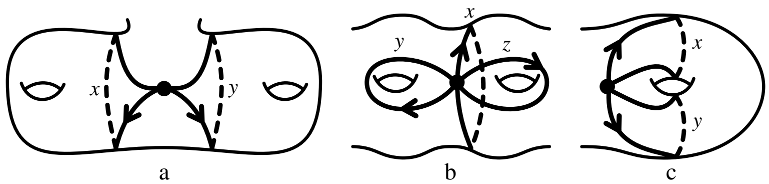


Figure 2: *a. x and y are strongly essentially separate b. $\{x,y\}$ is strongly essentially separate from $\{x,z\}$. c. x and y are essentially separate but not strongly essentially separate*

We thus want to determine which pairs of elements evaluate to zero under $\omega(\cdot,\cdot)$. This will require the following definitions.

Definition. Consider sets of curves $s = \{x_1,\ldots,x_n\} \subset \pi_1(\Sigma_g)$ and $s' = \{y_1,\ldots,y_m\} \subset \pi_1(\Sigma_g)$.

- The sets s and s' are *essentially separate* if there exist connected subsurfaces X and X' of Σ_g with the following properties.
  - Both X and X' contain the basepoint.
  - $\Sigma_g = X \cup X'$ and $X \cap X' = \partial X = \partial X'$.
  - $s \subset \mathrm{Im}(\pi_1(X) \to \pi_1(\Sigma_g))$ and $s' \subset \mathrm{Im}(\pi_1(X') \to \pi_1(\Sigma_g))$.
- The sets s and s' are *strongly essentially separate* if we can choose X and X' as above such that both X and X' have exactly one boundary component (which necessarily contains the basepoint).

See Figure 2 for examples of curves that are essentially separate and strongly essentially separate.

This brings us to the following key definition. Recall that $H_L = \mathrm{H}_1(\Sigma_g;\mathbb{Z}/L)$.

Definition. Define $\mathscr{S}_g \subset \mathrm{H}_1(K_g;\mathbb{Q}) \times C_g$ to equal $\mathscr{S}_g(1) \cup \mathscr{S}_g(2)$, where the $\mathscr{S}_g(i)$ are as follows. To simplify our notation, we will denote $\pi_1(\Sigma_g)$ by π.

$$\mathscr{S}_g(1) = \{(\langle\!\langle x\rangle\!\rangle^v, \langle\!\langle y\rangle\!\rangle^{v'}) \mid v,v' \in H_L,\, x \in K_g,\, y \in [\pi,\pi], \text{ and } x \text{ and } y \text{ are essentially separate}\},$$

$$\mathscr{S}_g(2) = \{(\langle\!\langle x\rangle\!\rangle^v, \langle\!\langle y,z^L\rangle\!\rangle^{v'}) \mid v,v' \in H_L,\, x \in K_g,\, y,z \in \pi,\, z \text{ can be realized by a simple closed nonseparating curve, and } \{z\} \text{ and } \{x,y\} \text{ are strongly essentially separate}\}.$$

We now prove the following.

Lemma 5.3. *For $(x,y) \in \mathscr{S}_g$, we have $\omega(x,y) = 0$.*

Proof. We must deal with both $\mathscr{S}_g(1)$ and $\mathscr{S}_g(2)$.

Step 1. $\omega(\langle\!\langle x\rangle\!\rangle^v, \langle\!\langle y\rangle\!\rangle^{v'}) = 0$ *for* $(\langle\!\langle x\rangle\!\rangle^v, \langle\!\langle y\rangle\!\rangle^{v'}) \in \mathscr{S}_g(1)$.

Since x and y are essentially separate, they can be freely homotoped to disjoint curves. This implies that any two lifts of x and y to the cover of Σ_g corresponding to K_g can be homotoped so as to be disjoint. Examining the formula for $\omega(\cdot,\cdot)$ in Lemma 3.2, this immediately implies that $\omega(\langle\!\langle x\rangle\!\rangle^v, \langle\!\langle y\rangle\!\rangle^{v'}) = 0$, as desired.

Step 2. $\omega(\langle\!\langle x\rangle\!\rangle^v, \langle\!\langle y,z^L\rangle\!\rangle^{v'}) = 0$ *for* $(\langle\!\langle x\rangle\!\rangle^v, \langle\!\langle y,z^L\rangle\!\rangle^{v'}) \in \mathscr{S}_g(2)$.

Recall that if $w \in \pi_1(\Sigma_g)$, then $\overline{w}$ denotes the element of $H_L = \mathrm{H}_1(\Sigma_g;\mathbb{Z}/L)$ associated to w. By Lemma 5.1, we have

$$\langle\!\langle y,z^L\rangle\!\rangle^{v'} = \langle\!\langle z^L\rangle\!\rangle^{v'+\overline{y}} - \langle\!\langle z^L\rangle\!\rangle^{v'}.$$

Since x and z^L are essentially separate, an argument similar to the argument in Step 1 shows that

$$\omega(\langle\!\langle x\rangle\!\rangle^v, \langle\!\langle y,z^L\rangle\!\rangle^{v'}) = \omega(\langle\!\langle x\rangle\!\rangle^v, \langle\!\langle z^L\rangle\!\rangle^{v'+\overline{y}}) - \omega(\langle\!\langle x\rangle\!\rangle^v, \langle\!\langle z^L\rangle\!\rangle^{v'}) = 0 - 0 = 0,$$

as desired. □

Remark. In Step 2 of the above, we only used the fact that z is essentially separate from x. The remainder of the assumptions on elements of $\mathscr{S}_g(2)$ will be later used in the proof of Lemma 5.5.

It follows that the set $\{x\otimes y \mid (x,y)\in \mathscr{S}_g\}$ is contained in $\ker(\partial)$. This is not everything, however. Since $\mathrm{Mod}_g^1(L)$ acts trivially on I_g, the kernel of ∂ also contains $x\otimes y - f(x)\otimes f(y)$ for $x\in \mathrm{H}_1(K_g;\mathbb{Z})$ and $y\in C_g$ and $f\in \mathrm{Mod}_g^1(L)$. Define $\mathscr{K}_g < \mathrm{H}_1(K_g;\mathbb{Z})\otimes C_g$ be the span of the set

$$\{x\otimes y \mid (x,y)\in \mathscr{S}_g\}\cup\{x\otimes y - f(x)\otimes f(y) \mid x\in \mathrm{H}_1(K_g;\mathbb{Z}),\, y\in C_g,\, f\in \mathrm{Mod}_g^1(L)\}$$

and let $Q_g = \mathrm{H}_1(K_g;\mathbb{Z})\otimes C_g/\mathscr{K}_g$. Since $\mathscr{K}_g\subset \ker(\partial)$, the map $\partial : \mathrm{H}_1(K_g;\mathbb{Z})\otimes C_g\to I_g$ induces a map $\psi : Q_g\to I_g$. Following preliminary results in §6, we will prove the following lemma in §7.

Lemma 5.4. *For $g\geq 3$, the map ψ is an isomorphism.*

It follows that

$$\mathscr{K}_g = \ker(\partial) = \mathrm{Im}(\mathrm{H}_1(K_g;C_{g,1})\to \mathrm{H}_1(K_g;C_g)).$$

To prove Lemma 3.3, which asserts that the map $\mathrm{H}_1(K_g;C_{g,1})\to \mathrm{H}_1(\mathrm{Mod}_g^1(L);C_g)$ is the zero map, it therefore suffices to prove the following lemma, whose proof is in §8.

Lemma 5.5. *For $g\geq 4$, the image of $\mathscr{K}_g$ in $\mathrm{H}_1(\mathrm{Mod}_g^1(L);C_g)$ is zero.*

This completes the outline of the proof of Lemma 3.3 (and our outline of the remainder of the paper). However, we also owe the reader a proof of Lemma 3.4, which asserts that

$$(C_g)_{\pi_1(\Sigma_g)} = (C_{g,1})_{\pi_1(\Sigma_g)} = 0$$

for $g\geq 3$.

Proof of Lemma 3.4. We can extend the long exact sequence (8) to the right to get an exact sequence

$$\mathrm{H}_1(K_g;\mathbb{Z})\otimes C_g \xrightarrow{\partial} I_g \longrightarrow (C_{g,1})_{K_g} \longrightarrow C_g \longrightarrow 0.$$

Lemma 5.4 implies that ∂ is surjective, so we deduce that $(C_{g,1})_{K_g}\cong C_g$. It is thus enough to prove that $(C_g)_{\pi_1(\Sigma_g)} = 0$. Lemma 2.2 implies that there is a $\pi_1(\Sigma_g)$-invariant decomposition

$$\mathrm{H}_1(K_g;\mathbb{Q})\cong C_g\oplus \mathrm{H}_1(\Sigma_g;\mathbb{Q}).$$

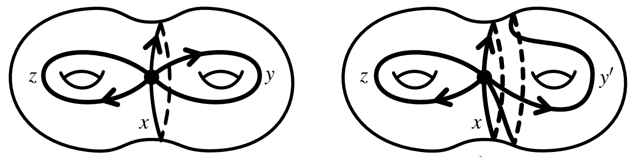
Also, Lemma 2.3 implies that $(\mathrm{H}_1(K_g;\mathbb{Q}))_{\pi_1(\Sigma_g)}\cong \mathrm{H}_1(\Sigma_g;\mathbb{Q})$. We conclude that $(C_g)_{\pi_1(\Sigma_g)} = 0$. $\square$

6 Generators and relations for Q_g

This section contains preliminaries for the proof of Lemma 5.4. We begin in §6.1 by introducing the notion of the intersection pattern of curves, which will play an important role in both this section and in §8. Next, in §6.2, we introduce certain important elements $X(v,w_1,w_2)$ of Q_g. We calculate the image of $X(v,w_1,w_2)$ under ψ in §6.3. We show that the $X(v,w_1,w_2)$ span Q_g in §6.4. Finally, in §6.5, we determine some relations between these elements.

Lemma 5.4 is proven in §7 below. The proof is essentially a lengthy calculation with generators and relations.

Throughout this section, let $\eta : \mathrm{H}_1(K_g;\mathbb{Z})\otimes C_g\to Q_g$ be the projection.



a b

Figure 3: *The curves $\{x,y,z\}$ in a have the same oriented intersection pattern as the curves $\{x,y',z\}$ in b.*

6.1 Intersection patterns

We will have to perform some detailed calculations with elements of $\pi_1(\Sigma_g)$. These calculations will depend on certain pictures of curves on the surface, and in this section we will establish some vocabulary for this. We begin with the following definition.

Definition. Let S be a surface, possibly with boundary. Assume that S has a fixed basepoint. An embedding $i : S \to \Sigma_g$ is a *simple embedding* if i takes the basepoint of S to the basepoint of Σ_g and all components of $\Sigma_g \setminus i(S)$ have one boundary component.

Remark. We allow $S = \Sigma_g$ and $i = \mathrm{id}$.

Remark. The key property of simple embeddings is as follows. Let $i : S \to \Sigma_g$ be a simple embedding. Then if γ is a simple closed separating curve on S, then $i(\gamma)$ is a simple closed separating curve on Σ_g.

Next, we make the following definition.

Definition. Let S be a surface, possibly with boundary, and let $\{x'_1,\ldots,x'_k\} \subset \pi_1(S)$. We will say that a set $\{x_1,\ldots,x_k\} \subset \pi_1(\Sigma_g)$ of curves *has the same unoriented intersection pattern* as $\{x'_1,\ldots,x'_k\}$ if there is a simple embedding $f : S \to \Sigma_g$ such that $f_*(x'_i) = x_i^{\pm 1}$ for all $1 \leq i \leq k$. If f can be chosen such that $f(x'_i) = x_i$ for all $1 \leq i \leq k$, then we will say that that the curves *have the same oriented intersection pattern*.

Remark. In what follows, the surface S and the curves $\{x'_1,\ldots,x'_k\}$ will often be given by pictures. To avoid cluttering the pictures, we will often depict boundary components via gaps in their edges. For instance, there are boundary components at the top and bottom of Figure 4.a below.

We will frequently assert without proof that a set of curves has a given (un)oriented intersection pattern. In all these cases, the assertion will be a trivial consequence of the "change of coordinates" principle from [8, §1.3]. Rather than give a formal description of this principle, we will illustrate it with a concrete example (there are many more examples in [8, §1.3]). Namely, we will prove that the curves $\{x,y,z\}$ in Figure 3.a have the same oriented intersection pattern as the curves $\{x,y',z\}$ in Figure 3.b (we remark that $y' = y^{-1}x$).

The proof is as follows. The union of the curves in Figure 3.a (resp. Figure 3.b) forms an oriented graph Γ_1 (resp. Γ_2) embedded in Σ_2 with one vertex (the basepoint) and three loops labeled with $\{x,y,z\}$ (resp. $\{x,y',z\}$). There is an isomorphism $f : \Gamma_1 \to \Gamma_2$ taking the edge labeled x to the edge labeled x, the edge labeled y to the edge labeled y', and the edge labeled z to the edge labeled z. The embedding of Γ_i in Σ_2 induces a cyclic order on the oriented edges entering and leaving the single vertex, and the isomorphism f respects these cyclic orderings. This implies that f extends to a diffeomorphism $f' : N_1 \to N_2$, where N_i is a regular neighborhood of Γ_i. An Euler characteristic computation shows that the components of $\Sigma_2 \setminus N_i$ are diffeomorphic, and we thus obtain a basepoint-preserving diffeomorphism $f'' : \Sigma_2 \to \Sigma_2$ such that $(f'')_*(x) = x$ and $(f'')_*(y) = y'$ and $(f'')_*(z) = z$, as desired.

