Exercises 3
Monday 6/15/17

9. Show that if k is odd $\mathcal{M}_k = 0$. [Hint: Write the functional equation for the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.]

10. A modular form $f \in \mathcal{M}_{2k}$ of weight[1] $2k$ with q-expansion

$$f(z) = a_0 + a_1 q + a_2 q^2 + \cdots$$

is said to be a **cusp form** if $a_0 = 0$. Denote by $\mathcal{S}_{2k} \subset \mathcal{M}_{2k}$ the sub-vector space of cusp forms. Show that

$$\mathcal{M}_{2k} = \mathbb{C} \cdot E_{2k} \oplus \mathcal{S}_{2k}.$$

11. (a) Let $2 < 2k < 12$. Show that $\mathcal{S}_{2k} = 0$. [Hint: $E_{2k} \in \mathcal{M}_{2k}$ and you already know a bound on the dimension of $\mathcal{M}_{2k}$.]
    (b) Show that $\mathcal{M}_2 = 0$. [Hint: Are there any everywhere holomorphic differentials on the sphere?]
    (c) Show that $\mathcal{M}_k = 0$ if $k \leq 0$. [Hint: Multiply by Δ.]

12. Consider the modular form $\Delta = q \prod_{n \geq 1} (1 - q^n)^{24}$ of weight 12.
    (a) Show that Δ has no zeros in $\mathcal{H}$.
    (b) Suppose f is a cusp form (see the previous exercise). Show that $\dfrac{f}{\Delta}$ is holomorphic in $\mathcal{H}$ and at ∞.
    (c) Deduce that $\mathcal{M}_k \xrightarrow{\cdot\Delta} \mathcal{S}_{k+12}$ is a bijection.
    (d) Conclude that

$$\dim \mathcal{M}_{2k} = \begin{cases} \lfloor 2k/12 \rfloor + 1 & 2k \not\equiv 2 \pmod{12} \\ \lfloor 2k/12 \rfloor & 2k \equiv 2 \pmod{12} \end{cases}.$$

[1] The previous exercise shows that there are no odd weight modular forms