p-adic Galois Representations
Math 162b Winter 2012
Lecture Notes

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Contents

Sources These lecture notes are mashups of various sources, with some added clarifications where I couldn't follow the argument. Evidently the material presented here is treated by these sources, and in most cases it will be lifted without acknowledgement from the "text-books" for the convenience of exposition.

Lecture 1
2012-01-04

See overview notes.

Lecture 2
2012-01-06

1 Local Class Field Theory

In square brackets I give the section numbers from the math 160b (winter 2012) course notes.

1.1 Local fields

Let K be a field of characteristic 0 with nonarchimedean valuation $v : K^\times \to \mathbb{R}$ (e.g., $\mathbb{Q}_p$ or a finite extension of $\mathbb{Q}_p$, or an algebraic extension of $\mathbb{Q}_p$, etc.) Write $v(0) = \infty$.

Denote by $\mathcal{O}_K = \{x \in K | v(x) \geq 0\}$, and $\mathfrak{m}_K = \{x \in K | v(x) > 0\}$. Then $\mathcal{O}_K^\times = \ker v$ and let $k_K = \mathcal{O}_K/\mathfrak{m}_K$ be the residue field. The ring $\mathcal{O}_K$ is a PID and if v is discrete, i.e., $\operatorname{Im} v \subset \mathbb{R}$ is discrete, then there exists a uniformizer ϖ_K such that $\mathfrak{m}_K = (\varpi_K)$.

The field K has a topology given by the norm $|x|_K = (\#k_K)^{-v(x)}$ (if k_K is not finite, replace it by any real number > 1). K being complete means completeness in this topology.

1.1.1 Hensel's lemma

[Math 160b Winter 2012: §I.2]

Lemma 1.1. *Let K be complete with respect to v, let $P \in \mathcal{O}_K[X]$ be monic and let $\bar{c} \in \mathcal{O}_K$ such that $P(\bar{c}) \equiv 0 \pmod{\mathfrak{m}_K}$ but $P'(\bar{c}) \not\equiv 0 \pmod{\mathfrak{m}_K}$. Then there exists $c \in \mathcal{O}_K$ such that $c \equiv \bar{c} \pmod{\mathfrak{m}_K}$ and $P(c) = 0$.*

Remark 1.

1. The standard application is the existence of a Teichmüller homomorphism $\omega : k_K^\times \to \mathcal{O}_K^\times$ such that $\omega(x) \equiv x \pmod{\mathfrak{m}_K}$.
2. This construction is later generalized by Witt vectors.

1.1.2 Krasner's lemma

[Math 160b Winter 2012: Problem set 2]

Lemma 1.2. *Let K be complete with respect to v, and let $\alpha, \beta \in \overline{K}$. If $v(\beta - \alpha) > v(\sigma(\alpha) - \alpha)$ for all $\sigma \in G_{K(\alpha)/K}$ then $\alpha \in K(\beta)$.*

Remark 2.

1. The standard application is to showing that if two polynomials are sufficiently close p-adically then they have isomorphic splitting fields.
2. This can be use to show that there are finitely many local field extensions of a certain degree.
3. Conceptually, it is the first instance where approximating in the p-adic world does not lead to loss of information.

1.2 Newton polygons

[Math 160b Winter 2012: §I.3]

1.2.1 Definition

Let K be a field with valuation v. For a polynomial $f = \sum_{k=0}^{d} f_k X^k \in K[X]$ the Newton polygon NP_f is the lower convex hull of the points $(i, v(f_i))$ and $(0, \infty)$ and (d, ∞).

Definition 1.3. A slope of f is a slope of a segment of NP_f.

1.2.2 Newton polygons and products

Theorem 1.4. *Let K and v be as before.*

1. *Let $f, g \in K[X]$ such that all slopes of f are less than all slopes of g. Then NP_{fg} is the concatenation of NP_f and NP_g.*
2. *If $(d, v(f_d))$ is a vertex of NP_h where $h \in K[X]$ has degree $n > d > 0$ then there exist polynomials $f, g \in K[X]$ such that $h = fg$ and $\mathrm{NP}_f = \mathrm{NP}_h|_{[0,d]}$ and $\mathrm{NP}_g = \mathrm{NP}_h|_{[d,n]}$.*
3. *If NP_f is pure of slope α, i.e., it consists of a segments of slope α, then all the roots of f have valuation $-\alpha$.*

Remark 3.

1. Used to study ramification of local fields.
2. Useful for finding uniformizers. For example, $\zeta_{p^n} - 1$ can be shown to be a uniformizer of $\mathbb{Q}_p(\zeta_{p^n})$ by analyzing the Newton polygon of its minimal polynomial.
3. Can be generalized to Newton polygons of power series, which we'll use to study log (which then will be used to study the fundamental exact sequence and extensions of p-adic Galois representations).

1.3 Ramification of local fields

[Math 160b Winter 2012: §II.2]

1.3.1 Ramification

If $L/K/\mathbb{Q}_p$ are finite extensions write $f_{L/K} = [k_L : k_K]$ be the inertia index and $e_{L/K} = [v_K(L^\times) : v_K(K^\times)]$ be the ramification index.

Definition 1.5. Say that L/K is

- unramified if $e_{L/K} = 1$;
- totally ramified if $f_{L/K} = 1$;
- tamely ramified if $p \nmid e_{L/K}$;
- wildly ramified if $p \mid e_{L/K}$.

Note that these can be made sense of even for infinite extensions.

Theorem 1.6. *Let $L/K/\mathbb{Q}_p$ be finite extensions.*

1. $f_{L/K} e_{L/K} = [L : K]$.

2. *The field $K^{\mathrm{ur}} = K(\omega(\overline{k_K}^{\times}))$ is the maximal unramified extension of K, and $\overline{K}/K^{\mathrm{ur}}$ is totally ramified with Galois group I_K, the inertia subgroup.*

3. *The field $K^{\mathrm{t}} = K^{\mathrm{ur}}(\varpi_K^{1/n} | p \nmid n)$ is the maximal tamely ramified extension of K, and $\overline{K}/K^{\mathrm{t}}$ is totally wildly ramified with Galois group P_K, the wild inertia subgroup.*

4. *Have an exact sequence $1 \to I_K \to G_K \to G_{k_K} \to 1$ and Frob_K will denote both the topological generator of $G_{K^{\mathrm{ur}}/K} \cong G_{k_K} \cong \mathrm{Frob}_K^{\widehat{\mathbb{Z}}}$ and some lift to G_K, well-defined up to conjugation.*

5. *Writing $I_{L/K} = G_{L/L\cap K^{\mathrm{ur}}}$ and $P_{L/K} = G_{L/L\cap K^{\mathrm{t}}}$ have $1 \to I_{L/K} \to G_{L/K} \to G_{k_L/k_K} \to 1$. Moreover, L/K is unramified if and only if $I_{L/K} = \{1\}$ and L/K is tamely ramified if and only if $P_{L/K} = \{1\}$.*

Example 1.7. $K = \mathbb{Q}_p(\zeta_p)$ is totally ramified over $\mathbb{Q}_p$ because $v_p(\zeta_p - 1) = \frac{1}{p-1} = \frac{1}{[K:\mathbb{Q}_p]}$ so $e_{K/\mathbb{Q}_p} = [K : \mathbb{Q}_p]$ so $f_{K/\mathbb{Q}_p} = 1$.

1.3.2 Ramification filtrations

[Math 160b Winter 2012: §III.1] The subgroups $G_{L/K} \supset I_{L/K} \supset P_{L/K}$ of more and more complex elements of the Galois group fit into a ramification filtration.

Definition 1.8. If L/K is finite for $u \geq -1$ the lower ramification filtration groups are

$$G_{L/K,u} = \{\sigma \in G_{L/K} | v_L(\sigma(x) - x) \geq u + 1, \forall x \in \mathcal{O}_L\}$$

Theorem 1.9. 1. *$G_{L/K,u} = G_{L/K,\lceil u \rceil}$.*

2. *$G_{L/K,-1} = G_{L/K}$.*

3. *$G_{L/K,0} = I_{L/K}$.*

4. *$G_{L/K,1} = P_{L/K}$.*

5. *For $u >> 0$ have $G_{L/K,u} = \{1\}$.*

Definition 1.10. For L/K finite consider $\phi_{L/K} : [-1, \infty) \to [-1, \infty)$ given by

$$\phi_{L/K}(x) = \int_0^x \frac{du}{[G_{L/K,0} : G_{L/K,u}]}$$

which is a piece-wise linear function, of slope 1 on the interval $[-1, 0]$, and slope $1/e_{L/K}$ for $x >> 0$.

Definition 1.11. The upper ramification filtration groups are

$$G^u_{L/K} = G_{L/K,\phi^{-1}_{L/K}(u)}$$

Theorem 1.12 (Herbrand)**.** *Let $L/M/K$ be finite extensions*

1. *$G^u_{M/K} = G^u_{L/K}/(G^u_{L/K} \cap G_{L/M})$.*

2. *$\phi_{L/K} = \phi_{M/K} \circ \phi_{L/M}$.*

Remark 4. Theorem 1.12 allows one to make sense of G^u_K (but not of the lower filtration).

Theorem 1.13 (Hasse-Arf)**.** *If L/K is a finite abelian extension then $G^u_{L/K} = G^{\lfloor u \rfloor}_{L/K}$, i.e., the jumps in the upper filtration are at integers. In other words, the y-coordinates of the vertices of the graph of $\phi_{L/K}$ are integers.*

Example 1.14. If $F = \mathbb{Q}_p$ and $F_\infty = F(\zeta_{p^\infty})$ then F_∞/F is totally ramified, abelian, with Galois group $G_{F_\infty/F} \cong \mathbb{Z}_p^\times$ and $G^n_{F_\infty/F} \cong 1 + p^n\mathbb{Z}_p$.

1.3.3 Different

Definition 1.15. If L/K is a finite extension then

- The inverse different is $\mathcal{D}_{L/K}^{-1} = \{x \in L | \operatorname{Tr}_{L/K}(x\mathcal{O}_L) \subset \mathcal{O}_K\}$ is a fractional ideal of L containing $\mathcal{O}_L$.
- The different is $\mathcal{D}_{L/K}$ is the inverse of $\mathcal{D}_{L/K}^{-1}$, i.e., $\mathcal{D}_{L/K} = \{x \in L | x\mathcal{D}_{L/K}^{-1} \subset \mathcal{O}_L\}$.

Remark 5. The different measures the ramification of local field extensions.

Theorem 1.16. *Let L/K be a finite extension.*

1. $v_L(\mathcal{D}_{L/K}) = \int_{-1}^{\infty} (\#G_{L/K,u} - 1)du.$
2. $v_K(\mathcal{D}_{L/K}) = \int_{-1}^{\infty} \left(1 - \frac{1}{\#G_{L/K}^u}\right) du.$
3. *If I is an ideal of L then* $v_K(\operatorname{Tr}_{L/K}(I)) = \lfloor v_K(I\mathcal{D}_{L/K}) \rfloor$

1.4 Main results of local class field theory

1.4.1 The Weil group

Recall that by Theorem 1.6 $1 \to I_K \to G_K \to G_{k_K} \to 1$ where $G_{k_K} \cong \operatorname{Frob}_K^{\widehat{\mathbb{Z}}}$.

Definition 1.17. The Weil group W_K is the preimage via the projection map of $\operatorname{Frob}_K^{\mathbb{Z}}$, with the topology that makes I_K open and $\operatorname{Frob}_K^{\mathbb{Z}}$ discrete.

1.4.2 The main results

Theorem 1.18. *Let $K/\mathbb{Q}_p$ be a finite extension.*

1. *There exists an injective homomorphism* $\operatorname{rec}_K : K^\times \hookrightarrow G_K^{\text{ab}}$, *such that:*
2. $K^\times \cong W_K^{\text{ab}}$, $\mathcal{O}_K^\times \cong I_K^{\text{ab}}$ *and for* $n \geq 1$, $1 + \mathfrak{m}_K^n \cong G_K^{\text{ab},n}$.
3. *If L/K is finite then* $\operatorname{rec}_L(x) = \operatorname{rec}_K(N_{L/K}(x))$.

Remark 6. 1. This identifies the ramification filtration on G_K^{ab} with the Lie filtration on $K^\times$.

2. This is a general phenomenon, if the Galois group is a p-adic Lie group then the upper filtration and the Lie filtration are "the same".

Definition 1.19. The cyclotomic character $\chi_{\text{cycl}} : G_K \to \mathbb{Z}_p^\times$ is given by the condition that $g(\zeta_{p^n} = \zeta_{p^n}^{\chi_{\text{cycl}}(g)}$. Alternatively, χ_{cycl} can be obtained by lifting $I_K \to I_K^{\text{ab}} \cong \mathcal{O}_K^\times \xrightarrow{N_{K/\mathbb{Q}_p}} \mathbb{Z}_p^\times$ to G_K.

1.5 Galois cohomology

1.5.1 Continuous cohomology

Definition 1.20. Let G be a (pro)finite group and M a topological group with a continuous G-action. Set

$$H^0(G, M) = M^G$$
$$H^1(G, M) = \{f : G \to M \text{ continuous} | f(gh) = f(g)g(f(h))\}/\sim$$

where $f \sim h$ if for some $m \in M$ one has $h(g) = mf(g)g(m)^{-1}$.

Remark 7. If M is abelian then $H^i(G, M) = R^i M^G$ is the right derived functor as usual.

1.5.2 Inflation-restriction sequence

Theorem 1.21. *Let $H \subset G$ be a normal subgroup of a profinite group and let M be a topological group with G action. Then one has an "exact" sequence*

$$1 \to H^1(G/H, M^H) \to H^1(G, M) \to H^1(H, M)^{G/H}$$

where exactness is categorical.

Remark 8. If M is an abelian group this follows from the usual 5-term exact sequence obtained from the Hochschild-Serre spectral sequence.

1.5.3 Examples

Proposition 1.22. *1. If G is procyclic generated by g then*

$$H^0(G, M) = M^g$$
$$H^1(G, M) = M/(g-1)M$$

2. (Hilbert 90) If L/K is finite then

$$H^1(G_{L/K}, L^\times) = 0$$
$$H^1(G_{L/K}, L) = 0$$
$$H^1(G_{L/K}, \mathrm{GL}(n, L)) = 0$$
$$H^1(G_{L/K}, \mathrm{M}_{n\times n}(L)) = 0$$

Lecture 3
2012-01-09

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2 $\mathbb{C}_p$-representations

2.1 The field $\mathbb{C}_p$

Definition 2.1. For a p-adic field K let $\mathbb{C}_K = \widehat{\overline{K}}$. If $K \subset \overline{\mathbb{Q}}_p$ write $\mathbb{C}_K = \mathbb{C}_p$.

Proposition 2.2. *1. $\mathbb{C}_p \neq \overline{\mathbb{Q}}_p$, i.e., $\overline{\mathbb{Q}}_p$ is not complete.*

2. $\mathbb{C}_p$ is algebraically closed.

Proof. 1. See problem set 2.

Choose a_n roots of unity, such that $a_n \in \mathbb{Q}_p^{\mathrm{ur}}$, $a_{n-1} \in \mathbb{Q}_p(a_n)$ and $[\mathbb{Q}_p(a_n) : \mathbb{Q}_p(a_{n-1})] > n$. For example could take $a_n = \zeta_{q^{(n!)^2}}$ where $q \neq p$ is a prime. Let $\alpha = \sum_{n=1}^{\infty} a_n p^n \in \mathbb{C}_p$ and assume that $\alpha \in \overline{\mathbb{Q}}_p$. Let $m = [\mathbb{Q}_p(\alpha) : \mathbb{Q}_p]$ and let $\alpha_m = \sum_{n=1}^{m} a_n p^n$. Choose a Galois extension $M/\mathbb{Q}_p$ containing α, α_m and a_m.

Since $[M : \mathbb{Q}_p(a_{m-1})] \geq [\mathbb{Q}_p(a_m) : \mathbb{Q}_p(a_{m-1})] > m$ one may find $\sigma_1, \ldots, \sigma_{m+1} \in G_{M/\mathbb{Q}_p(a_{m-1})}$ such that $\sigma_i(a_m)$ are all distinct.

Clearly $v_p(\alpha - \alpha_m) \geq m+1$ and thus for all i one has $v_p(\sigma_i(\alpha) - \sigma_i(\alpha_m)) \geq m+1$. Also, for $i \neq j$ we have $v_p(\sigma_i(\alpha_m) - \sigma_j(\alpha_m)) = v_p(\sigma(a_m) - \sigma_j(a_m)) + m$. Since a_m is the root of a polynomial which is separable mod p, it follows that $\sigma_i(a_m) \not\equiv \sigma_j(a_m) \pmod p$ and so $v_p(\sigma(a_m) - \sigma_j(a_m)) = 0$.

Putting things together we get

$$v_p(\sigma_i(\alpha) - \sigma_j(\alpha)) = v_p(\sigma_i(\alpha) - \sigma_i(\alpha_m) + \sigma_i(\alpha_m) - \sigma_j(\alpha_m) + \sigma_j(\alpha_m) - \sigma_j(\alpha))$$

and in the latter $v_p(\sigma_i(\alpha)-\sigma_i(\alpha_m)), v_p(\sigma_j(\alpha_m)-\sigma_j(\alpha)) \geq m+1$ but $v_p(\sigma_i(\alpha_m)-\sigma_j(\alpha_m)) = m$ and so $v_p(\sigma_i(\alpha) - \sigma_j(\alpha)) = m$ and so $\sigma_i(\alpha) \neq \sigma_j(\alpha)$. But then α has $m+1$ distinct conjugates, contradicting that α has degree m over $\mathbb{Q}_p$.

2. Let $\alpha \in \overline{\mathbb{C}_p}$ and WLOG $v(\alpha) \geq 0$. Let $f = X^n + a_{n-1}X^{n-1} + \cdots + a_0 \in \mathbb{C}_p[X]$ be its minimal polynomial. Let $c = \max v_p(\alpha_i - \alpha)$ where $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_n$ are the roots of f. Choose $g \in \overline{\mathbb{Z}}_p[X]$ an approximation of f such that if $g(X) = X^n + b_{n-1}X^{n-1} + \cdots + b_0$ then for all i we have $v_p(a_i - b_i) > nc$, which is always possible as $\mathbb{C}_p$ is the completion of $\overline{\mathbb{Q}_p}$. Let $\beta_1, \ldots, \beta_n$ be the roots of g.

   Then since $v(\alpha) \geq 0$ we have $\sum v_p(\alpha - \beta_i) = v_p(g(\alpha)) = v_p(g(\alpha) - f(\alpha)) \geq \min v_p(a_i - b_i) > nc$. Thus for some i, $v_p(\alpha - \beta_i) > c$. By Krasner's lemma 1.2 it follows that $\alpha \in \mathbb{C}_p(\beta_i) = \mathbb{C}_p$.

□

2.2 Ax-Sen-Tate and Galois invariants

Definition 2.3. Let G be a profinite group and R a topological ring with an action of G. Then $\mathrm{Rep}_R(G)$ will consist of finite free R-modules M with semilinear actions of G, i.e., an action of $g \in G$ on M such that if $\alpha \in R$ and $m \in M$ then $g(\alpha m) = g(\alpha)g(m)$.

Definition 2.4. If R is a topological ring with an action of G_K (say $R = \mathbb{Q}_p$, or $R = \overline{\mathbb{Q}_p}$, or $R = \mathbb{C}_p$) and $\eta : G_K \to R^\times$ is a character, let $R(\eta) \in \mathrm{Rep}_R(G_K)$ be the one dimensional representation with basis e_η described by $g(\alpha e_\eta) := g(\alpha)\eta(g)e_\eta$, for $\alpha \in R$.

Write $\mathbb{Q}_p(n) = \mathbb{Q}_p(\chi^n_{\mathrm{cycl}})$ and $\mathbb{C}_p(n) = \mathbb{C}_p(\chi^n_{\mathrm{cycl}})$.

The goal of the next few sections is to study $H^0(G_K, \mathbb{C}_p(\eta))$ and $H^1(G_K, \mathbb{C}_p(\eta))$ for certain η including χ^n_{cycl}.

2.2.1 A lemma on roots of polynomials

Lemma 2.5. *Let $f \in \overline{\mathbb{Q}_p}[X]$ be monic of degree n such that all roots have valuation $\geq u$.*

1. *If $n = p^k n_o$ with $p \nmid n_0$ then $f^{(p^k)}$ has a root β with $v(\beta) \geq u$.*

2. *If $n = p^{k+1}$ then $f^{(p^k)}$ has a root β with $v(\beta) \geq u - \dfrac{v(p)}{p^k(p-1)}$.*

Proof. Let $f(X) = X^n + a_{n-1}X^{n-1} + \cdots a_0$. By Theorem 1.4 all slopes of NP_f are $\leq -u$ so $a_{n-i} \geq iu$. Write $q = p^k$, we have

$$\frac{f^{(q)}}{q!} = \sum_{i=0}^{n-q} \binom{n-i}{q} a_{n-i} X^{n-i-q}$$

whose roots have product $\prod \beta = \pm a_q / \binom{n}{q}$. Therefore $\sum v(\beta) = v(a_q) - v\left(\binom{n}{q}\right)$ so there exists a root β such that

$$\begin{aligned} v(\beta) &\geq \frac{v(a_q)}{n-q} - \frac{1}{n-q} v\left(\binom{n}{q}\right) \\ &\geq u - -\frac{1}{n-q} v\left(\binom{n}{q}\right) \end{aligned}$$

But from problem set 1 one has that

$$v\left(\binom{n}{q}\right) = \begin{cases} 0 & n = p^k n_0 \\ v(p) & n = p^{k+1} \end{cases}$$

and the conclusion follows. □

2.2.2 Approximations of algebraic numbers

If $\alpha \in \overline{K}$ write $\Delta_K(\alpha) = \min v(\sigma(\alpha) - \alpha)$ where $\sigma \in G_{K(\alpha)/K}$. While Krasner's lemma says that if an algebraic element is very close to an element then it lies in the field generated by that element, the following lemma will say that no matter what finite field extension one chooses one can find a sufficiently good approximation in that field to any given algebraic element.

Lemma 2.6. *Let $K/\mathbb{Q}_p$ be a finite extension and let $\alpha \in \overline{K}$. Then there exists $\beta \in K$ such that*

$$v(\alpha - \beta) \geq \Delta_K(\alpha) - \frac{v(p)}{(p-1)^2}$$

Proof. In fact we'll show that one may find β such that

$$v(\alpha - \beta) \geq \Delta_K(\alpha) - \sum_{i=1}^{\lfloor \log_p n \rfloor} \frac{v(p)}{p^{i-1}(p-1)}$$

which implies the lemma.

Let $Q(X)$ be the minimal polynomial of α over K. We'll show by induction over $\deg Q$. The base case, when $\deg Q = 1$ is immediate as then one can take $\beta = \alpha$.

Now for the inductive step. Let $P(X) = Q(X + \alpha)$ which has roots $\sigma(\alpha) - \alpha$ for $\alpha \in G_{K(\alpha)/K}$. By definition, all the roots of P have valuation $\geq \Delta_K(\alpha)$. Let $n = \deg Q$ and let $n = p^k n_0$ or $n = p^{k+1}$ with $q = p^k$ as in Lemma 2.5. Thus there exists a root $\widetilde{\beta}$ of $P^{(q)}$ such that

$$v(\widetilde{\beta}) \geq \begin{cases} \Delta_K(\alpha) & n = p^k n_0 \\ \Delta_K(\alpha) - \frac{v(p)}{p^k(p-1)} & n = p^{k+1} \end{cases}$$

Let $\beta = \widetilde{\beta} + \alpha$ be a root of $Q^{(q)}$ such that

$$v(\beta - \alpha) \geq \begin{cases} \Delta_K(\alpha) & n = p^k n_0 \\ \Delta_K(\alpha) - \frac{v(p)}{p^k(p-1)} & n = p^{k+1} \end{cases}$$

Note that

$$\begin{aligned} v(\sigma(\beta) - \beta) &= v(\sigma(\beta) - \sigma(\alpha) + \sigma(\alpha) - \alpha + \alpha - \beta) \\ &\geq \min v(\sigma(\beta) - \sigma(\alpha)), v(\sigma(\alpha) - \alpha), v(\alpha - \beta) \\ &\geq \min \Delta_K(\alpha), v(\alpha - \beta) \\ &\geq \begin{cases} \Delta_K(\alpha) & n = p^k n_0 \\ \Delta_K(\alpha) - \frac{v(p)}{p^k(p-1)} & n = p^{k+1} \end{cases} \end{aligned}$$

as $v(\sigma(\beta) - \sigma(\alpha)) = v(\alpha - \beta)$.

By the inductive hypothesis applied to $Q^{(q)}$ of degree $n - q$ one may find $\gamma \in K$ such that

$$v(\beta - \gamma) \geq \Delta_K(\beta) - \sum_{i=1}^{\lfloor \log_p (n-q) \rfloor} \frac{v(p)}{p^{i-1}(p-1)}$$

Then if $n = p^k n_0$ one has $\lfloor \log_p(n-q) \rfloor = \lfloor \log_p n \rfloor$ so

$$v(\beta - \gamma) \geq \Delta_K(\alpha) - \sum_{i=1}^{\lfloor \log_p n \rfloor} \frac{v(p)}{p^{i-1}(p-1)}$$

and if $n = p^{k+1}$ then $\lfloor \log_p(n-q) \rfloor = k$ while $\lfloor \log_p n \rfloor = k+1$. Therefore

$$\begin{aligned} v(\beta - \gamma) &\geq \Delta_K(\alpha) - \frac{v(p)}{p^k(p-1)} - \sum_{i=1}^{k} \frac{v(p)}{p^{i-1}(p-1)} \\ &= \Delta_K(\alpha) - \sum_{i=1}^{k+1} \frac{v(p)}{p^{i-1}(p-1)} \\ &= \Delta_K(\alpha) - \sum_{i=1}^{\lfloor \log_p n \rfloor} \frac{v(p)}{p^{i-1}(p-1)} \end{aligned}$$

and the inductive step follows. ☐

2.2.3 Galois invariants: the Ax-Sen-Tate lemma

Theorem 2.7. *Let L/K be an algebraic extension. Then $\mathbb{C}_p^{G_L} = \widehat{L}$. In particular, if L/K is finite then $\mathbb{C}_p^{G_L} = L$.*

Proof. Let v be a valuation on L and let $x \in \mathbb{C}_p^{G_L}$. Choose $\alpha_n \in \overline{\mathbb{Q}_p}$ such that $x = \lim_{n\to\infty} \alpha_n$.

For $\sigma \in G_L$ have

$$\begin{aligned} v(\sigma(\alpha_n) - \alpha_n) &= v(\sigma(\alpha_n - x) - (\alpha_n - x)) \\ &\geq \min v(\sigma(\alpha_n - x)), v(\alpha_n - x) \\ &= v(\alpha_n - x) \end{aligned}$$

and therefore $\Delta_L(\alpha_n) \geq v(\alpha_n - x)$.

By Lemma 2.6 it follows that one may find $\beta_n \in L$ such that $v(\beta_n - \alpha_n) \geq \Delta_L(\alpha_n) - \frac{v(p)}{(p-1)^2}$. But then

$$\begin{aligned} v(x - \beta_n) &= v(x - \alpha_n + \alpha_n - \beta_n) \\ &\geq \min v(x - \alpha_n), v(\alpha_n - \beta_n) \\ &\geq \min v(x - \alpha_n), \Delta_L(\alpha_n) - \frac{v(p)}{(p-1)^2} \\ &\geq v(x - \alpha_n) - \frac{v(p)}{(p-1)^2} \end{aligned}$$

which goes to infinity so $x = \lim_{n\to\infty} \beta_n \in \widehat{L}$.

Conversely, if $x \in \widehat{L}$ such that $x = \lim \beta_n$ then for $g \in G_L$ one has $g(\beta_n) = \beta_n$. Since G_L acts continuously it follows that $g(x) = g(\lim \beta_n) = \lim g(\beta_n) = \lim \beta_n = x$ so $x \in \mathbb{C}_p^{G_L}$. ☐

Lecture 4
2012-01-11

2.3 Ramification estimates and Tate periods

The goal of this section is the study of $H^0(G_K, \mathbb{C}_p(n))$ and $H^1(G_K, \mathbb{C}_p(n))$. The idea is to first restrict to $\ker \chi_{\text{cycl}} = G_{K_\infty}$ where $K_\infty = K(\zeta_{p^\infty})$ is the p^∞ cyclotomic extension. Then $\mathbb{C}_p(n)^{G_{K_\infty}} = \widehat{K_\infty}(n)$ by the Ax-Sen-Tate Theorem 2.7. One first studies the ramification of K_∞ and then approximates $\widehat{K_\infty}$ by finite extensions.

2.3.1 Cyclotomic extensions

Let $F = \mathbb{Q}_p$ and K/F be a finite extension. Write $K_n = K(\zeta_{p^n})$, $H_K = G_{K_\infty}$, $\Gamma_K = G_{K_\infty/K}$, $\Gamma_{K_n} = G_{K_\infty/K_n}$ and similarly for F; we have the following diagram of Galois groups

$$\begin{tikzcd}
\overline{K} \arrow[r, dash] \arrow[d, dash, "H_K"'] & \overline{F} \arrow[d, dash, "H_F"] \\
K_\infty \arrow[r, dash] \arrow[d, dash, "\Gamma_{K_n}"'] & F_\infty \arrow[d, dash, "1+p^n\mathbb{Z}_p"] \\
K_n \arrow[r, dash] \arrow[d, dash] & F_n \arrow[d, dash, "(\mathbb{Z}/p^n)^\times"] \\
K \arrow[r, dash] \arrow[uu, dash, bend left=60, "\Gamma_K"] & F \arrow[uu, dash, bend right=60, "\mathbb{Z}_p^\times"']
\end{tikzcd}$$

Lemma 2.8. *1. The cyclotomic character factors through* $\chi_{\mathrm{cycl}} : \Gamma_K \hookrightarrow \Gamma_F \cong \mathbb{Z}_p^\times$.