6.2 The elements $X(v,w_1,w_2)$

The purpose of this section is to introduce certain elements $X(v,w_1,w_2)$ in Q_g. The key will be the following lemma. In it, recall that if $x \in \pi_1(\Sigma_g)$, then $\bar{x}$ denotes the element of $H_L = \mathrm{H}_1(\Sigma_g;\mathbb{Z}/L)$ associated to x.

Lemma 6.1. *Fix $w \in H_L = \mathrm{H}_1(\Sigma_g;\mathbb{Z}/L)$. For $1 \leq i \leq 2$, let $x_i, y_i, z_i \in \pi_1(\Sigma_g)$ be such that $\{x_i,y_i,z_i\}$ has the same oriented intersection pattern as the curves $\{x,y,z\}$ in Figure 4.a. Assume that $\bar{y}_1 = \bar{y}_2$ and $\bar{z}_1 = \bar{z}_2$. Then $\eta(\langle\!\langle x_1 \rangle\!\rangle \otimes \langle\!\langle y_1,z_1 \rangle\!\rangle^w) = \eta(\langle\!\langle x_2 \rangle\!\rangle \otimes \langle\!\langle y_2,z_2 \rangle\!\rangle^w)$.*

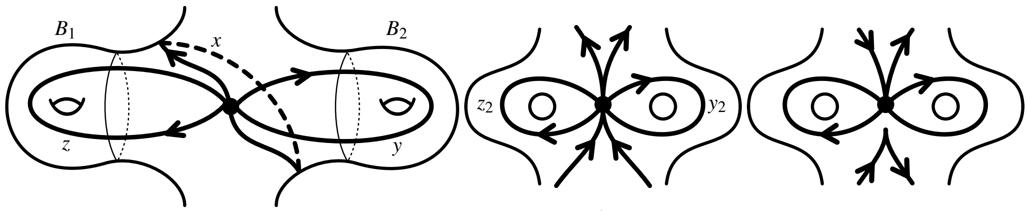


Figure 4: *a. The configuration of curves such that $\phi(\langle\langle x\rangle\rangle \otimes \langle\langle y,z\rangle\rangle^v) = X(v,w_1,w_2)$. Also, the one-holed tori B_1 and B_2 will be needed in the proof of Lemma 6.3. The central four-holed sphere in the picture will be called A in that proof. b. The curves $f(x_1)$ and x_2 leave at the top and come back at the bottom. c. The product $f(x_1)x_2^{-1}$ is essentially disjoint from $[y_2,z_2]$. The orientations of the "top" and "bottom" piece depend on the manner in which $f(x_1)$ and x_2 leave and come back to the basepoint*

Proof. It is easy to see that there exists some $f \in \mathrm{Mod}_g^1(L)$ such that $f(y_1) = y_2$ and $f(z_1) = z_2$ (the proof of this is a slight variation on the proof of [20, Proposition 6.7], which proves the analogous result for unbased curves). We then have

$$\eta(\langle\langle x_1\rangle\rangle \otimes \langle\langle y_1,z_1\rangle\rangle^w) = \eta(\langle\langle f(x_1)\rangle\rangle \otimes \langle\langle f(y_1),f(z_1)\rangle\rangle^w) = \eta(\langle\langle f(x_1)\rangle\rangle \otimes \langle\langle y_2,z_2\rangle\rangle^w).$$

The curves $\{f(x_1),y_2,z_2\}$ have the same oriented intersection pattern as the curves $\{x,y,z\}$ in Figure 4.a. Moreover (see Figures 4.b–c), the curves $f(x_1)x_2^{-1}$ and $[y_2,z_2]$ are essentially separate, so we conclude that

$$\eta(\langle\langle f(x_1)\rangle\rangle \otimes \langle\langle y_2,z_2\rangle\rangle^w) = \eta(\langle\langle x_2\rangle\rangle \otimes \langle\langle y_2,z_2\rangle\rangle^w),$$

as desired. □

We will need the following definition. Let $i(\cdot,\cdot)$ be the $\mathbb{Z}/L$-valued algebraic intersection pairing on H_L.

Definition. A k-element set $\{w_1,\ldots,w_k\} \subset H_L$ will be said to be *isotropic* if $i(w_i,w_j) = 0$ for all $1 \leq i,j \leq k$ and *unimodular* if $\langle w_1,\ldots,w_k\rangle$ is direct summand of H_L that is isomorphic to a k-dimensional free $\mathbb{Z}/L$-submodule.

It is clear that if the curves $\{x,y,z\}$ have the same oriented intersection pattern as the curves in Figure 4.a, then $\{\overline{y},\overline{z}\} \subset H_L$ is isotropic and unimodular. The converse is true as well. This will require the following lemma.

Lemma 6.2. *For some $n,m \geq 0$, let $\{w_1,\ldots,w_n,w'_1,\ldots,w'_m\} \subset H_L$ be a unimodular set. Assume that*

$$i(w_i,w_j) = i(w'_{i'},w'_{j'}) = 0 \quad \textit{and} \quad i(w_i,w_{i'}) = \begin{cases} 1 & \textit{if } i = i' \\ 0 & \textit{if } i \neq i' \end{cases}$$

for $1 \leq i,j \leq n$ and $1 \leq i',j' \leq m$. There then exists a set $\{\alpha_1,\ldots,\alpha_n,\alpha'_1,\ldots,\alpha'_m\}$ of unbased oriented simple closed curves on Σ_g with the following properties.

- *The $\mathbb{Z}/L$-homology class of α_i is w_i for $1 \leq i \leq n$ and the $\mathbb{Z}/L$-homology class of α'_i is w'_i for $1 \leq i \leq m$.*
- *α_i and α'_i intersect once for $1 \leq i \leq \min(n,m)$. Otherwise, the curves $\{\alpha_1,\ldots,\alpha_n,\alpha'_1,\ldots,\alpha'_m\}$ are pairwise disjoint.*

Proof. Identical to the proof of [19, Lemma A.3]. □

If $\{w_1,w_2\} \subset H_L$ is isotropic and unimodular, then Lemma 6.2 says that we can find unbased, disjoint simple closed curves Y and Z such that the $\mathbb{Z}/L$-homology classes of Y and Z are w_1 and w_2, respectively. Connecting Y and Z to the basepoint in an appropriate way, we find $y,z \in \pi_1(\Sigma_g)$ such that $\overline{y} = w_1$ and $\overline{z} = w_2$ and $\{y,z\}$ has the same oriented intersection pattern as the curves in Figure 4.a. It is then clear that we can find some $x \in \pi_1(\Sigma_g)$ such that $\{x,y,z\}$ has the same oriented intersection pattern as the curves in Figure 4.a.

We now introduce notation for the elements of Q_g we have been discussing.

Definition. For $v, w_1, w_2 \in H_L$ such that $\{w_1, w_2\}$ is isotropic and unimodular, define $X(v, w_1, w_2) = \eta(\langle\!\langle x \rangle\!\rangle \otimes \langle\!\langle y, z \rangle\!\rangle^v) \in Q_g$, where $x, y, z \in \pi_1(\Sigma_g)$ have the same oriented intersection pattern as the curves in Figure 4.a and $w_1 = \bar{y}$ and $w_2 = \bar{z}$.

The paragraph before the definition shows that we can find appropriate $\{x, y, z\}$, and Lemma 6.1 implies that $X(v, w_1, w_2)$ only depends on $\{v, w_1, w_2\}$.

6.3 The image of $X(v, w_1, w_2)$ under ψ

Recall that ψ is the natural map $Q_g \to I_g$ induced by the Reidemeister pairing $\omega(\cdot,\cdot)$ from Lemma 3.2. We now prove the following.

Lemma 6.3. *Consider $v, w_1, w_2 \in H_L$ such that $\{w_1, w_2\}$ is isotropic and unimodular. Then* $\psi(X(v, w_1, w_2)) = [\![v]\!] - [\![v + w_1]\!] - [\![v + w_2]\!] + [\![v + w_1 + w_2]\!]$.

Proof. Let $x, y, z \in \pi_1(\Sigma_g)$ be curves with the same oriented intersection pattern as the curves in Figure 4.a such that $\bar{y} = w_1$ and $\bar{z} = w_2$. We then have $X(v, w_1, w_2) = \eta(\langle\!\langle x \rangle\!\rangle \otimes \langle\!\langle y, z \rangle\!\rangle^v)$, and the lemma is equivalent to proving that

$$\omega(\langle\!\langle x \rangle\!\rangle, \langle\!\langle y, z \rangle\!\rangle^v) = [\![v]\!] - [\![v + \bar{y}]\!] - [\![v + \bar{z}]\!] + [\![v + \bar{y} + \bar{z}]\!].$$

Let $\rho : S_g^K \to \Sigma_g$ be the cover corresponding to K_g. The group of deck transformations is thus H_L. Let $B_1 \subset \Sigma_g$ (resp. $B_2 \subset \Sigma_g$)) be the one-holed torus on the left (resp. right) side of Figure 4.a. Also, let $A \subset \Sigma_g$ be the four-holed sphere "between" B_1 and B_2 in Figure 4.a. We then have the following.

- $\rho^{-1}(A)$ is the disjoint union of $|H_L| = L^{2g}$ four-holed spheres each of which projects homeomorphically onto A.
- $\rho^{-1}(B_i)$ is the disjoint union of $|H_L|/L^2 = L^{2g-2}$ components. If $\tilde{B}_i$ is one of those components, then $\tilde{B}_i$ is an L^2-holed torus and $\rho|_{\tilde{B}_i} : \tilde{B}_i \to B_i$ is a cover with deck group $(\mathbb{Z}/L)^2$.

The homology class $\langle\!\langle y, z \rangle\!\rangle^v$ on S_g^K can be realized by a simple closed curve γ as in Figure 5. If $\tilde{A}$ is the component of $\rho^{-1}(A)$ containing the basepoint, then this simple closed curve does the following.

- Beginning in the component $[\![v]\!] \cdot \tilde{A}$ of $\rho^{-1}(A)$, it goes through a component of $\rho^{-1}(B_2)$ to arrive in $[\![v + \bar{y}]\!] \cdot \tilde{A}$.
- It then goes through a component of $\rho^{-1}(B_1)$ to arrive in $[\![v + \bar{y} + \bar{z}]\!] \cdot \tilde{A}$.
- It then goes through a component of $\rho^{-1}(B_2)$ to arrive in $[\![v + \bar{z}]\!] \cdot \tilde{A}$.
- It finally goes through a component of $\rho^{-1}(B_1)$ to arrive back in $[\![v]\!] \cdot \tilde{A}$.

Let $\tilde{x}$ be the lift of x contained in $\tilde{A}$. As is evident from Figure 5, the curve γ intersects four different H_L-translates of $\tilde{x}$, two with positive sign and two with negative sign. Examining these intersections, we see that

$$\omega(\langle\!\langle x \rangle\!\rangle, \langle\!\langle y, z \rangle\!\rangle^v) = [\![v]\!] - [\![v + \bar{y}]\!] - [\![v + \bar{z}]\!] + [\![v + \bar{y} + \bar{z}]\!],$$

as desired. $\square$

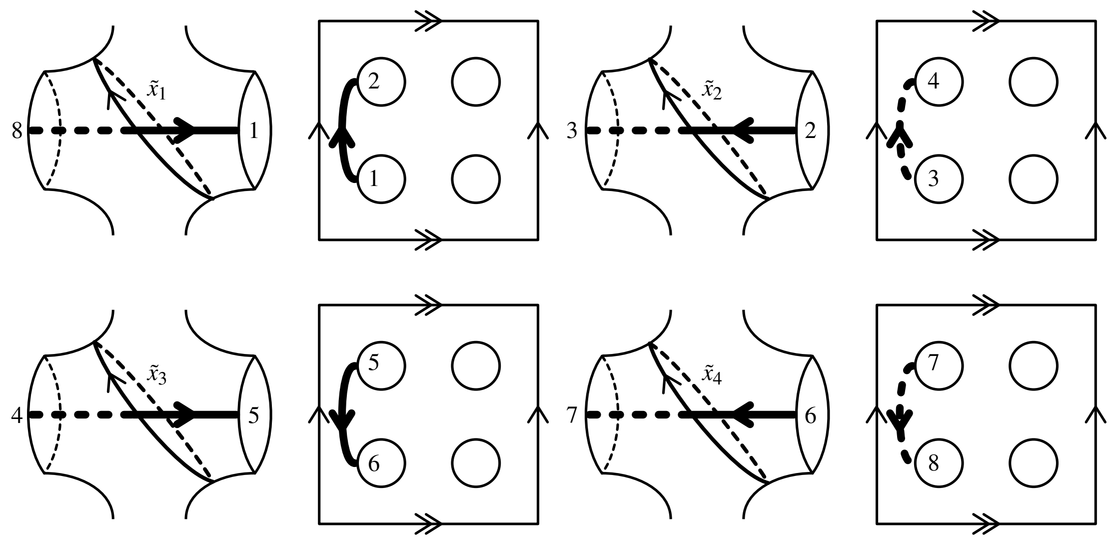


Figure 5: *The curve γ in S_g^K whose homology class is $\langle\!\langle y,z\rangle\!\rangle^v$. The dark portions are the lifts of $y^{\pm 1}$ and the dashed portions are the lifts of $z^{\pm 1}$. The numbers at the end of each segment indicate where the curve goes next. The 4-holed spheres are components of $\rho^{-1}(A)$ and the L^2-holed tori (depicted here for $L=2$; the sides of the squares should be glued up in the indicated ways) are components of $\rho^{-1}(B_1)$ and $\rho^{-1}(B_2)$. The curve γ intersects four lifts $\tilde{x}_1,\ldots,\tilde{x}_4$ of $\rho^{-1}(x)$. If $\tilde{x}$ is the lift of x starting at the basepoint, then $\tilde{x}_1 = [\![v]\!]\cdot\tilde{x}$ and $\tilde{x}_2 = [\![v+\bar{y}]\!]\cdot\tilde{x}$ and $\tilde{x}_3 = [\![v+\bar{y}+\bar{z}]\!]\cdot\tilde{x}$ and $\tilde{x}_4 = [\![v+\bar{z}]\!]\cdot\tilde{x}$*

6.4 Q_g is spanned by the $X(v,w_1,w_2)$

In this section, we prove that the $X(v,w_1,w_2)$ span Q_g. The proof will use the following lemma.

