Errata to "The Birch and Swinnerton-Dyer Conjecture for Abelian Varieties over Number Fields"

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1 Lemmas 5.11 and 5.12

I am grateful to Remke Kloosterman from Humboldt University in Berlin and Stefan Keil for pointing out some mistakes in Lemmas 5.11 and 5.12. I paste his comments below.

To produce a counterexample to Lemma 5.12: Suppose $E/\mathbb{Q}$ and $E'/\mathbb{Q}$ are elliptic curves such that there is a cyclic isogeny $\phi : E \to E'$ of prime degree p, $E(\mathbb{Q})$ has odd rank r and both $E(\mathbb{Q})$ and $E'(\mathbb{Q})$ are torsion free. In this case Lemma 5.12 would state that $|\operatorname{coker}(\phi)| = |\operatorname{coker}(\phi^\vee)|$. Now, there is an exact sequence

$$0 \to E'(\mathbb{Q})/\phi(E(\mathbb{Q})) \to E/pE- > E(\mathbb{Q})/\phi^\vee(E'(\mathbb{Q})) \to 0$$

(cf. Remark X.4.7 in Silverman, Arithmetic of elliptic curves) In particular, $|\operatorname{coker}(\phi)||\operatorname{coker}(\phi^\vee)| = p^r$. Since r is odd we have that $|\operatorname{coker}(\phi)| \neq |\operatorname{coker}(\phi^\vee)|$.

The actual mistake in the proof is probably that you assume that there is a basis for the free part of $B(K)$, consisting of elements that are in $\phi(A(K))$. This might not be the case. If you take this into account then the statement of lemma 5.11 and of lemma 5.12 would change by some factor, which cancels in the proof of invariance of BSD under isogeny.