2. *There exists an integer* n_K *such that* $\chi_{\mathrm{cycl}}(\Gamma_K) \supset 1 + p^{n_K}\mathbb{Z}_p$.

3. *For* $n \geq n_K$ *one has* $\chi_{\mathrm{cycl}}(\Gamma_{K_n}) \cong 1 + p^n\mathbb{Z}_p$ *and* $K_n \cap F_\infty = F_n$.

Proof. 1. The map $\Gamma_K \to \Gamma_F$ given by restriction to F_∞ is injective; otherwise, let $g \in \Gamma_K$ nontrivial. Then there exists n large enough such that $g \notin \Gamma_{K_n}$ (since Γ_{K_n} form a basis around 1 in Γ_K) and so g does not fix ζ_{p^n}. But then g cannot be trivial in Γ_F. That χ_{cycl} factors through this map follows since χ_{cycl} is trivial on $H_K = G_{K_\infty}$ and $\Gamma_K \cong G_K/H_K$.

2. Since χ_{cycl} is continuous it follows that $\chi_{\mathrm{cycl}}(\Gamma_K)$ is compact. The logarithm $\log : 1 + p^2\mathbb{Z}_p \to p^2\mathbb{Z}_p$ is a continuous homomorphism so again by continuity it follows that $\log(\chi_{\mathrm{cycl}}(\Gamma_K) \cap (1 + p^2\mathbb{Z}_p))$ is a closed subgroup of $\mathbb{Z}_p$. But then it must also be open. Since log is continuous this implies that $\chi_{\mathrm{cycl}}(\Gamma_K) \cap (1 + p^2\mathbb{Z}_p))$ is open and so $\chi_{\mathrm{cycl}}(\Gamma_K)$ contains some $1 + p^{n_K}\mathbb{Z}_p$.

3. For $n \geq n_K$ we have $\chi_{\mathrm{cycl}}(\Gamma_{F_n}) = 1 + p^n\mathbb{Z}_p \subset \chi_{\mathrm{cycl}}(\Gamma_K)$ as above.

   The injection $\Gamma_K \hookrightarrow \Gamma_F$ given an injection $\Gamma_{K_n} \hookrightarrow \Gamma_{F_n}$. But Γ_K also surjects onto Γ_{F_n} by the above. If $g \in \Gamma_K$ maps to an element of Γ_{F_n} it must fix ζ_{p^n} so $g \in \Gamma_{K_n}$ and so Γ_{K_n} surjects onto Γ_{F_n} and so the two groups are isomorphic. This implies that $\chi_{\mathrm{cycl}}(\Gamma_{K_n}) \cong 1 + p^n\mathbb{Z}_p$.

   Now $K_n \cap F_\infty = F_\infty^{\Gamma_{K_n}} = F_\infty^{\Gamma_{F_n}} = F_n$.

□

2.3.2 Ramification in cyclotomic extensions

Having compared the Galois groups of the cyclotomic extensions of K and F we now proceed to compare their upper ramification filtrations.

Lemma 2.9. *Let* $F = \mathbb{Q}_p$ *and* K/F *finite.*

1. *For* $n \geq n_K$ *the extension* K_{n+1}/K_n *is totally ramified of degree* p.

2. $[K_n : F_n]$ *is decreasing and* $G_{K_n/F_n} = G_{K_\infty/F_\infty}$ *for* n *large enough.*

3. *There exists* u_K *such that if* $n \geq n_K$ *and* $u \geq u_K$ *then* $G^u_{K_n/F_{n_K}} \cong G^u_{F_n/F_{n_K}}$.

Proof. 1. For $n \geq n_K$ we have by Lemma 2.8

$$\begin{aligned} G_{K_{n+1}/K_n} &\cong G_{F_{n+1}/F_n} \\ &= I_{F_{n+1}/F_n} \\ &= \{g \in G_{F_{n+1}/F_n} | v(gx - x) > 0, \forall v(x) \geq 0\} \\ &= \{g \in G_{K_{n+1}/K_n} | v(gx - x) > 0, \forall v(x) \geq 0\} \\ &= I_{K_{n+1}/K_n} \end{aligned}$$

where the second line follows because F_{n+1}/F_n is totally ramified of degree p. Thus K_{n+1}/K_n is totally ramified, and the degree is p.

2. $[K_n : F_n] = [KF_n : FF_n]$ decreases and stabilizes.

3. Let u_K such that $G^{u_K}_{K_{n_K}/F_{n_K}} = \{1\}$ (Theorem 1.9). Also recall by Lemma 2.8 for $n \geq n_K$ have $K_n \cap F_\infty = F_n$ so $G_{K_n/F_{n_K}} \cong G_{F_n/F_{n_K}} \times G_{K_{n_K}/F_{n_K}}$.

   By Herbrand's Theorem 1.12 it follows that for $u \geq u_K$ we have $G^u_{K_{n_K}/F_{n_K}} \cong G^u_{K_n/F_{n_K}}/(G^u_{K_n/F_{n_K}} \cap G_{K_n/K_{n_K}})$ and since the former is trivial it must be that $G^u_{K_n/F_{n_K}} \hookrightarrow G_{K_n/K_{n_K}} \cong G_{F_n/F_{n_K}}$. But another application of Herbrand gives that $G^u_{F_n/F_{n_K}} \cong G^u_{K_n/F_{n_K}}/(G^u_{K_n/F_{n_K}} \cap G_{K_n/F_n})$ so $G^u_{K_n/F_{n_K}}$ surjects onto $G^u_{F_n/F_{n_K}} \subset G_{F_n/F_{n_K}}$. Therefore the conclusion follows, having already shown injection.

□

Lemma 2.10. *1. The sequence $\{p^n v_p(\mathcal{D}_{K_n/F_n})\}$ is bounded.*

2. There exist a constant c and a bounded sequence a_n such that

$$v_p(\mathcal{D}_{K_n/F}) = n + c + \frac{a_n}{p^n}$$

Proof. 1. We may assume that $n \geq n_K$. Then

$$\begin{aligned} v_p(\mathcal{D}_{K_n/F_n}) &= v_p(\mathcal{D}_{K_n/F_{n_K}} - v_p(\mathcal{D}_{F_n/F_{n_K}} \\ &= \frac{1}{e_{F_{n_K}/F}} \int_{-1}^{\infty} \left(\frac{1}{\#G^u_{F_n/F_{n_K}}} - \frac{1}{\#G^u_{K_n/F_{n_K}}} \right) \\ &= \frac{1}{e_{F_{n_K}/F}} \int_{-1}^{u_K} \left(\frac{1}{\#G^u_{F_n/F_{n_K}}} - \frac{1}{\#G^u_{K_n/F_{n_K}}} \right) \\ &\leq \frac{1}{e_{F_{n_K}/F}} \int_{-1}^{u_K} \frac{1}{\#G^u_{F_n/F_{n_K}}} \end{aligned}$$

where the second line follows from Theorem 1.16 and the third from Lemma 2.9.

Now we have that $G_{F_n/F_{n_K},v} = G_{F_n/F,v} \cap G_{F_n/F_{n_K}}$ and $G_{F_n/F,v} = G^{\phi_{F_n/F}(v)}_{F_n/F} = G_{F_n/F_{\lfloor \phi_{F_n/F}(v) \rfloor}}$. From Theorem 1.12 we get that (using that $G^u_{F_n/F} = G_{F_n/F_{\lfloor u \rfloor}}$)

$$\begin{aligned} G^u_{F_n/F_{n_K}} &= G_{F_n/F_{n_K}, \phi^{-1}_{F_n/F_{n_K}}(u)} \\ &= G_{F_n/F_{\lfloor \phi_{F_n/F} \circ \phi^{-1}_{F_n/F_{n_K}}(u) \rfloor}} \cap G_{F_n/F_{n_K}} \\ &= G_{F_n/F_{\lfloor \phi_{F_{n_K}/F}(u) \rfloor}} \cap G_{F_n/F_{n_K}} \\ &= G_{F_n/F_{\max(\lfloor \phi_{F_{n_K}/F}(u) \rfloor, n_K)}} \end{aligned}$$

so

$$\#G^u_{F_n/F_{n_K}} = p^{n-\max(\lfloor \phi_{F_{n_K}/F}(u)\rfloor, n_K)}$$

and thus

$$\begin{aligned}
v_p(\mathcal{D}_{K_n/F_n}) &\le \frac{1}{e_{F_{n_K}/F}} \int_{-1}^{u_K} p^{\max(\lfloor \phi_{F_{n_K}/F}(u)\rfloor, n_K)-n} du \\
p^n v_p(\mathcal{D}_{K_n/F_n}) &\le \frac{1}{e_{F_{n_K}/F}} \int_{-1}^{u_K} p^{\max(\lfloor \phi_{F_{n_K}/F}(u)\rfloor, n_K)} du
\end{aligned}$$

and the right hand side is independent of n.

2. See problem set 2.

   We compute

$$\begin{aligned}
v_p(\mathcal{D}_{F_n/F}) &= \int_0^\infty \left(1 - \frac{1}{\#G^u_{F_n/F}}\right) \\
&= \sum_{i=0}^{n} \left(1 - \frac{1}{\#G_{F_n/F_i}}\right) \\
&= \sum_{i=0}^{n} \left(1 - p^{i-n}\right) \\
&= n + 1 - \frac{p}{p-1}\left(1 - \frac{1}{p^{n+1}}\right)
\end{aligned}$$

   and the result follows from the fact that $v_p(\mathcal{D}_{K_n/F}) = v_p(\mathcal{D}_{K_n/F_n}) + v_p(\mathcal{D}_{F_n/F})$. □

Lecture 5

2012-01-18

2.3.3 Almost etaleness

Theorem 2.11. *Let $L/K/\mathbb{Q}_p$ be finite extensions. Then $\mathrm{Tr}_{L_\infty/K_\infty}(\mathfrak{m}_{L_\infty}) \supset \mathfrak{m}_{K_\infty}$.*

Proof. For $m \ge n \ge \max(n_K, n_L)$ we know that $G_{L_m/K_m} \cong G_{L_n/K_n}$ so

$$\mathrm{Tr}_{L_\infty/K_\infty}(\mathfrak{m}_{L_n}) = \mathrm{Tr}_{L_n/K_n}(\mathfrak{m}_{L_n}) = \mathfrak{m}_{K_n}^{c_n}$$

where the exponent c_n can be computed using Theorem 1.16 as

$$\begin{aligned}
c_n &= \lfloor v_{K_n}(\mathfrak{m}_{L_n}\mathcal{D}_{L_n/K_n})\rfloor \\
&= \lfloor v_{K_n}(\mathfrak{m}_{L_n}) + e_{K_n/F} v_p(\mathcal{D}_{L_n/K_n})\rfloor \\
&= \lfloor e_{L_n/K_n} + e_{K_n/F_n} e_{F_n/F}(v_p(\mathcal{D}_{L_n/F}) - v_p(\mathcal{D}_{K_n/F}))\rfloor
\end{aligned}$$

Now $e_{L_n/K_n} \le [L_n : K_n] \le [L : K]$ and $e_{K_n/F_n} \le [K_n : F_n] \le [K : F]$ and $e_{F_n/F} = p^{n-1}(p-1)$ since F_n/F is totally ramified. It now follows from Lemma 2.10 that c_n is bounded by some constant c. Thus $\mathfrak{m}_{K_n}^c \subset \mathrm{Tr}_{L_\infty/K_\infty}(\mathfrak{m}_{L_\infty})$ for all n.

Let $x \in \mathfrak{m}_{K_\infty}$ and let $x \in \mathfrak{m}_{K_m}$ for some m. Since e_{K_n/K_m} is unbounded, for $n >> 0$ one has $e_{K_n/K_m} > c$ so $x \in \mathfrak{m}_{K_m} \subset \mathfrak{m}_{K_n}^c \subset \mathrm{Tr}_{L_\infty/K_\infty}(\mathfrak{m}_{L_\infty})$. □

Corollary 2.12. *Let $K/\mathbb{Q}_p$ be a finite extension.*

1. *Every finite extension of K_∞ is of the form $L_\infty = LK_\infty$ for a finite extension L/K.*
2. *If L/K is finite there exists $\alpha \in L_\infty$ such that $\mathrm{Tr}_{L_\infty/K_\infty}(\alpha) = 1$ and $v(\alpha) > -v(\varpi_K)$ where ϖ_K is a uniformizer for K.*

Proof. 1. See problem set 2.

2. From Theorem 2.11 there exists $\widetilde{\alpha}$ such that $v(\widetilde{\alpha}) > 0$ and $\mathrm{Tr}_{L_\infty/K_\infty}(\widetilde{\alpha}) = \varpi_K$. Let $\alpha = \widetilde{\alpha}/\varpi_K$ in which case $\mathrm{Tr}_{L_\infty/K_\infty}(\alpha) = 1$ and $v(\alpha) > -v(\varpi_K)$.

□

2.3.4 Normalized Traces

Definition 2.13. For $n \geq n_K$ and $x \in K_{n+k}$ define $\mathrm{pr}_n(x) = p^{-k}\,\mathrm{Tr}_{K_{n+k}/K_n}(x)$ which is independent of k since K_{n+k+1}/K_{n+k} is cyclic of degree p by Lemma 2.9.

Lemma 2.14. *Let $n \geq n_K$ and $x \in K_{n+k}$. Then*

$$v_p(\mathrm{pr}_n(x)) \geq v_p(x) - \frac{\alpha_n}{p^n}$$

where α_n is a bounded sequence.

Proof. Using Theorem 1.16 in the fourth line and Lemma 2.10 in the seventh line one has

$$\begin{aligned}
v_p(\mathrm{pr}_n(x)) &= -k + v_p(\mathrm{Tr}_{K_{n+k}/K_n}(x)) \\
&= -k + v_p(\mathrm{Tr}_{K_{n+k}/K_n}(\mathfrak{m}_{K_{n+k}}^{v_{K_{n+k}}(x)})) \\
&= -k + e_{K_n/F}^{-1}\lfloor v_{K_n}(\mathfrak{m}_{K_{n+k}}^{v_{K_{n+k}}(x)}\mathcal{D}_{K_{n+k}/K_n})\rfloor \\
&> -k + e_{K_n/F}^{-1}(v_{K_n}(\mathfrak{m}_{K_{n+k}}^{v_{K_{n+k}}(x)}) + v_{K_n}(\mathcal{D}_{K_{n+k}/K_n}) - 1) \\
&= -k + e_{K_n/F}^{-1}(v_{K_n}(x) + e_{K_n/F}v_p(\mathcal{D}_{K_{n+k}/K_n}) - 1) \\
&= -k + v_p(x) + \big(v_p(\mathcal{D}_{K_{n+k}/F}) - v_p(\mathcal{D}_{K_n/F})\big) - e_{K_n/F}^{-1} \\
&= v_p(x) - k + n + k + c + \frac{a_{n+k}}{p^{n+k}} - n - c - \frac{a_n}{p^n} - \frac{1}{e_{K_n/F_n}e_{F_n/F}} \\
&= v_p(x) - \frac{\alpha_n}{p^n}
\end{aligned}$$

where

$$\begin{aligned}
\alpha_n &= a_n - \frac{a_{n+k}}{p^k} + \frac{p^n}{e_{K_n/F_n}e_{F_n/F}} \\
&= a_n - \frac{a_{n+k}}{p^k} + \frac{p}{e_{K_n/F_n}(p-1)}
\end{aligned}$$

which is bounded since a_n is bounded and $e_{K_n/F_n} \leq [K_n : F_n]$ stabilizes to $[K_\infty : F_\infty]$.

□

Corollary 2.15. *For $n \geq n_K$ the function pr_n is uniformly continuous on K_∞ and thus extends to a continuous function $\mathrm{pr}_n : \widehat{K_\infty} \to K_n$.*

Proof. Lemma 2.14 implies

$$|\,\mathrm{pr}_n(x) - \mathrm{pr}_n(y)| = |\,\mathrm{pr}_n(x-y)| \leq |x-y|p^{\alpha_n p^{-n}} < C|x-y|$$

where C is some constant. □

Remark 9. Write $K_n^{\perp} = \{x \in \widehat{K_\infty} | \operatorname{pr}_n(x) = 0\}$. Since pr_n is an idempotent we get a decomposition $\widehat{K_\infty} = K_n \oplus K_n^{\perp}$.

Proposition 2.16. *For $n \geq n_K$ and $x \in \widehat{K_\infty}$:*

1. $v_p(\operatorname{pr}_n(x)) \geq v_p(x) - \alpha_n p^{-n}$;
2. $x = \lim_{n\to\infty} \operatorname{pr}_n(x)$;
3. pr_n *commutes with the action of* $\Gamma_K = G_{K_\infty/K}$.

Proof. 1. The function pr_n is continuous on $\widehat{K_\infty}$ and the inequality follows from Lemma 2.14.

2. Fix n. Since $x \in \widehat{K_\infty}$ for every $C > 0$ we may choose m and $x_{n+m} \in K_{n+m}$ such that $v_p(x - x_{n+m}) > C$. Since $x_{n+m} = \operatorname{pr}_{n+m+j}(x_{n+m})$ it follows that

$$\begin{aligned} v_p(x - \operatorname{pr}_{n+m}(x)) &= v_p(x - x_{n+m} + \operatorname{pr}_{n+m}(x_{n+m}) - \operatorname{pr}_{n+m}(x)) \\ &\geq \min v_p(x - x_{n+m}), v_p(\operatorname{pr}_{n+m}(x - x_{n+m})) \\ &\geq \min C, C - \alpha_{n+m} p^{-(n+m)} \\ &= C - \alpha_{n+m} p^{-(n+m)} \\ &> C - \alpha_{n+m} p^{-n} \end{aligned}$$

Since $\alpha_{n+m} p^{-n}$ is bounded as $m \to \infty$, making $C \to \infty$ gives that $x = \lim_{m\to\infty} \operatorname{pr}_{n+m}(x)$.

3. Let γ be a topological generator of Γ_K, which is procyclic. Then for $n + k > n \geq n_K$ the group G_{K_{n+k}/K_n} is cyclic generated by some power γ^s. Thus

$$\gamma \operatorname{pr}_n(x) = p^{-k} \gamma \sum_i (\gamma^s)^i(x) = p^{-k} \sum_i (\gamma^s)^i(\gamma(x)) = \operatorname{pr}_n(\gamma(x))$$

□

2.3.5 Tate periods: degree 0

Lemma 2.17. *Let $\eta : G_K \to \mathbb{Z}_p^\times$ be a finite order character. Then $\mathbb{C}_p(\eta) \cong \mathbb{C}_p$ as G_K-modules.*

Proof. Let L/K be finite such that $\eta(G_L) = 1$. Then η factors through $G_{L/K}$ and $\eta \in \operatorname{Hom}(G_{L/K}, \mathbb{Z}_p^\times) = H^1(G_{L/K}, \mathbb{Z}_p^\times)$ where $\mathbb{Z}_p^\times$ has trivial Galois action. But $H^1(G_{L/K}, \mathbb{Z}_p^\times) \to H^1(G_{L/K}, L^\times) = 0$ by Hilbert 90 so there exists $\xi \in L^\times$ such that $\eta(g) = \xi^{-1} g(\xi)$.

Then $\mathbb{C}_p(\eta) \to \mathbb{C}_p$ given by $\alpha e_\eta \mapsto \xi\alpha$ is a G_K equivariant isomorphism. □

Theorem 2.18. *Let $K/\mathbb{Q}_p$ be a finite extension and let $\eta : G_K \to \mathbb{Z}_p^\times$ be a continuous character such that $\eta(H_K) = 1$, where $H_K = G_{K_\infty}$. (For example, $\eta = \chi_{\text{cycl}}^n$ where $n \in \mathbb{Z}$.) Then*

$$H^0(G_K, \mathbb{C}_p(\eta)) = \mathbb{C}_p(\eta)^{G_K} = \begin{cases} 0 & \eta \text{ has infinite order} \\ K & \eta \text{ has finite order} \end{cases}$$

Proof. Suppose that $\mathbb{C}_p(\eta)^{G_K}$ is nonempty and that it contains αe_η where $\alpha \in \mathbb{C}_p$ is nonzero. Then for $g \in G_K$:

$$\alpha e_\eta = g(\alpha e_\eta) = g(\alpha)\eta(g) e_\eta$$

so $g(\alpha) = \eta(\alpha)^{-1}\alpha$.

If $h \in H_K$ then $\eta(h) = 1$ by assumption so we deduce that $g(\alpha) = \alpha$ so $\alpha \in \mathbb{C}_p^{H_K} = \widehat{K_\infty}$ by Ax-Sen-Tate.

By Proposition 2.16 $\alpha = \lim_{n\to\infty} \mathrm{pr}_n(\alpha)$ and since $g \in \Gamma_K$ commutes with pr_n it follows that

$$g(\mathrm{pr}_n \alpha) = \mathrm{pr}_n g(\alpha) = \mathrm{pr}_n(\eta(g)^{-1}\alpha) = \eta(g)^{-1} \mathrm{pr}_n \alpha$$

so we conclude that

$$\eta(g) = \frac{\mathrm{pr}_n \alpha}{g(\mathrm{pr}_n \alpha)}$$

But choosing $g \in \Gamma_{K_n}$, which invaries $\mathrm{pr}_n \alpha \in K_n$ gives that $\eta(\Gamma_{K_n}) = 1$ so $\eta(G_{K_n}) = 1$ so η would have to have finite order.

The second part follows from Lemma 2.17 as if η has finite order then $\mathbb{C}_p(\eta)^{G_K} = \mathbb{C}_p^{G_K} = K$ by Ax-Sen-Tate. $\square$

Lecture 6
2012-01-20

2.3.6 Topological generators

Let γ be a topological generator for Γ_K and γ_n a topological generator for Γ_{K_n}. Then $\gamma_n = \gamma^s$ for some s and for $n \geq n_K$ one has $\gamma_{n+k} = \gamma_n^{p^k}$.

Lemma 2.19 (Lemma A)**.** *If $x \in K_\infty$ then $v_p((1-\gamma_n^m)(x)) \geq v_p((1-\gamma_n)(x))$.*

Proof. Indeed

$$\begin{aligned} v_p((1-\gamma_n^m)(x)) &= v_p((\sum_{i=0}^{m-1} \gamma_n^i)(1-\gamma_n)(x)) \\ &= v_p(\sum_{i=0}^{m-1} \gamma_n^i(1-\gamma_n)(x)) \\ &\geq \min v_p(\gamma_n^i(1-\gamma_n)(x)) \\ &= v_p((1-\gamma_n)(x)) \end{aligned}$$

$\square$

Lemma 2.20 (Lemma B)**.** *If $x \in K_m$ where $m > n \geq n_K$ then*

$$v_p(x - \mathrm{pr}_n(x)) \geq v_p((1-\gamma_n)(x)) - 1 - \sum_{k=n}^{m-1} \frac{\alpha_k}{p^k}$$

Proof. We show this by induction. The base case, when $m = n+1$ is

$$\begin{aligned} v_p(x - \mathrm{pr}_n(x)) &= v_p(px - \mathrm{Tr}_{K_{n+1}/K_n}(x)) - 1 \\ &= v_p(\sum_{i=1}^{p-1}(1-\gamma_n^i)(x)) - 1 \\ &\geq \min v_p((1-\gamma_n^i)(x)) - 1 \\ &\geq v_p((1-\gamma_n)(x)) - 1 \end{aligned}$$

where the last line follows from the previous lemma.

Now for the inductive step. Suppose known for $m = n+k$ and now we look at $x \in K_{m+1}$. Then $\mathrm{Tr}_{K_{m+1}/K_m}(x) \in K_m$ and by the inductive hypothesis we have

$$v_p(\mathrm{Tr}_{K_{m+1}/K_m}(x) - \mathrm{pr}_n(\mathrm{Tr}_{K_{m+1}/K_m}(x))) \geq v_p((1-\gamma_n)(\mathrm{Tr}_{K_{m+1}/K_m}(x))) - 1 - \sum_{k=n}^{m-1} \frac{\alpha_k}{p^k}$$

But

$$\begin{aligned} v_p((1-\gamma_n)(\mathrm{Tr}_{K_{m+1}/K_m}(x))) &= v_p(\mathrm{Tr}_{K_{m+1}/K_m}((1-\gamma_n)(x))) \\ &= v_p(\mathrm{pr}_m((1-\gamma_n)(x))) + 1 \\ &\geq v_p((1-\gamma_n)(x)) + 1 - \frac{\alpha_m}{p^m} \end{aligned}$$

where the last line follows from Proposition 2.16. We deduce that

$$v_p(\mathrm{Tr}_{K_{m+1}/K_m}(x) - \mathrm{pr}_n(\mathrm{Tr}_{K_{m+1}/K_m}(x))) \geq v_p((1-\gamma_n)(x)) - \sum_{k=n}^{m} \frac{\alpha_k}{p^k}$$

Finally

$$\begin{aligned} v_p(x - \mathrm{pr}_n(x)) &= v_p(x - \frac{1}{p}\mathrm{Tr}_{K_{m+1}/K_m}(x) + \frac{1}{p}(\mathrm{Tr}_{K_{m+1}/K_m}(x) - p\,\mathrm{pr}_n(x))) \\ &\geq \min v_p(x - \mathrm{pr}_m(x)), v_p(\frac{1}{p}(\mathrm{Tr}_{K_{m+1}/K_m}(x) - p\,\mathrm{pr}_n(x))) - 1 \\ &\geq \min v_p((1-\gamma_n)(x)) - 1, v_p((1-\gamma_n)(x)) - 1 - \sum_{k=n}^{m} \frac{\alpha_k}{p^k} \\ &= v_p((1-\gamma_n)(x)) - 1 - \sum_{k=n}^{m} \frac{\alpha_k}{p^k} \end{aligned}$$

where in the third line we use the inductive hypothesis for K_{m+1}/K_m. □

Proposition 2.21 (Proposition A)**.** *Let $n \geq n_K$. The operator $1-\gamma_n$ is bijective on $K_n^{\perp}$, its inverse $(1-\gamma_n)^{-1}$ is continuous and the operator norm $||(1-\gamma_n)^{-1}||$ is bounded independent of n.*

Proof. Since γ_n is a generator of Γ_{K_n}, the kernel of $1-\gamma_n$ on $\widehat{K_\infty}$ is $\widehat{K_\infty}^{\Gamma_{K_n}} = K_n$ by Ax-Sen-Tate. Thus the kernel $1-\gamma_n$ on $K_n^{\perp}$ is $K_n^{\perp} \cap K_n = \{0\}$ so the operator is injective.

For $m \geq n$ the linear map $1-\gamma_n$ is injective on the finite dimensional vector space $K_m \cap K_n^{\perp}$ and so it is surjective. Let $y \in K_m \cap K_n^{\perp}$ which by surjectivity can be written as $y = (1-\gamma_n)(x)$ for $x \in K_m \cap K_n^{\perp}$. Lemma 2.20 applied to x gives

$$v_p(x - \mathrm{pr}_n(x)) \geq v_p((1-\gamma_n)(x)) - 1 - \sum_{k=n}^{m-1} \frac{\alpha_k}{p^k}$$

Since $\mathrm{pr}_n(x) = 0$ as $x \in K_n^{\perp}$ and $x = (1-\gamma_n)^{-1}(x)$ we deduce

$$v_p((1-\gamma_n)^{-1}(y)) \geq v_p(y) - C$$

where

$$C = 1 + \sum_{k=n}^{\infty} \frac{\alpha_k}{p^k}$$

which is a number as α_k are bounded. Therefore on $K_m \cap K_n^{\perp}$ we have

$$\begin{aligned} ||(1-\gamma_n)^{-1}|| &= \sup \frac{|(1-\gamma_n)^{-1}(y)|}{|y|} \\ &\leq |p|^C \end{aligned}$$

so the operator norm $||(1-\gamma_n)^{-1}||$ on $K_m \cap K_n^{\perp}$ is bounded independent of n and m. Therefore $(1-\gamma_n)^{-1}$ extends to a continuous function on $K_n^{\perp}$ of norm independent of n. □

Proposition 2.22 (Proposition B)**.** *Let $\eta : \Gamma_K \to \overline{\mathbb{Z}_p}^\times$ be an infinite order continuous character and let γ be a topological generator of Γ_K. Then $1 - \gamma : \widehat{K_\infty}(\eta) \to \widehat{K_\infty}(\eta)$ is surjective.*

Proof. Let C be the uniform bound on $||(1-\gamma_n)^{-1}||$ on $K_n^\perp$ from Proposition 2.21. Since the character η is continuous and Γ_{K_n} form a neighborhood basis of the identity in Γ_K it follows that $\lim_{n\to\infty} \eta(\gamma_n) = 1$ and so for $n >> 0$ we have $|1 - \eta(\gamma_n)| < C^{-1}$. Therefore $\left|\left|\dfrac{1-\eta(\gamma_n)}{1-\gamma_n}\right|\right| < 1$ and so on $K_n^\perp$ everything converges in the following computation:

$$\begin{aligned}\frac{1}{1-\gamma_n\eta(\gamma_n)} &= \frac{1}{(1-\gamma_n)\left(1+\left(\frac{1-\eta(\gamma_n)}{1-\gamma_n}\right)\gamma_n\right)}\\ &= (1-\gamma_n)^{-1}\sum_{j\geq 0}\left(\gamma_n(1-\eta(\gamma_n))(1-\gamma_n)^{-1}\right)^j\end{aligned}$$

Therefore $1 - \gamma_n\eta(\gamma_n) : K_n^\perp \to K_n^\perp$ is surjective, as its inverse is well-defined. This is equivalent to saying that $1 - \gamma_n : K_n^\perp(\eta) \to K_n^\perp(\eta)$ is surjective.

Now on $K_n(\eta)$. Since η has infinite order it follows that for all γ_n, $\eta(\gamma_n) \neq 1$. Since γ_n invaries K_n we get that on K_n, $1 - \gamma_n = 1 - \eta(\gamma_n) \neq 0$ and so it is an injective map on a finite dimensional vector space, thus also surjective.