Lemma 6.4 ([19, Lemma A.1])**.** *Fix $g \geq 1$ and set $\pi = \pi_1(\Sigma_g)$. The group $[\pi,\pi]$ is then generated by the set*

$$\{\gamma \mid \gamma \in \pi \text{ can be realized by a simple closed separating curve}\}.$$

We now prove our lemma.

Lemma 6.5. *For $g \geq 1$, the vector space Q_g is spanned by the set*

$$\{X(v,w_1,w_2) \mid v,w_1,w_2 \in H_L,\ \{w_1,w_2\} \text{ is isotropic and unimodular}\}.$$

Proof. Define

$$C_g(\mathbb{Z}) = \ker(\mathrm{H}_1(K_g;\mathbb{Z}) \longrightarrow \mathrm{H}_1(\Sigma_g;\mathbb{Z})).$$

Our proof will have three steps. As notation, for $q,q' \in Q_g$, write $q \equiv q'$ if q and q' are equal modulo $\eta(C_g(\mathbb{Z})\otimes C_g)$.

Step 1. *Let $x,y,z \in \pi_1(\Sigma_g)$ be such that z can be realized by a simple closed nonseparating curve and $\{z\}$ is strongly essentially separate from $\{x,y\}$. Also, let $v \in H_L$. Then $\eta(\langle\!\langle x^L\rangle\!\rangle \otimes \langle\!\langle y,z\rangle\!\rangle^v) \equiv 0$.*

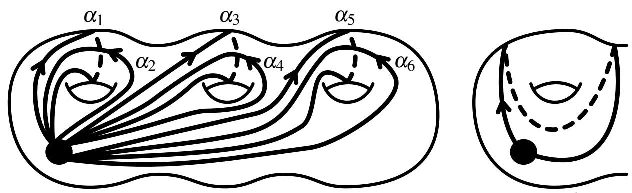
Since $\mathrm{Mod}_g^1(L)$ contains all inner automorphisms of $\pi_1(\Sigma_g)$, we can use the $\mathrm{Mod}_g^1(L)$-invariance of Q_g to deduce that

$$\eta(\langle\!\langle x^L\rangle\!\rangle \otimes \langle\!\langle y,z\rangle\!\rangle^{v+j\cdot\bar{z}}) = \eta(\langle\!\langle z^{-j}x^Lz^j\rangle\!\rangle \otimes \langle\!\langle y,z\rangle\!\rangle^v) \equiv \eta(\langle\!\langle x^L\rangle\!\rangle \otimes \langle\!\langle y,z\rangle\!\rangle^v).$$

for all $j \in \mathbb{Z}$. Consequently,

$$\eta(\langle\!\langle x^L\rangle\!\rangle \otimes \langle\!\langle y,z\rangle\!\rangle^v) \equiv \frac{1}{L}\sum_{j=0}^{L-1}\eta(\langle\!\langle x\rangle\!\rangle^L \otimes \langle\!\langle y,z\rangle\!\rangle^{v+j\cdot\bar{z}}) = \frac{1}{L}\eta(\langle\!\langle x^L\rangle\!\rangle \otimes \langle\!\langle y,z^L\rangle\!\rangle^v) = 0.$$

This last equality follows from the fact that $(\langle\!\langle x^L\rangle\!\rangle, \langle\!\langle y,z^L\rangle\!\rangle^v) \in \mathscr{S}_g(2)$



a b

Figure 6: *a. A standard basis for* $\pi_1(\Sigma_g)$ *b. The curve* $[\alpha_1, \alpha_2]$.

Step 2. $\eta(C_g(\mathbb{Z}) \otimes C_g) = Q_g$.

Our goal is to show that $q \equiv 0$ for all $q \in Q_g$. To do this, it is enough to show that $\eta(\langle\!\langle \alpha_1^L \rangle\!\rangle \otimes c) \equiv 0$ for all $\alpha_1 \in \pi_1(\Sigma_g)$ that can be realized by a simple closed nonseparating curve and all $c \in C_g$. Extend α_1 to a standard basis $\{\alpha_1, \ldots, \alpha_{2g}\}$ for $\pi_1(\Sigma_g)$ as in Figure 6.a. The vector space C_g is spanned by the set

$$S = \{\langle\!\langle \alpha_i, \alpha_j \rangle\!\rangle^v \mid 1 \leq i < j \leq 2g, \, v \in H_L\}.$$

Consider $\langle\!\langle \alpha_i, \alpha_j \rangle\!\rangle^v \in S$. If $i \geq 3$ or $(i,j) = (1,2)$, then $[\alpha_i, \alpha_j]$ is essentially separate from α_1^L (see Figure 6.b for $(i,j) = (1,2)$) and thus $\eta(\langle\!\langle x^L \rangle\!\rangle \otimes \langle\!\langle \alpha_i, \alpha_j \rangle\!\rangle^v) = 0$. Otherwise, $i \in \{1,2\}$ and $j > 2$. It follows that $\{\alpha_j\}$ is strongly essentially separate from $\{\alpha_1, \alpha_i\}$, and thus Step 1 implies $\eta(\langle\!\langle \alpha_1^L \rangle\!\rangle \otimes \langle\!\langle \alpha_i, \alpha_j \rangle\!\rangle^v) \equiv 0$.

Step 3. $\eta(C_g(\mathbb{Z}) \otimes C_g)$ *is generated by the set*

$$\{X(v, w_1, w_2) \mid v, w_1, w_2 \in H_L, \ \{w_1, w_2\} \text{ is isotropic and unimodular}\}.$$

Set

$$S = \{\gamma \in \pi_1(\Sigma_g) \mid \gamma \neq 1, \, \gamma \text{ can be realized by a simple closed separating curve}\}.$$

Lemma 6.4 implies that $C_g(\mathbb{Z})$ is generated by $\{\langle\!\langle \gamma \rangle\!\rangle \mid \gamma \in S\}$. Consider $\gamma \in S$. Let X_1 and X_2 be the two surfaces into which γ cuts Σ_g. Order them so that X_1 lies to the right of γ and X_2 to the left. We can then find a basis $B_\gamma^1 \cup B_\gamma^2$ for $\pi_1(\Sigma_g)$ with the following properties.

- For $\delta \in B_\gamma^i$, we have $\delta \in \mathrm{Im}(\pi_1(X_i) \to \pi_1(\Sigma_g))$.
- Consider $\delta_1 \in B_\gamma^1$ and $\delta_2 \in B_\gamma^2$. The curves $\{\gamma, \delta_1, \delta_2\}$ then have the same oriented intersection pattern as the curves in Figure 4.a.

It follows that C_g is spanned by the set $U_\gamma \cup V_\gamma$, where

$$U_\gamma = \{\langle\!\langle \delta_1, \delta_2 \rangle\!\rangle^v \mid \delta_i \in B_\gamma^i, \, v \in H_L\} \quad \text{and} \quad V_\gamma = \{\langle\!\langle \delta, \delta' \rangle\!\rangle^v \mid \text{there exists } i \text{ such that } \delta, \delta' \in B_\gamma^i, \, v \in H_L\}.$$

We then have that $C_g(\mathbb{Z}) \otimes C_g$ is spanned by the set $Z \cup W$, where

$$Z = \{\langle\!\langle \gamma \rangle\!\rangle \otimes c \mid \gamma \in S, \, c \in U_\gamma\} \quad \text{and} \quad W = \{\langle\!\langle \gamma \rangle\!\rangle \otimes c \mid \gamma \in S, \, c \in V_\gamma\}.$$

For $\langle\!\langle \gamma \rangle\!\rangle \otimes \langle\!\langle \delta_1, \delta_2 \rangle\!\rangle^v \in Z$, we have $\eta(\langle\!\langle \gamma \rangle\!\rangle \otimes \langle\!\langle \delta_1, \delta_2 \rangle\!\rangle^v) = X(v, \overline{\delta}_1, \overline{\delta}_2)$. For $\langle\!\langle \gamma \rangle\!\rangle \otimes \langle\!\langle \delta, \delta' \rangle\!\rangle^v \in W$, the curves γ and $[\delta, \delta']$ are essentially separate, so $\eta(\langle\!\langle \gamma \rangle\!\rangle \otimes \langle\!\langle \delta, \delta' \rangle\!\rangle^v) = 0$. The desired result follows. $\square$

6.5 Relations between the $X(w, v_1, v_2)$

The goal of this section is to prove the following lemma, which gives relations between the $X(v, w_1, w_2)$.

Lemma 6.6. *Let* $\{w_1, w_2\} \subset \mathrm{H}_1(\Sigma_g; \mathbb{Z}/L)$ *be an isotropic and unimodular set. Then the following hold for all* $v \in \mathrm{H}_1(\Sigma_g; \mathbb{Z}/L)$.

1. $X(v, w_1, w_2) = X(v, w_2, w_1)$

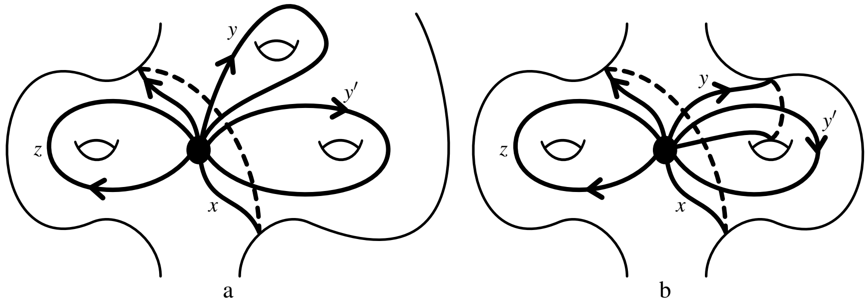


Figure 7: *a.* $i(y,y')=0$ *and the sets of curves* $\{x,y,z\}$ *and* $\{x,y',z\}$ *and* $\{x,yy',z\}$ *have the same oriented intersection pattern as the curves in Figure 4.a* *b.* $i(y,y')=-1$ *and the sets of curves* $\{x,y,z\}$ *and* $\{x,y',z\}$ *and* $\{x,yy',z\}$ *and* $\{x,y(y')^{-1},z\}$ *have the same oriented intersection pattern as the curves in Figure 4.a*

2. $X(v,-w_1,w_2) = -X(v-w_1,w_1,w_2)$
3. $\sum_{i=0}^{L-1} X(v+i\cdot w_1,w_1,w_2)=0$ *and* $\sum_{i=0}^{L-1} X(v+i\cdot w_2,w_1,w_2)=0$
4. *Let* $w_3 \in \mathrm{H}_1(\Sigma_g;\mathbb{Z}/L)$ *be such that* $\{w_1,w_2,w_3\}$ *is unimodular, such that* $i(w_2,w_3)=0$*, and such that* $-1 \le i(w_1,w_3) \le 1$*. Then* $X(v,w_1+w_3,w_2) = X(v,w_1,w_2)+X(v+w_1,w_3,w_2)$*.*

Remark. It is instructive to check (using the formula in Lemma 6.3) that each of these relations is taken to 0 by ψ.

Proof of Lemma 6.6. Let $x,y,z \in \pi_1(\Sigma_g)$ be curves such that $\{x,y,z\}$ has the same oriented intersection pattern as the curves in Figure 4.a and such that $\overline{y}=w_1$ and $\overline{z}=w_2$. Hence $X(v,w_1,w_2)=\eta(\langle\!\langle x\rangle\!\rangle \otimes \langle\!\langle y,z\rangle\!\rangle^v)$.

For item 1, observe that if we flip y and z, then our curves no longer have the same oriented intersection pattern as the curves in Figure 4.a (they do have the same unoriented intersection pattern). To restore the correct orientations, we must reverse x. In other words,

$$X(v,w_2,w_1)=\eta(\langle\!\langle x^{-1}\rangle\!\rangle \otimes \langle\!\langle z,y\rangle\!\rangle^v) = \eta((-\langle\!\langle x\rangle\!\rangle)\otimes(-\langle\!\langle y,z\rangle\!\rangle^v)) = \eta(\langle\!\langle x\rangle\!\rangle \otimes \langle\!\langle y,z\rangle\!\rangle^v) = X(v,w_1,w_2),$$

as desired.

