Exercises 4
Monday 6/16/17

13. Recall that $\mathcal{M}_{2k} \cdot \mathcal{M}_{2l} \subset \mathcal{M}_{2k+2l}$ under usual multiplications of functions. This implies that the vector space $\mathcal{M} = \mathbb{C} \oplus \mathcal{M}_2 \oplus \mathcal{M}_4 \oplus \cdots$ has a natural ring structure. Show that
$$\mathcal{M} = \mathbb{C}[E_4, E_6]$$

14. (This exercise is not interesting to do by hand, try your hands at it with Sage when you're up for a little experimentation) Recall the modular forms
$$\begin{aligned}
\Delta &= q - 24q^2 + 252q^3 + \cdots \\
E_4 &= 1 + 240q + 2160q^2 + 6720q^3 + \cdots \\
E_6 &= 1 - 504q - 16632q^2 - 122976q^3 + \cdots \\
E_8 &= 1 + 480q + 61920q^2 + 1050240q^3 + \cdots \\
E_{10} &= 1 - 264q - 135432q^2 - 5196576q^3 + \cdots \\
\frac{691}{65520}E_{12} &= \frac{691}{65520} + q + 2049q^2 + 177148q^3 + \cdots \\
E_4^3 &= 1 + 720q + 179280q^2 + 16954560q^3 + \cdots \\
E_6^2 &= 1 - 1008q + 220752q^2 + 16519104q^3 + \cdots
\end{aligned}$$
Show that
$$\begin{aligned}
\Delta &= \frac{E_4^3 - E_6^2}{1728} \\
E_8 &= E_4^2 \\
E_{10} &= E_4 E_6 \\
\frac{691}{65520}E_{12} - \Delta &= \frac{691}{156}\left(\frac{E_4^3}{720} - \frac{E_6^2}{1008}\right)
\end{aligned}$$
[Hint: To find linear relations you only need the first several coefficients in the q-expansions.]

15. Recall that $\sigma_k(n) = \sum_{d|n} d^k$. Use the previous exercise to show that
$$\sigma_7(n) = \sigma_3(n) + 120\sum_{k=1}^{n-1}\sigma_3(k)\sigma_3(n-k)$$
$$11\sigma_9(n) = 21\sigma_5(n) - 10\sigma_3(n) + 5040\sum_{k=1}^{n-1}\sigma_3(k)\sigma_5(n-k)$$

16. Let $f(z)$ be a modular form of weight $k \geq 2$. Consider the set V_{k-2} the set of homogeneous polynomials $P(X, Y)$ of degree $k-2$ on which the group $\mathrm{SL}(2, \mathbb{Z})$ acts on the left via $g \bullet P(X, Y) = P(aX +$

$bY, cX + dY)$ and on the right via $P(X,Y) \star g = g^{-1} \bullet P(X,Y)$. Let $\omega(z) = f(z)(-X + zY)^{k-2} dz$ be a 1-differential on the upper half plane with values in V_{k-2}.

Show that

$$\omega(g \cdot z) = \omega(z) \star g.$$

This implies that ω yields a holomorphic differential on $X = \mathcal{H}/\operatorname{SL}(2,\mathbb{Z})$ with values in V_{k-2}. When $k = 2$ this yields the differential $f(z)dz$ that I mentioned in lecture. (The correct way to interpret this result is that it interprets modular forms as Betti cohomology classes in H^1 rather than coherent cohomology classes in H^0.)

17. Let $\Gamma \subset \operatorname{SL}(2,\mathbb{Z})$ be a finite index subgroup. One can show that the topological quotient $\mathcal{H}/\Gamma$ is naturally a Riemann surface and that $X_\Gamma = \mathcal{H} \cup \mathbb{P}^1\mathbb{Q}/\Gamma$ is naturally a compact Riemann surface obtained by attaching to $\mathcal{H}/\Gamma$ finitely many points at infinity called cusps. One can define $\mathcal{M}_k(\Gamma)$ analogously to $\mathcal{M}_k$ with two differences: (a) the functional equation $f(g \cdot z) = (cz + d)^k f(z)$ is only required for $g \in \Gamma$ and (b) the function $f(z)$ has to be holomorphic at each of the points at infinity.

    Show that

    $$\dim_{\mathbb{C}} \mathcal{M}_k(\Gamma) \leq \frac{k \operatorname{vol}(X_\Gamma, dxdy/y^2)}{4\pi} + 1 = \frac{k[\operatorname{SL}(2,\mathbb{Z}) : \Gamma]}{12} + 1$$

    [Hint: The same proof works as in the case $\Gamma = \operatorname{SL}(2,\mathbb{Z})$.]

18. Let $r(n)$ be the number of ways to write n as

    $$n = x^2 + y^2 + z^2 + t^2$$

    where $x, y, z, t \in \mathbb{Z}$. The purpose of this exercise is to show that

    $$\boxed{r(n) = 8 \sum_{d \mid n, 4 \nmid d} d}$$

    First, some notation and black boxes. Let $\Gamma = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{SL}(2,\mathbb{Z}) \mid 4 \mid c\}$ and $G_2 = \sum_{(m,n) \neq (0,0)} \frac{1}{(mz+n)^2}$.

    While G_{2k} converges absolutely and is a holomorphic modular form if $k \geq 2$, G_2 converges (not absolutely) with

    $$\mathcal{E}_2 = \frac{G_2}{-8\pi^2} = -\frac{1}{24} + \sum_{n \geq 1} \sigma_1(n) q^n$$

    This holomorphic function is NOT modular[1] because it does not satisfy the functional equations. You may take for granted the following:

    - For each integer t, $\mathcal{E}_{2,t}(z) = \mathcal{E}_2(z) - t\mathcal{E}_2(tz)$ is a modular form in $\mathcal{M}_2(\Gamma)$.
    - Consider the holomorphic function $\theta = \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 z} = \sum_{n \in \mathbb{Z}} q^{n^2}$. Then $\theta^4 \in \mathcal{M}_2(\Gamma)$.
    - $\dim \mathcal{M}_2(\Gamma) = 2$ is generated by $\mathcal{E}_{2,2}$ and $\mathcal{E}_{2,4}$.

    Show that

    (a) $\theta^4 = 8\mathcal{E}_{2,4}$.

    (b) $\theta^4 = \sum_{n \geq 0} r(n) q^n$.

    (c) Conclude the above formula for $r(n)$.

[1] The nonholomorphic function $\mathcal{E}_2(z) + \frac{1}{8\pi \operatorname{Im} z}$ is modular. It is an example of a nearly "holomorphic" modular form. Geometrically holomorphic modular forms correspond to holomorphic differentials and nearly holomorphic modular forms correspond, via Hodge theory, to more general Betti cohomology classes on modular curves.