We conclude that $1 - \gamma_n : \widehat{K_\infty}(\eta) \to \widehat{K_\infty}(\eta)$ is surjective. But $\gamma_n = \gamma^s$ for some s and $1 - \gamma_n = (1-\gamma)\left(\sum_{i=0}^{s-1}\gamma^i\right)$ so $1 - \gamma$ is necessarily surjective as well.

□

2.3.7 Galois cohomology of $\widehat{K_\infty}$

This section computes the continuous cohomology groups in degree 1 of the group $H_K = G_{K_\infty}$, which will later feed into an inflation-restriction sequence. We start with a lemma on approximations of cocycles.

Lemma 2.23.
1. *If $M \in H^1(H_K, p^n\mathcal{O}_{\mathbb{C}_p})$ there exists $x \in p^{n-1}\mathcal{O}_{\mathbb{C}_p}$ such that the cohomologous cocycle $g \mapsto M(g) + g(x) - x \in H^1(H_K, p^{n+1}\mathcal{O}_{\mathbb{C}_p})$.*
2. *If $n \geq 2$ and $M \in H^1(H_K, 1 + \varpi_K^n \mathrm{M}_d(\mathcal{O}_{\mathbb{C}_p}))$ then there exists a matrix $N \in 1 + \varpi_K^{n-1}\mathrm{M}_d(\mathcal{O}_{\mathbb{C}_p})$ such that the cohomologous cocycle $g \mapsto N^{-1}M(g)g(N) \in H^1(H_K, 1 + \varpi_K^{n+1}\mathrm{M}_d(\mathcal{O}_{\mathbb{C}_p}))$.*

Proof.
1. Note that $p^{n+2}\mathcal{O}_{\mathbb{C}_p}$ is open in $p^n\mathcal{O}_{\mathbb{C}_p}$ and so the quotient $p^n\mathcal{O}_{\mathbb{C}_p}/p^{n+2}\mathcal{O}_{\mathbb{C}_p}$ is discrete. Therefore the kernel of $M : H_K \to p^n\mathcal{O}_{\mathbb{C}_p}/p^{n+2}\mathcal{O}_{\mathbb{C}_p}$ is open and by Corollary 2.12 it must contain a subgroup of the form H_L where L/K is a finite extension, and increasing L we may also assume that L/K is Galois. Thus $M(H_L) \subset p^{n+2}\mathcal{O}_{\mathbb{C}_p}$.

   Let $\alpha \in L_\infty$ from Corollary 2.12 such that $\mathrm{Tr}_{L_\infty/K_\infty}(\alpha) = 1$ and $v(\alpha) > -v(\varpi_K) > -v(p)$. For a set T of representatives of H_K/H_L in H_K let

$$x_T = \sum_{g\in T} g(\alpha)M(g)$$

If $h \in H_L$ and $hT = \{hg | g \in T\}$ then we compute

$$
\begin{aligned}
h(x_T) &= h\left(\sum_{g \in T} g(\alpha) M(g)\right) \\
&= \sum_{g \in T} (hg)(\alpha) h(M(g)) \\
&= \sum_{g \in T} (hg)(\alpha)(M(hg) - M(g)) \\
&= \sum_{hg \in hT} (hg)(\alpha) M(hg) - \left(\sum_{hg \in hT} (hg)(\alpha)\right) M(g) \\
&= x_{hT} - M(g)
\end{aligned}
$$

where in the last line we used $\displaystyle\sum_{hg \in hT} (hg)(\alpha) = \mathrm{Tr}_{L_\infty/K_\infty}(\alpha) = 1$.

Also note that

$$
\begin{aligned}
x_{hT} - x_T &= \sum_{g \in T} ((hg)(\alpha) M(hg) - \sum_{g \in T} g(\alpha) M(g) \\
&= \sum_{g \in T} (gg^{-1}hg)(\alpha) M(gg^{-1}hg) - \sum_{g \in T} g(\alpha) M(g) \\
&= \sum_{g \in T} g(\alpha)(M(g) + g(M(g^{-1}hg))) - \sum_{g \in T} g(\alpha) M(g) \\
&= \sum_{g \in T} g(\alpha) g(M(g^{-1}hg)) \\
&\equiv 0 \pmod{p^{n+1}}
\end{aligned}
$$

Here the third line follows from the fact that H_L is normal in H_K by choice of L and thus $g^{-1}hg$ acts trivially on α, and from the cocycle condition on M; the last line follows from the fact that $v(\alpha) > -1$ and $v(M(g^{-1}hg)) \geq n + 2$.

Finally, $M(g) + g(x_T) - x_T = x_{hT} - x_T \in H^1(H_K, p^{n+1}\mathcal{O}_{\mathbb{C}_p})$.

2. Note that $1 + \varpi_K^{n+2} \mathrm{M}_d(\mathcal{O}_{\mathbb{C}_p})$ is open in $1 + \varpi_K^n \mathrm{M}_d(\mathcal{O}_{\mathbb{C}_p})$ so as before $(1 + \varpi_K^n \mathrm{M}_d(\mathcal{O}_{\mathbb{C}_p}))/(1 + \varpi_K^{n+2} \mathrm{M}_d(\mathcal{O}_{\mathbb{C}_p}))$ is discrete and so there exists L/K finite (Galois) such that $M(H_K) \subset 1 + \varpi_K^{n+2} \mathrm{M}_d(\mathcal{O}_{\mathbb{C}_p})$.

   As before, for representatives T of H_K/H_L in H_K let $N_T = \displaystyle\sum_{g \in T} g(\alpha) M(g)$, where α is as before. Note that if one writes $M(g) = 1 + \varpi_K^n X(g)$ then

$$
\begin{aligned}
N_T &= \sum_{g \in T} g(\alpha)(1 + \varpi_K^n X(g)) \\
&= \mathrm{Tr}_{L_\infty/K_\infty}(\alpha) + \varpi_K^n \sum_{g \in T} g(\alpha) X(g) \\
&\equiv 1 \pmod{\varpi_K^{n-1}}
\end{aligned}
$$

   because $v(\alpha) > -1$ and $X(g) \in \mathrm{M}_d(\mathcal{O}_{\mathbb{C}_p})$.

   As before we compute

$$
g(N_T) = M(g)^{-1} N_{hT}
$$

and thus

$$\begin{aligned} N_T^{-1} M(g) g(N_T) &= N_T^{-1} N_{hT} \\ &= 1 + N_T^{-1}(N_{gT} - N_T) \\ &\equiv 1 \pmod{\varpi_K^{n+1}} \end{aligned}$$

because as before we have

$$N_{gT} - N_T \equiv 0 \pmod{\varpi_K^{n+1}}$$

□

Lecture 7
2012-01-23

The main result of this section is the following:

Proposition 2.24. *We have*

1. $H^1(H_K, \mathbb{C}_p) = \{1\}$.
2. $H^1(H_K, \mathrm{GL}(d, \mathbb{C}_p)) = \{1\}$.

Proof. 1. Let $M \in H^1(H_K, \mathbb{C}_p)$. By continuity there exists $n_0 \in \mathbb{Z}$ such that $\operatorname{Im} M \subset p^{n_0}\mathcal{O}_{\mathbb{C}_p}$. Applying Lemma 2.23 successively we obtain for $n \geq n_0$ elements $x_n \in p^{n-1}\mathcal{O}_{\mathbb{C}_p}$ such that

$$M(g) + \sum_{n=n_0}^{m} (g(x_n) - x_n) \in p^{m+1}\mathcal{O}_{\mathbb{C}_p}$$

But then $x = \sum_{n=n_0}^{\infty} x_n$ converges and $M(g) + g(x) - x \in p^m \mathcal{O}_{\mathbb{C}_p}$ for all m and so it must be trivial. Therefore $M(g) = x - g(x)$ so it is the trivial cocycle.

2. Let $M \in H^1(H_K, \mathrm{GL}(d, \mathbb{C}_p))$. By continuity, since H_K is compact, it must be that $\operatorname{Im} M$ is also compact. But $(1 + \varpi_K^2 \mathrm{M}_d(\mathcal{O}_{\mathbb{C}_p})) \cap \operatorname{Im} M$ is an open subgroup of the compact $\operatorname{Im} M$ and so $\operatorname{Im} M / ((1 + \varpi_K^2 \mathrm{M}_d(\mathcal{O}_{\mathbb{C}_p})) \cap \operatorname{Im} M)$ is discrete. Therefore we may find L/K finite Galois such that $M(H_L) \subset 1 + \varpi_K^2 \mathrm{M}_d(\mathcal{O}_{\mathbb{C}_p})$.

   As before, applying Lemma 2.23 successively we obtain for $n \geq 2$ matrices $N_n \in 1 + \varpi_K^{n-1} \mathrm{M}_d(\mathcal{O}_{\mathbb{C}_p})$ and $N = \prod_{n \geq 2} N_n$ will converge giving $M(g) = N g(N)^{-1}$ the trivial cocycle in $H^1(H_L, 1 + \varpi_K^2 \mathrm{M}_d(\mathcal{O}_{\mathbb{C}_p}))$.

   Consider now the inflation-restriction sequence (Theorem 1.21):

$$1 \to H^1(H_K/H_L, \mathrm{GL}(d, \mathbb{C}_p)^{H_L}) \xrightarrow{\text{inf}} H^1(H_K, \mathrm{GL}(d, \mathbb{C}_p)) \xrightarrow{\text{res}} H^1(H_L, \mathrm{GL}(d, \mathbb{C}_p))$$

   We have already established that $\operatorname{res}(M) = 1$ and so $M = \inf(N)$ for some $N \in H^1(H_K/H_L, \mathrm{GL}(d, \mathbb{C}_p)^{H_L}) = H^1(G_{L_\infty/K_\infty}, \mathrm{GL}(d, \widehat{L_\infty}))$ (an application of Ax-Sen-Tate). But L_∞/K_∞ is a finite extension and therefore $H^1(G_{L_\infty/K_\infty}, \mathrm{GL}(d, \widehat{L_\infty})) = \{1\}$ by Hilbert 90 which gives $M = \inf(N) = 1$.

□

2.3.8 Tate periods: degree 1

Theorem 2.25. *Let $\eta : G_K \to \mathbb{Z}_p^\times$ be a continuous character such that $\eta(H_K) = 1$. Then*

$$H^1(G_K, \mathbb{C}_p(\eta)) = \begin{cases} 0 & \eta \text{ has infinite order} \\ K \cdot \log \chi_{\mathrm{cycl}} & \eta \text{ has finite order} \end{cases}$$

where $\log : \mathbb{Z}_p^\times \to \mathbb{Z}_p$ *is the logarithm map from Problem set 2.*

Proof. By inflation restriction for $H_K \subset G_K$ and $\Gamma_K \cong G_K/H_K$ we get

$$1 \to H^1(\Gamma_K, \mathbb{C}_p(\eta)^{H_K}) \to H^1(G_K, \mathbb{C}_p(\eta)) \to H^1(H_K, \mathbb{C}_p(\eta))$$

But $\mathbb{C}_p(\eta)^{H_K} = \widehat{K_\infty}(\eta)$ by Ax-Sen-Tate and $H^1(H_K, \mathbb{C}_p(\eta)) = H^1(H_K, \mathbb{C}_p) = 0$ by Proposition 2.24 and the fact that $\eta(H_K) = 1$. We conclude that

$$\begin{aligned} H^1(G_K, \mathbb{C}_p(\eta)) &\cong H^1(\Gamma_K, \widehat{K_\infty}(\eta)) \\ &\cong \widehat{K_\infty}(\eta)/(1-\gamma)\widehat{K_\infty}(\eta) \end{aligned}$$

where γ is a topological generator of the procyclic group Γ_K (by Proposition 1.22).

When η has infinite order Proposition 2.22 implies that $1-\gamma$ is surjective on $\widehat{K_\infty}(\eta)$ and so that $H^1(G_K, \mathbb{C}_p(\eta)) \cong \widehat{K_\infty}(\eta)/(1-\gamma)\widehat{K_\infty}(\eta) = 0$.

When η has finite order Lemma 2.17 shows that $\mathbb{C}_p(\eta) \cong \mathbb{C}_p$ so we need to compute $H^1(G_K, \mathbb{C}_p) \cong H^1(\Gamma_K, \widehat{K_\infty}) \cong \widehat{K_\infty}/(1-\gamma)\widehat{K_\infty}$. But we've seen that $1-\gamma$ is surjective on $K_n^\perp$ in the proof of Proposition 2.22 and so $\widehat{K_\infty}/(1-\gamma)\widehat{K_\infty} = K_n/(1-\gamma)K_n$.

Consider the map $K \to K_n/(1-\gamma)K_n$ obtained by inclusion and then projection. If the map were not injective one could find a nonzero $x \in K$ such that $x = (1-\gamma)(y)$ for some $y \in K_n$. Let $\gamma_n = \gamma^s$ in which case we would have $[K_n : K]x = \mathrm{Tr}_{K_n/K}(x) = (1 + \gamma + \cdots + \gamma^{s-1})(x) = (1-\gamma^s)(y) = 0$ as $y \in K_n$ which is fixed by $\gamma_n = \gamma^s$. Therefore the map is injective.

Since $\Gamma_K/\Gamma_{K_n} \cong G_{K_n/K}$ and Γ_{K_n} fixes K_n, inflation-restriction gives an exact sequence

$$1 \to H^1(G_{K_n/K}, K_n) \to H^1(\Gamma_K, K_n) \to H^1(\Gamma_{K_n}, K_n)^{G_{K_n/K}}$$

But Hilbert 90 shows that $H^1(G_{K_n/K}, K_n) = 0$ and so we get an injection

$$K_n/(1-\gamma)K_n \cong H^1(\Gamma_K, K_n) \hookrightarrow H^1(\Gamma_{K_n}, K_n)^{G_{K_n/K}} \cong (K_n/(1-\gamma_n)K_n)^{G_{K_n/K}} = K_n^{G_{K_n/K}} = K$$

and it follows from the explicit construction of the isomorphism $H^1(\Gamma_K, K_n) \cong K_n/(1-\gamma)K_n$ that if $x \in K$ has image x in $K_n/(1-\gamma)K_n$ then $\mathrm{res}(x) \in K$ is equal to x.

Explicitly, the isomorphism $K \cong K_n/(1-\gamma)K_n \cong H^1(\Gamma_K, K_n)$ is given by

$$x \mapsto (\gamma^r \mapsto (1 + \gamma + \cdots \gamma^{r-1})(x))$$

which is cohomologous to $\gamma^r \mapsto rx$. But this can also be written as

$$\gamma^r \mapsto \frac{\log \chi_{\mathrm{cycl}}(\gamma^r)}{\log \chi_{\mathrm{cycl}}(\gamma)} x$$

which is a scalar multiple of $g \mapsto \log \chi_{\mathrm{cycl}}(g)$. $\square$

Corollary 2.26. *If $m \neq n$ and $V \in \mathrm{Rep}_{\mathbb{C}_p}(G_K)$ such that $0 \to \mathbb{C}_p(m) \to V \to \mathbb{C}_p(n) \to 0$ then $V = \mathbb{C}_p(m) \oplus \mathbb{C}_p(n)$.*

Proof. Twist by $\mathbb{C}_p(-n)$ and get $0 \to \mathbb{C}_p(m-n) \to V(-n) \to \mathbb{C}_p \to 0$ which extensions are in bijection with cohomology classes in $H^1(G_K, \mathbb{C}_p(m-n)) = 0$. $\square$

Remark 10. We will see later that the Tate curve provides a nonsplit sequences $0 \to \mathbb{Q}_p(1) \to V \to \mathbb{Q}_p \to 0$ which doesn't even split over $\overline{\mathbb{Q}_p}$.

2.3.9 Tate periods: degree ≥ 2

This section was not covered in class, and was included here at the same time as §5.6, since it is necessary for Proposition 5.66. These results are adapted from [3, 14.3.1, 14.3.2].

Lemma 2.27. *Let $n \geq 2$ and $m \in \mathbb{Z}$.*

1. *If $M \in H^n(H_K, p^m\mathcal{O}_{\mathbb{C}_p})$ there exists a cochain $N \in C^{n-1}(H_K, p^{n-1}\mathcal{O}_{\mathbb{C}_p})$ such that $M - dN \in H^n(H_K, p^{m+1}\mathcal{O}_{\mathbb{C}_p})$.*
2. *$H^n(H_K, \mathbb{C}_p) = 0$.*

Proof. 1. Let L/K finite and let α such that $\mathrm{Tr}_{L_\infty/K_\infty}(\alpha) = 1$ with $v(\alpha) > -v(p)$ as in the proof of Lemma 2.23. For a set of representatives T of H_K/H_L let

$$N_T(g_1, \dots, g_{n-1}) := (-1)^n \sum_{h\in T} (g_1 \cdots g_{n-1}h)(\alpha) M(g_1, \dots, g_{n-1}, h)$$

We now compute

$$\begin{aligned}
(dN_T)(g_1, \dots, g_n) &= g_1(N_T(g_2, \dots, g_n)) + \sum_{j=1}^{n-1} (-1)^j N_T(g_1, \dots, g_jg_{j+1}, \dots, g_n) + (-1)^n N_T(g_1, \dots, g_{n-1}) \\
&= (-1)^n \sum_{h\in T} g_1 \cdots g_n h(\alpha) \left(g_1(M(g_2, \dots, g_n, h)) + (-1)^j \sum_{j=1}^{n-1} M(g_1, \dots, g_jg_{j+1}, \dots, g_n, h) \right) \\
&+ (-1)^n N_T(g_1, \dots, g_{n-1}) \\
&= (-1)^n \sum_{h\in T} g_1 \cdots g_n h(\alpha) \left((dM)(g_1, g_2, \dots, g_n, h) + (-1)^n M(g_1, \dots, g_n)\right. \\
&- \left.(-1)^n M(g_1, \dots, g_{n-1}, g_nh)\right) + (-1)^n N_T(g_1, \dots, g_{n-1}) \\
&\overset{\mathrm{Tr}\,\alpha=1, dM=0}{=} - \sum_{h\in T} g_1 \cdots g_n h(\alpha) M(g_1, \dots, g_{n-1}, g_nh) + M(g_1, \dots, g_n) \\
&+ (-1)^n N_T(g_1, \dots, g_{n-1}) \\
(M - dN_T)(g_1, \dots, g_n) &= \sum_{h\in T} g_1 \cdots g_{n-1} \left(g_nh(\alpha)M(g_1, \dots, g_{n-1}, g_nh) - h(\alpha)M(g_1, \dots, g_{n-1}, h)\right)
\end{aligned}$$

but H_L is normal in H_K so $h^{-1}g_nh \in H_L$ acts trivially on α so we may rewrite the above as

$$(M - dN_T)(g_1, \dots, g_n) = \sum_{h\in T} g_1 \cdots g_{n-1}h(\alpha) \left(M(g_1, \dots, g_{n-1}, g_nh) - M(g_1, \dots, g_{n-1}, h)\right)$$

Now we choose the finite extension L/K. For each $h \in H_K$ by continuity of the cochain M there exists a finite extension L_h/K such that for $g_1, \dots, g_n \in H_{L_h}$ we have

$$M(g_1, \dots, g_{n-1}, g_nh) - M(g_1, \dots, g_{n-1}, h) \in p^{n+2}\mathcal{O}_{\mathbb{C}_p}$$

Since H_{L_h} is open, $U_h = (1, \dots, 1, h)H_{L_h}$ is open in H_K^n. But the opens U_h cover the compact $1 \times \cdots \times 1 \times H_K$ and so there exists a finite subcover $U_{h_1}, \dots, U_{h_k}$. Let $L = L_{h_1} \cdots L_{H_k}$. Then for all $h \in H_K$ and $g_1, \dots, g_n \in H_L$ there exists an h_i such that $h, g_nh \in H_{L_{h_i}}h_i$ in which case

$$\begin{aligned}
M(g_1, \dots, g_{n-1}, g_nh) - M(g_1, \dots, g_{n-1}, h_i) &\in p^{n+2}\mathcal{O}_{\mathbb{C}_p} \\
M(g_1, \dots, g_{n-1}, h) - M(g_1, \dots, g_{n-1}, h_i) &\in p^{n+2}\mathcal{O}_{\mathbb{C}_p}
\end{aligned}$$

and so

$$M(g_1, \ldots, g_{n-1}, g_n h) - M(g_1, \ldots, g_{n-1}, h) \in p^{n+2}\mathcal{O}_{\mathbb{C}_p}$$

Since $v(\alpha) > -v(p)$ we deduce that $(M - dN_T)(g_1, \ldots, g_n) \in p^{n+1}\mathcal{O}_{\mathbb{C}_p}$ and $N_T(g_1, \ldots, g_{n-1}) \in p^{n-1}\mathcal{O}_{\mathbb{C}_p}$.

2. Let $M \in H^n(H_K, \mathbb{C}_p)$. By continuity there exists m_0 such that $M \in H^n(H_K, p^{m_0}\mathcal{O}_{\mathbb{C}_p})$. Using the first part for each $m \geq m_0$ we construct $N_m \in C^n(H_K, p^{m-1}\mathcal{O}_{\mathbb{C}_p})$ such that $M - \sum_{k=m_0}^m dN_k \in H^n(H_K, p^{m+1}\mathcal{O}_{\mathbb{C}_p})$. But then $N = \sum_{k=m_0}^{\infty} N_k$ converges and $M - dN \in H^n(H_K, p^{m+1}\mathcal{O}_{\mathbb{C}_p})$ for all $m \geq m_0$ and so $M = dN$ is trivial as a cohomology class.

□

2.4 Sen theory

2.4.1 Hodge-Tate representations

We start with a bit of notation. For a representation $V \in \mathrm{Rep}_{\mathbb{C}_p}(G_K)$ and an integer n let $V\{n\} = V(n)^{G_K} = \{v \in V | g(v) \cong \chi_{\mathrm{cycl}}(g)^{-n}v, \forall g \in G_K\}$. The beginnings of Hodge-Tate theory is the following lemma due to Serre and Tate:

Lemma 2.28 (Serre-Tate)**.** *Let $V \in \mathrm{Rep}_{\mathbb{C}_p}(G_K)$. Then there exists a natural map*

$$\xi_V : \bigoplus_n (\mathbb{C}_p(-n) \otimes V\{n\}) \to V$$

and the map ξ_V is injective.

Proof. For immediate proof see [3, Lemma 2.3.1]. The statement will follow from the formalism of admissible representations, specifically Theorem 3.7. □

The map ξ_V is called a comparison map and nice things happen when it is an isomorphism:

Definition 2.29. A representation $V \in \mathrm{Rep}_{\mathbb{C}_p}(G_K)$ is said to be Hodge-Tate if the comparison map ξ_V is an isomorphism. The Hodge-Tate weights of a Hodge-Tate representation V are the integers n such that $V\{n\} \neq 0$.

Remark 11. If $V \in \mathrm{Rep}_{\mathbb{C}_p}(G_K)$ in the Grothendieck ring is a sum of $\mathbb{C}_p(n)$ with distinct n then Corollary 2.26 implies that V is a Hodge-Tate representation. However, it is not necessarily true if the Hodge-Tate weights are not distinct, and there are extensions $0 \to \mathbb{C}_p \to V \to \mathbb{C}_p \to 0$ with V not Hodge-Tate.

Remark 12. Hodge-Tate representations can be thought of in two ways: the first, as forming a category, is a special instance of a category of "admissible" representations, which we study in the next section; the second, which we pursue in the remainder of this section, is as special types of $\mathbb{C}_p$ representations where a certain matrix (the Sen operator) is diagonalizable with integer eigenvalues. Both points of view are crucial in p-adic Hodge theory, the first one because it leads to many classes of admissible representations, such as de Rham, crystalline, semistable, etc, while the second because it ties the Galois representations arising from geometry to p-adic differential equations.

2.4.2 Galois descent

The Galois descent procedure of Sen theory is that $\mathbb{C}_p$ representations should be the same thing as $\widehat{K_\infty}$ representations.

Lemma 2.30. *$H^1(G_K, \mathrm{GL}(d, \mathbb{C}_p)) \cong H^1(\Gamma_K, \mathrm{GL}(d, \widehat{K_\infty}))$.*

Proof. This follows from the inflation-restriction sequence and Proposition 2.24. □

Lecture 8
2012-01-25

2.4.3 Decompletion

Consider the natural map $H^1(\Gamma_K, \mathrm{GL}(d, K_\infty)) \to H^1(\Gamma_K, \mathrm{GL}(d, \widehat{K_\infty}))$. The main result of this section is that this map is an isomorphism, a procedure known as "decompletion" because it takes Galois representations over the completion $\widehat{K_\infty}$ to Galois representations over K_∞. The way to show this is to use Tate's normalized maps to approximate cocycles over the infinite cyclotomic extension by finite ones, where completion does nothing. As such, this section is a careful combination of the approximation estimates from the previous sections on the normalized traces.

Lemma 2.31. *1. The limit $\varinjlim H^1(\Gamma_K, \mathrm{GL}(d, K_n))$ is identified with isomorphism classes of $V \in \mathrm{Rep}_{K_\infty}(\Gamma_K)$ for which there exists some n and $V_n \in \mathrm{Rep}_{K_n}(\Gamma_n)$ such that $V = K_\infty \otimes_{K_n} V_n$.*

2. The natural map $\varinjlim H^1(\Gamma_K, \mathrm{GL}(d, K_n)) \to H^1(\Gamma_K, \mathrm{GL}(d, K_\infty))$ is an isomorphism

Proof. This is [3, Exercise 14.4.4] □

Lemma 2.32. *Let K, v be a field with valuation. For a matrix $M = (m_{ij}) \in \mathrm{M}_d(K)$ define $v(M) = \min_{i,j} v(m_{ij})$.*

1. *Then*

$$v(MN) \geq v(M) + v(N)$$
$$v(M + N) \geq \min(v(M), v(N))$$

2. *With the notation of Lemma 2.20, if $M \in \mathrm{M}_d(K_m^\perp)$ then*

$$v_p(M) > v_p((1 - \gamma_m)(M)) - C$$

where C is some constant larger than $1 + \sum_{k=n}^{\infty} \frac{\alpha_k}{p^k} < \infty$.

Proof. 1. Straightforward.

2. Since $M \in \mathrm{M}_d(K_m^\perp)$ we have $M = M - \mathrm{pr}_m(M)$ and so by Lemma 2.20

$$\begin{aligned} v_p(M) &= v_p(M - \mathrm{pr}_m(M)) \\ &= \min v_p(m_{ij} - \mathrm{pr}_m(m_{ij})) \\ &> \min v_p((1 - \gamma_m)(m_{ij}) - C \\ &= v_p((1 - \gamma_m)(M)) - C \end{aligned}$$

□

We next prove a lemma on approximating matrices over $\widehat{K_\infty}$ with matrices over K_n.

Lemma 2.33. *Let $A = \max(\alpha_n p^{-n})$ and let C as in the previous proposition. Let $M \in \mathrm{M}_d(\widehat{K_\infty})$, which we will approximate by matrices over K_n, for n large enough.*

1. *Suppose $M \in \mathrm{GL}(d, \widehat{K_\infty})$ is a matrix such that $\gamma_n(M) = U_1 M U_2$ where $U_1, U_2 \in 1 + p^C \mathrm{M}_d(K_n)$. Then $M \in \mathrm{GL}(d, K_n)$.*

2. *Assume that M can be written as a sum $M = 1 + M_n + M_\infty$ where $M_n \in \mathrm{M}_d(K_n)$ and $M_\infty \in \mathrm{M}_d(\widehat{K_\infty})$ (here M_∞ can be thought of as the defect of writing M over K_n) with the property that $v_p(M_n) > A + 2C$ and $v_p(M_\infty) \geq v_p(M_n) + A$. Then there exists $B \in \mathrm{M}_d(\widehat{K_\infty})$ such that $v_p(B - 1) \geq v_p(M_\infty) - A - C$ and $B^{-1} M \gamma_n(B) = 1 + N_n + N_\infty$ where $N_n \in \mathrm{M}_d(K_n)$, $N_\infty \in \mathrm{M}_d(\widehat{K_\infty})$ such that $v_p(N_n) \geq v_p(M_n)$ and $v_p(N_\infty) \geq v_p(M_\infty) + v_p(M_n) - A - 2C$ (the new defect N_∞ has smaller norm than the old defect M_∞).*

3. *If $v_p(M-1) > 2A+2C$ there exists $B \in \mathrm{M}_d(\widehat{K_\infty})$ such that $v_p(B-1) \geq v_p(M-1) - A - C$ and $B^{-1}M\gamma_n(B) \in \mathrm{M}_d(K_n)$.*

Proof. 1. Write $M_n = M - \mathrm{pr}_n(M) \in \mathrm{M}_d(K_n^{\perp})$. Then, since γ_n commutes with pr_n we have

$$\begin{aligned}(1-\gamma_n)(M_n) &= M_n - U_1 M_n U_2 \\ &= (U_1-1)M_n(U_2-1) - (U_1-1)M_nU_2 - U_1M_n(U_2-1) \\ v_p((1-\gamma_n)(M_n)) &\geq \min v_p((U_1-1)M_n(U_2-1)), v_p((U_1-1)M_nU_2), \\ &\quad v_p(U_1M_n(U_2-1)) \\ &\geq v_p(M_n) + C\end{aligned}$$

If $M_n \neq 0$ then this contradicts Lemma 2.32 and so $M = \mathrm{pr}_n(M) \in \mathrm{M}_d(K_n)$.

Similarly one may show that $M^{-1} = \mathrm{pr}_n(M^{-1})$ and so that $M \in \mathrm{GL}(d, K_n)$

2. Let $N = (1-\gamma_n)^{-1}(M_\infty - \mathrm{pr}_n(M_\infty))$ which exists as $M_\infty - \mathrm{pr}_n(M_\infty) \in \mathrm{M}_d(K_n^{\perp})$. Let $N_n = M_n + \mathrm{pr}_n(M_\infty)$, $B = 1 + N$ and $N_\infty = (M-1)\gamma_n(N) - N(M-1) - NM\gamma_n(N) + (B^{-1}+N-1)M\gamma_n(B)$. Checking that these choices work is a matter of applying Lemma 2.20 and Proposition 2.16. For details see [3, Lemma 14.2.4].