For item 2, observe that the set of curves $\{x,y^{-1}x,z\}$ has the same oriented intersection pattern as $\{x,y,z\}$ (see the example in §6.1). Also, since $\overline{x}=0$, we have $\overline{y^{-1}x}=-\overline{y}=-w_1$. Thus we can apply Lemma 5.1 to get that

$$\begin{aligned} X(v,-w_1,w_2) &= \eta(\langle\!\langle x\rangle\!\rangle \otimes \langle\!\langle y^{-1}x,z\rangle\!\rangle^v) = \eta(\langle\!\langle x\rangle\!\rangle \otimes \langle\!\langle y^{-1},z\rangle\!\rangle^v)+\eta(\langle\!\langle x\rangle\!\rangle \otimes \langle\!\langle x,z\rangle\!\rangle^{v-w_1}) \\ &= -\eta(\langle\!\langle x\rangle\!\rangle \otimes \langle\!\langle y,z\rangle\!\rangle^{v-w_1})+\eta(\langle\!\langle x\rangle\!\rangle \otimes \langle\!\langle x\rangle\!\rangle^{v-w_1})-\eta(\langle\!\langle x\rangle\!\rangle \otimes \langle\!\langle x\rangle\!\rangle^{v-w_1+w_2}). \end{aligned}$$

Since x is essentially separate from itself, the last two terms vanish and this equals

$$-\eta(\langle\!\langle x\rangle\!\rangle \otimes \langle\!\langle y,z\rangle\!\rangle^{v-w_1}) = -X(v-w_1,w_1,w_2),$$

as desired.

For item 3, item 1 implies that it is enough to prove that $\sum_{i=0}^{L-1} X(v+i\cdot w_2,w_1,w_2)=0$. Observe that $(\langle\!\langle x\rangle\!\rangle, \langle\!\langle y,z^L\rangle\!\rangle^v) \in \mathscr{S}_g(2)$, so by Lemma 5.1 we have

$$0=\eta(\langle\!\langle x\rangle\!\rangle \otimes \langle\!\langle y,z^L\rangle\!\rangle^v) = \sum_{i=0}^{L-1}\eta(\langle\!\langle x\rangle\!\rangle \otimes \langle\!\langle y,z\rangle\!\rangle^{v+i\cdot\overline{z}}) = \sum_{i=0}^{L-1} X(v+i\cdot w_2,w_1,w_2),$$

as desired.

We conclude with item 4. Here we will have to change our curves x and y and z. We will prove shortly that we can find $x,y,y',z \in \pi_1(\Sigma_g)$ with the following properties.

- $\bar{y} = w_1$ and $\bar{y}' = w_3$ and $\bar{z} = w_2$.
- The sets of curves $\{x,y,z\}$ and $\{x,y',z\}$ and $\{x,yy',z\}$ each have the same oriented intersection pattern as the curves in Figure 4.a.

See Figures 7.a,b. Assuming this for the moment, the proof is completed by appealing to Lemma 5.1 to deduce that

$$\begin{aligned} X(v,w_1+w_3,w_2) &= \eta(\langle\!\langle x\rangle\!\rangle \otimes \langle\!\langle yy',z\rangle\!\rangle^v) = \eta(\langle\!\langle x\rangle\!\rangle \otimes \langle\!\langle y,z\rangle\!\rangle^v) + \eta(\langle\!\langle x\rangle\!\rangle \otimes \langle\!\langle y',z\rangle\!\rangle^{v+\bar{y}}) \\ &= X(v,w_1,w_2) + X(v+w_1,w_3,w_2). \end{aligned}$$

It remains to prove the above claim. By Lemma 6.2, we can find unbased oriented simple closed curves Y and Y' and Z on Σ_g with the following properties.

- The $\mathbb{Z}/L$-homology classes of Y, Y', and Z are w_1, w_3, and w_2, respectively.
- Z is disjoint from Y and Y'. Also, Y and Y' are disjoint if $i(w_1,w_3) = 0$ and intersect once if $i(w_1,w_3) = \pm 1$.

We can then connect Y, Y', and Z to the basepoint to obtain curves $y,y',z \in \pi_1(\Sigma_g)$ such that $\{y,y',z\}$ has the same oriented intersection pattern as the curves in Figure 7.a (if $i(w_1,w_3)=0$) or 7.b (if $i(w_1,w_3) = -1$) or 7.b with the orientation of the curve y' reversed (if $i(w_1,w_3)=1$). It is then clear that we can find $x \in \pi_1(\Sigma_g)$ such that $\{x,y,y',z\}$ has the indicated oriented intersection pattern, as desired. $\square$

7 The map $\psi : Q_g \to I_g$ is an isomorphism

The purpose of this section is to prove that the map $\psi : Q_g \to I_g$ is an isomorphism. The actual proof is in §7.2. This is proceeded by §7.1, which constructs a generating set V for Q_g that is slightly smaller than the generating set determined in §6.4.

Throughout this section, we will freely use the main results of §6 (i.e. Lemmas 6.3 and 6.5 and 6.6).

7.1 A smaller generating set

Fix a symplectic basis $B = \{a_1,b_1,\ldots,a_g,b_g\}$ for H_L. Define $V = V_1 \cup V_2 \subset Q_g$, where the V_i are as follows.

$$\begin{aligned} V_1 &= \{X(v,s_1,s_2) \mid v \in H_L,\ s_1,s_2 \in B \text{ distinct},\ i(s_1,s_2)=0\}, \\ V_2 &= \{X(v,s_1,s_1+es_2) \mid v \in H_L),\ s_1,s_2 \in B \text{ distinct},\ e \in \{-1,1\},\ i(s_1,s_2)=0\}, \end{aligned}$$

The goal of this section is to prove the following.

Lemma 7.1. *For $g \geq 4$, the vector space Q_g is spanned by V.*

We begin with the following relations in Q_g.

Lemma 7.2. *Fix $g \geq 1$, and let $\{s_1,s_2,s_3\} \subset H_L$ be a unimodular set such that $i(s_1,s_3) = i(s_2,s_3) = 0$ and $-1 \leq i(s_1,s_2) \leq 1$. Then for all $v \in H_L$ we have the following two relations.*

$$\begin{aligned} X(v,s_2,s_3) &= X(v+s_1,s_2,s_3) + X(v,s_1,s_3) - X(v+s_2,s_1,s_3), \\ X(v-s_1,s_2,s_3) &= X(v,s_2,s_3) + X(v-s_1,s_1,s_3) - X(v-s_1+s_2,s_1,s_3). \end{aligned}$$

Proof. The first relation follows from the fact that

$$X(v,s_1+s_2,s_3) = X(v,s_1,s_3) + X(v+s_1,s_2,s_3) \quad \text{and} \quad X(v,s_1+s_2,s_3) = X(v,s_2,s_3) + X(v+s_2,s_1,s_3).$$

The second follows from the first via the substitution $v \mapsto v - s_1$. $\square$

We next show that $\langle V \rangle$ contains several other classes of elements. Define

$$\begin{aligned} V_3 = \{X(v, s_1 + es_2, s_3 + e's_4) \mid v \in H_L,\ s_1, s_2, s_3, s_4 \in B \text{ distinct},\ e, e' \in \{-1, 1\}, \\ i(s_1, s_3) = \pm 1,\ i(s_1 + es_2, s_3 + e's_4) = 0\}, \\ V_4 = \{X(v, s_1 + es_2, s_1 + es_2 + e's_3) \mid v \in H_L,\ s_1, s_2, s_3 \in B \text{ distinct},\ e, e' \in \{-1, 1\}, \\ i(s_1, s_2) = \pm 1,\ i(s_1, s_3) = i(s_2, s_3) = 0\}. \end{aligned}$$

We then have the following.

Lemma 7.3. *For $g \geq 1$, we have $V_3, V_4 \subset \langle V \rangle$.*

Proof. For $x, y \in Q_g$, write $x \equiv y$ if x and y are equal modulo $\langle V_1, V_2 \rangle$. First consider $X(v, w_1, w_2) \in V_3$. Our goal is to show that $X(v, w_1, w_2) \equiv 0$. For concreteness, we will do the case $X(v, w_1, w_2) = X(v, a_i + b_j, b_i + a_j)$ for some $1 \leq i, j \leq g$ with $i \neq j$; the other cases are similar.

Observe first that Lemma 7.2 implies that

$$X(v, b_i + a_j, a_i + b_j) = X(v + a_i, b_i + a_j, a_i + b_j) + X(v, a_i, a_i + b_j) - X(v + b_i + a_j, a_i, a_i + b_j).$$

Since $X(v, a_i, a_i + b_j), X(v + b_i + a_j, a_i, a_i + b_j) \in V_2$, we deduce that $X(v, a_i + b_j, b_i + a_j) \equiv X(v + a_i, a_i + b_j, b_i + a_j)$. In a similar manner, we have $X(v + a_i, a_i + b_j, b_i + a_j) \equiv X(v + (a_i + b_j), a_i + b_j, b_i + a_j)$. Iterating this, we obtain $X(v, a_i + b_j, b_i + a_j) \equiv X(v + k(a_i + b_j), a_i + b_j, b_i + a_j)$ for all $k \in \mathbb{Z}$. But this implies that

$$X(v, a_i + b_j, b_i + a_j) \equiv \frac{1}{L} \sum_{k=0}^{L-1} X(v + k(a_i + b_j), a_i + b_j, b_i + a_j) = 0,$$

as desired.

Now consider $X(v', w_1', w_2') \in V_4$. We will show that $X(v', w_1', w_2')$ can be written as a a linear combination of elements of $V_1 \cup V_2 \cup V_3$. For concreteness, we will do the case $X(v', w_1', w_2') = X(v, a_i + b_i, a_i + b_i + a_j)$ for some $1 \leq i, j \leq g$ with $i \neq j$; the other cases are similar. In this case, we have

$$\begin{aligned} X(v, a_i + b_i, a_i + b_i + a_j) =& X(v, (a_i + b_j) + (b_i - b_j), a_i + b_i + a_j) \\ =& X(v, a_i + b_j, a_i + b_i + a_j) + X(v + a_i + b_j, b_i - b_j, a_i + b_i + a_j) \\ =& X(v, a_i + b_j, a_i) + X(v + a_i, a_i + b_j, b_i + a_j) \\ &+ X(v + a_i + b_j, b_i - b_j, b_i) + X(v + a_i + b_j + b_i, b_i - b_j, a_i + a_j), \end{aligned}$$

as desired. $\square$

Next, say that $v \in H_L$ is a *simple element of length at most k* if it can be written as

$$v = \sum_{i=1}^{g} (c_i a_i + d_i b_i)$$

for some $c_i, d_i \in \{-1, 0, 1\}$ such that at most k of the c_i and d_i are nonzero. Define

$$\begin{aligned} V' = \{X(v, w_1, w_2) \mid v, w_1, w_1 \in H_L,\ \{w_1, w_2\} \text{ is isotropic and unimodular, and} \\ w_1 \text{ is a simple element of length at most 3} \end{aligned}$$

We then have the following

Lemma 7.4. *For $g \geq 1$, we have $V' \subset \langle V \rangle$.*

Proof. An easy case-by-case check shows that one can use the "bilinearity relations" (relations 1, 2, and 4 in Lemma 6.6) to express every element of V' as a linear combination of elements of $V \cup V_3 \cup V_4$, and thus via Lemma 7.3 as a linear combination of elements of V. $\square$

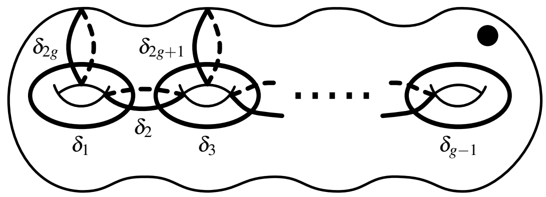


Figure 8: *Generators for* Mod_g^1

We finally prove Lemma 7.1.

Proof of Lemma 7.1. The mapping class group Mod_g^1 acts on both H_L and Q_g. The action on H_L is transitive on pairs $\{w_1, w_2\}$ of vectors that are isotropic and unimodular. It follows that the Mod_g^1-orbit of the set V contains every element $X(v, w_1, w_2)$. Lemma 6.5 says that these generate Q_g, so we conclude that it is enough to show that $\langle V \rangle$ is invariant under Mod_g^1. Let $\{\delta_1, \ldots, \delta_{2g+1}\}$ be the simple closed curves depicted in Figure 8. The set

$$S = \{T_{\delta_i} \mid 1 \leq i \leq 2g+1\}$$

then generates Mod_g^1 (see [8]). It is enough to prove that $s \cdot X(v, w_1, w_2) \in \langle V \rangle$ for $s \in S^{\pm 1}$ and $X(v, w_1, w_2) \in V$. However, it is easy to see that $s(w_1)$ is a simple element of length at most 3, so $s \cdot X(v, w_1, w_2) \in V'$ and the desired result follows from Lemma 7.4. $\square$

7.2 ψ is an isomorphism

In this section, we prove Lemma 5.4, which we recall asserts that for $g \geq 3$, the map $\psi : Q_g \to I_g$ is an isomorphism. Our proof is lengthy, but the basic idea is as follows.

- Using the relations in Q_g, we will show that the set $\langle V \rangle$ is generated by a set containing $\dim \mathbb{Q}[H_L] - 1$ elements.
- By carefully examining the image of ψ, we will show that $\mathbb{Q}[H_L] = \psi(\langle V \rangle) + \langle [\![0]\!] \rangle$.

A simple dimension count will then establish the lemma.

Some parts of our proof will be by induction on g. We will thus need notation for $\mathbb{Q}[H_L]$ which takes g into account, so define $\mathscr{B}_g = \mathbb{Q}[\mathrm{H}_1(\Sigma_g; \mathbb{Z}/L)]$.