3. We will use the second part to construct better and better approximations of M over K_n. Let $M_n^{(1)} = 0$ and $M_\infty^{(1)} = M - 1$. Applying the first part of this lemma recursively we get matrices B_k such that

   - $v_p(B_k - 1) \geq (k+1)(v_p(M-1) - 2A - 2C) + A + C$ and
   - $(B_0 \cdots B_k)^{-1} M\gamma_m(B_0 \cdots B_k) = 1 + M_n^{(k)} + M_\infty^{(k)}$ with $v_p(M_n^{(k)}) \geq v_p(M-1) - A$ and $v_p(M_\infty^{(k)}) \geq v_p(M-1) + k(v_p(M-1) - 2A - 2C)$.
     Let $B = \lim\limits_{k\to\infty} B_0 \cdots B_k$ (which converges by the condition on B_k) in which case we would have $B^{-1}M\gamma_m(B) = 1 + M_n + M_\infty$ with $M_n \in \mathrm{M}_d(K_n)$ and M_∞ infinitely divisible by p and so $M_\infty = 0$. For details see [3, Lemma 14.2.5].

□

Proposition 2.34. *The natural map $H^1(\Gamma_K, \mathrm{GL}(d, K_\infty)) \to H^1(\Gamma_K, \mathrm{GL}(d, \widehat{K_\infty}))$ is injective.*

Proof. By Lemma 2.31 it is enough to show that $H^1(\Gamma_K, \mathrm{GL}(d, K_n)) \hookrightarrow H^1(\Gamma_K, \mathrm{GL}(d, \widehat{K_\infty}))$. Suppose this map is not injective, i.e., there exist cocycles $U, U' \in H^1(\Gamma_K, \mathrm{GL}(d, K_n))$ such that they become cohomologous over $\widehat{K_\infty}$, i.e., there exists a matrix $B \in \mathrm{GL}(d, \widehat{K_\infty})$ such that for $g \in \Gamma_K$ one has $U'(g) = B^{-1}U(g)g(B)$ which we rewrite at $g(B) = U(g)^{-1}BU'(g)$.

The cocycles U and U' are continuous and Γ_{K_m} form a neighborhood basis of the identity in Γ_K so for $m >> 0$ one has $U(\gamma_m), U'(\gamma_m) \in 1 + p^C \,\mathrm{M}_d(\mathcal{O}_{\mathbb{C}_p})$ where C is as in Lemma 2.32. These choices imply that $v_p(U(\gamma_m)) = v_p(U'(\gamma_m)) = 0$ and that $v_p(U(\gamma_m)^{-1} - 1) \geq C$ and $v_p(U'(\gamma_m) - 1) \geq C$.

Applying the first part of Lemma 2.33 we deduce that $B \in \mathrm{GL}(d, K_m)$, which shows that U and U' are cohomologous in $H^1(\Gamma_K, \mathrm{GL}(d, K_\infty))$. □

Theorem 2.35. *We have $H^1(\Gamma_K, \mathrm{GL}(d, K_\infty)) \cong H^1(\Gamma_K, \mathrm{GL}(d, \widehat{K_\infty}))$.*

Proof. Injectivity is the content of Proposition 2.34. Now surjectivity. Let $U \in H^1(\Gamma_K, \mathrm{GL}(d, \widehat{K_\infty}))$. For $m >> 0$ we have $v_p(U(\gamma_m) - 1) > 2A + 2C$ as Γ_{K_m} form a neighborhood basis of identity in Γ_K. Now the the third part of Lemma 2.33 shows that $U'(\gamma_m) = B^{-1}U(\gamma_m)\gamma_m(B) \in \mathrm{GL}(d, K_m)$.

If γ is a topological generator of Γ_K we still need to show that $U'(\gamma)$ is defined over K_∞. Recall that $\gamma_m = \gamma^s$ for some s and thus $\gamma\gamma_m = \gamma_m\gamma$. Thus $U'(\gamma)\gamma(U'(\gamma_m)) = U'(\gamma\gamma_m) = U'(\gamma_m\gamma) = U'(\gamma_m)\gamma_m(U'(g))$ and so $\gamma_m(U'(\gamma)) = U'(\gamma_m)^{-1}U'(\gamma)\gamma(U'(\gamma_m))$. Appyling the first part of Lemma 2.33 gives that $U'(\gamma) \in \mathrm{GL}(d, \widehat{K_m})$ as well. This implies that $U' \in H^1(\Gamma_K, \mathrm{GL}(d, K_m))$ and U' is cohomologous to U in $H^1(\Gamma_K, \mathrm{GL}(\widehat{K_\infty}))$. □

2.4.4 Sen Theory

This section is an overview of the main results of Sen theory, with no proofs, as they are easily readable in [3].

Theorem 2.36. *Let $V \in \mathrm{Rep}_{\mathbb{C}_p}(G_K)$ of dimension $d \geq 1$. Then one can find uniquely $\mathrm{D}_{\mathrm{Sen}}(V) \in \mathrm{Rep}_{K_\infty}(\Gamma_K)$ a K_∞ submodule of V such that $\widehat{K_\infty} \otimes_{K_\infty} \mathrm{D}_{\mathrm{Sen}}(V) \cong V^{H_K}$ (and thus that $\mathbb{C}_p \otimes_{K_\infty} \mathrm{D}_{\mathrm{Sen}}(V) \cong V$). Moreover, $\mathrm{D}_{\mathrm{Sen}}(V)$ descends to some K_n.*

Proof. This is [3, Theorem 15.1.2]. Existence follows from Theorem 2.35 and the interpretation in Lemma 2.31 of the cohomology of $\mathrm{GL}(d)$ as isomorphism classes of Galois representations. ☐

Proposition 2.37. *It can be shown that $\mathrm{D}_{\mathrm{Sen}} : \mathrm{Rep}_{\mathbb{C}_p}(G_K) \to \mathrm{Rep}_{K_\infty}(\Gamma_K)$ is a fully faithful functor that respect direct sums and tensor products.*

Proof. This is [3, Lemma 15.1.3] and [3, Proposition 15.1.4]. The compatibility properties follow from the uniqueness of $\mathrm{D}_{\mathrm{Sen}}$. ☐

Theorem 2.38. *For $D \in \mathrm{Rep}_{K_\infty}(\Gamma_K)$ there exists a unique K_∞-linear operator $\Theta_{D,\mathrm{Sen}}$, called the Sen operator, which has the property that for all $v \in D$ for $m >> 0$ and $g \in \Gamma_{K_m}$ one has $g(v) = \exp(\log(\chi_{\mathrm{cycl}}(g))\Theta_{D,\mathrm{Sen}})(v)$.*

Proof. Fixing a basis of D let $U \in H^1(\Gamma_K, \mathrm{GL}(d, K_\infty))$ be the cocycle describing the action of Γ_K on this basis. Then if U is defined over some K_n then one may check that

$$\Theta_{D,\mathrm{Sen}}(v) = \frac{\log(U(\gamma_n)(v))}{\log(\chi_{\mathrm{cycl}}(\gamma_n))}$$

gives a well-defined operator. Then one must shrink Γ_{K_n} to Γ_{K_m} where m depends on v in order to make sense of exp, which has a smaller radius of convergence than log (see problem set 2). ☐

Proposition 2.39. *The Galois representation of Γ_K on $D \in \mathrm{Rep}_{K_\infty}(\Gamma_K)$ seem to be, via Theorem 2.38, encoded in the Sen operator. Let $\mathcal{S}_{K_\infty}$ be the category of finite dimensional K_∞ vector space with K_∞-linear endomorphisms. While the functor taking the Γ_K representation D to $(D, \Theta_{D,\mathrm{Sen}}) \in \mathcal{S}_{K_\infty}$ is neither fully faithful nor essentially surjective, it does however detect isomorphisms in the sense that $D_1, D_2 \in \mathrm{Rep}_{K_\infty}(\Gamma_K)$ are isomorphic if and only if $(D_1, \Theta_{D_1,\mathrm{Sen}})$ and $(D_2, \Theta_{D_2,\mathrm{Sen}})$ are isomorphic.*

Proof. This is [3, Corollary 15.1.13] ☐

Corollary 2.40. *Let $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$. Then $\mathbb{C}_p \otimes_K (\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{G_K} \cong \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ if and only V is potentially unramified, i.e., there exists a finite extension L/K such that I_L acts trivially on V.*

Proof. We will only show this in the case when H_K acts trivially on V, the general case requiring the study of the cohomology of $\ker V$ (assuming that V is not potentially trivial) instead of H_K, via the general Tate-Sen formalism. Writing $W = \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ suppose $\mathbb{C}_p \otimes_K W^{G_K} \cong W$. Thus $\mathrm{D}_{\mathrm{Sen}}(W) \cong \mathrm{D}_{\mathrm{Sen}}(\mathbb{C}_p \otimes_K W^{G_K})$ where on the right hand side G_K acts only on $\mathbb{C}_p$ coefficients, but not on a basis of W^{G_K}. This gives that the associated cocycle to V is trivial, being cohomologous to $B^{-1}g(B)$ for $B \in \mathrm{GL}(d, \mathbb{C}_p)$. The triviality of the action of H_K gives $B \in \mathrm{GL}(d, \widehat{K_\infty})$. If $\mathrm{D}_{\mathrm{Sen}}$ descends to K_n get that $B \in \mathrm{GL}(d, K_n)$ and so get $K_n \otimes_{\mathbb{Q}_p} V \otimes K_n^{\dim V}$. But then restricting to G_{K_n} gives trivial action on V. ☐

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It turns out that while the functor $\mathrm{D}_{\mathrm{Sen}}$ is not fully faithful the category $\mathcal{S}_{K_\infty}$ is sufficiently rich to detect Hodge-Tate representations:

Proposition 2.41. *Let $V \in \mathrm{Rep}_{\mathbb{C}_p}(G_K)$. Then V is Hodge-Tate if and only if $\Theta_{\mathrm{D}_{\mathrm{Sen}}(V),\mathrm{Sen}}$ is diagonalizable with integer eigenvalues. In that case these eigenvalues are the Hodge-Tate weights.*

Example 2.42. We have seen that the nonvanishing of $H^1(G_K, \mathbb{C}_p) = K \log \chi_{\text{cycl}}$ gives a nontrivial extension $0 \to \mathbb{C}_p \to V \to \mathbb{C}_p \to 0$ on which G_K acts via the matrix $\begin{pmatrix} 1 & \log \chi_{\text{cycl}} \\ & 1 \end{pmatrix}$. This representation, not being isomorphic to $\mathbb{C}_p \oplus \mathbb{C}_p$ is not Hodge-Tate, and

$$\Theta_{\mathrm{D}_{\mathrm{Sen}},\mathrm{Sen}}(V) = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}$$

which follows from the formula defining the Sen operator.

3 Admissible Representations

3.1 The category of Hodge-Tate representations

3.1.1 Basics

Recall that $W \in \mathrm{Rep}_{\mathbb{C}_p}(G_K)$ was defined to be Hodge-Tate if the comparison morphism

$$\xi_W : \bigoplus_n \mathbb{C}_p(-n) \otimes_{\mathbb{C}_p} W \to W$$

was an isomorphism. Write $\mathrm{Rep}^{\mathrm{HT}}_{\mathbb{C}_p}(G_K)$ for the full subcategory of $\mathrm{Rep}_{\mathbb{C}_p}(G_K)$ consisting of objects which are Hodge-Tate and let $\mathrm{Rep}^{\mathrm{HT}}_{\mathbb{Q}_p}(G_K)$ be the full subcategory of $\mathrm{Rep}^{\mathrm{HT}}_{\mathbb{Q}_p}(G_K)$ consisting of objects W such that $W \otimes_{\mathbb{Q}_p} \mathbb{C}_p \in \mathrm{Rep}^{\mathrm{HT}}_{\mathbb{C}_p}(G_K)$.

For $W \in \mathrm{Rep}_{\mathbb{C}_p}(G_K)$ the K-vector spaces $W\{n\}$ can be thought of as the graded pieces of the graded vector space $\bigoplus_n W\{n\}$. Therefore we now introduce some categories of vector spaces.

3.1.2 Graded vector spaces

Let GrVect_K be the category of graded vector spaces, i.e., of vector spaces W over K together with a grading $W = \bigoplus_n \mathrm{gr}^n W$ where $\mathrm{gr}^n W$ is a K-subvector space. Morphisms in this category are morphisms $f : W_1 \to W_2$ of vector spaces such that $f : \mathrm{gr}^n W_1 \to \mathrm{gr}^n W_2$ for all n. A graded ring is a ring R with a grading $\mathrm{gr}^\bullet R$ such that $1 \in \mathrm{gr}^0 R$ and $\mathrm{gr}^m R \cdot \mathrm{gr}^n R \subset \mathrm{gr}^{m+n} R$.

The category GrVect_K has

1. direct sums,
2. tensor products: $(W_1, \mathrm{gr}^\bullet W_1) \otimes_K (W_2, \mathrm{gr}^\bullet W_2) = (W_1 \otimes_K W_2, \mathrm{gr}^\bullet(W_1 \otimes_K W_2))$ where $\mathrm{gr}^n(W_1 \otimes_K W_2) = \sum_{i+j=n} \mathrm{gr}^i W_1 \otimes_K \mathrm{gr}^j W_2$,
3. linear duals: $(W, \mathrm{gr}^\bullet W)^\vee = (W^\vee, \mathrm{gr}^\bullet W^\vee)$ where $\mathrm{gr}^n W^\vee = (\mathrm{gr}^{-n} W)^\vee)$,
4. kernels: if $T : W' \to W$ is a morphism in GrVect_K then $(\ker T, \mathrm{gr}^\bullet(\ker T) \in \mathrm{GrVect}_K$ where $\mathrm{gr}^n(\ker T) = \ker T \cap \mathrm{gr}^n W'$,
5. cokernels: if $W' \to W$ is a morphism in GrVect_K then $(\operatorname{coker} T', \mathrm{gr}^\bullet(\operatorname{coker} T)) \in \mathrm{GrVect}_K$ where $\mathrm{gr}^n(\operatorname{coker} T) = (\mathrm{gr}^n W + T(W'))/T(W')$.

The category GrVect_K is an abelian category.

In concordance with the notation of [3, §2.4], we will write $K\langle n\rangle$ for the graded vector space K with $\mathrm{gr}^m K = K$ if $m = n$ and 0 otherwise. Then $K\langle m\rangle \otimes_K K\langle n\rangle \cong K\langle m+n\rangle$ and $K\langle m\rangle \cong K\langle -m\rangle$.

3.1.3 Filtered modules

Let R be a commutative ring. Let FilMod_R be the category of (separated and exhaustive) filtered R-modules, i.e., of R-modules W together with descending filtrations $\ldots \supset \mathrm{Fil}^n W \supset \mathrm{Fil}^{n+1} W \supset \ldots$ such that $\mathrm{Fil}^n W = W$ for $n << 0$ (exhaustive) and $\mathrm{Fil}^n W = 0$ for $n >> 0$ (separated). Morphisms in this caregory are morphisms $f : W_1 \to W_2$ of R-modules such that $f : \mathrm{Fil}^n W_1 \to \mathrm{Fil}^n W_2$ for all n. A filtered ring is a ring R with a separated and exhaustive filtration $\mathrm{Fil}^n R$ such that $1 \in \mathrm{Fil}^0 R$ and $\mathrm{Fil}^m R \cdot \mathrm{Fil}^n R \subset \mathrm{Fil}^{m+n} R$. If K is a vector space write $\mathrm{FilVect}_K$ instead of FilMod_K.

The category $\mathrm{FilVect}_K$ has

1. direct sums,
2. tensor products: $(W_1, \mathrm{Fil}^\bullet W_1) \otimes_R (W_2, \mathrm{Fil}^\bullet W_2) = (W_1 \otimes_R W_2, \mathrm{Fil}^\bullet(W_1 \otimes_R W_2))$ where $\mathrm{Fil}^n(W_1 \otimes_R W_2) = \displaystyle\sum_{i+j=n} \mathrm{Fil}^i W_1 \otimes_R \mathrm{Fil}^j W_2$,
3. linear duals: $(W, \mathrm{Fil}^\bullet W)^\vee = (W^\vee, \mathrm{Fil}^\bullet(W^\vee))$ where $\mathrm{Fil}^n(W^\vee) = \{v^\vee \in W^\vee \,|\, \mathrm{Fil}^{1-n} W \subset \ker v^\vee\}$.
4. kernels: if $T : W' \to W$ is a morphism in $\mathrm{FilVect}_K$ then $(\ker T, \mathrm{Fil}^\bullet(\ker T)) \in \mathrm{FilVect}_K$ where $\mathrm{Fil}^n(\ker T) = \ker T \cap \mathrm{Fil}^n W'$,
5. cokernels: if $T : W' \to W$ is a morphism in $\mathrm{FilVect}_K$ then $(\ker T, \mathrm{Fil}^\bullet(\mathrm{coker}\, T)) \in \mathrm{FilVect}_K$ where $\mathrm{Fil}^n(\mathrm{coker}\, T) = (\mathrm{Fil}^n W + T(W'))/T(W')$.

However, $\mathrm{FilVect}_K$ is not an abelian category.

There is a functor $\mathrm{gr}^\bullet : \mathrm{FilVect}_K \to \mathrm{GrVect}_K$ taking $(W, \mathrm{Fil}^\bullet W)$ to $(W, \mathrm{gr}^\bullet W)$ where $\mathrm{gr}^n W = \mathrm{Fil}^n W / \mathrm{Fil}^{n+1} W$.

If $W \in \mathrm{FilVect}_K$ and $d \in \mathbb{Z}$ let $W[d] \in \mathrm{FilVect}_K$ be the vector space W with the filtration $\mathrm{Fil}^n W[d] = \mathrm{Fil}^{n+d} W$. Then $W[n]^\vee = W^\vee[-n]$.

3.1.4 Reformulating the comparison morphism

We have not proven Lemma 2.28 but will reformulate it in a way which will make it amenable to the notation of Theorem 3.7.

Let $\mathrm{B_{HT}} = \bigoplus_n \mathbb{C}_p(n)$ which is a graded ring (via $\mathbb{C}_p(m) \otimes_{\mathbb{C}_p} \mathbb{C}_p(n) \cong \mathbb{C}_p(m+n)$) with a G_K action. Then we may obtain the graded vector space $\displaystyle\bigoplus_n W\{n\} = (\mathrm{B_{HT}} \otimes_{\mathbb{C}_p} W)^{G_K} \in \mathrm{GrVect}_K$. Then we get a natural morphism

$$\alpha_{\mathrm{HT},W} : \mathrm{B_{HT}} \otimes_K \left(\bigoplus_n W\{n\} \right) \to \mathrm{B_{HT}} \otimes_{\mathbb{C}_p} W$$

which is simply $\alpha_{\mathrm{HT},W} = \displaystyle\bigoplus_n \xi_W(n)$ where $\xi_W(n)$ is the Tate twist of ξ_W by χ^n_{cycl}. Then the injectivity of ξ_W will follow from that of $\alpha_{\mathrm{HT},W}$ from Theorem 3.7. In fact one may recover $\xi_W = \mathrm{gr}^0 \alpha_{\mathrm{HT},W}$.

As a matter of preliminary notation we write (now for $W \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$, and $D \in \mathrm{GrVect}_K$)

$$\mathrm{D_{HT}}(W) = (\mathrm{B_{HT}} \otimes_{\mathbb{Q}_p} W)^{G_K}$$
$$\mathrm{V_{HT}}(D) = \mathrm{gr}^0(\mathrm{B_{HT}} \otimes_K D)$$

which of course makes sense as GrVect_K has tensor products, in which case $\xi_W : \mathrm{V_{HT}}(\mathrm{D_{HT}}(W)) \to W$.

The formalism of admissible representations will generalize the functors (we don't know they are functors yet) $\mathrm{D_{HT}}$ and $\mathrm{V_{HT}}$ to more general rings than $\mathrm{B_{HT}}$, rings which we describe next.

3.2 Regular rings

The setup is the following: G is a group, F is a field, B is an integral F-algebra with an action of G, and B^G is a field.

Definition 3.1. The algebra B is (F, G)-regular if the following two conditions are satisfied:

1. $(\operatorname{Frac} B)^G = B^G$, and
2. if $b \in B$ is nonzero and $F \cdot b$ is G-stable then $b \in B^\times$.

Example 3.2. If B is a field then B is (F, G)-regular.

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Proposition 3.3. *The ring* B_{HT} *is* $(\mathbb{Q}_p, G_K)$*-regular.*

Proof. Certainly $\mathrm{B}_{\mathrm{HT}}^{G_K} = K$ (by Ax-Sen-Tate) is a field. To tackle the first condition of regularity it is useful to consider the isomorphism $\mathrm{B}_{\mathrm{HT}} \cong \mathbb{C}_p[T]$ where G_K acts on T via $g(T) = \chi_{\mathrm{cycl}}(g)T$. Then $\operatorname{Frac} \mathrm{B}_{\mathrm{HT}} \subset \mathbb{C}_p(\!(T)\!)$ so we need to check that $\mathbb{C}_p(\!(T)\!)^{G_K} = K$. But if $f(T) = \sum_{n \geq n_0} a_n T^n$ is G_K invariant then $a_n \in \mathbb{C}_p(n)^{G_K}$ for all n and so $f(T) = a_0 \in \mathbb{C}_p^{G_K} = K = \mathrm{B}_{\mathrm{HT}}^{G_K}$ by Ax-Sen-Tate.

Now for the second condition of regularity, let $f(T) \in \mathrm{B}_{\mathrm{HT}} \neq 0$ give a G_K stable line $f(T)\mathbb{Q}_p$ on which G_K acts via a character $\eta : G_K \to \mathbb{Q}_p^\times$. Thus

$$\begin{aligned} g\Big(\sum_{n=0}^{\infty} a_n T^n\Big) &= \sum_n g(a_n) \chi_{\mathrm{cycl}}^n(g) T^n \\ &= \sum_n \eta(g) a_n T^n \end{aligned}$$

which gives $a_n \in \mathbb{C}_p(\chi_{\mathrm{cycl}}^n \eta^{-1})^{G_K}$. Suppose that for some n one has $a_n \neq 0$ which implies that $\dim_K \mathbb{C}_p(\chi_{\mathrm{cycl}}^n \eta^{-1})^{G_K} = 1 = \dim \mathbb{C}_p(\chi_{\mathrm{cycl}}^n \eta^{-1})$. But then Corollary 2.40 implies that $\chi_{\mathrm{cycl}}^n \eta^{-1}$ is potentially unramified. Since χ_{cycl} is infinitely ramified it follows that this condition can be satisfied only for one n.

However, we did not give a proof of Corollary 2.40 and an alternative is the following. Since $a_n \neq 0$ it follows that $\mathbb{C}_p(\chi_{\mathrm{cycl}}^n \eta^{-1})^{G_K} \neq 0$. Therefore $\mathrm{D}_{\mathrm{Sen}}(\mathbb{C}_p(\chi_{\mathrm{cycl}}^n \eta^{-1})) = K_\infty$ and so $\mathbb{C}_p(\chi_{\mathrm{cycl}}^n \eta^{-1}) \cong \mathbb{C}_p$. But then if $m \neq n$ then $\mathbb{C}_p(\chi_{\mathrm{cycl}}^m \eta^{-1})^{G_K} \cong \mathbb{C}_p(m-n)^{G_K} = 0$ by the Ax-Sen-Tate theorem and so $a_m = 0$.

The final conclusion is that $f(T) = a_n T^n$ which is invertible. $\square$

3.3 Admissible representations

Definition 3.4. Suppose $V \in \operatorname{Rep}_F(G)$ and B is a (F, G)-regular ring. Then

$$\mathrm{D}_B(V) = (B \otimes_F V)^G$$

is the associated Dieudonné module.

Remark 13. One has a natural comparison morphism

$$\alpha_{B,V} : B \otimes_E \mathrm{D}_B(V) \cong B \otimes_E (B \otimes_F V)^G \to B \otimes_E (B \otimes_F V) \cong (B \otimes_E B) \otimes_F V \to B \otimes_F V$$

Proposition 3.5. 1. *Let B be a (F, G)-regular ring and $V \in \operatorname{Rep}_F(G)$. Then $\alpha_{B,V}$ is injective.*

2. $\dim_E \mathrm{D}_B(V) \leq \dim_F V$ *with equality if and only if $\alpha_{B,V}$ is an isomorphism.*

Proof. 1. Let $d_1, \ldots, d_n$ be the smallest number of linearly independent elements in $\mathrm{D}_B(V)$ such that there exist $b_i \neq 0$ with $\alpha_{B,V}(\sum b_i \otimes_E d_i) = \sum b_i d_i = 0$. Since the d_i are G-invariant for all $g \in G$ get $\sum g(b_i) d_i = 0$. Dividing by b_1 get

$$d_1 + \sum \frac{b_i}{b_1} d_i = 0$$
$$d_1 + \sum g\left(\frac{b_i}{b_1}\right) = 0$$

Subtracting and using the minimality of n we get that for all g

$$\frac{b_i}{b_1} = g\left(\frac{b_i}{b_1}\right)$$

so $\dfrac{b_i}{b_1} \in (\operatorname{Frac} B)^G = B^G = E$ by regularity. But then

$$\sum_{i=2}^{n} \frac{b_i}{b_1} d_i = 0$$

is an identity in $\mathrm{D}_B(V)$ which contradicts the independence of d_i.

2. The above proof also shows that $\alpha_{B,V} \otimes_B \operatorname{Frac} B$ is injective and a comparison of $\operatorname{Frac} B$ dimensions gives that $\dim_E \mathrm{D}_B(V) \leq \dim_F V$. Suppose now that $\dim_E \mathrm{D}_B(V) \leq \dim_F V$. Let e_i be an E-basis of $\mathrm{D}_B(V)$ and v_j an F-basis of V and let $\alpha_V(e) = Av$ where A is a matrix with $\det A \in (\operatorname{Frac} B)^\times$. To show that α_V is an isomorphism (over B not only over $\operatorname{Frac} B$) is suffices to show that $\det A \in B^\times$ and for that it is enough to show that $F \cdot \det A$ is G-stable, by regularity. But the e_i are G-invariant and so $\det \alpha_{B,V}(e_1 \wedge \ldots \wedge e_d) = \det A v_1 \wedge \ldots v_d$ is also G-invariant. Thus G acts on $F \cdot \det A$ via the inverse of the determinant of the action of G on V, and so $F \cdot \det A$ is G-stable.

□

Definition 3.6. A representation $V \in \mathrm{Rep}_F(G)$ is said to be B-admissible if α_V is an isomorphism. Write $\mathrm{Rep}_F^B(G)$ for the full subcategory of $\mathrm{Rep}_F(G)$ of representations which are B-admissible.

Theorem 3.7. *Let B be (F, G)-regular and $V \in \mathrm{Rep}_F(G)$.*

1. $\mathrm{D}_B : \mathrm{Rep}_F^B(G) \to \mathrm{Vect}_E$ *is a covariant, exact and faithful functor to the category of finite dimensional E-vector spaces.*

2. *Every subrepresentation or quotient of a B-admissible representation is B-admissible.*

3. *If V, V' are B-admissible then $\mathrm{D}_B(V) \otimes_E \mathrm{D}_B(V') \cong \mathrm{D}_B(V \otimes_F V')$ and $V \otimes_F V'$ is also B-admissible.*

4. *Exterior and symmetric powers preserve B-admissibility and commute with D_B.*

5. *If V is B-admissible then $V^\vee$ is B-admissible and $\mathrm{D}_B(V) \otimes_E \mathrm{D}_B(V^\vee) \to \mathrm{D}_B(F) = E$ is a perfect duality.*

Proof. 1. D_B is clearly covariant. To show exactness on $\mathrm{Rep}_F^B(G)$, note that if $0 \to U \to V \to W \to 0$ is an exact sequence of F-representations then also $0 \to B \otimes_F U \to B \otimes_F V \to B \otimes_F W \to 0$ is exact and thus $0 \to B \otimes_E \mathrm{D}_B(U) \to B \otimes_E \mathrm{D}_B(V) \to B \otimes_E \mathrm{D}_B(W) \to 0$ is exact and so $0 \to \mathrm{D}_B(U) \to \mathrm{D}_B(V) \to \mathrm{D}_B(W) \to 0$ is exact as E-vector spaces. That D_B is faithful follows from the fact that if $\mathrm{D}_B(f) = 0$ then $f = 0$ on $B \otimes_F V$ and by left exactness of tensoring with B it follows that $f = 0$ on V.

2. Let $0 \to V' \to V \to V'' \to 0$ such that V is B-admissible. Then by left exactness of D_B on $\mathrm{Rep}_F(G)$ it follows that $\dim_E \mathrm{D}_B(V) \leq \dim_E \mathrm{D}_B(V') + \dim_E \mathrm{D}_B(V'')$. But the left hand side is $\dim_F V = \dim_F V' + \dim_F V''$ while the right hand side is $\leq \dim_F V' + \dim_F V''$ by Proposition 3.5. Thus V' and V'' are both B-admissible.

3. The image of $\mathrm{D}_B(V) \otimes_E \mathrm{D}_B(V') \to (B \otimes_F V) \otimes_E (B \otimes_F V') \to B \otimes_F (V \otimes_F V')$ is G=invariant and so factors through $\mathrm{D}_B(V \otimes V')$. Since $\dim_E \mathrm{D}_B(V) \otimes_E \mathrm{D}_B(V') = \dim_F V \dim_F V'$ while $\dim_E \mathrm{D}_B(V \otimes V') \leq \dim_F V \dim_F V'$ it is enough to show that this map is injective. It is enough to check this after tensoring with B (as B is an algebra over a field) in which case the map is $B \otimes_E \mathrm{D}_B(V) \otimes_E \mathrm{D}_B(V') \cong (B \otimes_E \mathrm{D}_B(V)) \otimes_B (B \otimes_E \mathrm{D}_B(V')) \to B \otimes_E (V \otimes_F V')$ is simply $\alpha_V \otimes_B \alpha_{V'}$.