To make the calculations a bit more palatable, we will break this down into several steps. We first determine what ψ does to V_1.

Lemma 7.5. *Fix $g \geq 1$. Set $\mathscr{B}_g^1 = \langle \{[\![ca_i + db_i]\!] \mid c, d \in \mathbb{Z}/L,\ 1 \leq i \leq g\} \rangle$. Then the map $\psi|_{\langle V_1 \rangle}$ is injective and $\mathscr{B}_g = \psi(\langle V_1 \rangle) \oplus \mathscr{B}_g^1$.*

Proof. The proof will by induction on g. For the base case $g = 1$, the set V_1 is empty and the assertion is trivial. Assume now that $g \geq 2$ and that the lemma is true for all smaller g. Define

$$\begin{aligned} V_1^I &= \{X(v, s_1, s_2) \in V_1 \mid s_1, s_2 \notin \{a_g, b_g\}\}, \\ V_1^A &= \{X(v, a_g, s) \mid X(v, a_g, s) \in V_1\}, \\ V_1^B &= \{X(v, b_g, s) \mid X(v, b_g, s) \in V_1\}, \end{aligned}$$

so $V = V_1^I \cup V_1^A \cup V_1^B$. The proof will consist of three steps.

Step 1. *Set $\mathscr{B}_g^2 = \langle \{[\![ca_i + db_i + ea_g + fb_g]\!] \mid c, d, e, f \in \mathbb{Z}/L,\ 1 \leq i \leq g-1\} \rangle$. Then the map $\psi|_{\langle V_1^I \rangle}$ is injective and $\mathscr{B}_g = \psi(\langle V_1^I \rangle) \oplus \mathscr{B}_g^2$.*

For $e,f\in\mathbb{Z}/L$, define

$$\begin{aligned}
\mathscr{B}_g(e,f) &= \langle\{[\![v+ea_g+fb_g]\!] \mid v\in\langle a_1,b_1,\dots,a_{g-1},b_{g-1}\rangle\}\rangle,\\
\mathscr{B}_g^2(e,f) &= \langle\{[\![ca_i+db_i+ea_g+fb_g]\!] \mid c,d\in\mathbb{Z}/L,\ 1\le i\le g-1\}\rangle,\\
V_1^I(e,f) &= \{X(v+ea_g+fb_g,s_1,s_2)\in V_1 \mid s_1,s_2\notin\{a_g,b_g\},\ v\in\langle a_1,b_1,\dots,a_{g-1},b_{g-1}\rangle\}.
\end{aligned}$$

Observe that

$$\mathscr{B}_g=\bigoplus_{e,f\in\mathbb{Z}/L}\mathscr{B}_g(e,f)\quad\text{and}\quad \mathscr{B}_g^2=\bigoplus_{e,f\in\mathbb{Z}/L}\mathscr{B}_g^2(e,f)\quad\text{and}\quad V_1^I=\bigsqcup_{e,f\in\mathbb{Z}/L}V_1^I(e,f).$$

Moreover, for all $e,f\in\mathbb{Z}/L$ we have $\psi(V_1^I(e,f))\subset\mathscr{B}_g(e,f)$. We conclude that it is enough to prove that $\psi|_{\psi(\langle V_1^I(e,f)\rangle)}$ is injective and $\mathscr{B}_g(e,f)=\psi(\langle V_1^I(e,f)\rangle)\oplus\mathscr{B}_g^2(e,f)$ for all $e,f\in\mathbb{Z}/L$.

Consider $e,f\in\mathbb{Z}/L$. The map $[\![v]\!]\mapsto[\![v+ea_g+fb_g]\!]$ induces an isomorphism $\rho:\mathscr{B}_{g-1}\to\mathscr{B}_g(e,f)$ that restricts to an isomorphism $\mathscr{B}_{g-1}^1\cong\mathscr{B}_g^2(e,f)$. Let $V_{1,g-1}\subset Q_{g-1}$ be the $(g-1)$-dimensional analogue of V_1 and $\psi_{g-1}:Q_{g-1}\to\mathscr{B}_{g-1}$ be the $(g-1)$-dimensional analogue of ψ. The map $X(v,w_1,w_2)\mapsto X(v+ea_g+fb_g,w_1,w_2)$ induces a homomorphism $Q_{g-1}\to Q_g$ that restricts to a surjection $\rho':\langle V_{1,g-1}\rangle\to\langle V_1^I(e,f)\rangle$. Observe that the diagram

$$\begin{tikzcd}
\langle V_{1,g-1}\rangle \arrow[r, "\psi_{g-1}"] \arrow[d, "\rho'"'] & \mathscr{B}_{g-1} \arrow[d, "\rho"'] \\
\langle V_1^I(e,f)\rangle \arrow[r, "\psi"] & \mathscr{B}_g(e,f)
\end{tikzcd}$$

commutes. By the induction hypothesis, $\psi_{g-1}|_{\langle V_{1,g-1}\rangle}$ is injective and $\mathscr{B}_{g-1}=\psi_{g-1}(\langle V_{1,g-1}\rangle)\oplus\mathscr{B}_{g-1}^1$. We conclude that ρ' is injective and thus an isomorphism. Moreover, $\psi|_{\langle V_1^I(e,f)\rangle}$ is injective and $\mathscr{B}_g(e,f)=\psi(\langle V_1^I(e,f)\rangle)\oplus\mathscr{B}_g^2(e,f)$, as desired.

Step 2. *Set $\mathscr{B}_g^3=\langle\{[\![ca_i+db_i+fb_g]\!],\ [\![ea_g+fb_g]\!] \mid c,d,e,f\in\mathbb{Z}/L,\ 1\le i\le g-1\}\rangle$. Then the map $\psi|_{\langle V_1^I,V_1^A\rangle}$ is injective and $\mathscr{B}_g=\psi(\langle V_1^I,V_1^A\rangle)\oplus\mathscr{B}_g^3$.*

Define

$$V_1^{A,1}=\{X(ca_i+db_i+ea_g+fb_g,a_g,s)\mid 1\le i\le g-1,\ c,d,e,f\in\mathbb{Z}/L,\ s\in\{a_i,b_i\}\}\subset V_1^A.$$

We claim that $\langle V_1^I,V_1^A\rangle=\langle V_1^I,V_1^{A,1}\rangle$. Indeed, consider $X(w,a_g,s)\in V_1^A$. By Lemma 7.2, for any $s'\in\{a_1,b_1,\dots,a_{g-1},b_{g-1}\}$ with $s'\neq s$ and $i(s,s')=0$, we have

$$X(w-s',a_g,s)=X(w,a_g,s)+X(w-s',s',s)-X(w-s'+a_g,s',s).$$

Hence modulo $\langle V_1^I\rangle$, we have $X(w,a_g,s)$ equal to $X(w-s',a_g,s)$. Iterating this, modulo $\langle V_1^I\rangle$ we have $X(w,a_g,s)$ equal to an element of $V_1^{A,1}$, as desired.

For $x\in\mathbb{Z}/L$, we will denote by $|x|$ the unique integer representing x with $0\le|x|<L-1$. Noting that $\psi(V_1^{A,1})\subset\mathscr{B}_g^2$, we claim that $\mathscr{B}_g^2=\psi(\langle V_1^{A,1}\rangle)+\mathscr{B}_g^3$. Indeed, assume that there is some $[\![ca_i+db_i+ea_g+fb_g]\!]\in\mathscr{B}_g^2$ that is not in $\psi(\langle V_1^{A,1}\rangle)+\mathscr{B}_g^3$. Choose $[\![ca_i+db_i+ea_g+fb_g]\!]$ such that $|c|+|d|+|e|$ is minimal among elements with this property. By assumption we must have $|e|$ and one of $|c|$ or $|d|$ (say $|c|$) nonzero. Setting $w'=ca_i+db_i+ea_g+fb_g$, we then have $X(w'-a_i-a_g,a_g,a_i)\in V_1^{A,1}$ and

$$[\![w']\!]-\psi(X(w'-a_i-a_g,a_g,a_i))=[\![w'-a_i]\!]+[\![w'-a_g]\!]-[\![w'-a_i-a_g]\!].$$

We conclude that one of $[\![w'-a_i]\!]$, $[\![w'-a_g]\!]$, or $[\![w'-a_i-a_g]\!]$ is not in $\psi(\langle V_1^{A,1}\rangle)+\mathscr{B}_g^3$, contradicting the minimality of $|c|+|d|+|e|$.

Now define

$$\begin{aligned}V_1^{A,2} =& \{X(ca_i+db_i+ea_g+fb_g,a_g,a_i) \mid 1\le i\le g-1,\, c,d,e,f\in\mathbb{Z}/L\}\\ &\cup\{X(db_i+ea_g+fb_g,a_g,b_i)\mid 1\le i\le g-1,\, d,e,f\in\mathbb{Z}/L\}\subset V_1^{A,1},\\ V_1^{A,3} =& \{X(v,a_g,s)\in V_1^{A,2}\mid \text{the } s \text{ and } a_g\text{-coordinates of } v \text{ do not equal } L-1\}\subset V_1^{A,2}.\end{aligned}$$

We will prove that $\langle V_1^{A,3}\rangle = \langle V_1^{A,1}\rangle$. Using the third relation in Lemma 6.6, we see that $\langle V_1^{A,3}\rangle = \langle V_1^{A,2}\rangle$. It is thus enough to prove that $\langle V_1^{A,1}\rangle = \langle V_1^{A,2}\rangle$. An element of $V_1^{A,1}\setminus V_1^{A,2}$ is of the form $X(w'',a_g,b_i)$. Lemma 7.2 says that

$$X(w''-a_i,b_i,a_g) = X(w'',b_i,a_g)+X(w''-a_i,a_i,a_g)-X(w''-a_i+b_i,a_i,a_g).$$

Iterating this, we conclude that modulo $\langle V_1^{A,2}\rangle$, we have $X(w'',a_g,b_i)$ equal to an element of $V_1^{A,2}$, as desired.

We deduce from the above two paragraphs that $\mathscr{B}_g^2 = \psi(\langle V_1^{A,3}\rangle)+\mathscr{B}_g^3$. Since $V_1^{A,3}$ contains

$$\begin{aligned}(g-1)(L^2(L-1)^2+L(L-1)^2) &= ((g-1)(L^2-1)L^2+L^2)-((g-1)(L^2-1)L+L^2)\\ &= \dim(\mathscr{B}_g^2)-\dim(\mathscr{B}_g^3)\end{aligned}$$

elements, we obtain that $\psi|_{\langle V_1^{A,3}\rangle}$ is injective and $\mathscr{B}_g^2 = \psi(\langle V_1^{A,3}\rangle)\oplus\mathscr{B}_g^3$. By Step 1, the fact that $\langle V_1^I,V_1^A\rangle = \langle V_1^I,V_1^{A,1}\rangle$, and the fact that $\langle V_1^{A,3}\rangle = \langle V_1^{A,1}\rangle$, we conclude that $\psi|_{\langle V_1^I,V_1^A\rangle}$ is injective and $\mathscr{B}_g = \psi(\langle V_1^I,V_1^A\rangle)\oplus\mathscr{B}_g^3$, as desired.

Step 3. *Recall that $\mathscr{B}_g^1 = \langle\{[\![ca_i+db_i]\!] \mid c,d\in\mathbb{Z}/L,\ 1\le i\le g\}\rangle$. The map $\psi|_{\langle V_1^I,V_1^A,V_1^B\rangle}$ is injective and $\mathscr{B}_g = \psi(\langle V_1^I,V_1^A,V_1^B\rangle)\oplus\mathscr{B}_g^1$.*

The argument for this step is very similar to the argument in Step 2, so we only sketch it. Define

$$\begin{aligned}V_1^{B,1} =& \{X(ca_i+db_i+fb_g,b_g,s)\mid 1\le i\le g-1,\, c,d,f\in\mathbb{Z}/L,\, s\in\{a_i,b_i\}\}\subset V_1^B,\\ V_1^{B,2} =& \{X(ca_i+db_i+fb_g,b_g,a_i)\mid 1\le i\le g-1,\, c,d,f\in\mathbb{Z}/L\}\\ &\cup\{X(db_i+fb_g,b_g,b_i)\mid 1\le i\le g-1,\, d,f\in\mathbb{Z}/L\}\subset V_1^{B,1},\\ V_1^{B,3} =& \{X(v,b_g,s)\in V_1^{B,2}\mid \text{the } s \text{ and } b_g\text{-coordinates of } v \text{ do not equal } L-1\}\subset V_1^{B,2}.\end{aligned}$$

Noting that $\psi(V_1^{B,1})\subset\mathscr{B}_g^3$, arguments similar to those in Step 2 show that $\langle V_1^I,V_1^A,V_1^B\rangle = \langle V_1^I,V_1^A,V_1^{B,1}\rangle$, that $\mathscr{B}_g^3 = \psi(\langle V_1^{B,1}\rangle)+\mathscr{B}_g^1$, that $\langle V_1^{B,1}\rangle = \langle V_1^{B,2}\rangle$, and that $\langle V_1^{B,2}\rangle = \langle V_1^{B,3}\rangle$.