4. If V is B-admissible then $V^{\otimes r}$ is B-admissible by the previous part. Then $\wedge^r V$ is a quotient and so is B-admissible. Similarly for $\operatorname{Sym}^r V$.

5. Let V be B-admissible of dimension d. There exists a natural isomorphism

$$\det(V^\vee) \otimes_F \wedge^{d-1} V \cong V^\vee$$

given by $(x_1 \wedge \ldots \wedge x_d) \otimes (y^2 \wedge \ldots \wedge y_d) \mapsto (y_1 \mapsto \det(x_i(y_j)))$. Therefore it is enough to show that $\det(V^\vee)$ is B-admissible. Therefore it is enough to show this part for $d = 1$ as $\det V$ is B-admissible.

Let $V = F \cdot e$ with respect to which the action of G is given by the character $\eta : G \to F^\times$. Let $\mathrm{D}_B(V) = E \cdot (b \otimes e)$ for some $b \in B$ and G-invariance of $b \otimes e$ gives $\eta(g) = b\eta(b)^{-1}$. Then $\mathrm{D}_B(V^\vee) = E \cdot (b^{-1} \otimes e^\vee)$.

The pairing arises as the composition $\mathrm{D}_B(V) \otimes_E \mathrm{D}_B(V^\vee) \cong \mathrm{D}_B(V \otimes V^\vee) \to \mathrm{D}_B(F) = E$. Its perfectness is immediate when $\dim_F V = 1$. In general, the perfectness of the pairing is equivalent to the perfectness of the pairing $\wedge^d \mathrm{D}_B(V) \otimes_E \wedge^d \mathrm{D}_B(V^\vee) \cong \mathrm{D}_B(\wedge^d V \otimes_F \wedge^d V^\vee) \to E$. (A bilinear pairing given by a matrix A is perfect if and only if $\det A \neq 0$.) The perfectness of the latter follows from the one dimensional case.

$\square$

Example 3.8. 1. Hodge-Tate representations are the B_{HT}-admissible ones.

2. By Corollary 2.40 a $\mathbb{Q}_p$ representation V is $\mathbb{C}_p$-admissible if and only if it is potentially unramified.

3. A much easier statement, which boils down to Hilbert 90, is that V is $\overline{\mathbb{Q}_p}$-admissible if and only if V is potentially trivial.

4. There is a $\mathbb{Q}_p$ subalgebra $\mathrm{B}_{\mathrm{Sen}}$ of $\mathbb{C}_p[\![u]\!]$ (where G_K acts semilinearly via $g(u) = u + \log \chi_{\mathrm{cycl}}(g)$) such that $\mathrm{D}_{\mathrm{Sen}} = \mathrm{D}_{\mathrm{B}_{\mathrm{Sen}}}$. See the third problem set.

5. Other examples for B, some to be studied, are B_{dR} giving de Rham representation, $\mathrm{B}_{\mathrm{cris}}$ giving crystalline representations, B_{st} giving semistable representations, E^{sep} giving $\mathbb{F}_p$ representations, $\widehat{\mathcal{O}_{\mathcal{E}}^{\mathrm{ur}}}$ giving $\mathbb{Z}_p$ representations and $\widehat{\mathcal{E}^{\mathrm{ur}}}$ giving $\mathbb{Q}_p$ representations.

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2012-02-01

3.4 Hodge-Tate again

Since the ring B_{HT} is $(\mathbb{Q}_p, G_K)$ is regular we get the functor D_{HT} satisfying all the properties of the previous section. However, B_{HT} is also a graded ring.

Definition 3.9. Let $\operatorname{gr}^n \mathrm{D}_{\mathrm{HT}}(V) = V\{n\}$. Then $(\mathrm{D}_{\mathrm{HT}}(V), \operatorname{gr}^\bullet \mathrm{D}_{\mathrm{HT}}(V)) \in \mathrm{GrVect}_K$. For $D \in \mathrm{GrVect}_K$ let $\mathrm{V}_{\mathrm{HT}}(D) = \operatorname{gr}^0(\mathrm{B}_{\mathrm{HT}} \otimes_K D)$.

Proposition 3.10. *The functors $\mathrm{D}_{\mathrm{HT}} : \mathrm{Rep}^{\mathrm{HT}}_{\mathbb{C}_p}(G_K) \to \mathrm{GrVect}_K$ and $\mathrm{V}_{\mathrm{HT}} : \mathrm{GrVect}_K \to \mathrm{Rep}^{\mathrm{HT}}_{\mathbb{C}_p}(G_K)$ are quasi-inverse and they provide an equivalence of categories.*

Proof.

$$\begin{aligned}\mathrm{V_{HT}}(\mathrm{D_{HT}}(V)) &= \mathrm{gr}^0(\mathrm{B_{HT}} \otimes_K \mathrm{D_{HT}}(V)) \\ &\cong \mathrm{gr}^0(\mathrm{B_{HT}} \otimes_{\mathbb{C}_p} W) \\ &\cong \mathrm{gr}^0(\mathrm{B_{HT}}) \otimes_{\mathbb{C}_p} W \\ &\cong W\end{aligned}$$

and if $(D, \mathrm{gr}^\bullet D) \in \mathrm{GrVect}_K$ one has

$$\begin{aligned}\mathrm{D_{HT}}(\mathrm{V_{HT}}(D)) &= \oplus_{r,q}(\mathbb{C}_p(r) \otimes_{\mathbb{C}_p} \mathbb{C}_p(-q) \otimes_K \mathrm{gr}^q D)^{G_K} \\ &= \oplus_{r,q}(\mathbb{C}_p(r-q) \otimes_{\mathbb{C}_p} \mathrm{gr}^q D)^{G_K} \\ &= \oplus_r \mathrm{gr}^r D \\ &= D\end{aligned}$$

by Ax-Sen-Tate □

Remark 14. $\mathrm{D_{HT}} : \mathrm{Rep}^{\mathrm{HT}}_{\mathbb{Q}_p}(G_K) \to \mathrm{GrVect}_K$ however is not an equivalence of categories. In fact all potentially unramified $\mathbb{Q}_p$ representations have the same image via $\mathrm{D_{HT}}$ by Corollary 2.40.

4 de Rham Representations

Proposition 3.10 does not apply in the case of $\mathbb{Q}_p$ representations as nonisomorphic $\mathbb{Q}_p$ representations can become isomorphic over $\mathbb{C}_p$. A finer period ring is necessary for $\mathbb{Q}_p$ representations. What properties it should have? Inspiration comes from algebraic de Rham cohomology. It should be a filtered ring, with residue field $\mathbb{C}_p$ and graded ring $\mathrm{B_{HT}}$. Moreover, if $\mathfrak{m}$ is the maximal ideal one wants the extension $\mathfrak{m}/\mathfrak{m}^2 \to \mathrm{B}/\mathfrak{m}^2 \to \mathrm{B}/\mathfrak{m}$ to be nonsplit in order to account for the Tate curve:

The Tate curve (of some parameter q) E_q is an elliptic curve over $\mathbb{Q}_p$ with multiplicative reduction at p. Its Tate module gives $V = T_pE_q \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \in \mathrm{Rep}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p})$ such that $0 \to \mathbb{Q}_p(1) \to V \to \mathbb{Q}_p \to 0$ is not split. In fact it does not split over $\overline{\mathbb{Q}_p}$, whereas it splits over $\mathbb{C}_p$ (and therefore is Hodge-Tate).

4.1 Witt Vectors

The Witt vectors construction is supposed to generalize the Teichmüller lift $[\cdot] : k_K^\times \to \mathcal{O}_{K \cap \mathbb{Q}_p^{\mathrm{ur}}}$.

4.1.1 Definitions

Definition 4.1. Let A be a ring and consider a chain of ideals $A \supset I_1 \supset I_2 \supset \dots$ such that $A/I_1 = R$ is an $\mathbb{F}_p$ algebra and such that $I_n \cdot I_m \subset I_{n+m}$. The ring A is said to be a p-ring if it is separated and complete for the topology defined by (I_n).

Definition 4.2. A is a strict p-ring if it is a p-ring and moreover if p is not nilpotent in A.

Example 4.3.

1. $\mathbb{Z}_p$ with $I_n = p^n\mathbb{Z}_p$ is a strict p-ring with residue field $\mathbb{F}_p$, which is perfect.
2. If $K/\mathbb{Q}_p$ is finite then $\mathcal{O}_K$ with $I_n = p^n\mathcal{O}_K$ is a strict p-ring with residue field $\mathcal{O}_K/(p)$. It is perfect if and only if $K/\mathbb{Q}_p$ is unramified. Choosing instead $I_n = \varpi_K^n\mathcal{O}_K$ gives a p-ring with perfect residue field k_K. It is strict if and only if $K/\mathbb{Q}_p$ is unramified.
3. $\mathcal{O}_{\mathbb{C}_p}$ with $I_n = p^n\mathcal{O}_{\mathbb{C}_p}$ is a strict p-ring.
4. If J is any index set let $S_J = \mathbb{Z}_p[X_j^{p^{-m}}]_{j \in J, m \geq 0}$ and let $\widehat{S_J} = \varprojlim S_J/p^nS_J$. Then $\widehat{S_J}$ is a strict p-ring with perfect residue field $\overline{S_J} = \mathbb{F}_p[X_j^{p^{-m}}]_{j \in J, m \geq 0}$.

4.1.2 Perfect rings

Lemma 4.4. *Let A be a p-ring with residue ring R which is an $\mathbb{F}_p$-algebra. Let $x = (x_0, x_1, \ldots)$ with $x_i \in R$ such that $x_{i+1}^p = x_i$ and let $\widehat{x_i} \in A$ be any lift of x_i. Then $(\widehat{x_n}^{p^n})_n$ converges to some $\psi(x) \in A$ which only depends on x.*

Proof. Check by induction that $\widehat{x_n}^{p^k} \equiv \widehat{x_{n-1}}^{p^{k-1}} \pmod{I_k}$. Indeed, for $k = 1$ this follows from $x_n^p = x_{n-1}$. Then suppose $\widehat{x_n}^{p^k} = \widehat{x_{n-1}}^{p^{k-1}} + y$ for $y \in I_k$. Then

$$\begin{aligned}
\widehat{x_n}^{p^{k+1}} &= (\widehat{x_{n-1}}^{p^{k-1}} + y)^p \\
&= \widehat{x_{n-1}}^{p^k} + \sum_{i=1}^{p} \binom{p}{i} \widehat{x_{n-1}}^{ip^{k-1}} y^{p-i}
\end{aligned}$$

Since $p \in I_1$ (as $p = 0$ in R) the first term of the sum will be in $I_1 \cdot I_k \subset I_{k+1}$ while all the others will contain y^2 and so will be in $I_k \cdot I_k \subset I_{2k} \subset I_{k+1}$. Thus $(\widehat{x_n}^{p^n})$ will be a Cauchy sequence and so converges to some limit $\widehat{x}$. If $(\widehat{x_n}')$ is another choice of lifts then the same proof applies to $(\widehat{x_1}^p, (\widehat{x_2}')^{p^2}, \widehat{x_3}^{p^3}, (\widehat{x_4}')^{p^4}, \ldots)$ which will then have the common limit $\psi(x)$. ∎

Corollary 4.5. *If R is perfect and $\alpha \in R$ then let $[\alpha] = \psi(x_\alpha)$ where $x_\alpha = (\alpha, \alpha^{1/p}, \alpha^{1/p^2}, \ldots)$ exists and is unique by perfectness. This lift is the Teichmüller lift of α.*

Lemma 4.6. *If A is a string p-ring with perfect residue ring R then every $\alpha \in A$ can be written uniquely as*

$$\alpha = \sum_{n \geq 0} p^n [\alpha_n]$$

for $\alpha_n \in R$.

Proof. See problem set 3. ∎

4.1.3 Universal Witt polynomials

Lemma 4.7. *Let $[\cdot]$ be the Teichmuller lift from the perfect $\mathbb{F}_p$ algebra $\overline{S} = \mathbb{F}_p[X_i^{p^{-m}}, Y_i^{p^{-m}}]_{i,m \geq 0} \to \widehat{S} = \varprojlim \mathbb{Z}_p[X_i^{p^{-m}}, Y_i^{p^{-m}}]_{i,m \geq 0}/(p^n)$. There exist "polynomials" $\overline{S}_i, \overline{P}_i \in \overline{S}$ such that*

$$\sum_{i \geq 0} p^i [X_i] + \sum_{i \geq 0} p^i [Y_i] = \sum_{i \geq 0} p^i [\overline{S}_i]$$

$$\left(\sum_{i \geq 0} p^i [X_i] \right) \left(\sum_{i \geq 0} p^i [Y_i] \right) = \sum_{i \geq 0} p^i [\overline{P}_i]$$

Proof. This follows from Lemma 4.6 as $\widehat{S}$ is a strict p-ring with residue algebra $\overline{S}$. For example

$$\begin{aligned}
\overline{S}_0(X_0, Y_0) &= X_0 + Y_0 \\
\overline{S}_1(X_0, X_1, Y_0, Y_1) &= X_1 + Y_1 + \frac{1}{p}((X_0^{1/p} + Y_0^{1/p})^p - X_0 - Y_0)
\end{aligned}$$

and so on. ∎

Remark 15. The polynomials $\overline{S}_i$ are homogeneous of degree 1 in $X_0, \ldots, X_i$ and again of degree 1 in $Y_0, \ldots, Y_i$.

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Lemma 4.8. *Let J be a set of indices and A a strict p-ring with perfect residue ring R. Let $\overline{\theta} : \overline{S}_J \to R$ be a homomorphism and $\widetilde{\theta} : \overline{S}_J \to A$ be any multiplicative lift of $\overline{\theta}$. Then there exists a unique ring homomorphism $\widehat{\theta} : \widehat{S}_J \to A$ such that $\theta([x]) = \widetilde{\theta}(x)$.*

Proof. Define $\theta : S_J \to A$ to be the ring homomorphism such that $\theta(X_i^{p^{-m}}) = \widetilde{\theta}(X_i^{p^{-m}})$ for all i and m. Now θ is uniformly continuous so extends to a ring homomorphism on $\widehat{S}_J$. One may check by induction, using the multiplicativity of $\widetilde{\theta}$ as in the proof of Lemma 4.4, that for $x \in \overline{S}_J$

$$\theta([x]^{p^k}) - \widetilde{\theta}(x^{p^k}) \in p^{k+1}\widehat{S}_J$$

(The base case uses the fact that $\overline{\theta}$ is a ring homomorphism.) Applying this for $x^{p^{-n}} \in \overline{S}_J$ and $k = n$ shows that $\theta([x]) - \widetilde{\theta}(x) \in p^{n+1}\widehat{S}_J$ for all n which gives $\theta([x]) = \widetilde{\theta}(x)$.

Uniqueness follows from the formula

$$\theta(\sum_{i \geq 0} p^i [f_i]) = \sum_{i \geq 0} p^i \widetilde{\theta}(f_i)$$

□

Proposition 4.9. *Let A be a p-ring with perfect residue ring R. If $x = (x_0, \ldots)$ and $y = (y_0, \ldots)$ are tuples of elements in R then*

$$\sum_{i \geq 0} p^i [x_i] + \sum_{i \geq 0} p^i [y_i] = \sum_{i \geq 0} p^i [\overline{S}_i(x, y)]$$

$$\left(\sum_{i \geq 0} p^i [x_i]\right)\left(\sum_{i \geq 0} p^i [y_i]\right) = \sum_{i \geq 0} p^i [\overline{P}_i(x, y)]$$

Proof. Consider the ring homomorphism $\overline{\theta} : \overline{S} \to R$ given by $\overline{\theta}(X_i^{p^{-m}}) = x_i^{p^{-m}}$ and $\overline{\theta}(Y_i^{p^{-m}}) = y_i^{p^{-m}}$ and let $\widetilde{\theta}$ be defined multiplicatively by $\widetilde{\theta}(x) = [x]$. Then Lemma 4.8 gives $\theta : \widehat{S} \to A$ such that $\theta([x]) = \widetilde{\theta}(x)$. Then the two formulae follow immediately from Lemma 4.7:

$$\begin{aligned}
\sum_{i \geq 0} p^i [x_i] + \sum_{i \geq 0} p^i [y_i] &= \theta(\sum_{i \geq 0} p^i [X_i] + \sum_{i \geq 0} p^i [Y_i]) \\
&= \theta(\sum_{i \geq 0} p^i [\overline{S}_i]) \\
&= \sum_{i \geq 0} p^i \widetilde{\theta}(\overline{S}_i) \\
&= \sum_{i \geq 0} p^i [\overline{S}_i(x, y)]
\end{aligned}$$

and similarly for the product formula. □

4.1.4 Witt Vectors

Lemma 4.10. *If A is a strict p-ring with residue ring R and if I is a perfect ideal of R then*

$$\mathrm{W}(I) = \{\sum_{i \geq 0} p^i [x_i] | x_i \in I\}$$

is a closed ideal of A and $A/\mathrm{W}(I)$ is a strict p-ring with residue ring R/I.

Proof. $\mathrm{W}(I)$ is closed under addition because if $x = \sum p^i[x_i]$ and $y = \sum p^i[y_i]$ then $x + y = \sum p^i[\overline{S}_i(x, y)]$ and $\overline{S}_i(x, y) \in I$ as $\overline{S}_i$ is a homogeneous polynomial. Now the polynomials $\overline{P}_i$ have monomials which contain both x-s and y-s as $0 = 0 \cdot y = \sum p^i[0] \sum p^i[y_i] = \sum p^i[\overline{P}_i(0, y)]$ and so $\overline{P}_i$ cannot contain terms with only y-s. Therefore if $x \in I$ and $y \in A$ then $\overline{P}_i(x, y) \in I$ and so $xy \in \mathrm{W}(I)$.

We now have $(A/\mathrm{W}(I))/(p) = A/(\mathrm{W}(I) + pA) = (A/pA)/(\mathrm{W}(I)/(p)) = R/I$. The only thing left to check is that p is not nilpotent in $A/\mathrm{W}(I)$, i.e., that $p^k \notin \mathrm{W}(I)$ for all $k \geq 0$. But $p^k = p^0[0] + \cdots + p^{k-1}[0] + p^k[1] + p^{k+1}[0] + \cdots$ so $p^k \in \mathrm{W}(I)$ if and only if $1 \in I$. ☐

Theorem 4.11 (Witt vectors)**.** *If R is a perfect ring of characteristic p there exists a unique strict p-ring $\mathrm{W}(R)$ with residue ring R. If A is a p-ring with residue ring S and $\overline{\theta} : R \to S$ is a ring homomorphism and $\widetilde{\theta} : R \to A$ is a multiplicative lift of $\overline{\theta}$ then get a ring homomorphism $\theta : \mathrm{W}(R) \to A$ such that $\theta([x]) = \widetilde{\theta}(x)$.*

Proof. Consider a presentation of $R \cong \overline{S}_J/I$ for a perfect ideal I. (For example could choose $J = R$ with I all the relevant relations.) Let $\mathrm{W}(R) = \widehat{S}_J/\mathrm{W}(I)$. Since $\widehat{S}_J$ is a strict p-ring with residue ring $\overline{S}_J$ and I is a perfect ideal it follows that $\mathrm{W}(R)$ is a strict p-ring with residue ring $\widehat{S}_J/\mathrm{W}(I) = R$.

From Lemma 4.8 there exists a lift of the composition $\overline{S}_J \to R \to A$ to $\widehat{S}_J \to A$ and it can be checked that this induces $\theta : \widehat{S}_J/\mathrm{W}(I) \to A$. ☐

4.2 Perfections and the ring R

We would like to construct Witt vectors starting with the ring $\mathcal{O}_{\mathbb{C}_p}/(p)$, which is not perfect. To remedy this we will study perfections.

4.2.1 R

Definition 4.12. For an $\mathbb{F}_p$-algebra A let $\mathrm{R}(A) = \varprojlim_{x \mapsto x^p} A$.

Remark 16. If A is perfect then $\mathrm{R}(A) = A$ and the inverse map is $a \mapsto (a, a^{1/p}, a^{1/p^2}, \ldots)$.

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2012-02-06

Lemma 4.13. *$\mathrm{R}(\mathcal{O}_{\mathbb{C}_p}/(p))$ can be described as $\{(x^{(0)}, x^{(1)}, \ldots) | x_i \in \mathcal{O}_{\mathbb{C}_p}, x_{i+1}^p = x_i\}$ where the ring structure on the latter is given by*

$$(x^{(i)})(y^{(i)}) = (x^{(i)}y^{(i)})$$
$$(x^{(i)} + y^{(i)}) = (\lim_j (x^{(i+j)} + y^{(i+j)})^{p^j})$$
$$-(x^{(i)}) = ((-1)^p x^{(i)})$$

Proof. Let A be the latter ring and consider the morphism $A \to \mathrm{R}(\mathcal{O}_{\mathbb{C}_p}/(p))$ given by $(x^{(i)}) \mapsto (x^{(i)} \mod p)$. This is clearly a ring homomorphism and an inverse is given by $(x_i) \mapsto (\psi(x_i))$ where ψ is the lift from Lemma 4.4. (Explicitly $\psi(x_i) = \lim_j \widehat{x_{i+j}}^{p^j}$ for some lifts $\widehat{x_{i+j}}$ to $\mathcal{O}_{\mathbb{C}_p}$.) The formula for negation follows from the formula for addition. ☐

Corollary 4.14. *The ring $\mathrm{R} = \mathrm{R}(\mathcal{O}_{\mathbb{C}_p}/(p))$ is a domain.*

Proof. That R is a domain follows from Lemma 4.13 as $\mathcal{O}_{\mathbb{C}_p}$ is a domain. ☐

Proposition 4.15. *Let $v_{\mathrm{R}}((x^{(0)}, \ldots)) = v_p(x^{(0)})$. Then*

1. *v_{R} is a valuation on R;*

2. *if $x, y \in \mathrm{R}$ such that $v_{\mathrm{R}}(x) \geq v_{\mathrm{R}}(y)$ then there exists $z \in \mathrm{R}$ such that $x = yz$;*

3. *R is complete and separated with respect to v_{R};*

4. *$v_{\mathrm{R}}((x_0, \ldots)) = \lim_{n\to\infty} p^n v_p(\widehat{x_n})$ for lifts $\widehat{x_n} \in \mathcal{O}_{\mathbb{C}_p}$ where $x_n \in \mathcal{O}_{\mathbb{C}_p}/(p)$.*

Proof. 1. Renormalizing we may assume that $x^{(0)} \in \mathcal{O}_{\mathbb{C}_p}$, which will then be a lift of $\mathcal{O}_{\mathbb{C}_p}/(p)$. The multiplicativity of v_{R} is clear and we would have to show the nonarchimedean property:

$$\begin{aligned} v_{\mathrm{R}}(x+y) &= v_p((x+y)^{(0)}) \\ &= v_p(\lim_{m\to\infty}(x^{(m)} + y^{(m)})^{p^m}) \\ &= \lim_{m\to\infty} p^m v_p(x^{(m)} + y^{(m)}) \\ &\geq \lim_{m\to\infty} p^m \min(v_p(x^{(m)}), v_p(y^{(m)})) \\ &= \min(v_{\mathrm{R}}(x), v_{\mathrm{R}}(y)) \end{aligned}$$

2. It follows that $v_p(x^{(0)}/y^{(0)}) \geq 0$ so $x/y = (x^{(0)}/y^{(0)}, x^{(1)}/y^{(1)}, \ldots) \in \mathrm{R}$.

3. If $x = (x_0, x_1, \ldots) \in \varprojlim \mathcal{O}_{\mathbb{C}_p}/(p)$ then $v_{\mathrm{R}}(x) \geq n$ if and only if $x_0 = x_1 = \ldots = x_n = 0$ which implies that limits exist in $\varprojlim \mathcal{O}_{\mathbb{C}_p}/(p) = \mathrm{R}$ and that the topology is separated.

4. This follows from Lemma 4.13 as $x^{(0)} = \psi(x_0) = \lim \widehat{x_n}^{p^n}$.

$\square$

Lemma 4.16. *If $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \ldots) \in \mathrm{R}$ then $v_{\mathrm{R}}(\varepsilon - 1) = \dfrac{p}{p-1}$.*

Proof. As before we have

$$\begin{aligned} v_{\mathrm{R}}(\varepsilon - 1) &= \lim_{n\to\infty} p^n v_p(\zeta_{p^n} + (-1)^{(n)}) \\ &= \lim_{n\to\infty} p^n v_p(\zeta_{p^n} + (-1)^p) \end{aligned}$$

where the second line follows from the negation formula in Lemma 4.13. Recall that

$$v_p(\zeta_{p^n} - 1) = \frac{1}{p^{n-1}(p-1)}$$

and so $v_2(\zeta_{2^n} + 1) = v_2(\zeta_{2^n} - 1 + 2) = v_2(\zeta_{2^n} - 1) = \dfrac{1}{2^{n-1}}$. Therefore in both cases $p = 2$ and $p > 2$ we get

$$\begin{aligned} v_{\mathrm{R}}(\varepsilon - 1) &= \lim_{n\to\infty} \frac{p^n}{p^{n-1}(p-1)} \\ &= \frac{p}{p-1} \end{aligned}$$

$\square$

4.2.2 Frac R

Theorem 4.17. *Via the isomorphism $\operatorname{Frac}\mathrm{R} = \{(x^{(0)}, x^{(1)}, \ldots) | x_i \in \mathbb{C}_p, x_{i+1}^p = x_i\}$ as above $\operatorname{Frac}\mathrm{R}$ is an algebraically closed field of characteristic p.*

Proof. The explicit description for $\operatorname{Frac} \mathrm{R}$ follows from the fact that R is the subset of x with $v_{\mathrm{R}}(x) \geq 0$ and the (noncanonical) element $\widetilde{p} = (p, p^{1/p}, \ldots)$ has positive valuation.

Consider a polynomial $P(X) \in \operatorname{Frac} \mathrm{R}[X]$ and we need to show that $P(X)$ has roots in $\operatorname{Frac} \mathrm{R}$. Multiplying by a suitable power of $[p]$ we may assume that $P(X) \in \mathrm{R}[X]$. Write $P(X) = X^d + a_{d-1}X^{d-1} + \cdots + a_0$ and write $a_k = \varprojlim a_{k,n}$ where $a_{k,n} \in \mathcal{O}_{\mathbb{C}_p}/(p)$. Let $P_n(X) = X^d + a_{d-1,n}X^{d-1} + \cdots + a_{0,n} \in \mathcal{O}_{\mathbb{C}_p}[X]$ which then satisfies that $P_n(X)^p = P_{n-1}(X^p)$. Let $\widetilde{P}_n(X) \in \mathcal{O}_{\mathbb{C}_p}[X]$ be a lift of $P_n(X)$, which will then have roots $\alpha_{n,1}, \ldots, \alpha_{n,d}$. The relationship $P_n(X)^p = P_{n-1}(X^p)$ implies that $\widetilde{P}_{n-1}(\alpha_{n,i}^p) \equiv 0 \pmod p$ for all i and so $\prod_{j=1}^{d}(\alpha_{n,i}^p - \alpha_{n-1,j}) \in (p)$. Thus for at least one j it must be that $v_p(\alpha_{n,i}^p - \alpha_{n-1,j}) \geq \dfrac{1}{d}$. By induction, using the binomial formula, it's easy to see that $v_p(\alpha_{n,i}^{p^k} - \alpha_{n-1,j}^{p^{k-1}}) \geq \dfrac{k}{d}$ and so $(\alpha_{n,i}^{p^{d-1}})^p \equiv \alpha_{n-1,j}^{p^{d-1}} \pmod p$ which means that

$$\{\alpha_{n,i}^{p^{d-1}} | 1 \leq i \leq d\}^p \equiv \{\alpha_{n-1,i}^{p^{d-1}} | 1 \leq i \leq d\} \pmod p$$

and so after a reordering of the roots it follows that $\varprojlim \alpha_{n,i}^{p^{d-1}} \pmod p \in \operatorname{Frac} \mathrm{R}$. These will be the roots of P. $\square$

4.2.3 Actions on R

Definition 4.18. The Galois group G_K acts on R via $g((x^{(0)}, x^{(1)}, \ldots)) = (g(x^{(0)}), g(x^{(1)}), \ldots)$.

Definition 4.19. There is a Frobenius map φ on R given by $\varphi((x^{(0)}, x^{(1)}, \ldots)) = ((x^{(0)})^p, x^{(0)}, x^{(1)}, \ldots)$.

Lemma 4.20. *We have*

1. $\mathrm{R}^{\varphi=1} = \mathbb{F}_p$. *(In fact* $\mathrm{R}^{\varphi^r=1} = \mathbb{F}_{p^r}$.*)*
2. $\mathrm{R}^{G_K} = k_K$.

Proof.

1. Clearly $\mathrm{R}^{\varphi=1} = \{(x, x, \ldots) | x^p = x\} \cong \mathbb{F}_p$.
2. Let $x = (x^{(0)}, x^{(1)}, \ldots)$ be G_K-invariant with $x^{(i)} \in \mathcal{O}_{\mathbb{C}_p}$. Then for all $g \in G_K$ we have $x^{(i)} \in \mathbb{C}_p^{G_K} = K$. Also, using $(x^{(i+1)})^p = x^{(i)}$ it follows that $v_p(x^{(i)}) = p^{-i}v_p(x^{(0)})$ and since $x^{(i)} \in K$, and therefore has v_p valuation in $e_{K/\mathbb{Q}_p}^{-1}\mathbb{Z}$ it must be that $x^{(i)} \in \mathcal{O}_{\mathbb{C}_p}^\times$. Write $x^{(i)} = [u_i](1 + \varpi_K y_i)$ where $u_i \in k_K$ and $y_i \in \mathcal{O}_K$. Then using that $x^{(i)} = (x^{(i+j)})^{p^j}$ we get that in $1 + \mathfrak{m}_K$ we have $1 + \varpi_K y_i = (1 + \varpi_K y_{i+j})^{p^j}$. Choosing j large enough we get a contradiction unless $y_i = 0$ which implies that $x^{(i)} = [u_i]$ and so, since k_K is perfect, we get that $\mathrm{R}^{G_K} \cong k_K$.