We deduce that $\mathscr{B}_g^3 = \psi(\langle V_1^{B,3}\rangle)+\mathscr{B}_g^1$. Since $V_1^{B,3}$ contains

$$\begin{aligned}(g-1)((L-1)^2L+(L-1)^2) &= ((g-1)(L^2-1)L+L^2)-(g(L^2-1)+1)\\ &= \dim(\mathscr{B}_g^3)-\dim(\mathscr{B}_g^1)\end{aligned}$$

elements, we obtain that $\psi|_{\langle V_1^{B,3}\rangle}$ is injective and $\mathscr{B}_g^3 = \psi(\langle V_1^{B,3}\rangle)\oplus\mathscr{B}_g^1$. By Step 2 and the identities

$$\langle V_1^I,V_1^A,V_1^B\rangle = \langle V_1^I,V_1^A,V_1^{B,1}\rangle \quad\text{and}\quad \langle V_1^{B,3}\rangle = \langle V_1^{B_1}\rangle,$$

we conclude that $\psi|_{\langle V_1^I,V_1^A,V_1^B\rangle}$ is injective and $\mathscr{B}_g = \psi(\langle V_1^I,V_1^A,V_1^B\rangle)\oplus\mathscr{B}_g^1$, as desired. □

Our final lemma is a further relation in Q_g.

Lemma 7.6. *Let $\{a_1',b_1',a_2',b_2'\}$ be a unimodular subset of $\mathrm{H}_1(\Sigma_g;\mathbb{Z}/L)$ with $i(a_1',b_1') = i(a_2',b_2') = 1$ and $i(a_1',a_2') = i(a_1',b_2') = i(b_1',a_2') = i(b_1',b_2') = 0$. Then for all $v\in\mathrm{H}_1(\Sigma_g;\mathbb{Z}/L)$ we have*

$$\begin{aligned}&X(v,a_1',a_2')-X(v+b_1',a_1',a_2')-X(v+b_2',a_1',a_2')+X(v+b_1'+b_2',a_1',a_2')\\ &\quad= X(v,b_1',b_2')-X(v+a_1',b_1',b_2')-X(v+a_2',b_1',b_2')+X(v+a_1'+a_2',b_1',b_2')\end{aligned}$$

Proof. The group $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts on Q_g, and there exists some $f \in \mathrm{Sp}_{2g}(\mathbb{Z})$ such that $f(a_i') = a_i$ and $f(b_i') = b_i$ for $i = 1,2$. We can therefore assume that $a_i' = a_i$ and $b_i' = b_i$ for $i = 1,2$. But an easy calculation shows that ψ takes both sides of our relation to the same element of $\mathscr{B}_g$, so the lemma follows from Lemma 7.5. $\square$

We can now prove Lemma 5.4.

Proof of Lemma 5.4. Define $Q_g' = Q_g/\langle V_1 \rangle$ and $\mathscr{B}_g' = \mathscr{B}_g/\psi(\langle V_1 \rangle)$. We have an induced map $\psi' : Q_g' \to \mathscr{B}_g'$. Using the direct sum decomposition of Lemma 7.5, we will identify $\mathscr{B}_g'$ with the subspace

$$\langle \{ [\![ca_i + db_i]\!] \mid c,d \in \mathbb{Z}/L,\ 1 \leq i \leq g \} \rangle$$

of $\mathscr{B}_g$. Letting $V_2' \subset Q_g'$ be the image of $V_2 \subset Q_g$, Lemmas 7.1 and 7.5 say that it is enough to prove that $\psi'|_{\langle V_2' \rangle}$ is injective and $\mathscr{B}_g' = \psi'(\langle V_2' \rangle) \oplus \langle [\![0]\!] \rangle$.

Let $\phi : Q_g \to Q_g'$ be the projection. The proof will require seven claims. It follows the same pattern as Steps 2 and 3 of the proof of Lemma 7.5. In Claims 1–3 and 5–6, we will obtain a "minimal" size generating set for $\langle V_2' \rangle$. In Claims 4 and 7, we will show that $\mathscr{B}_g' = \psi'(\langle V_2' \rangle) + \langle [\![0]\!] \rangle$. A dimension count will then establish the lemma.

Claim 1. *Let $s, s_1, s_2 \in B$ satisfy $s \neq s_1, s_2$ and $i(s,s_1) = i(s,s_2) = 0$. Then for all $v \in \mathrm{H}_1(\Sigma_g;\mathbb{Z}/L)$ and $e_1, e_2 \in \{-1,1\}$, we have $\phi(X(v,s,s+e_1s_1)) = \phi(X(v,s,s+e_2s_2))$.*

Proof of Claim. For $1 \leq i \leq 2$, using the second relation in Lemma 6.6, we get that

$$X(v+e_1s_1+s, e_2s_2, s), X(v+e_2s_2+s, e_1s_1, s) \in \langle V_1 \rangle.$$

Hence

$$\phi(X(v, e_1s_1+e_2s_2+s, s)) = \phi(X(v, e_1s_1+s, s) + X(v+e_1s_1+s, e_2s_2, s)) = \phi(X(v,s,s+e_1s_1))$$

and

$$\phi(X(v, e_1s_1+e_2s_2+s, s)) = \phi(X(v, e_2s_2+s, s) + X(v+e_2s_2+s, e_1s_1, s)) = \phi(X(v,s,s+e_2s_2)).$$

The claim follows. $\square$

In light of Claim 1, we will denote by $Y(v,s)$ the image in V_2' of $X(v,s,s+e's')$, where $e' \in \{-1,1\}$ and $s' \in B$ are arbitrary elements such that $X(v,s,s+e's') \in V_2$.

Claim 2. *Consider $Y(v,s) \in V_2'$. Pick $1 \leq i \leq g$ such that $s \in \{a_i, b_i\}$. Write $v = v_1 + v_2$ with $v_1 \in \langle a_i, b_i \rangle$ and $v_2 \in \langle \{a_j, b_j \mid j \neq i\} \rangle$. Then $Y(v,s) = Y(v_1,s)$.*

Proof of Claim. Consider $s' \in \{a_j, b_j \mid j \neq i\}$. It is enough to show that $Y(v-s',s) = Y(v,s)$. Pick $s'' \in \{a_j, b_j \mid j \neq i\}$ such that $s'' \neq s'$ and $i(s',s'') = 0$ (this uses the fact that $g \geq 3$). Observe that $Y(v,s) = \phi(X(v,s,s+s''))$ and $Y(v-s',s) = \phi(X(v-s',s,s+s''))$. Lemma 7.2 says that

$$X(v-s', s, s+s'') = X(v,s,s+s'') + X(v-s', s', s+s'') - X(v-s'+s, s', s+s''). \tag{9}$$

For w equal to $v-s'$ or $v-s'+s$, we have

$$X(w, s', s+s'') = X(w, s+s'', s') = X(w,s,s') + X(w+s, s'', s') \in \langle V_1 \rangle.$$

Applying ϕ to both sides of (9), we thus obtain that $Y(v,s) = Y(v-s',s)$, as desired. $\square$

Claim 3. *For all $1 \leq i \leq g$ and $v \in \langle a_i, b_i \rangle$, we have*

$$Y(v,a_i) - 2Y(v+b_i,a_i) + Y(v+2b_i,a_i) = Y(v,b_i) - 2Y(v+a_i,b_i) + Y(v+2a_i,b_i).$$

Proof of Claim. Pick $1 \le j < k \le g$ such that $i \ne j,k$ (this uses the fact that $g \ge 3$). Lemma 7.6 applied with $(a_1', b_1', a_2', b_2') = (a_i + a_k, b_i - b_j, a_i + a_j, b_i - b_k)$ says that

$$\begin{aligned}
&X(v, a_i + a_k, a_i + a_j) - X(v + b_i - b_j, a_i + a_k, a_i + a_j) - X(v + b_i - b_k, a_i + a_k, a_i + a_j) \\
&\quad + X(v + b_i - b_j + b_i - b_k, a_i + a_k, a_i + a_j) \\
=&X(v, b_i - b_j, b_i - b_k) - X(v + a_i + a_k, b_i - b_j, b_i - b_k) - X(v + a_i + a_j, b_i - b_j, b_i - b_k) \\
&\quad + X(v + a_i + a_k + a_i + a_j, b_i - b_j, b_i - b_k).
\end{aligned} \tag{10}$$

Since $X(v + a_i, a_k, a_i + a_j) \in \langle V_1 \rangle$, we have that

$$\phi(X(v, a_i + a_k, a_i + a_j)) = \phi(X(v, a_i, a_i + a_j) + X(v + a_i, a_k, a_i + a_j)) = Y(v, a_i).$$

Similarly, we have $\phi(X(v + b_i - b_j, a_i + a_k, a_i + a_j)) = Y(v + b_i - b_j, a_i)$. By Claim 2, this equals $Y(v + b_i, a_i)$. Continuing in this manner, we deduce that ϕ maps (10) to the desired relation between the $Y(\cdot,\cdot)$. $\square$

For the next claim, recall that we are using Lemma 7.5 to identify $\mathscr{B}_g' = \mathscr{B}_g / \psi(\langle V_1 \rangle)$ with the subspace $\langle \{ [\![c a_i + d b_i]\!] \mid c, d \in \mathbb{Z}/L,\ 1 \le i \le g \} \rangle$ of $\mathscr{B}_g$.

Claim 4. *For some $1 \le i \le g$, let $s \in \{a_i, b_i\}$ and $v \in \langle a_i, b_i \rangle$. Then $\psi'(Y(v,s)) = [\![v]\!] - 2[\![v+s]\!] + [\![v+2s]\!]$.*

Proof of Claim. Let $\rho : \mathscr{B}_g \to \mathscr{B}_g'$ be the projection. Pick $1 \le j \le g$ such that $j \ne i$. Observe that $Y(v,s) = \phi(X(v, s, s - a_j))$ and $X(v + s - a_j, a_j, s) \in V_1$. Thus $\psi'(Y(v,s))$ equals

$$\begin{aligned}
\rho(\psi(X(v, s - a_j, s))) &= \rho(\psi(X(v, s - a_j, s) + X(v + s - a_j, a_j, s))) \\
&= \rho(([\![v]\!] - [\![v + s - a_j]\!] - [\![v+s]\!] + [\![v + 2s - a_j]\!]) \\
&\qquad + ([\![v + s - a_j]\!] - [\![v+s]\!] - [\![v + 2s - a_j]\!] + [\![v+2s]\!]) \\
&= \rho([\![v]\!] - 2[\![v+s]\!] + [\![v+2s]\!]).
\end{aligned}$$

Since $v, v+s, v+2s \in \langle a_i, b_i \rangle$, this equals $[\![v]\!] - 2[\![v+2]\!] + [\![v+2s]\!]$, as desired. $\square$

For some $1 \le i \le g$, consider $s \in \{a_i, b_i\}$ and $v \in \langle a_i, b_i \rangle$. Making use of Claim 4, an easy induction establishes that for $n \ge 1$, we have

$$\psi'(\sum_{k=1}^{n} k \cdot Y(v + (k-1)s, s)) = [\![v]\!] - (n+1)[\![v + ns]\!] + n[\![v + (n+1)s]\!].$$

In particular, setting $Z(v,s) = \sum_{k=1}^{L} k \cdot Y(v + (k-1)s, s)$, we have

$$\psi'(Z(v,s)) = L[\![v+s]\!] - L[\![v]\!]. \tag{11}$$

We now prove the following.

Claim 5. *For all $1 \le i \le g$ and $v \in \langle a_i, b_i \rangle$, we have*

$$Z(v, a_i) - 2Z(v + b_i, a_i) + Z(v + 2b_i, a_i) = L \cdot Y(v + a_i, b_i) - L \cdot Y(v, b_i).$$

Proof of Claim. Observe that $Z(v, a_i) - 2Z(v + b_i, a_i) + Z(v + 2b_i, a_i)$ equals

$$\sum_{k=1}^{L} k \cdot (Y(v + (k-1)a_i, a_i) - 2Y(v + (k-1)a_i + b_i, a_i) + Y(v + (k-1)a_i + 2b_i, a_i)).$$

By Claim 3, this equals

$$\sum_{k=1}^{L} k \cdot (Y(v + (k-1)a_i, b_i) - 2Y(v + k a_i, b_i) + Y(v + (k+1)a_i, b_i)).$$

An argument similar to the argument used to calculate ψ' of $Z(\cdot,\cdot)$ then shows that this equals $L \cdot Y(v + a_i, b_i) - L \cdot Y(v, b_i)$, and we are done. $\square$

Claim 6. *We have*

$$\langle V_2' \rangle = \langle \{Y(ca_i + db_i, a_i) \mid 1 \le i \le g,\ c,d \in \mathbb{Z}/L,\ c \ne L-1\} \\ \cup \{Y(db_i, b_i) \mid 1 \le i \le g,\ d \in \mathbb{Z}/L,\ d \ne L-1\} \rangle.$$

Proof of Claim. Claim 2 implies that

$$\langle V_2' \rangle = \{Y(v,s) \mid v \in \langle a_i, b_i \rangle \text{ and } s \in \{a_i, b_i\} \text{ for some } 1 \le i \le g\}. \tag{12}$$

If $v \in \{a_i, b_i\}$ for some $1 \le i \le g$, then

$$Z(v, a_i) \in \langle \{Y(ca_i + db_i, a_i) \mid 1 \le i \le g,\ c,d \in \mathbb{Z}/L\} \rangle.$$

We can therefore use the relation in Claim 5 to reduce (12) to

$$\langle V_2' \rangle = \langle \{Y(ca_i + db_i, a_i) \mid 1 \le i \le g,\ c,d \in \mathbb{Z}/L\} \\ \cup \{Y(db_i, b_i) \mid 1 \le i \le g,\ d \in \mathbb{Z}/L\} \rangle. \tag{13}$$

Finally, if $v \in \langle a_i, b_i \rangle$ and $s \in \{a_i, b_i\}$ for some $1 \le i \le g$, then from the third relation in Lemma 6.6, we obtain the relation $\sum_{k=0}^{L-1} Y(v + k \cdot s, s) = 0$ in Q_g'. This allows us to reduce (13) to the claimed generating set, and we are done. □

Claim 7. $\mathscr{B}_g' = \langle \psi'(V_2') \rangle + \langle [\![0]\!] \rangle$.

Proof of Claim. Consider $c, d \ge 0$ with $(c,d) \ne 0$. With the convention that an empty sum of abelian group elements is the zero element, we can use (11) to get that

$$\begin{aligned} \frac{1}{L}\psi'\Big(\sum_{j=1}^{c} Z(ja_i, a_i) + \sum_{k=1}^{d} Z(ca_i + kb_i, b_i)\Big) &= ([\![ca_i]\!] - [\![0]\!]) + ([\![ca_i + db_i]\!] - [\![ca_i]\!]) \\ &= [\![ca_i + db_i]\!] - [\![0]\!]. \end{aligned}$$

Here the first equality follows from the fact that the indicated sums become telescoping sums after applying ψ'. The claim follows. □

Observe now that the generating set for $\langle V_2' \rangle$ given by Claim 6 has

$$g(L(L-1) + L - 1) = g(L^2 - 1) = \dim(\mathscr{B}_g') - 1$$

elements, so by Claim 7 we have that $\psi'|_{\langle V_2' \rangle}$ is injective and $\mathscr{B}_g' = \langle \psi'(V_2') \rangle \oplus \langle [\![0]\!] \rangle$, as desired. □

8 Killing off $\mathscr{S}_g$

This section is devoted to the proof of Lemma 5.5. The proof itself is contained in §8.2. This is proceeded by §8.1, which contains a technical lemma about essentially separate curves.

8.1 Separating essentially separate curves

This section is devoted to the proof of the following lemma.

Lemma 8.1. *Consider $v, v' \in H_L$ and $x, y \in K_g$ such that x and y are essentially separate and $\langle\!\langle y \rangle\!\rangle \in C_g$. There then exists some $n \ge 1$ and $v_1, v_1', \ldots, v_n, v_n' \in H_L$ and $x_1, y_1, \ldots, x_n, y_n \in K_g$ with the following properties.*

1. $\langle\!\langle x \rangle\!\rangle^{v} \otimes \langle\!\langle y \rangle\!\rangle^{v'} = \sum_{i=1}^{n} \langle\!\langle x_i \rangle\!\rangle^{v_i} \otimes \langle\!\langle y_i \rangle\!\rangle^{v_i'}$.

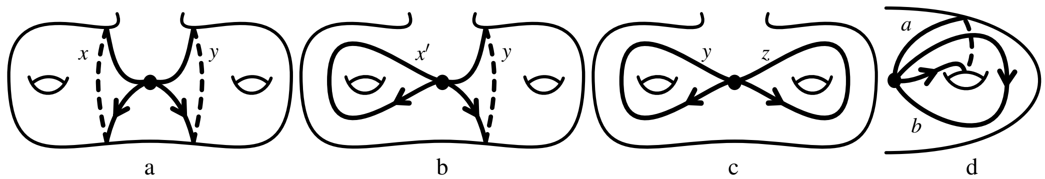


Figure 9: *a,b. The intersection patterns needed for Lemma 8.1 c. A nonseparating figure eight. d. $[a,b]$ is a genus 1 separating curve.*

2. *For all $1 \leq i \leq n$, one of the following two conditions is satisfied.*

   (a) $\{x_i, y_i\}$ has the same unoriented intersection pattern as the curves in Figure 9.a, or

   (b) $x_i = (x_i')^L$, where $\{x_i', y_i\}$ has the same unoriented intersection pattern as the curves in Figure 9.b.

Remark. The second condition in the conclusion of Lemma 8.1 implies that $\langle\!\langle y_i \rangle\!\rangle \in C_g$.

To prove Lemma 8.1, we will need a definition and a lemma.

Definition. A curve $x \in \pi_1(\Sigma_g)$ is a *genus k separating curve* if it can be realized by a simple closed curve that separates Σ_g into two pieces, one of which is homeomorphic to $\Sigma_{k,1}$. Two curves $y, z \subset \pi_1(\Sigma_g)$ form a *nonseparating figure eight* if they have the same unoriented intersection pattern as the curves in Figure 9.c.

Remark. If $a, b \in \pi_1(\Sigma_g)$ have the same unoriented intersection pattern as the curves in Figure 9.d, then $[a,b] \in \pi_1(\Sigma_g)$ is a genus 1 separating curve.

Lemma 8.2. *Let $S \subset \Sigma_g$ be a subsurface such that the basepoint lies on ∂S and $\Sigma_g \setminus S$ is connected. Define $P_S = \mathrm{Im}(\pi_1(S) \to \pi_1(\Sigma_g))$ and $H_S = \mathrm{Im}(\mathrm{H}_1(S;\mathbb{Z}/L) \to H_L)$. Next, define the following sets.*

$$
\begin{aligned}
U_1 &= \{\langle\!\langle x^L \rangle\!\rangle \mid x \in P_S \text{ is a simple closed nonseparating curve}\},\\
U_2 &= \{\langle\!\langle x \rangle\!\rangle^v \mid v \in H_S,\ x \in P_S \text{ is a genus } 1 \text{ separating curve}\},\\
U_3 &= \{\langle\!\langle y,z \rangle\!\rangle^v \mid v \in H_S,\ y,z \subset P_S \text{ form a nonseparating figure eight}\}.
\end{aligned}
$$

Then the following hold.

1. *$C_g \cap \mathrm{H}_1(K_g \cap P_S;\mathbb{Q})$ is spanned by $U_2 \cup U_3$ and $\mathrm{H}_1(K_g \cap P_S;\mathbb{Q})$ is spanned by $U_1 \cup U_2 \cup U_3$*
2. *If the genus of S is positive, then $C_g \cap \mathrm{H}_1(K_g \cap P_S;\mathbb{Q})$ is spanned by U_2 and $\mathrm{H}_1(K_g \cap P_S;\mathbb{Q})$ is spanned by $U_1 \cup U_2$.*

Proof. Define

$$K_S = \ker(\pi_1(S) \to \mathrm{H}_1(S;\mathbb{Z}/L)) \qquad \text{and} \qquad C_S = \ker(\mathrm{H}_1(K_S;\mathbb{Q}) \to \mathrm{H}_1(S;\mathbb{Q})).$$

Since $\Sigma_g \setminus S$ is connected, the map $\mathrm{H}_1(S;\mathbb{Z}/L) \to \mathrm{H}_1(\Sigma_g;\mathbb{Z}/L)$ is injective. Using the commutative diagram

$$
\begin{array}{ccc}
\pi_1(S) & \longrightarrow & \pi_1(\Sigma_g) \\
\downarrow & & \downarrow \\
\mathrm{H}_1(S;\mathbb{Z}/L) & \longrightarrow & \mathrm{H}_1(\Sigma_g;\mathbb{Z}/L)
\end{array}
$$

we deduce that the natural maps

$$K_S \longrightarrow K_g \cap P_S \qquad \text{and} \qquad C_S \longrightarrow C_g \cap \mathrm{H}_1(K_g \cap P_S;\mathbb{Q})$$

are surjective.

Define

$$\begin{aligned}
V_1' &= \{x^L \in K_S \mid x \in \pi_1(S) \text{ can be realized by a simple closed curve that is either} \\
&\qquad \text{nonseparating or freely homotopic to a boundary component}\}, \\
V_2' &= \{x^w \in K_S \mid x, w \in \pi_1(S) \text{ and } x \text{ maps to a genus 1 separating curve in } \pi_1(\Sigma_g)\}, \\
V_3' &= \{[y,z]^w \in K_S \mid y,z,w \in \pi_1(S) \text{ and } \{y,z\} \text{ maps to a nonseparating figure} \\
&\qquad \text{eight in } \pi_1(\Sigma_g)\}.
\end{aligned}$$

There is a short exact sequence

$$1 \longrightarrow [\pi_1(S), \pi_1(S)] \longrightarrow K_S \longrightarrow L \cdot \mathrm{H}_1(S;\mathbb{Z}) \longrightarrow 1,$$

where $L \cdot \mathrm{H}_1(S;\mathbb{Z})$ denotes the subgroup $\{L \cdot v \mid v \in \mathrm{H}_1(S;\mathbb{Z})\}$. The subgroup of K_S generated by V_1' surjects onto $L \cdot \mathrm{H}_1(S;\mathbb{Z})$. Also, making use of a standard basis for $\pi_1(S)$, one can easily check that $V_2' \cup V_3'$ generates $[\pi_1(S), \pi_1(S)]$. Finally, the proof of [19, Theorem A.3] shows that if the genus of S is positive, then $[\pi_1(S), \pi_1(S)]$ is generated by V_2'. The upshot of this is that K_S is generated by $V_1' \cup V_2' \cup V_3'$ in all cases and by $V_1' \cup V_2'$ if the genus of S is positive.

Defining V_i to be the image of V_i' in $\mathrm{H}_1(K_S;\mathbb{Q})$, we obtain that $\mathrm{H}_1(K_S;\mathbb{Q})$ (resp. C_S) is spanned by $V_1 \cup V_2 \cup V_3$ (resp. $V_2 \cup V_3$) in all cases and by $V_1 \cup V_2$ (resp. V_2) if the genus of S is positive. The set V_i maps to U_i under the natural map $\mathrm{H}_1(K_S;\mathbb{Q}) \to \mathrm{H}_1(K_g \cap P_S;\mathbb{Q})$. Since this map and the restricted map $C_S \to C_g \cap \mathrm{H}_1(K_g \cap P_S;\mathbb{Q})$ are surjections, the lemma follows □

Proof of Lemma 8.1. Since x and y are essentially separate, we can decompose Σ_g into the union of two connected subsurfaces S_1 and S_2 with the following properties.

1. S_1 and S_2 both contain the basepoint.
2. $S_1 \cap S_2 = \partial S_1 = \partial S_2$.
3. $x \in \mathrm{Im}(\pi_1(S_1) \to \pi_1(\Sigma_g))$ and $y \in \mathrm{Im}(\pi_1(S_2) \to \pi_1(\Sigma_g))$.

Applying Lemma 8.2 to each S_i, we can write

$$\langle\!\langle x \rangle\!\rangle = \sum_{i=1}^{k} \langle\!\langle x_i \rangle\!\rangle^{v_i} \qquad \text{and} \qquad \langle\!\langle y \rangle\!\rangle = \sum_{j=1}^{k'} \langle\!\langle y_j \rangle\!\rangle^{v_j'},$$

where $v_i, v_j' \in H_L$ and x_i and y_j satisfy the following conditions.

- $x_i \in \mathrm{Im}(\pi_1(S_1) \to \pi_1(\Sigma_g))$ and $y_j \in \mathrm{Im}(\pi_1(S_2) \to \pi_1(\Sigma_g))$ for all i and j. In particular, the curves x_i and y_j are essentially separate for all i and j.
- x_i is either a genus 1 separating curve, the commutator of a nonseparating figure eight, or z^L for some simple closed nonseparating curve z.
- y_j is either a genus 1 separating curve or the commutator of a nonseparating figure eight.

We then have

$$\langle\!\langle x \rangle\!\rangle^{v} \otimes \langle\!\langle y \rangle\!\rangle^{v'} = \sum_{i=1}^{k} \sum_{j=1}^{k'} \langle\!\langle x_i \rangle\!\rangle^{v+v_i} \otimes \langle\!\langle y_j \rangle\!\rangle^{v'+v_j'}.$$

We are almost done – the only problem is that x_i or y_j might be the commutator of a nonseparating figure eight for some i or j. However, if that happens, then we may perform the above procedure again to the pair $\{x_i, y_j\}$, but this time it is easy to see that we may ensure that the genera of S_1 and S_2 are both positive, in which case we do not need to use nonseparating figure eights. □

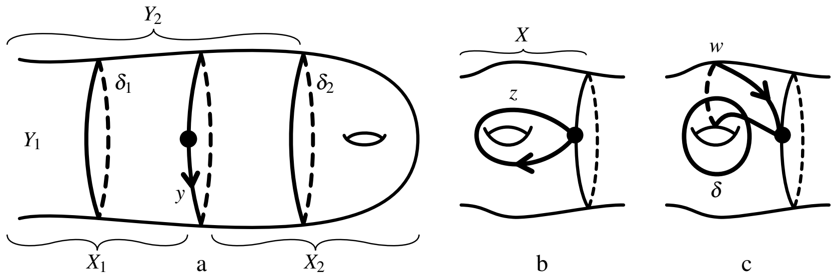


Figure 10: *a. Figure needed to deal with elements of $\mathscr{S}_g(1)$. b. The curve z can be realized by a simple closed nonseparating curve and lies entirely in $X \subset \Sigma_g$. c. $T_\delta^{eL}(w) = z^L w$. The sign $e = \pm 1$ depends on the orientation of z; with the orientation depicted in the figure, we have $e = 1$.*

8.2 The proof of Lemma 5.5

In this final section, we prove Lemma 5.5. We begin by briefly recalling its statement. The set

$$\mathscr{K}_g < \mathrm{H}_1(K_g;\mathbb{Z}) \otimes C_g = \mathrm{H}_1(K_g;C_g)$$

is the span of the set

$$\{x \otimes y \mid (x,y) \in \mathscr{S}_g\} \cup \{x \otimes y - f(x) \otimes f(y) \mid x \in \mathrm{H}_1(K_g;\mathbb{Z}),\, y \in C_g,\, f \in \mathrm{Mod}_g^1(L)\}.$$

Here $\mathscr{S}_g$ is the set defined in §5.2. Lemma 5.5 asserts that for $g \geq 4$, the image of $\mathscr{K}_g$ in $\mathrm{H}_1(\mathrm{Mod}_g^1(L);C_g)$ is zero.