$\square$

4.2.4 W(R)

Since R is a perfect $\mathbb{F}_p$ algebra we may construct the Witt vectors W(R) and get something well-behaved.

Lemma 4.21. *There is a ring homomorphism* $\theta : \mathrm{W}(\mathrm{R}) \to \mathcal{O}_{\mathbb{C}_p}$ *given by*

$$\theta(\sum p^n[c_n]) = \sum c_n^{(0)}p^n$$

where $c_n \in \mathrm{R}$.

Proof. Recall that $x \in \mathrm{R}$ can be described either as $x = (x^{(0)}, \ldots)$ with $x^{(i)} \in \mathcal{O}_{\mathbb{C}_p}$ or as $x = \varprojlim x_i$ with $x_i \in \mathcal{O}_{\mathbb{C}_p}/(p)$ and $x \mapsto x^{(0)}$ is a multiplicative lift of $x \mapsto x_0$. Now the existence of θ as a ring homomorphism given by the formula above follows from Theorem 4.11 applied to $\overline{\theta}(x) = x_0$ and $\widetilde{\theta}(x) = x^{(0)}$. $\square$

Definition 4.22.

1. The Galois group G_K acts on W(R) via $g(\sum p^k[x_k]) = \sum p^k[g(x_k)]$ for $x_k \in \mathrm{R}$.

2. Frobenius acts via $\varphi(\sum p^k[x_k]) = \sum p^k[\varphi(x_k)]$.

Lemma 4.23. *We have*

1. $\mathrm{W}(\mathrm{R})^{\varphi=1} = \mathrm{W}(\mathrm{R}^{\varphi=1}) = \mathrm{W}(\mathbb{F}_p) = \mathbb{Z}_p$.
2. $\mathrm{W}(\mathrm{R})^{G_K} = \mathrm{W}(\mathrm{R}^{G_K}) = \mathrm{W}(k_K) = \mathcal{O}_{K_0}$ *where* $K_0 = K \cap \mathbb{Q}_p^{\mathrm{ur}}$ *is the maximal unramified subfield of* K.

Proof. Follows from definitions and Lemma 4.20. □

Lecture 14
2012-02-08

4.3 B_{dR}

4.3.1 $\ker\theta$

For $x \in \mathrm{W}(\mathrm{R})$ let $\overline{x}$ be the image in R.

Proposition 4.24. *If* $\alpha \in \ker\theta$ *is such that* $v_{\mathrm{R}}(\overline{\alpha}) = 1$ *then* $\ker\theta = (\alpha)$ *is principal.*

Proof. First, note that $\ker\theta \cap p^n\,\mathrm{W}(\mathrm{R}) = p^n \ker\theta$ as $\mathbb{C}_p$ is torsion free. If $\theta(x) = 0$ then $\overline{x}^{(0)} \in (p)$ (as if $\theta(\sum p^n[c_n]) = \sum p^n c_n^{(0)} = 0$ then $p \mid c_0^{(0)}$). Therefore $v_{\mathrm{R}}(\overline{x}) = v_p(\overline{x}^{(0)}) \geq 1 = v_{\mathrm{R}}(\overline{\alpha})$ so there exists $\overline{y} \in \mathrm{R}$ such that $\overline{x} = \overline{\alpha y}$. This implies that $x \equiv \alpha y \pmod p$ for some $y \in \mathrm{W}(\mathrm{R})$. Suppose we can write $x = \alpha y_n + p^n v_n$ where $y_n, v_n \in \mathrm{R}$ (so $n \geq 1$). Then $\theta(p^n v_n) = 0$ and so $v_n = \ker\theta$. We just showed that we may write $v_n = \alpha w + pz$ and so writing $y_{n+1} = y_n + p^n w$ and $v_{n+1} = z$ it follows that $x = \alpha y_{n+1} + p^{n+1} v_{n+1}$. Then αy_n converges to some element of $\mathrm{W}(\mathrm{R})$ of the form αy_∞ (as $\mathrm{W}(\mathrm{R})$ is complete) which makes $x \in (\alpha)\,\mathrm{W}(\mathrm{R})$. □

Example 4.25. For example we may choose

1. $\alpha = [\widetilde{p}] - p$ or
2. $\alpha = \dfrac{[\varepsilon] - 1}{[\varepsilon^{1/p}] - 1}$

Proof. 1. Note that $\widetilde{p} \in \mathrm{R}$ lifts to $[\widetilde{p}] = [\widetilde{p}] + p\cdot 0 + p^2 \cdot 0 + \cdots \in \mathrm{W}(\mathrm{R})$ while $p = 0 + p \cdot 1 + p^2 \cdot 0 + \cdots \in \mathrm{W}(\mathrm{R})$. Therefore $\theta([\widetilde{p}]) = [\widetilde{p}^{(0)}] = [p] = p$ while $\theta(p) = p \cdot [1^{(0)}] = p$. Thus $\theta([\widetilde{p}] - p) = 0$. Also the image of $[\widetilde{p}] - p$ in R is $\widetilde{p}$ which has v_{R} valuation 1 and so $\ker\theta = ([\widetilde{p}] - p)\,\mathrm{W}(\mathrm{R})$.

2. Since $\theta([\varepsilon] - 1) = [\varepsilon^{(0)}] - 1 = 0$ but $\theta([\varepsilon^{1/p}] - 1) \neq 0$ by the same computation it follows that $\alpha \in \ker\theta$. Also, from Lemma 4.16 we get that $v_{\mathrm{R}}(\overline{\alpha}) = v_{\mathrm{R}}(\varepsilon - 1) - v_{\mathrm{R}}(\varepsilon^{1/p} - 1) = \dfrac{p}{p-1} - \dfrac{1}{p-1} = 1$ and so $\ker\theta = (\alpha)\,\mathrm{W}(\mathrm{R})$.

□

Lemma 4.26. *The* G_K *equivariant surjective ring homomorphism* $\theta : \mathrm{W}(\mathrm{R}) \to \mathcal{O}_{\mathbb{C}_p}$ *extends to a* G_K *equivariant surjective ring homomorphism* $\theta_{\mathbb{Q}} : \mathrm{W}(\mathrm{R})[1/p] \to \mathbb{C}_p$.

1. *Show that* $\mathrm{W}(\mathrm{R}) \cap (\ker\theta_{\mathbb{Q}})^k = (\ker\theta)^k$ *for all* k.
2. *Show that* $\mathrm{W}(\mathrm{R})[1/p]$ *is separated for the* $\ker\theta_{\mathbb{Q}}$*-adic topology, i.e., that* $\cap(\ker\theta_{\mathbb{Q}})^k = 0$.

Proof. 1. The extension $\theta_{\mathbb{Q}}$ is given by $\theta_{\mathbb{Q}}(\sum_{n \geq -m} p^n[c_n]) = \sum_{c \geq -m} p^n c_n^{(0)}$ and so agrees with θ on $\mathrm{W}(\mathrm{R})$, which implies the statement.

2. Suppose $x \in \cap(\ker\theta_{\mathbb{Q}})^n \subset \mathrm{W(R)}[1/p]$. For some k have $p^k x \in \mathrm{W(R)}$ and so $x \in \cap(\ker\theta)^n[1/p]$. So it is enough to show that $\cap(\ker\theta)^n = 0$.

   Any element x in $\cap(\ker\theta)^k$ is divisible by arbitrary powers of $[\widetilde{p}] - p \in \ker\theta$. But $[\widetilde{p}] - p = (\widetilde{p}, -1, \dots) \in \mathrm{W(R)}$ and so the image of x in R will be divisible by arbitrary powers of $\widetilde{p}$ in R. But R is complete and separated for v_{R} and so the image of x in R will be 0. Thus we may write $x = px'$ for $x' \in \mathrm{W(R)}$. Let α be a generator of $\ker\theta$. By Proposition 4.24 it follows that $\alpha = \sum p^n[\alpha_n]$ with $v_p(\alpha_0^{(0)}) = 1$ and so the image of α in R is not 0. Since $x \in (\ker\theta)^n$ there exists $y_n \in \mathrm{R}$ such that $x = px' = \alpha^n y_n$. Since the image of α in R is not 0, and R is a domain, it follows that the image of y_n in R is 0, and so $y_n = py_n'$. This gives $x' = \alpha^n y_n'$ and so $x' \in \cap(\ker\theta)^n$. Repeating this argument shows that x is divisible by arbitrary powers of p, and so is 0.

□

4.3.2 $\mathrm{B}_{\mathrm{dR}}^+$ and B_{dR}

Definition 4.27. Set $\mathrm{B}_{\mathrm{dR}}^+ = \varprojlim_{n\to\infty} \mathrm{W(R)}[1/p]/(\ker\theta)^n$. Projecting to the first factor gives $\theta_{\mathrm{dR}}^+ : \mathrm{B}_{\mathrm{dR}}^+ \twoheadrightarrow \mathrm{W(R)}[1/p]/\ker\theta \cong \mathbb{C}_p$.

Proposition 4.28. *The ring $\mathrm{B}_{\mathrm{dR}}^+$ is a complete discrete valuation ring, with maximal ideal $\ker\theta$, residue field $\mathbb{C}_p$ and uniformizer any choice of generator of the principal ideal $\ker\theta$.*

Proof. Let $\omega = \dfrac{[\varepsilon]-1}{[\varepsilon^{1/p}]-1}$ be a generator of $\ker\theta$. By construction every element of $\mathrm{B}_{\mathrm{dR}}^+$ can be written as a sum $x = x_0 + x_1\omega + x_2\omega^2 + \cdots$ with $x_i \in \mathrm{W(R)}[1/p]$. Let k be the smallest nonzero coefficient x_k in which case $x \in \omega^k \mathrm{B}_{\mathrm{dR}}^+ - \omega^{k+1}\mathrm{B}_{\mathrm{dR}}^+$ which gives a valuation on B_{dR}^+. This turns $\mathrm{B}_{\mathrm{dR}}^+$ into a complete discrete valuation ring for that valuation. □

Remark 17. Frobenius φ on $\mathrm{W(R)}[1/p]$ does not stabilize $\ker\theta$ and thus does not extend to $\mathrm{B}_{\mathrm{dR}}^+$.

Definition 4.29. Define $\mathrm{B}_{\mathrm{dR}} = \operatorname{Frac}\mathrm{B}_{\mathrm{dR}}^+$. It is equipped with a G_K-stable filtration $\operatorname{Fil}^n\mathrm{B}_{\mathrm{dR}} = \mathfrak{m}_{\mathrm{B}_{\mathrm{dR}}^+}^n$.

Lecture 15
2012-02-10

Lemma 4.30. *The series*

$$\log([\varepsilon]) = \sum_{n\geq 1}(-1)^{n-1}\frac{([\varepsilon]-1)^n}{n}$$

1. *converges in the dvr topology on $\mathrm{B}_{\mathrm{dR}}^+$ to a uniformizer t.*
2. *If $g \in G_K$ then $g(t) = \chi_{\mathrm{cycl}}(g)t$.*

Proof. 1. The coefficients $\frac{(-1)^{n-1}}{n}$ belong to $\mathrm{W(R)}[1/p]$ and $([\varepsilon]-1)^n \in (\ker\theta)^n$ and so t converges. To see that t is a uniformizer note that $\dfrac{t}{1-[\varepsilon]} \in (\mathrm{B}_{\mathrm{dR}})^\times$ from the definition. Also, recall from Example 4.25 that $\ker\theta$ is generated by $\dfrac{1-[\varepsilon]}{1-[\varepsilon^{1/p}]}$. Since $\theta(1-[\varepsilon^{1/p}]) \neq 0$ it follows that $1-[\varepsilon^{1/p}] \in (\mathrm{B}_{\mathrm{dR}})^\times$ and so

$$\frac{t}{\frac{1-[\varepsilon]}{1-[\varepsilon^{1/p}]}} = \frac{t}{1-[\varepsilon]}(1-[\varepsilon^{1/p}])$$

is a product of two units which implies that $\ker\theta = (t)$.

2. The equality $g(t) = \chi_{\mathrm{cycl}}(g)t$ makes sense in $\mathrm{B}_{\mathrm{dR}}^+$ endowed with the dvr topology with respect to which t makes sense. However, we will prove this equality by showing that the two sides are both equal to $\log([\varepsilon]^{\chi_{\mathrm{cycl}}(g)})$, which does not converge in the dvr topology. To make sense of this auxilliary quantity and to show that the two sides of the equality are equal to this auxilliary formula, we need to endow $\mathrm{B}_{\mathrm{dR}}^+$ with a different topology, in which the p-adic nature of the exponent $\chi_{\mathrm{cycl}}(g)$ interacts with the dvr nature of the logarithm.

   To begin with let $a \in \mathbb{Z}$. The formal power series $a\log(1+x)$ and $\log((1+x)^a) := \log(1+((1+x)^a-1))$ are equal for $a \in \mathbb{Z}_p$ and so they agree modulo x^n for all n. Plugging in $x = [\varepsilon] - 1$ we get

   $$a\log([\varepsilon]) \equiv \log([\varepsilon]^a) \pmod{([\varepsilon]-1)^n}$$

   for all $n \geq 0$. Thus the two power series are equal in the dvr topology on $\mathrm{B}_{\mathrm{dR}}^+$.

   To extend this identity to $a \in \mathbb{Z}_p$ it is enough to construct a topology on $\mathrm{B}_{\mathrm{dR}}^+$ in which $\log(1+x) = \sum(-1)^{n-1}x^n/n$ converges, is G_K-equivariant and continuous. Then for $a = \sum_{i\geq 0} a_i p^i$ we have

   $$\log((1+x)^{\sum_{i=0}^k a_i p^i}) = \left(\sum_{i=0}^k a_i p^i\right)\log(1+x)$$

   and the result follows by continuity. Such a topology is constructed in [3, Exercise 4.5.3]) by letting open sets in $\mathrm{W}(\mathrm{R})[1/p]$ be of the form

   $$U_{N,\mathfrak{a}} = \bigcup_{j>-N} (p^{-j}\,\mathrm{W}(\mathfrak{a}^{p^j}) + p^N\,\mathrm{W}(\mathrm{R}))$$

   and giving $\mathrm{B}_{\mathrm{dR}}^+$ the inverse limit topology, i.e., the coarsest topology making all projection maps $\mathrm{B}_{\mathrm{dR}}^+ \to \mathrm{W}(\mathrm{R})[1/p]/(\ker\theta)^n$ continuous for all n. Relative to this topology as well the ring $\mathrm{B}_{\mathrm{dR}}^+$ is complete.

   For $g \in G_K$ we know that $g([\varepsilon]) = [g(\varepsilon)] = [\varepsilon^{\chi_{\mathrm{cycl}}(g)}] = [\varepsilon]^{\chi_{\mathrm{cycl}}(g)}$ by definition of χ_{cycl} and so

   $$g(t) = g(\log([\varepsilon]) = \log(g([\varepsilon])) = \log([\varepsilon]^{\chi_{\mathrm{cycl}}(g)}) = \chi_{\mathrm{cycl}}(g)t$$

□

Corollary 4.31. $\mathrm{B}_{\mathrm{dR}} = \mathrm{B}_{\mathrm{dR}}[1/t]$ *and* $\mathrm{Fil}^i\,\mathrm{B}_{\mathrm{dR}} := t^i\,\mathrm{B}_{\mathrm{dR}}^+$ *for* $i \in \mathbb{Z}$ *is a* G_K*-stable filtration giving* $\mathrm{gr}^\bullet\,\mathrm{B}_{\mathrm{dR}} \cong \mathrm{B}_{\mathrm{HT}}$.

Proof. The first part follows because t is a uniformizer, while the second part follows from the fact that G_K acts on t via a scalar. The last part of the statement follows from the fact that $t^{i+1}\,\mathrm{B}_{\mathrm{dR}}^+/t^i\,\mathrm{B}_{\mathrm{dR}}^+$, a one dimensional $\mathbb{C}_p$ vector space, has G_K action via χ_{cycl}^i and so $\mathrm{gr}^i\,\mathrm{B}_{\mathrm{dR}} \cong \mathbb{C}_p(i) = \mathrm{gr}^i\,\mathrm{B}_{\mathrm{HT}}$. □

Remark 18. The reason for introducting B_{dR} is that the p-adic etale cohomology of smooth projective varieties over K are p-adic Galois representations which are B_{dR}-admissible, and there are examples where this is not true for smaller subrings of B_{dR}.

4.3.3 Cohomology of B_{dR}

Since B_{dR} is a field it is automatically $(\mathbb{Q}_p, G_K)$-regular; the goal of this section is to compute the G_K invariants in order to compute the target category of $\mathrm{D}_{\mathrm{B}_{\mathrm{dR}}}$.

Lemma 4.32. *If* $i \neq 0$ *then*

1. $H^1(G_K, t^{i+1}\,\mathrm{B}_{\mathrm{dR}}^+) = H^1(G_K, t^i\,\mathrm{B}_{\mathrm{dR}}^+)$,

2. $(t^{i+1}\,\mathrm{B}_{\mathrm{dR}}^{+})^{G_K} = (t^i\,\mathrm{B}_{\mathrm{dR}}^{+})^{G_K}$,

3. $H^1(G_K, t\,\mathrm{B}_{\mathrm{dR}}^{+}) = 0$,

4. $(\mathrm{B}_{\mathrm{dR}})^{G_K} = (\mathrm{B}_{\mathrm{dR}}^{+})^{G_K}$,

5. $(t\,\mathrm{B}_{\mathrm{dR}}^{+})^{G_K} = 0$.

Proof. 1. Lemma 4.30 implies that $t^{i+1}\,\mathrm{B}_{\mathrm{dR}}^{+}/t^i\,\mathrm{B}_{\mathrm{dR}}^{+}$, a one dimensional $\mathbb{C}_p$ vector space, has G_K action via χ_{cycl}^i and so we get a G_K equivariant exact sequence

$$0 \to t^{i+1}\,\mathrm{B}_{\mathrm{dR}}^{+} \to t^i\,\mathrm{B}_{\mathrm{dR}}^{+} \to \mathbb{C}_p(i) \to 0$$

The cohomology long exact sequence gives

$$H^0(G_K, \mathbb{C}_p(i)) \to H^1(G_K, t^{i+1}\,\mathrm{B}_{\mathrm{dR}}^{+}) \to H^1(G_K, t^i\,\mathrm{B}_{\mathrm{dR}}^{+}) \to H^1(G_K, \mathbb{C}_p(i))$$

and the statement follows from Ax-Sen-Tate and Theorem 2.25.

2. The previous long exact sequence also gives

$$0 \to H^0(G_K, t^{i+1}\,\mathrm{B}_{\mathrm{dR}}^{+}) \to H^0(G_K, t^i\,\mathrm{B}_{\mathrm{dR}}^{+}) \to H^0(G_K, \mathbb{C}_p(i))$$

and the statement follows from Ax-Sen-Tate.

3. If $M \in H^1(G_K, t^i\,\mathrm{B}_{\mathrm{dR}}^{+})$ then by the first part there exists $y \in t^i\,\mathrm{B}_{\mathrm{dR}}^{+}$ such that $g \mapsto M(g) + g(y) - y \in H^1(G_K, t^{i+1}\,\mathrm{B}_{\mathrm{dR}}^{+})$. Starting with $M_1 = M \in H^1(G_K, t\,\mathrm{B}_{\mathrm{dR}}^{+})$ obtain a sequence $M_i \in H^1(G_K, t^i\,\mathrm{B}_{\mathrm{dR}}^{+})$ obtained recursively as $M_{i+1}(g) = M_i(g) + g(y_i) - y_i$ where $y_i \in t^i\,\mathrm{B}_{\mathrm{dR}}^{+}$. Letting $y = \sum y_i$, which converges in $\mathrm{B}_{\mathrm{dR}}^{+}$ we get that $M(g) = M_1(g) + g(y) - y \in H^1(G_K, t^i\,\mathrm{B}_{\mathrm{dR}}^{+})$ for all $i \geq 0$. Now the separatedeness of $\mathrm{B}_{\mathrm{dR}}^{+}$ implies that M is trivial and so $M_1 = 0$.

4. If $x \in (\mathrm{B}_{\mathrm{dR}})^{G_K}$ then there exists $i \in \mathbb{Z}$ such that $x \in t^i\,\mathrm{B}_{\mathrm{dR}}^{+}$. Applying the second part recursively we get the statement.

5. Applying the second part we get recursively that $(t\,\mathrm{B}_{\mathrm{dR}}^{+})^{G_K} \subset t^i\,\mathrm{B}_{\mathrm{dR}}^{+}$ for each $i \geq 1$, which immediately implies the statement.

□

Theorem 4.33. *We have* $(\mathrm{B}_{\mathrm{dR}})^{G_K} = (\mathrm{B}_{\mathrm{dR}}^{+})^{G_K} = K$.

Proof. From $0 \to t\,\mathrm{B}_{\mathrm{dR}}^{+} \to \mathrm{B}_{\mathrm{dR}}^{+} \to \mathbb{C}_p \to 0$ we get

$$0 \to (t\,\mathrm{B}_{\mathrm{dR}}^{+})^{G_K} \to (\mathrm{B}_{\mathrm{dR}}^{+})^{G_K} \to \mathbb{C}_p^{G_K} \to H^1(G_K, t\,\mathrm{B}_{\mathrm{dR}}^{+})$$

Lemma 4.32 implies that this can be rewritten as

$$0 \to 0 \to (\mathrm{B}_{\mathrm{dR}}^{+})^{G_K} \to K \to 0$$

and the statement follows. □

Remark 19. The above apparently also implies a theorem of Colmez that $\overline{K} \subset \mathrm{B}_{\mathrm{dR}}^{+}$ is dense (but not so in B_{dR}).

Proposition 4.34. [1] *If* $n \in \mathbb{Z}$ *then* $H^1(G_K, t^n\,\mathrm{B}_{\mathrm{dR}}^{+}) \twoheadrightarrow H^1(G_K, \mathbb{C}_p(n))$.

Proof. This is tricky, and this "proof" is a sketch. Assuming one has a good theory of continuous cohomology in degree ≥ 2 which gives long exact sequences and commutes with limits then this would follow from $H^2(G_K, t^n\,\mathrm{B}_{\mathrm{dR}}^{+}) = 0$. This can be proven akin to Lemma 4.32 using $H^2(G_K, \mathbb{C}_p(n)) = 0$, which can be deduced from Lemma 2.27 similarly to Theorem 2.25. □

[1] not covered in class, needed for §5.6

4.4 De Rham representations

4.4.1 Filtered vector spaces

The ring B_{dR} is $(\mathbb{Q}_p, G_K)$ regular and we denote by $\mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_K)$ the category of B_{dR}-admissible representations. Then $\mathrm{D}_{\mathrm{dR}} : \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_K) \to \mathrm{Vect}_K$ will have all the properties of Theorem 3.7, but we would like to enrich the target category as we did for Hodge-Tate representations.

Definition 4.35. If $V \in \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_K)$ let $\mathrm{Fil}^i \mathrm{D}_{\mathrm{dR}}(V) := (t^i \mathrm{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$. This gives $\mathrm{D}_{\mathrm{dR}} : \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_K) \to \mathrm{FilVect}_K$.

Proposition 4.36.

1. *The functor D_{dR} to $\mathrm{FilVect}_K$ is exact, faithful, respects direct sums, tensor products, subobjects and quotients, and thus duals and symmetric and exterior powers.*
2. *The comparison B_{dR}-linear isomorphism $\alpha_{\mathrm{dR},V} : \mathrm{B}_{\mathrm{dR}} \otimes_K \mathrm{D}_{\mathrm{dR}}(V) \to \mathrm{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$ and its inverse are isomorphisms of filtered vector spaces.*

Proof.

1. This is a tedious exercise using dimension comparisons, for details see [3, Proposition 6.3.3].
2. By construction $\alpha_{\mathrm{dR},V}$ gives an isomorphism of filtered vector spaces so we now show the same for $\alpha^{-1}_{\mathrm{dR},V}$. An inductive argument reduces this to showing that $\mathrm{gr}^\bullet(\alpha_{\mathrm{dR},V})$ is an isomorphism of graded vector spaces. But this is $\alpha_{\mathrm{HT},V}$ which is such an isomorphism of graded vector spaces by construction.

□

Lecture 16
2012-02-13

Example 4.37. Let $n \in \mathbb{Z}$.

1. $\mathrm{D}_{\mathrm{dR}}(\mathbb{Q}_p(n)) = Kt^{-n}$ and so $\mathbb{Q}_p(n)$ is de Rham.
2. A representation $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ is de Rham if and only if $V(n) = V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(n)$ is de Rham.

Proof.

1. We have $(\mathrm{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(n))^{G_K} \cong (t^n \mathrm{B}_{\mathrm{dR}})^{G_K} \cong Kt^{-n}$.
2. $\mathrm{D}_{\mathrm{dR}}(V(n)) \cong \mathrm{D}_{\mathrm{dR}}(V) \otimes_K \mathrm{D}_{\mathrm{dR}}(\mathbb{Q}_p(n)) \cong \mathrm{D}_{\mathrm{dR}}(V) \otimes_K Kt^{-n}$ and the result follows by dimension comparison.

□

4.4.2 Comparison with Hodge-Tate

Lemma 4.38. *If $V \in \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_K)$ then $\mathrm{gr}^\bullet \mathrm{D}_{\mathrm{dR}}(V) \cong \mathrm{D}_{\mathrm{HT}}(V)$ and so V is also Hodge-Tate.*

Proof. Since the filtration on B_{dR} is G_K-stable it follows that

$$\begin{aligned}\mathrm{Fil}^i \mathrm{D}_{\mathrm{dR}}(V)/\mathrm{Fil}^{i+1} \mathrm{D}_{\mathrm{dR}}(V) &\cong (\mathrm{gr}^i \mathrm{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K} \\ &\cong \mathrm{gr}^i \mathrm{D}_{\mathrm{HT}}(V)\end{aligned}$$

and the result follows. □

4.4.3 Base change

For clarity we write $\mathrm{D}_{\mathrm{dR},K}(V) = (\mathrm{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$. To deal with base change for D_{dR} we need Galois descent.

Lemma 4.39 (Galois descent)**.** *Let $L/K/\mathbb{Q}_p$ be either finite Galois or $L = \widehat{K^{\mathrm{ur}}}$. Let $V \in \mathrm{Rep}_L(G_{L/K})$. Then $L \otimes_K V^{G_{L/K}} \cong V$.*

Proof. First, the case when L/K is finite. The natural multiplication map $L \otimes_K V^{G_{L/K}} \to V$ is surjective because otherwise there would exist $0 \neq \lambda \in V^* = \mathrm{Hom}_L(V, L)$ such that $\lambda(L \otimes_K V^{G_{L/K}}) = 0$. For any $a \in L$ and $v \in V$, $\sum_{g \in G_{L/K}} g(a \otimes v) \in V^{G_{L/K}}$ and so $\lambda(\sum g(a \otimes v)) = \sum g(a)\lambda(g(v)) = 0$. Picking v such that not all $\lambda(g(v))$ are 0 gives a linear relation between the $g \in G_{L/K}$ contradicting their linear independence. Now for injectivity. Suppose $e_1, \ldots, e_n$ is a K-basis of $V^{G_{L/K}}$. Suppose now that their image in V are not linearly independent over L, i.e., the map is not injective. Suppose $\sum x_i e_i = 0$ is a linear relation over L. Then for every $a \in L$ we have $\mathrm{Tr}_{L/K}(a \sum x_i e_i) = \sum \mathrm{Tr}_{L/K}(a x_i) e_i = 0$ and so by independence over K we get that $\mathrm{Tr}_{L/K}(a x_i) = 0$ for all $a \in L$. Since L/K is Galois, thus separable, the trace $\mathrm{Tr}_{L/K}$ does not vanish, implying that all the x_i are 0, contradicting our assumption.

Now suppose $L = \widehat{K^{\mathrm{ur}}}$ and let ϖ_K be a uniformizer of K and $\widehat{K^{\mathrm{ur}}}$. As usual, using the Baire category theorem, we can find a full rank $G_{\widehat{K^{\mathrm{ur}}}/K} \cong G_{k_K}$-stable lattice $\Lambda \subset V$. Then $\Lambda/\varpi_K\Lambda \in \mathrm{Rep}_{\overline{k_K}}(G_{k_K})$ and the action of G_{k_K} on $\Lambda/\varpi_K\Lambda$ has open stabilizers. This implies that $\Lambda/\varpi_K\Lambda = \varinjlim_{[l:k_K]<\infty} (\Lambda/\varpi_K\Lambda)^{G_l}$ and Galois descent in the finite case implies that $(\Lambda/\varpi_K\Lambda)^{G_l} \cong l \otimes_{k_K} (\Lambda/\varpi_K\Lambda)^{G_k}$, which in the limit gives $\overline{k_K} \otimes_{k_K} (\Lambda/\varpi_K\Lambda)^{G_{k_K}} \cong \Lambda/\varpi_K\Lambda$.