Let $\phi : \mathrm{H}_1(K_g;\mathbb{Z}) \otimes C_g \to \mathrm{H}_1(\mathrm{Mod}_g^1(L);C_g)$ be the natural map. Since inner automorphisms act trivially on homology, we have

$$\phi(x \otimes y) = \phi(f(x) \otimes f(y)) \tag{14}$$

for all $x \otimes y \in \mathrm{H}_1(K_g;\mathbb{Z}) \otimes C_g$ and $f \in \mathrm{Mod}_g^1(L)$. We must show that in addition we have $\phi(s) = 0$ for $s \in \mathscr{S}_g$. This will require several steps.

Step 1. *Consider $v, v' \in H_L$ and $x, y \in K_g$ such that x and y are essentially separate and $\langle\langle y \rangle\rangle \in C_g$, so $(\langle\langle x \rangle\rangle^v, \langle\langle y \rangle\rangle^{v'}) \in \mathscr{S}_g(1)$. Then $\phi(\langle\langle x \rangle\rangle^v \otimes \langle\langle y \rangle\rangle^{v'}) = 0$.*

By Lemma 8.1, we can assume that one of the following two conditions hold.

1. $\{x,y\}$ has the same unoriented intersection pattern as the curves in Figure 9.a, or
2. $x = (x')^L$, where $\{x',y\}$ has the same unoriented intersection pattern as the curves in Figure 9.b.

Thus y is a separating curve separating Σ_g into two subsurfaces X_1 and X_2 with $X_2 \cong \Sigma_{1,1}$ (see Figure 10.a). Also, x is in $\mathrm{Im}(\pi_1(X_1) \to \pi_1(\Sigma_g))$.

The group $\mathrm{Mod}_g^1(L)$ contains all inner automorphisms of $\pi_1(\Sigma_g)$, so using (14) we can assume that $v = 0$. We will produce a subgroup $\Gamma < \mathrm{Mod}_g^1(L)$ with the following three properties.

1. $\mathrm{H}_1(\Gamma;\mathbb{Q}) = 0$.
2. Γ acts trivially on $\langle\langle y \rangle\rangle$.
3. Γ contains the "point-pushing" mapping class corresponding to $x \in \pi_1(\Sigma_g)$.

This is enough to prove the desired claim. Indeed, since $\Gamma < \mathrm{Mod}_g^1(L)$, we have $f(v') = v'$ for all $f \in \Gamma$. This and item 2 imply that Γ fixes $\langle\langle y \rangle\rangle^{v'}$. There is thus a map $j : \mathrm{H}_1(\Gamma;\mathbb{Q}) \to \mathrm{H}_1(\mathrm{Mod}_g^1(L);C_g)$ corresponding to the inclusion $\Gamma \hookrightarrow \mathrm{Mod}_g^1(L)$ and the map $\mathbb{Q} \hookrightarrow C_g$ taking $1 \in \mathbb{Q}$ to $\langle\langle y \rangle\rangle^{v'}$. By item 3, the image of j contains $\phi(\langle\langle x \rangle\rangle^v \otimes \langle\langle y \rangle\rangle^{v'})$, so by item 1 we have $\phi(\langle\langle x \rangle\rangle^v \otimes \langle\langle y \rangle\rangle^{v'}) = 0$, as desired.

We construct Γ as follows. Let N be a regular neighborhood of y and let δ_i be the boundary component of N that is contained in X_i (see Figure 10.a). The curve δ_i separates Σ_g into two components. Let Y_i be the component of Σ_g cut along δ_i that intersects X_1. It follows that $Y_1 \cong \Sigma_{g-1,1}$ and $Y_2 \cong \Sigma^1_{g-1,1}$. Denoting the mapping class group of Y_i by $\mathrm{Mod}(Y_i)$, we have a Birman exact sequence

$$1 \longrightarrow \pi_1(\Sigma_{g-1,1}) \longrightarrow \mathrm{Mod}(Y_2) \longrightarrow \mathrm{Mod}_{g-1,1} \longrightarrow 1.$$

We have an isomorphism $\mathrm{Mod}_{g-1,1} \cong \mathrm{Mod}(Y_1)$, and this exact sequence splits via a map that identifies $f \in \mathrm{Mod}_{g-1,1}$ with a corresponding element of $\mathrm{Mod}(Y_1)$ and then extends f by the identity to an element of $\mathrm{Mod}(Y_2)$. Choosing such a splitting, we get a decomposition

$$\mathrm{Mod}(Y_2) = \pi_1(\Sigma_{g-1,1}) \rtimes \mathrm{Mod}_{g-1,1}\,.$$

Define

$$\Gamma = K_{g-1,1} \rtimes \mathrm{Mod}_{g-1,1}(L) < \pi_1(\Sigma_{g-1,1}) \rtimes \mathrm{Mod}_{g-1,1}\,.$$

It is clear that Γ contains the "point-pushing" mapping class corresponding to $x \in \pi_1(\Sigma_g)$. Also, the subgroup $\mathrm{Mod}_{g-1,1}(L) < \Gamma$ fixes $y \in \pi_1(\Sigma_g)$ and the subgroup $K_{g-1,1} < \Gamma$ acts on y by conjugation, and thus fixes $\langle\!\langle y \rangle\!\rangle$. We deduce that Γ acts trivially on $\langle\!\langle y \rangle\!\rangle$.

It remains to check that $\mathrm{H}_1(\Gamma;\mathbb{Q}) = 0$. From its semidirect product decomposition, we deduce that

$$\mathrm{H}_1(\Gamma;\mathbb{Q}) \cong \mathrm{H}_1(\mathrm{Mod}_{g-1,1}(L);\mathbb{Q}) \oplus (\mathrm{H}_1(K_{g-1,1};\mathbb{Q}))_{\mathrm{Mod}_{g-1,1}(L)}.$$

Since $g \geq 4$, Theorem 2.6 implies that $\mathrm{H}_1(\mathrm{Mod}_{g-1,1}(L);\mathbb{Q}) = 0$. Again using the fact that $g \geq 4$, Lemma 3.4 implies that $(\mathrm{H}_1(K_{g-1,1};\mathbb{Q}))_{\mathrm{Mod}_{g-1,1}(L)} = 0$, and we are done.

Step 2. *$\phi(a \otimes b) = 0$ if $(a,b) \in \mathscr{S}_g(2)$.*

Consider $(\langle\!\langle x \rangle\!\rangle^v, \langle\!\langle y, z^L \rangle\!\rangle^{v'}) \in \mathscr{S}_g(2)$, so z can be realized by a simple closed nonseparating curve and $\{z\}$ and $\{x,y\}$ are strongly essentially disjoint. We can thus find subsurfaces X and X' of Σ_g both of which contain the basepoint such that

$$z \in \mathrm{Im}(\pi_1(X) \to \pi_1(\Sigma_g)) \qquad \text{and} \qquad \{x,y\} \subset \mathrm{Im}(\pi_1(X') \to \pi_1(\Sigma_g)).$$

and such that

$$\Sigma_g = X \cup X' \qquad \text{and} \qquad X \cap X' = \partial X = \partial X'.$$

Additionally, the surfaces X and X' can be chosen such that both have only one boundary component.

As is shown in Figure 10.c, there exists some $w \in \mathrm{Im}(\pi_1(X) \to \pi_1(\Sigma_g))$ together with an unbased simple closed curve δ in $X \subset \Sigma_g$ such that $T_\delta^{eL}(w) = z^L w$ for some $e = \pm 1$. Since $\delta \subset X$, we have $T_\delta(x) = x$ and $T_\delta(y) = z$. Also, since $T_\delta^{eL} \in \mathrm{Mod}_g^1(L)$, it follows that T_δ^{eL} acts trivially on v and v'. This implies that T_δ^{eL} takes $\langle\!\langle x \rangle\!\rangle^v \otimes \langle\!\langle y, w \rangle\!\rangle^{v'}$ to

$$\begin{aligned} \langle\!\langle x \rangle\!\rangle^v \otimes \langle\!\langle y, z^L w \rangle\!\rangle^{v'} &= \langle\!\langle x \rangle\!\rangle^v \otimes \langle\!\langle y, z^L \rangle\!\rangle^{v'} + \langle\!\langle x \rangle\!\rangle^v \otimes \langle\!\langle y, w \rangle\!\rangle^{v'+L\cdot\bar{z}} \\ &= \langle\!\langle x \rangle\!\rangle^v \otimes \langle\!\langle y, z^L \rangle\!\rangle^{v'} + \langle\!\langle x \rangle\!\rangle^v \otimes \langle\!\langle y, w \rangle\!\rangle^{v'}. \end{aligned}$$

The first calculation here uses Lemma 5.1 and the second the fact that $L \cdot \bar{z} = 0$. Since $\mathrm{Mod}_g^1(L)$ acts trivially on the image of ϕ, we conclude that

$$\phi(\langle\!\langle x \rangle\!\rangle^v \otimes \langle\!\langle y, w \rangle\!\rangle^{v'}) = \phi(\langle\!\langle x \rangle\!\rangle^v \otimes \langle\!\langle y, w \rangle\!\rangle^{v'} + \langle\!\langle x \rangle\!\rangle^v \otimes \langle\!\langle y, z^L \rangle\!\rangle^{v'});$$

i.e. that $\phi(\langle\!\langle x \rangle\!\rangle^v \otimes \langle\!\langle y, z^L \rangle\!\rangle^{v'}) = 0$, as desired.

References

[1] J. Anderson and R. Villemoes, Degree one cohomology with twisted coefficients of the mapping class group, preprint 2007.

[2] J. Anderson and R. Villemoes, The first cohomology of the mapping class group with coefficients in algebraic functions on the $SL_2(\mathbb{C})$ moduli space, Algebr. Geom. Topol. **9** (2009), 1177–1199.

[3] J. Anderson and R. Villemoes, Cohomology of mapping class groups and the abelian moduli space, preprint 2009.

[4] J. S. Birman, *Braids, links, and mapping class groups*, Ann. of Math. Stud., 82, Princeton Univ. Press, Princeton, N.J., 1974.

[5] S. Boldsen, Improved homological stability for the mapping class group with integral or twisted coefficients, preprint 2009.

[6] K. S. Brown, *Cohomology of groups*, Corrected reprint of the 1982 original, Springer, New York, 1994.

[7] T. Church and A. Pixton, Separating twists and the Magnus representation of the Torelli group, to appear in Geom. Dedicata.

[8] B. Farb and D. Margalit, A primer on mapping class groups, preprint 2010.

[9] R. M. Hain, Torelli groups and geometry of moduli spaces of curves, in *Current topics in complex algebraic geometry (Berkeley, CA, 1992/93)*, 97–143, Cambridge Univ. Press, Cambridge.

[10] J. L. Harer, Stability of the homology of the mapping class groups of orientable surfaces, Ann. of Math. (2) **121** (1985), no. 2, 215–249.

[11] J. Hempel, Intersection calculus on surfaces with applications to 3-manifolds, Mem. Amer. Math. Soc. **43** (1983), no. 282, vi+48 pp.

[12] N. V. Ivanov, On the homology stability for Teichmüller modular groups: closed surfaces and twisted coefficients, in *Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991)*, 149–194, Contemp. Math., 150, Amer. Math. Soc., Providence, RI.

[13] D. Johnson, The structure of the Torelli group. I. A finite set of generators for $\mathscr{I}$, Ann. of Math. (2) **118** (1983), no. 3, 423–442.

[14] N. Kawazumi, On the stable cohomology algebra of extended mapping class groups for surfaces, in *Groups of Diffeomorphisms*, 383–400, Adv. Stud. Pure Math., 52, Math. Soc. Japan, Tokyo.

[15] E. Looijenga, Stable cohomology of the mapping class group with symplectic coefficients and of the universal Abel-Jacobi map, J. Algebraic Geom. **5** (1996), no. 1, 135–150.

[16] E. Looijenga, Prym representations of mapping class groups, Geom. Dedicata **64** (1997), no. 1, 69–83.

[17] I. Madsen and M. Weiss, The stable moduli space of Riemann surfaces: Mumford's conjecture, Ann. of Math. (2) **165** (2007), no. 3, 843–941.

[18] S. Morita, Families of Jacobian manifolds and characteristic classes of surface bundles. I, Ann. Inst. Fourier (Grenoble) **39** (1989), no. 3, 777–810.

[19] A. Putman, Cutting and pasting in the Torelli group, Geom. Topol. **11** (2007), 829–865.

[20] A. Putman, The second rational homology group of the moduli space of curves with level structures, preprint 2008.

[21] A. Putman, The Picard group of the moduli space of curves with level structures, preprint 2009.

[22] K. Reidemeister, Homotopiegruppen von Komplexen, Abh. Math. Sem. Hamburgischen Univ. **10** (1935), 211–215.

[23] K. Reidemeister, Complexes and homotopy chains, Bull. Amer. Math. Soc. **56** (1950), 297–307.

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