This implies that $H^1(G_{k_K}, \Lambda/\varpi_K\Lambda) = 0$ by Hilbert 90 and similarly that $H^1(G_{k_K}, \varpi_K^n\Lambda/\varpi_K^{n+1}\Lambda) = 0$ for $n \geq 0$. This then implies that $H^1(G_{k_K}, \Lambda) = 0$. We deduce that $\Lambda^{G_{k_K}}/\varpi_K\Lambda^{G_{k_K}} \cong (\Lambda/\varpi_K\Lambda)^{G_{k_K}}$. From here we get that $\Lambda^{G_{k_K}}$ is a finite free $\dim V$ dimensional $\mathcal{O}_K$-module. We then get that $\widehat{K^{\mathrm{ur}}} \otimes_{\mathcal{O}_K} \Lambda^{G_{k_K}} \cong V$ as desired. $\square$

Proposition 4.40. *If L/K is complete and discretely valued inside $\mathbb{C}_p$ then the natural map $L \otimes_K \mathrm{D}_{\mathrm{dR},K}(V) \to \mathrm{D}_{\mathrm{dR},L}(V)$ is an isomorphism. In particular, L/K can be finite, or L could be $\widehat{K^{\mathrm{ur}}}$.*

Proof. This follows from Lemma 4.39. See [3, Proposition 6.3.8]. $\square$

Example 4.41. If $\eta : G_K \to \mathbb{Z}_p^\times$ is a finite order character then there exists a finite extension L/K such that $\eta(G_L) = 1$. Therefore $\mathbb{Q}_p(\eta)$ is de Rham as a G_K representation as it is so as a G_L representation. Moreover, $\mathrm{D}_{\mathrm{dR},L}(\mathbb{Q}_p(\eta)) = L\langle 0 \rangle$ and so the same is true of $\mathrm{D}_{\mathrm{dR},K}$. Therefore, while D_{dR} is faithful, it is not fully so.

4.4.4 Characters

Example 4.42. Let $\eta : G_K \to \mathbb{Z}_p^\times$ be a continuous character. Then $\mathbb{Q}_p(\eta)$ is de Rham if and only if there exists an integer n such that $\chi_{\mathrm{cycl}}^n \eta$ is potentially unramified.

Proof. We can find $b^{-1} \otimes e_\eta \in \mathrm{D}_{\mathrm{dR}}(\mathbb{Q}_p(\eta)) = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\eta))^{G_K}$ if and only if $\eta(g) = b^{-1}g(b)$ for all $g \in G_K$, as we did in the proof of Theorem 2.18. Let $-n$ be the $\mathrm{B}_{\mathrm{dR}}^+$ valuation of b and so $t^n b \in \mathrm{B}_{\mathrm{dR}}^+ - t\,\mathrm{B}_{\mathrm{dR}}^+$. We get $(\chi_{\mathrm{cycl}}^n \eta)(g) t^n b = g(t^n b)$ and applying $\theta : \mathrm{B}_{\mathrm{dR}}^+ \to \mathbb{C}_p$, which commutes with the G_K-action, we get $(\chi_{\mathrm{cycl}}^n \eta)(g)\theta(t^n b) = g(\theta(t^n b))$. Since $t^i b \notin t\,\mathrm{B}_{\mathrm{dR}}^+$ it follows that $\theta(t^i b) \in \mathbb{C}_p - \{0\}$ and so $\theta(t^n b)^{-1} \otimes e_\eta \in \mathrm{D}_{\mathbb{C}_p}(\mathbb{Q}_p(\chi_{\mathrm{cycl}}^n \eta))$ (as in the proof of Theorem 2.18) showing that $\mathbb{Q}_p(\chi_{\mathrm{cycl}}^n \eta)$ is $\mathbb{C}_p$-admissible. The result then follows from Sen's theorem (Corollary 2.40). $\square$

5 Crystalline and Semistable Representations

5.1 $\mathrm{B}_{\mathrm{cris}}$

We will construct a K_0-subalgebra $\mathrm{B}_{\mathrm{cris}}$ of B_{dR} which, unlike B_{dR}) carries a Frobenius action.

5.1.1 $\mathrm{A}_{\mathrm{cris}}$

We will write $\xi = [\widetilde{p}] - p$ and $\omega = \dfrac{[\varepsilon] - 1}{[\varepsilon^{1/p}] - 1}$ be generators of $\ker\theta$.

Definition 5.1. Let

$$\mathrm{A}_{\mathrm{cris}} = \left\{ \sum_{n\geq 0} a_n \frac{\omega^n}{n!} | a_n \in \mathrm{W}(\mathrm{R}), a_n \to 0 \right\}$$

$$\mathrm{A}_{\mathrm{max}} = \left\{ \sum_{n\geq 0} a_n \frac{\omega^n}{p^n} | a_n \in \mathrm{W}(\mathrm{R}), a_n \to 0 \right\}$$

It is easy to see that these are rings on which G_K acts.

Definition 5.2. Let

$$\mathrm{B}^+_{\mathrm{cris}} = \mathrm{A}_{\mathrm{cris}}[1/p] \subset \mathrm{B}^+_{\mathrm{dR}}$$
$$\mathrm{B}^+_{\mathrm{max}} = \mathrm{A}_{\mathrm{max}}[1/p] \subset \mathrm{B}^+_{\mathrm{dR}}$$

Remark 20. The difference between cris and max is technical, and they will give the same notions of crystalline representations. The technical advantage if max is that p^n behaves p-adically predictably, whereas $n!$ less so. We will prove theorems for whichever of the two is more convenient.

5.1.2 t and $\mathrm{A}_{\mathrm{cris}}$

Proposition 5.3. *We have*

1. $t \in \mathrm{A}_{\mathrm{cris}}$.
2. $t^{p-1} \in p\,\mathrm{A}_{\mathrm{cris}}$.
3. *If* $x \in \ker\theta \cap \mathrm{A}_{\mathrm{cris}}$ *then* $x^n/n! \in \mathrm{A}_{\mathrm{cris}}$ *for all* $n \geq 1$. *In particular,* $t^n/n! \in \mathrm{A}_{\mathrm{cris}}$ *for all* $n \geq 1$.

Proof. 1. We have that

$$t = \sum_{n\geq 1} (-1)^{n-1} \frac{([\varepsilon]-1)^n}{n}$$
$$= \sum_{n\geq 1} (-1)^{n-1}(n-1)!([\varepsilon^{1/p}]-1)^n \frac{\omega^n}{n!}$$

which visibly is in $\mathrm{A}_{\mathrm{cris}}$.

2. To begin with note that $v_{\mathrm{R}}(\varepsilon-1) = \frac{p}{p-1}$ so $v_{\mathrm{R}}((\varepsilon-1)^{p-1}) = p = v_{\mathrm{R}}(\widetilde{p}^p)$ so there exists $r \in \mathrm{R}^\times$ such that $(\varepsilon-1)^{p-1} = \widetilde{p}^p r$. Thus $[\varepsilon-1]^{p-1} = [r](\xi+p)^p \equiv [r]\xi^p \pmod{p\,\mathrm{A}_{\mathrm{cris}}}$. But $\xi^p = p(p-1)!(\xi^p/p!) \in p\,\mathrm{A}_{\mathrm{cris}}$ so $[\varepsilon-1]^{p-1} \in p\,\mathrm{A}_{\mathrm{cris}}$. But $([\varepsilon]-1)^{p-1} - [\varepsilon-1]^{p-1} \in p\,\mathrm{W}(\mathrm{R}) \subset p\,\mathrm{A}_{\mathrm{cris}}$ and thus $\frac{([\varepsilon]-1)^{p-1}}{p} \in \mathrm{A}_{\mathrm{cris}}$. We have

$$t \equiv \sum_{n=1}^{p} (-1)^{n-1}(n-1)! \frac{([\varepsilon]-1)^n}{n!} \pmod{p\,\mathrm{A}_{\mathrm{cris}}}$$

since for $n \geq p+1$ we have $p \mid (n-1)!$. Thus

$$t \equiv ([\varepsilon]-1)(\sum_{n=1}^{p-1} (-1)^{n-1} \frac{([\varepsilon]-1)^{n-1}}{n} + (-1)^p \frac{([\varepsilon]-1)^{p-1}}{p}) \pmod{p\,\mathrm{A}_{\mathrm{cris}}}$$

and so there exist $a, b \in \mathrm{A}_{\mathrm{cris}}$ such that $t = a([\varepsilon]-1) + pb$. But the $t^{p-1} \equiv a^{p-1}([\varepsilon]-1)^{p-1} \equiv 0 \pmod{p\,\mathrm{A}_{\mathrm{cris}}}$.

3. We may write $a \in \ker\theta \cap \mathrm{A}_{\mathrm{cris}}$ as $a = \sum_{i\geq 1} a_i(\omega^i/i!)$ with $a_i \to 0$. For $N >> 0$ we have $n! \mid a_i$ for all $i \geq N$ and so to check $a^n/n! \in \mathrm{A}_{\mathrm{cris}}$ it is enough to check this for the partial sum $\sum_{i=1}^N a_i(\omega^i/i!)$. Writing $\gamma_n(x) = x^n/n!$ have

$$\gamma_n(x+y) = \sum_{i=0}^{n} \gamma_i(x)\gamma_{n-i}(y)$$
$$\gamma_m(\gamma_n(x)) = \frac{(mn)!}{m!(n!)^m}\gamma_{mn}(x)$$

and so checking that $\gamma_n(x+y) \in \mathrm{A}_{\mathrm{cris}}$ for $x, y \in \ker\theta \cap \mathrm{A}_{\mathrm{cris}}$ it is enough to do so for $\gamma_i(x)$ and $\gamma_{n-i}(y)$ for all i. Therefore it is enough to check $\gamma_n(a_i\omega^i/i!) \in \mathrm{A}_{\mathrm{cris}}$. But

$$\begin{aligned}\gamma_n(a_i\omega^i/i!) &= a_i^n\gamma_n(\gamma_i(\omega)) \\ &= a_i^n\frac{(ni)!}{n!(i!)^n}\gamma_{ni}(\omega) \\ &\in \mathrm{A}_{\mathrm{cris}}\end{aligned}$$

□

5.1.3 Regularity of $\mathrm{B}_{\mathrm{cris}}$

Definition 5.4. Let $\mathrm{B}_{\mathrm{cris}} = \mathrm{B}_{\mathrm{cris}}^+[1/t]$ and $\mathrm{B}_{\max} = \mathrm{B}_{\max}^+[1/t]$.

Remark 21. Note that $\mathrm{W}(k) \subset \mathrm{W}(\mathrm{R}) \subset \mathrm{A}_{\mathrm{cris}}$ and so $K_0 = \mathrm{W}(k)[1/p] \subset \mathrm{B}_{\mathrm{cris}}$.

Theorem 5.5. *There is an injection* $K \otimes_{K_0} \mathrm{B}_{\mathrm{cris}} \to \mathrm{B}_{\mathrm{dR}}$.

Proof. This is complicated. The original proof in [4, §4.1] is incorrect; see [3, Theorem 9.1.5]. □

Definition 5.6. We will set $\mathrm{Fil}^i(\mathrm{B}_{\mathrm{cris}}) := \mathrm{Fil}^i \mathrm{B}_{\mathrm{dR}} \cap \mathrm{B}_{\mathrm{cris}}$.

Remark 22. Note that $\mathrm{Fil}^0 \mathrm{B}_{\mathrm{cris}}$ contains and is not equal to $\mathrm{B}_{\mathrm{cris}}^+$.

Theorem 5.7. *The domains* $\mathrm{B}_{\mathrm{cris}}$ *and* $\mathrm{B}_{\max}$ *are* $(\mathbb{Q}_p, G_K)$*-regular with* $\mathrm{B}_{\mathrm{cris}}^{G_K} = \mathrm{B}_{\max}^{G_K} = K_0$.

Proof. We will do this for $\mathrm{B}_{\mathrm{cris}}$ only. First note that $K_0 \subset \mathrm{B}_{\mathrm{cris}}^{G_K} \subset (\operatorname{Frac}\mathrm{B}_{\mathrm{cris}})^{G_K}$. From the previous theorem we deduce that $K \otimes_{K_0} \operatorname{Frac}\mathrm{B}_{\mathrm{cris}} \hookrightarrow \mathrm{B}_{\mathrm{dR}}$ and so $K \otimes_{K_0} (\operatorname{Frac}\mathrm{B}_{\mathrm{cris}})^{G_K} = (K \otimes_{K_0} \operatorname{Frac}\mathrm{B}_{\mathrm{cris}})^{G_K} \hookrightarrow (\mathrm{B}_{\mathrm{dR}})^{G_K} = K$. This implies that $(\operatorname{Frac}\mathrm{B}_{\mathrm{cris}})^{G_K} = \mathrm{B}_{\mathrm{cris}}^{G_K} = K_0$ is a field.

For the second condition of regularity, pick $b \in \mathrm{B}_{\mathrm{cris}} - \{0\}$ such that $\mathbb{Q}_p b$ is G_K-stable. Since $\mathbb{Q}_p t$ is G_K stable and by construction t is invertible in $\mathrm{B}_{\mathrm{cris}}$ we may assume $b \in \mathrm{B}_{\mathrm{dR}}^+ - t\,\mathrm{B}_{\mathrm{dR}}^+$. Let $\overline{b} \neq 0$ be the image in $\mathrm{B}_{\mathrm{dR}}^+/t\,\mathrm{B}_{\mathrm{dR}}^+ = \mathbb{C}_p$ of b. Write $\eta : G_K \to \mathbb{Q}_p^\times$ for the character of G_K acting on the line $\mathbb{Q}_p b$; then η is continuous as it encodes the continuous action of G_K on the line $\mathbb{Q}_p\overline{b}$ where $\overline{b} \in \mathbb{C}_p$. As before, this shows that $\mathbb{Q}_p(\eta)$ is $\mathbb{C}_p$-admissible and by Sen's theorem (Corollary 2.40) we get that η is potentially unramified. Let L/K be a finite extension such that $\eta(I_L) = 1$ which implies that $b \in \widehat{L^{\mathrm{ur}}} = \mathbb{C}_p^{I_L}$.

Let $P \in \widehat{K^{\mathrm{ur}}}[X]$ be the minimal polynomial of $\overline{b}$ over $\widehat{K^{\mathrm{ur}}}$. Then for $g \in I_K$ it must be that $g(P(\overline{b})) = P(g(\overline{b})) = 0$ and so $g(\overline{b}) = \eta(g)\overline{b}$ is also a root of $P(X)$. Thus all the roots of $P(X)$ in $\mathbb{C}_p$ are of the form $\eta(g)\overline{b}$ for $g \in I_K$. Since $\widehat{K^{\mathrm{ur}}}$ is finite over $\widehat{\mathbb{Q}_p^{\mathrm{ur}}} = \mathrm{W}(\overline{k_K})[1/p]$ and since R is algebraically closed it follows that $\widehat{K^{\mathrm{ur}}} \subset \mathrm{W}(\mathrm{R})[1/p] \subset \mathrm{B}_{\mathrm{dR}}^+$ (but recall that there is no section to $\mathrm{B}_{\mathrm{dR}}^+ \twoheadrightarrow \mathbb{C}_p$) it follows that $P \in \mathrm{B}_{\mathrm{dR}}^+[X]$.

The polynomial P is separable over $\mathbb{C}_p$ and so Hensel's lemma applied to the complete dvr $\mathrm{B}_{\mathrm{dR}}^+$ shows that there exists β algebraic over $\mathrm{B}_{\mathrm{dR}}^+$ lifting $\bar{b}$ such that $P(\beta) = 0$. For $g \in I_K$ let $Q(X) = P(\eta(g)X)$ which has $\bar{b}$ as a root and has the same degree as the minimal polynomial P and so $Q(X) = P(X)$ in $\widehat{K^{\mathrm{ur}}}[X]$. Applying Hensel's lemma to the root $\bar{b}$ of $Q(X)$ shows there exists β_g algebraic over $\mathrm{B}_{\mathrm{dR}}^+$ such that β_g lifts $\bar{b}$. But then $\eta(g)\beta_g$ is a root of P lifting $\bar{b}$ and separability implies that $\eta(g)\beta_g = \beta$. This shows that the roots of $P \in \mathrm{B}_{\mathrm{dR}}^+[X]$ are of the form $\eta(g)\beta$ for $g \in I_K$. But since the coefficients of P are I_K-invariant the roots of P are also of the form $g(\beta)$. Since $g(\beta) - \eta(g)\beta \equiv 0 \pmod{t\,\mathrm{B}_{\mathrm{dR}}^+}$ it follows that $g(\beta) = \eta(g)\beta$.

Suppose $b \neq \beta$. Then $\mathbb{Q}_p(b-\beta)$ is a G_K-stable line in $\mathrm{B}_{\mathrm{dR}}^+$. Since $b - \beta \in \mathrm{Fil}^1\,\mathrm{B}_{\mathrm{dR}}^+$ let $r \geq 1$ be such that $b-\beta \in t^r\,\mathrm{B}_{\mathrm{dR}}^+ - t^{r+1}\,\mathrm{B}_{\mathrm{dR}}^+$. This gives a G_K-stable line $\mathbb{Q}_p(b-\beta)$ in $t^r\,\mathrm{B}_{\mathrm{dR}}^+/t^{r+1}\,\mathrm{B}_{\mathrm{dR}}^+ \cong \mathbb{C}_p(r)$ or alternatively that $\chi_{\mathrm{cycl}}^{-r}\eta$ is $\mathbb{C}_p$-admissible. But $r \geq 1$, η is finitely ramified and χ_{cycl} is infinitely ramified and so $\chi_{\mathrm{cycl}}^{-r}\eta$ is infinitely ramified and Sen's theorem shows that $\chi_{\mathrm{cycl}}^{-r}\eta$ cannot be $\mathbb{C}_p$ admissible.

Therefore $b = \beta$ and so $P(b) = 0$. Writing $P(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0$ with $a_0 \neq 0$ since P is irreducible we have $b^{-1} = -a_0^{-1}(b^{n-1} + a_{n-1}b^{n-2} + \cdots + a_1)$ and so $b \in \mathrm{B}_{\mathrm{cris}}^\times$. $\square$

Proposition 5.8. *A continuous character $\eta : G_K \to \mathbb{Q}_p^\times$ is crystalline, i.e., $\mathrm{D}_{\mathrm{B}_{\mathrm{cris}}}(\mathbb{Q}_p(\eta))$ is one dimensional if and only if $\eta = \chi_{\mathrm{cycl}}^n\mu$ for some integer n and unramified character μ.*

Proof. If $b \otimes e_\eta \in \mathrm{D}_{\mathrm{B}_{\mathrm{cris}}}(\mathbb{Q}_p(\eta))$ is nonzero then $\eta(g)g(b) = b$ and so $\mathbb{Q}_p \cdot b$ is G_K-stable. Rescaling by t^n for some integer n, the proof of Theorem 5.7 shows that b is algebraic over $\widehat{K^{\mathrm{ur}}}$. Let $L = \widehat{K_0^{\mathrm{ur}}}(b)$ be finite over $\widehat{K_0^{\mathrm{ur}}}$. Then Theorem 5.5 shows that $L \otimes_{L_0} \mathrm{B}_{\mathrm{cris}} \hookrightarrow \mathrm{B}_{\mathrm{dR}}$. But $L_0 = L \cap \widehat{K_0^{\mathrm{ur}}} = \widehat{K_0^{\mathrm{ur}}}$ and $L \subset \mathrm{B}_{\mathrm{cris}}$ so we get $L \otimes_{L_0} L \hookrightarrow \mathrm{B}_{\mathrm{dR}}$ which implies that $L \otimes_{L_0} L$ is a field and so $L = L_0$ giving $b \in \widehat{K_0^{\mathrm{ur}}}$ and so $\eta(I_K) = 1$ thus η is unramified. Rescaling back we get that $\eta = \chi_{\mathrm{cycl}}^n\mu$ for μ unramified. $\square$

5.1.4 Frobenius on $\mathrm{A}_{\mathrm{cris}}$

The Frobenius φ on $\mathrm{W(R)}[1/p]$ does not extend to B_{dR} because φ does not preserve $\ker\theta$. However one can define φ on $\mathrm{A}_{\mathrm{cris}}$ via a different description of the ring.

Writing $\mathrm{A}_{\mathrm{cris}}^0 = \mathrm{W(R)}[\omega^n/n!]_{n\geq 1}$ it can be shown that $\varprojlim A_{\mathrm{cris}}^0/p^n\,\mathrm{A}_{\mathrm{cris}}^0$ injects into $\mathrm{B}_{\mathrm{dR}}^+$ with image equal to $\mathrm{A}_{\mathrm{cris}}$ (this is hard, the content of [3, Proposition 9.1.1]).

Lemma 5.9. *The* $\mathrm{W(R)}$*-algebra* $\mathrm{A}_{\mathrm{cris}}^0 \subset \mathrm{W(R)}[1/p]$ *is stable under* φ.

Proof. Note that

$$\begin{aligned}
\varphi(\omega) &= \varphi(\sum_{i=0}^{p-1}[\varepsilon^{i/p}]) \\
&= p + \sum_{i=0}^{p-1}([\varepsilon^i] - 1) \\
&= p + \omega a
\end{aligned}$$

for some $a \in \mathrm{W(R)}$. Then

$$\begin{aligned}
\varphi(\frac{\omega^m}{m!}) &= \frac{(p+\omega a)^m}{m!} \\
&= \sum_{k=0}^{m} \frac{p^{m-i}a^i}{(m-i)!}\frac{\omega^i}{i!}
\end{aligned}$$

which belongs to $\mathrm{A}_{\mathrm{cris}}^0$ as $n! \mid p^n$ in $\mathbb{Z}_p$ for all n. $\square$

Definition 5.10. Let φ be the Frobenius endomorphism on $\mathrm{A}_{\mathrm{cris}}$ obtained as the p-adic completion of φ on $\mathrm{A}_{\mathrm{cris}}^0$. Also get φ on $\mathrm{B}_{\mathrm{cris}}^+$.

Lemma 5.11. *We have $\varphi(t) = pt$ and thus φ extends to an endomorphism of* $\mathrm{B}_{\mathrm{cris}}$.

Proof.

$$\begin{aligned}\varphi(t) &= \varphi(\sum_{n\geq 1}(-1)^{n-1}([\varepsilon]-1)^n/n) \\ &= \sum_{n\geq 1}(-1)^{n-1}([\varepsilon^p]-1)^n/n \\ &= \log([\varepsilon^p]) \\ &= pt\end{aligned}$$

□

Theorem 5.12. *The Frobenius $\varphi : \mathrm{A}_{\mathrm{cris}} \to \mathrm{A}_{\mathrm{cris}}$ is injective.*

Proof. This is hard. □

Lecture 18
2012-02-17

5.2 The fundamental exact sequence

The fundamental exact sequences compute the fixed points of Frobenius acting on various rings of periods. It is important for proving basic properties of $\mathrm{D}_{\mathrm{cris}}$ and D_{st}, but also for constructing the Bloch-Kato exponential and for working with ordinary Galois representations.

Theorem 5.13 (The fundamental exact sequence)**.** *The following sequences are exact*

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow \mathrm{B}_{\mathrm{cris}}^{\varphi=1} \longrightarrow \mathrm{B}_{\mathrm{dR}} / \mathrm{B}_{\mathrm{dR}}^+ \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow \mathrm{Fil}^0 \mathrm{B}_{\mathrm{cris}} \xrightarrow{\varphi-1} \mathrm{B}_{\mathrm{cris}} \longrightarrow 0$$

We will only prove the version for $\mathrm{B}_{\max}$ in Theorem 5.20, which is technically less involved, but whose proof contains all the relevant ideas:

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow \mathrm{B}_{\max}^{\varphi=1} \longrightarrow \mathrm{B}_{\mathrm{dR}} / \mathrm{B}_{\mathrm{dR}}^+ \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow \mathrm{Fil}^0 \mathrm{B}_{\max} \xrightarrow{\varphi-1} \mathrm{B}_{\max} \longrightarrow 0$$

5.2.1 Frobenius on W(R)

Definition 5.14. For a subring $A \subset \mathrm{A}_{\mathrm{cris}}$ we will write $I^{[r]}(A) = \cap_{n\geq 0}\varphi^{-n}(A \cap \mathrm{Fil}^r \mathrm{A}_{\mathrm{cris}})$ whre $\mathrm{Fil}^r \mathrm{A}_{\mathrm{cris}} = \mathrm{A}_{\mathrm{cris}} \cap \mathrm{Fil}^r \mathrm{B}_{\mathrm{cris}}$. We will write $I^{[r]} = I^{[r]}(\mathrm{A}_{\mathrm{cris}})$.

For simplicity we'll prove this for $\mathrm{B}_{\max}$ instead of $\mathrm{B}_{\mathrm{cris}}$.

Lemma 5.15. *We have $I^{[r]}(\mathrm{W(R)}) = ([\varepsilon]-1)^r\, \mathrm{W(R)}$.*

Proof. We first show the lemma for $r = 1$. Let $I_n = \left(\frac{[\varepsilon]-1}{[\varepsilon^{p^{-n}}]-1}\right) \mathrm{W(R)}$ in which case $\ker\theta = I_1$ and we first show that $\cap\varphi^{-n}(\ker\theta) = \cap I_n$.

If $x \in \cap \varphi^{-n}(\ker\theta)$ then $\theta(\varphi^n(x)) = 0$ for all x and so $\varphi^n(x) \in I_1$ giving $x \in \left(\frac{[\varepsilon^{p^{-n}}]-1}{[\varepsilon^{p^{-n-1}}]-1}\right) \mathrm{W(R)}$ and we write

$$x = \left(\frac{[\varepsilon^{p^{-n}}]-1}{[\varepsilon^{p^{-n-1}}]-1}\right) x_n$$

for $x_{n-1} \in \mathrm{W(R)}$. But then $\theta(\varphi^{n-1}(x)) = 0$ and so

$$\theta(\varphi^{n-1}\left(\frac{[\varepsilon^{p^{-n}}]-1}{[\varepsilon^{p^{-n-1}}]-1}\right))\theta(\varphi^{n-1}(x_n)) = 0$$

and this gives $\theta(\varphi^{n-1}(x_n)) = 0$. Applying the above we may write

$$x_{n-1} = \left(\frac{[\varepsilon^{p^{-(n-1)}}]-1}{[\varepsilon^{p^{-n}}]-1}\right) x_{n-1}$$

for $x_{n-2} \in \mathrm{W(R)}$ and continuing we get that

$$x = \left(\frac{[\varepsilon]-1}{[\varepsilon^{p^{-n-1}}]-1}\right) x_0 \in I_{n+1}$$

and so $\cap \varphi^{-n}(\ker\theta) \subset \cap I_n$. The reverse inclusion is obviously true.

It's now clear that $([\varepsilon]-1)\,\mathrm{W(R)} \subset \cap I_n$ so we only need to show the reverse inclusion to conclude the lemma for $r = 1$. If $x \in \cap I_n$ then writing $\overline{x}$ for the image of x in R we get that

$$v_{\mathrm{R}}(\overline{x}) \geq \frac{p}{p-1} - \frac{p}{p^n(p-1)}$$

for all $n \geq 1$ and so $v_{\mathrm{R}}(\overline{x}) \geq \frac{p}{p-1} = v_{\mathrm{R}}(\varepsilon - 1)$. Thus we may write $x = ([\varepsilon]-1)y + pz$ for $y, z \in \mathrm{W(R)}$ and it is easy to see that in that case we also get that $z \in \cap\varphi^{-n}(\ker\theta) = \cap I_n$. We then repeat and conclude that $x \in ([\varepsilon]-1)\,\mathrm{W(R)}$ as in the proof of Proposition 4.24.

The general statement now comes from the fact that

$$\cap\varphi^{-n}((\ker\theta)^r) = \cap(\varphi^{-n}(\ker\theta))^r$$

since φ is multiplicative. $\square$

5.2.2 Frobenius on $\mathrm{A}_{\max}$

Lemma 5.16. *If $x \in \mathrm{A}_{\max}$ such that $\theta(\varphi^n(x)) = 0$ (i.e., $x \in I^{[1]}\,\mathrm{A}_{\max}$) then $([\varepsilon]-1)/p \mid \varphi(x)$ in $\mathrm{A}_{\max}$.*

Proof. We compute

$$\begin{aligned} \varphi(\frac{\omega}{p}) &= \varphi(\frac{1}{p}\sum_{i=0}^{p-1}[\varepsilon^{i/(p-1)}]) \\ &= 1 + \sum_{i=0}^{p-1}\frac{[\varepsilon^{i/(p-1)}]-1}{p} \\ &\equiv 1 \pmod{\frac{[\varepsilon]-1}{p}\,\mathrm{A}_{\max}} \end{aligned}$$

Writing $x = \sum_{n\geq 0} a_n \frac{\omega^n}{p^n}$ we get that

$$\begin{aligned} \varphi^k(x) &\equiv \sum_{n\geq 0}\varphi^k(a_n) \pmod{\frac{[\varepsilon]-1}{p}\,\mathrm{A}_{\max}} \\ &\equiv \varphi^k(\sum a_n) \pmod{\frac{[\varepsilon]-1}{p}\,\mathrm{A}_{\max}} \end{aligned}$$

and so $\varphi^k(\sum a_n) = \varphi^k(x) + \frac{[\varepsilon]-1}{p}\alpha_k$ for some $\alpha_k \in \mathrm{A}_{\max}$. But then $\theta(\varphi^k(\sum a_n)) = 0$ by assumption on x and so $\varphi(\sum a_n) \in \mathrm{W(R)}$ is in the ideal generated by $[\varepsilon]-1$ by Lemma 5.15. Thus $\frac{[\varepsilon]-1}{p}$ divides $\varphi(x)$, since $[\varepsilon]-1$ divides $\varphi(\sum a_n)$. ☐

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5.2.3 Frobenius on $\mathrm{B}^+_{\max}$

Proposition 5.17. *If $x \in \mathrm{B}^+_{\max}$ such that for all $i \geq 0$ we have $\theta(\varphi^i(x)) = 0$ then $t \mid \varphi(x)$ in $\mathrm{B}^+_{\max}$.*

Proof. We may scale x by powers of p and assume $x \in \mathrm{A}_{\max}$. Then $\frac{[\varepsilon]-1}{p} \mid \varphi(x)$ in $\mathrm{A}_{\max}$ which implies that $[\varepsilon]-1 \mid \varphi(x)$ in $\mathrm{B}^+_{\max}$. Now

$$\frac{t}{[\varepsilon]-1} = 1 + \sum_{n\geq 2} \frac{(1-[\varepsilon])^{n-1}}{n} \in \mathrm{A}_{\max}$$

But $\frac{([\varepsilon]-1)^{n-2}}{n} \in \mathrm{A}_{\max}$ for $n \geq 2$ and so $\frac{t}{[\varepsilon]-1}$ is in fact a unit. Thus $t \mid [\varepsilon]-1 \mid \varphi(x)$ in $\mathrm{B}^+_{\max} = \mathrm{A}_{\max}[1/p]$. ☐

5.2.4 Frobenius invariants on $\mathrm{B}^+_{\max}$

Lemma 5.18. *We have*

1. $\mathrm{A}_{\max}^{\varphi=1} = \mathbb{Z}_p$.
2. $(\mathrm{B}^+_{\max})^{\varphi=1} = \mathbb{Q}_p$.

Proof.
1. Let $x \in \mathrm{A}_{cris}^{\varphi=1}$. We may write $x = \sum_{n\geq 0} x_n \frac{[\widetilde{p}]^n}{p^n}$ (as ω and $[\widetilde{p}]-p$ differ by a unit in $\mathrm{W(R)}$). Since $\varphi(x) = x$ it follows that for all $i \geq 1$ one has $\varphi^i(x) = x$ and so

$$\begin{aligned} x &= \varphi^i(x) \\ &= \varphi^i(x_0) + \sum_{n\geq 1} \varphi^i(x_n) \frac{[\widetilde{p}]^{p^i n}}{p^n} \\ &= \varphi^i(x_0) + \sum_{n\geq 1} \varphi^i(x_n) \frac{[\widetilde{p}]^{p^i n}}{p^{p^i n}} p^{n(p^i-1)} \end{aligned}$$

   In the topology on $\mathrm{A}_{\max}$ given by uniform convergence of the coefficients of the power series we get that

$$\lim_{i\to\infty} \sum_{n\geq 1} p^{n(p^i-1)} \varphi^i(x_n) \frac{[\widetilde{p}]^{p^i n}}{p^{p^i n}} = 0$$

   We conclude that $\lim\limits_{i\to\infty} \varphi^i(x) = \lim\limits_{i\to\infty} \varphi^i(x_0) \in \mathrm{W(R)}$ since $\mathrm{W(R)}$ is complete. But then $x \in \mathrm{W(R)}^{\varphi=1} = \mathbb{Z}_p$.

2. Follows from the first part after inverting p.

☐

Proposition 5.19. *For $k \geq 0$ there is an exact sequence*

$$0 \longrightarrow \mathbb{Q}_p t^k \longrightarrow (\mathrm{B}^+_{\max})^{\varphi=p^k} \longrightarrow \mathrm{B}^+_{\mathrm{dR}}/t^k\,\mathrm{B}^+_{\mathrm{dR}} \longrightarrow 0$$

Proof. When $k = 0$ this follows from Lemma 5.18. Suppose $k \geq 1$. Injectivity is clear so we now show by induction on k exactness in the middle. Let $x \in (\mathrm{B}^+_{\max})^{\varphi=p^k} \cap t^k \mathrm{B}^+_{\mathrm{dR}}$. We want to show that $x \in \mathbb{Q}_p t^k$. Since $t \mid t^k \mid x$ it follows that $\theta(\varphi^n(x)) = \theta(p^{nk}x) = 0$ for all $n \geq 0$ and so Proposition 5.17 implies that $t \mid \varphi(x)$ in $\mathrm{B}^+_{\max}$ (note that this is automatic in $\mathrm{B}^+_{\mathrm{dR}}$, but that is not enough). Writing $x = ty$ it follows that $y \in (\mathrm{B}^+_{\max})^{\varphi=p^{k-1}} \cap t^{k-1} \mathrm{B}^+_{\mathrm{dR}}$ which by the inductive hypothesis is $\mathbb{Q}_p t^{k-1}$. Thus $x \in \mathbb{Q}_p t^k$.

For surjectivity take for granted that $\log : 1 + \mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}} \to \mathbb{C}_p$ is surjective. (This is proven using Newton polygons for power series.) Let $x \in \mathrm{B}^+_{\mathrm{dR}}$ and let $\alpha \in 1 + \mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}}$ such that $(\log \alpha)^k = \theta(x)$.

Since the image of $[\alpha]-1$ in R has v_{R}-valuation > 0 there exists $k > 0$ such that $v_{\mathrm{R}}(([\alpha]-1)^k \mod p) \geq 1$ and so $([\alpha] - 1)^k \in \ker\theta$. Then

$$\begin{aligned}
\log([\alpha]) &= \sum_{n\geq 1}(-1)^{n-1}\frac{([\alpha]-1)^n}{n} \\
&= \sum_{n\geq 1}(-1)^{n-1}([\alpha]-1)^{(n \bmod k)}\frac{([\alpha]-1)^{\lfloor n/k\rfloor}}{n} \\
&= \sum_{n\geq 1}(-1)^{n-1}([\alpha]-1)^{(n \bmod k)}\frac{p^{\lfloor n/k\rfloor}}{n}\frac{([\alpha]-1)^{\lfloor n/k\rfloor}}{p^{\lfloor n/k\rfloor}}
\end{aligned}$$

and this convergen in $\mathrm{B}^+_{\max} = \mathrm{A}_{\max}[1/p]$ since $v_p(\frac{p^{\lfloor n/k\rfloor}}{n}) \to \infty$. Therefore $\log([\alpha]) \in \mathrm{B}^+_{\max}$ and $\varphi(\log([\alpha])) = p\log([\alpha])$ the same way we showed that $\varphi(t) = pt$.

Now $x - (\log[\alpha])^k \in \ker\theta$ (since $\theta(x) = (\log(\alpha))^k$). Thus we may write $x - (\log[\alpha])^k = ty$ for some $y \in \mathrm{B}^+_{\mathrm{dR}}$. By induction, there exists $z \in (\mathrm{B}^+_{\max})^{\varphi=p^{k-1}}$ mapping to y, i.e., $z \equiv y \pmod{t^{k-1}\mathrm{B}^+_{\mathrm{dR}}}$. Now $(\log[\alpha])^k + tz \equiv x \pmod{t^k \mathrm{B}_{\mathrm{dR}}}$ and

$$\varphi((\log[\alpha])^k + tz) = p^k((\log[\alpha])^k + tz)$$

and so $(\log[\alpha])^k + tz \in (\mathrm{B}^+_{\max})^{\varphi=p^k}$. $\square$

5.2.5 The fundamental sequence for $\mathrm{B}_{\max}$

We will prove the fundamental exact sequence for $\mathrm{B}_{\max}$ instead of $\mathrm{B}_{\mathrm{cris}}$, the latter being more technically involved.

Theorem 5.20. *The following sequences are exact*

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow \mathrm{B}^{\varphi=1}_{\max} \longrightarrow \mathrm{B}_{\mathrm{dR}} / \mathrm{B}^+_{\mathrm{dR}} \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow \mathrm{Fil}^0 \mathrm{B}_{\max} \xrightarrow{\varphi-1} \mathrm{B}_{\max} \longrightarrow 0$$

Proof. Dividing by t^k in Proposition 5.19 gives

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow (t^{-k}\mathrm{B}^+_{\max})^{\varphi=1} \longrightarrow t^{-k}\mathrm{B}^+_{\mathrm{dR}} / \mathrm{B}^+_{\mathrm{dR}} \longrightarrow 0$$

and taking a limit over k gives the first exact sequence.

Exactness in the middle in the first sequence gives

$$\begin{aligned}
\mathbb{Q}_p &= \ker(\varphi - 1 : \mathrm{B}_{\max} \to \mathrm{B}_{\mathrm{dR}} / \mathrm{B}^+_{\mathrm{dR}}) \\
&= \ker(\varphi - 1 : \mathrm{B}_{\max} \to \mathrm{B}_{\mathrm{dR}}) \cap \mathrm{B}^+_{\mathrm{dR}} \\
&= (\mathrm{B}_{\max})^{\varphi=1} \cap \mathrm{B}^+_{\mathrm{dR}} \\
&= (\mathrm{B}_{\max} \cap \mathrm{B}^+_{\mathrm{dR}})^{\varphi=1} \\
&= (\mathrm{Fil}^0 \mathrm{B}_{\max})^{\varphi=1}
\end{aligned}$$

which gives exactness in the middle for the second sequence. We only need to show that $1-\varphi : \operatorname{Fil}^0 \mathrm{B}_{\max} \to \mathrm{B}_{\max}$ is surjective. Let's first show that $1-\varphi : \mathrm{B}_{\max} \to \mathrm{B}_{\max}$ is surjective. Since $\mathrm{B}_{\max} = \cup_i t^{-i}\mathrm{B}^+_{\max}$ it is enough to show that $1-\varphi : t^{-i}\mathrm{B}^+_{\max} \to t^{-i}\mathrm{B}^+_{\max}$ is surjective which is equivalent to showing that $1-p^{-i}\varphi : \mathrm{B}^+_{\max} \to \mathrm{B}^+_{\max}$ is surjective.

Formally we have

$$\begin{aligned}
\frac{1}{1-p^{-i}\varphi} &= \sum_{k\geq 0}(p^{-i}\varphi)^k \\
-\frac{1}{p^{-i}\varphi(1-p^{i}\varphi^{-1})} &= -\frac{1}{p^{-i}\varphi}\sum_{k\geq 0}(p^{i}\varphi^{-1})^k \\
&= -\sum_{k\geq 1}(p^{i}\varphi^{-1})^k
\end{aligned}$$

and so

$$\frac{1}{1-p^{-i}\varphi} = -\sum_{k\geq 1}(p^{i}\varphi^{-1})^k$$

Now if $x \in \mathrm{B}^+_{\max}$ written as $x = \sum x_n \frac{[\widetilde{p}]^n}{p^n}$ then the previous formal equation gives

$$\begin{aligned}
x_0 &= -(1-p^{-i}\varphi)\sum_{k\geq 1}(p^{i}\varphi^{-i})^k(x_0) \\
&= (1-p^{-i}\varphi)(A)
\end{aligned}$$

which converges in the p-adic topology.

Similarly

$$\begin{aligned}
\sum_{n\geq 1} x_n\frac{[\widetilde{p}]^n}{p^n} &= (1-p^{-i}\varphi)\sum_{n\geq 1}\sum_{k\geq 0}(p^{-i}\varphi)^k(x_n\frac{[\widetilde{p}]^n}{p^n}) \\
&= (1-p^{-i}\varphi)\sum_{n\geq 1}\sum_{k\geq 0}\varphi^k(x_n)\frac{[\widetilde{p}]^{p^k n}}{p^{ik+n}} \\
&= (1-p^{-i}\varphi)\sum_{n\geq 1}\sum_{k\geq 0}\varphi^k(x_n)\frac{[\widetilde{p}]^{p^k n}}{p^{p^k n}}p^{p^k n-ik-n}
\end{aligned}$$

and this sum converges in $\mathrm{B}^+_{\max}$. Therefore $\sum_{n\geq 1} x_n\frac{[\widetilde{p}]^n}{p^n} = (1-p^{-i}\varphi)(B)$ and we deduce that $x = (1-p^{-i}\varphi)(A+B)$ with $A+B \in \mathrm{B}^+_{\max}$.

To see that the second exact sequence is exact we need to show that $1-\varphi : \mathrm{B}_{\max}\cap\mathrm{B}^+_{\mathrm{dR}} \to \mathrm{B}_{\max}$ is surjective. Let $x \in \mathrm{B}_{\max}$ and let $y \in \mathrm{B}_{\max}$ such that $(1-\varphi)(y) = x$. Also let $z \in \mathrm{B}^{\varphi=1}_{\max}$ mapping to $y \in \mathrm{B}_{\max} \subset \mathrm{B}_{\mathrm{dR}}$ in the first exact sequence. Then $y - z \in \mathrm{B}^+_{\mathrm{dR}}$ and $(1-\varphi)(y-z) = (1-\varphi)(y) = x$ and so $y-z$ is a preimage in $\operatorname{Fil}^0\mathrm{B}_{\max}$ as desired. □

Remark 23. It turns out that $\varphi(\mathrm{B}_{\max}) \subset \mathrm{B}_{\mathrm{cris}} \subset \mathrm{B}_{\max}$.

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2012-02-24

5.3 B_{st}

5.3.1 Definition

Definition 5.21. Let $\mathrm{B}_{\mathrm{st}}^{+} = \mathrm{B}_{\mathrm{cris}}^{+}[X]$ and $\mathrm{B}_{\mathrm{st}} = \mathrm{B}_{\mathrm{cris}}[X]$ endowed with Frobenius

$$\varphi(\sum a_n X^n) = \sum \varphi(a_n) p^n X^n$$

Galois action

$$g(\sum a_n X^n) = \sum g(a_n)(X + c(g)t)^n$$

where $g(\widetilde{p}) = \widetilde{p}\varepsilon^{c(g)}$ and monodromy $N = -d/dX$.

Lemma 5.22. *On B_{st} we have $N\varphi = p\varphi N$.*

5.3.2 Filtrations

Let $\log : \mathbb{C}_p \to \mathbb{C}_p$ given by $\log(p) = 0$. Then

$$\log([\widetilde{p}]) = \log(\frac{[\widetilde{p}]}{p}) = -\sum_{n\geq 1} \frac{(1 - [\widetilde{p}]/p)^n}{n} \in \mathrm{B}_{\mathrm{dR}}^{+}$$

Take for granted the following theorem:

Theorem 5.23. *The map $K \otimes_{K_0} \mathrm{B}_{\mathrm{st}} \to \mathrm{B}_{\mathrm{dR}}$ given by $a \otimes P(X) \mapsto aP(\log([\widetilde{p}]))$ is injective.*

Remark 24. This map is not canonical, depending on $\log([\widetilde{p}])$. However, it is G_K-equivariant.

Definition 5.24. Write $\mathrm{Fil}^k \mathrm{B}_{\mathrm{st}} = \mathrm{B}_{\mathrm{st}} \cap \mathrm{Fil}^k \mathrm{B}_{\mathrm{dR}}$ and $\mathrm{Fil}^k \mathrm{B}_{\mathrm{st}}^{+} = \mathrm{B}_{\mathrm{st}}^{+} \cap \mathrm{Fil}^k \mathrm{B}_{\mathrm{dR}}$.

5.3.3 Regularity

Proposition 5.25. *The ring B_{st} is $(\mathbb{Q}_p, G_K)$-regular with $\mathrm{B}_{\mathrm{st}}^{G_K} = K_0$.*

Proof. The first condition of regularity follows as in the case of $\mathrm{B}_{\mathrm{cris}}$ via Theorem 5.23. Now pick $b \in \mathrm{B}_{\mathrm{st}}$ different from 0 such that $\mathbb{Q}_p \cdot b$ is G_K-stable. Write $b = b_0 + b_1 X + \cdots + b_r X^r$ with $b_r \neq 0$. Let $\psi : G_K \to \mathbb{Q}_p^{\times}$ be the character encoding the G_K action on $\mathbb{Q}_p \cdot b$. Then

$$\begin{aligned} g(b) &= \psi(g) \sum_{i=0}^{r} b_i X^i \\ &= \sum_{i=0}^{r} g(b_i)(X + c(g)t)^i \end{aligned}$$

and comparing leading terms we get $\psi(g) b_r = g(b_r)$ which implies that ψ is continuous by looking at the image in $\mathbb{C}_p$. Moreover, it follows that $\mathbb{Q}_p \cdot b_r \subset \mathrm{B}_{\mathrm{cris}}$ is G_K-stable and so $b_r^{-1} \otimes e_\psi \in \mathrm{D}_{\mathrm{B}_{\mathrm{cris}}}(\mathbb{Q}_p(\eta))$ and Proposition 5.8 implies that $\psi = \chi_{\mathrm{cycl}}^{n}\mu$ for some unramified character μ. Replacing b by bt^{-n} we may assume that ψ is unramified. This implies that $b_r \in \mathrm{B}_{\mathrm{cris}}^{I_K} = \widehat{K_0^{\mathrm{ur}}}$, which is the maximal unramified subfield of $\widehat{K^{\mathrm{ur}}}$.

Comparing the coefficients of X^{r-1} in the formula we get

$$\psi(g) b_{r-1} = g(b_{r-1}) + r g(b_r) c(g) t$$

If $g \in I_K$ then $\psi(g) = 1$ and $g(b_r) = b_r$ so this becomes

$$b_{r-1} = g(b_{r-1}) + r b_r c(g) t$$

which can be rewritten as

$$\frac{b_{r-1}}{rb_r} - g\left(\frac{b_{r-1}}{rb_r}\right) = c(g)t$$

But $g(X) - X = c(g)t$ and so this becomes

$$g\left(X + \frac{b_{r-1}}{rb_r}\right) = X + \frac{b_{r-1}}{rb_r}$$

which implies that $X + \frac{b_{r-1}}{rb_r} \in \mathrm{B}_{\mathrm{st}}^{I_K} = \mathrm{B}_{\mathrm{cris}}^{I_K} \subset \mathrm{B}_{\mathrm{cris}}$. But $\frac{b_{r-1}}{rb_r} \in \mathrm{B}_{\mathrm{cris}}$ yet $X \notin \mathrm{B}_{\mathrm{cris}}$ and this contradiction implies that $r = 0$ and so $b = b_0 \in \mathrm{B}_{\mathrm{cris}}$ and so b is invertible by the regularity of $\mathrm{B}_{\mathrm{cris}}$. ☐

5.4 Filtered modules with Frobenius and monodromy

We now construct additive/abelian categories tha will be natural target categories for $\mathrm{D}_{\mathrm{cris}}$ and D_{st}.

5.4.1 Isocrystals

Definition 5.26. Let $\mathrm{Mod}_{K_0}^{\varphi}$ be the category of isocrystals over K_0. The objects are pairs (D, φ_D) of a finite dimensional K_0-vector space D and a bijective Frobenius-semilinear map $\varphi_D : D \to D$, i.e., if $\alpha \in K_0$ and $v \in D$ then $\varphi_D(\alpha v) = \sigma(\alpha)\varphi_D(v)$, where $\sigma \in G_{K_0/\mathbb{Q}_p}$ is a choice of Frobenius; the morphisms are morphisms of vector spaces commuting with the φ_D.

Remark 25. The category $\mathrm{Mod}_{K_0}^{\varphi}$ is abelian, with tensors and duals defined as follows:

- If $(D, \varphi_D), (D', \varphi_{D'}) \in \mathrm{Mod}_{K_0}^{\varphi}$ then $(D \otimes_{K_0} D', \varphi_D \otimes \varphi_{D'}) \in \mathrm{Mod}_{K_0}^{\varphi}$.
- If $(D, \varphi_D) \in \mathrm{Mod}_{K_0}^{\varphi}$ then $(D^{\vee}, \varphi_{D^{\vee}}) \in \mathrm{Mod}_{K_0}^{\varphi}$ where $\varphi_{D^{\vee}}(\lambda)(v) = \sigma(\lambda(\varphi_D^{-1}(v)))$.

Example 5.27. The basic example of isocrystal is obtained as follows: If $r > 0$ and s are integers let

$$D_{K_0,r,s} = K_0[X]/(K_0[X](X^r - p^s))$$

Then $D_{K_0,r,s}$ is a finite dimensional K_0-vector space (the division algorithm), and $\varphi_{D_{K_0,r,s}} : D_{K_0,r,s} \to D_{K_0,r,s}$ defined by $\varphi_{D_{K_0,r,s}}(P(X)) = X\sigma(P(X))$ is a bijection.

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Remark 26. If D is an isocrystal and $e_1, \ldots, e_n$ is a basis on which φ_D acts via the matrix A, and if $f = Be$ is another basis then φ_D acts on the basis f via the matrix $\sigma(B)AB^{-1}$. Therefore it does not make sense to talk about the characteristic polynomial of φ_D, and thus of eigenvalues of φ_D. In fact the eigenvalues of the matrix of φ_D relative to some basis don't even have v_p-valuations independent of the choice of basis. The isocrystals $D_{\widehat{\mathbb{Q}_p^{\mathrm{ur}}},r,s}$ over $\widehat{\mathbb{Q}_p^{\mathrm{ur}}}$ will function as "valuation $\frac{s}{r}$" eigenspaces for Frobenius.

Take the following theorem for granted:

Theorem 5.28. *The category* $\mathrm{Mod}_{\widehat{\mathbb{Q}_p^{\mathrm{ur}}}}^{\varphi}$ *is semisimple and the simple objects are (isomorphic to)* $D_{\widehat{\mathbb{Q}_p^{\mathrm{ur}}},r,s}$ *for* $(r, s) = 1$. *We will denote* $\Delta_{\frac{s}{r}} := D_{\widehat{\mathbb{Q}_p^{\mathrm{ur}}},r,s}$.

Definition 5.29. For $D \in \mathrm{Mod}_{K_0}^{\varphi}$, let $\widehat{D} := D \otimes_{K_0} \widehat{\mathbb{Q}_p^{\mathrm{ur}}}$ together with $\varphi_{\widehat{D}}(d \otimes x) := \varphi_D(d) \otimes \sigma_{\widehat{\mathbb{Q}_p^{\mathrm{ur}}}}(x)$ giving an object in $\mathrm{Mod}_{\widehat{\mathbb{Q}_p^{\mathrm{ur}}}}^{\varphi}$. The above theorem implies that we get a decomposition $\widehat{D} = \oplus \widehat{D}(\frac{s}{r})$ where $\widehat{D}(\frac{s}{r}) = \Delta_{\frac{s}{r}}^{e_{r,s}}$ for some nonnegative integer $e_{r,s}$. A rational $\frac{s}{r}$ such that $\widehat{D}(\frac{s}{r}) \neq 0$ is said to be a *slope* of D; the isocrystal D is said to be isoclinic if it has only one slope.

Proposition 5.30 (Slope decomposition). *Every $D \in \mathrm{Mod}_{K_0}^{\varphi}$ decomposes as a direct sum $\oplus_\alpha D(\alpha)$ where $\alpha \in \mathbb{Q}$ and $D(\alpha)$ is isoclinic with slope α.*

Proof. We have seen that there is a decomposition $\widehat{D} = \oplus \widehat{D}(\alpha)$ over $\widehat{\mathbb{Q}_p^{\mathrm{ur}}}$. There is a natural semilinear action of $G_{k_K} = G_K/I_K$ on the $\widehat{\mathbb{Q}_p^{\mathrm{ur}}}$-isocrystal Δ_α. Let $D(\alpha) := \widehat{D}(\alpha)^{G_{k_K}}$. By Galois descent (Lemma 4.39) it follows that $D(\alpha) \otimes_{K_0} \widehat{\mathbb{Q}_p^{\mathrm{ur}}} \cong \widehat{D}(\alpha)$ (as $\widehat{\mathbb{Q}_p^{\mathrm{ur}}}^{G_{k_K}} = K_0$). A dimension count now show that $D = \oplus D(\alpha)$ and that $D(\alpha)$ is isoclinic of slope α. $\square$

5.4.2 Newton Polygons

Definition 5.31. For $D \in \mathrm{Mod}_{K_0}^{\varphi}$ let $\alpha_0 < \alpha_1 < \ldots < \alpha_n$ be the slopes with multiplicities $\mu_0, \ldots, \mu_n$. The Newton polygon $P_N(D)$ of D is the convex polygon starting at $(0,0)$, consisting of $n+1$ segments, such that the i-th segment (for $i = 0, \ldots, n$) has horizontal length μ_i and slope α_i. We denote by $t_N(D)$ the y-coordinate of the rightmost endpoint of $P_N(D)$.

Figure 1: Figure copied from [3]

Lemma 5.32. *If $D \in \mathrm{Mod}_{K_0}^{\varphi}$ then the vertices of the Newton polygon t_N are integral.*

Proof. Let $D \otimes_{K_0} \widehat{\mathbb{Q}_p^{\mathrm{ur}}} = \oplus \Delta_{\alpha_i}^{\mu_i}$. If $\alpha_i = \frac{s_i}{r_i}$ then $\dim_{\widehat{\mathbb{Q}_p^{\mathrm{ur}}}} \Delta_{\alpha_i}^{\mu_i} = r_i\mu_i$ and so the segment of slope α_i and horizontal length $r_i\mu_i$ will have vertical length $s_i\mu_i \in \mathbb{Z}$. $\square$

Lemma 5.33. *If $(r,s) = 1$ and $(m,n) = 1$ then $\Delta_{\frac{s}{r}} \otimes_{\widehat{\mathbb{Q}_p^{\mathrm{ur}}}} \Delta_{\frac{n}{m}}$ is isoclinic.*

Proof. Write $\Delta_{\frac{s}{r}} \cong \widehat{\mathbb{Q}_p^{\mathrm{ur}}}[X]/(X^r - p^s)$ and $\Delta_{\frac{n}{m}} \cong \widehat{\mathbb{Q}_p^{\mathrm{ur}}}[Y]/(Y^m - p^n)$. Then

$$\begin{aligned}
\Delta_{\frac{s}{r}} \otimes_{\widehat{\mathbb{Q}_p^{\mathrm{ur}}}} \Delta_{\frac{n}{m}} &\cong \widehat{\mathbb{Q}_p^{\mathrm{ur}}}[X,Y]/(X^r - p^s, Y^m - p^n) \\
&= \bigoplus_{i=0}^{(r,m)-1} Y^i \widehat{\mathbb{Q}_p^{\mathrm{ur}}}[XY]/((XY)^{[r,m]} - p^{[r,m](\frac{s}{r}+\frac{n}{m})}) \\
&\cong \Delta_{\frac{s}{r}+\frac{n}{m}}^{(r,m)}
\end{aligned}$$

$\square$

Proposition 5.34. *We have*

1. $t_N(D \otimes_{K_0} D') = \dim_{K_0} D \cdot t_N(D') + t_N(D) \cdot \dim_{K_0} D'$,
2. $t_N(D) = t_N(\det D)$,
3. $t_N(D^\vee) = -t_N(D)$, *and*
4. *if $0 \to D \to D' \to D'' \to 0$ is exact in $\mathrm{Mod}_{K_0}^{\varphi}$ then $t_N(D') = t_N(D) + t_N(D'')$.*

Proof. 1. It is clear from the definition that $t_N(D \oplus D') = t_N(D) + t_N(D')$. Therefore, by induction, it is enough to show this for isoclinic isocrystals. But then by Lemma 5.33 we have

$$\begin{aligned} t_N(\Delta_{\frac{s}{r}} \otimes_{K_0} \Delta_{\frac{n}{m}}) &= (r,m) t_N(\Delta_{\frac{s}{r}+\frac{n}{m}}) \\ &= (r,m) \dim_{K_0} \Delta_{\frac{s}{r}+\frac{n}{m}} (\frac{s}{r} + \frac{n}{m}) \\ &= (r,m)[r,m](\frac{s}{r} + \frac{n}{m}) \\ &= sm + rn \\ &= \dim_{K_0} \Delta_{\frac{s}{r}} \cdot t_N(\Delta_{\frac{n}{m}}) + t_N(\Delta_{\frac{s}{r}}) \cdot \dim_{K_0} \Delta_{\frac{n}{m}} \end{aligned}$$

2. If D is isoclinic of slope $\alpha = \frac{s}{r}$ and dimension rd then $D^{\otimes rd}$ is isoclinic of slope sd by Lemma 5.33 and so $\det D$, a one dimensional subspace of $D^{\otimes rd}$, will be isoclinic of slope sd. Therefore $t_N(\det D) = sd = t_N(D)$. In general, if $D = \oplus D_\alpha$ where D_α is isoclinic of dimension d_α then $\det D = \wedge^{\sum d_\alpha}(\oplus D_\alpha) = \otimes_\alpha \wedge^{d_\alpha} D_\alpha$ and the result follows from the first part.

3. From the previous part it is enough to show this for one dimensional isocrystals of the form $\Delta_{\frac{s}{r}}$. But from the definition of duals in the category $\mathrm{Mod}^{\varphi}_{K_0}$ it follows that $\langle \sum \alpha_i X^i, \sum \beta_i Y^i \rangle = \sum \alpha_i \beta_i$ gives a well-defined Frobenius equivariant perfect pairing $K_0[X]/(X^r - p^s) \otimes K_0[Y]/(Y^r - p^{-s}) \to K_0$ which implies that $\Delta^\vee_{\frac{s}{r}} \cong \Delta_{-\frac{s}{r}}$. The conclusion follows.

4. Theorem 5.28 implies that the exact sequence splits over $\widehat{\mathbb{Q}_p^{\mathrm{ur}}}$ and the conclusion follows since the Newton polygon depends only on the isocrystal over $\widehat{\mathbb{Q}_p^{\mathrm{ur}}}$.

□

5.4.3 Filtered φ-modules and (φ, N)-modules

Definition 5.35. The category MF^{φ}_K of filtered φ-modules consists of triples $(D, \varphi_D, \mathrm{Fil}^\bullet D_K)$ such that $(D, \varphi_D) \in \mathrm{Mod}^{\varphi}_{K_0}$ and $(D_K, \mathrm{Fil}^\bullet D_K) \in \mathrm{FilVect}_K$, where $D_K := D \otimes_{K_0} K$. (Note that no compatibility between φ_D and $\mathrm{Fil}^\bullet D_K$ is required; in fact it wouldn't make sense to do so) Morphisms in the category are morphisms in the category $\mathrm{Mod}^{\varphi}_{K_0}$ such that the base change to K gives a morphism in $\mathrm{FilVect}_K$.

Remark 27. The category MF^{φ}_K is not abelian, but the fact that $\mathrm{Mod}^{\varphi}_{K_0}$ is abelian and §3.1.3 implies that in MF^{φ}_K there exist kernels, cokernels, image, coimage, short exact sequences, tensor products and duals.

Definition 5.36. A morphism in a category with image and coimage is said to be *strict* if the natural map from the coimage to the image is an isomorphism

Definition 5.37. The category $\mathrm{MF}^{\varphi,N}_K$ consists of tuples $(D, \varphi_D, N_D, \mathrm{Fil}^\bullet D_K)$ where $(D, \varphi_D, \mathrm{Fil}^\bullet D_K) \in \mathrm{MF}^{\varphi}_K$ and $N_D : D \to D$ is a K_0-linear morphism (called monodromy) such that $N_D \varphi_D = p \varphi_D N_D$. Morphisms in the category are morphisms in the category MF^{φ}_K which commute with N_D.

Remark 28. Much like MF^{φ}_K, the category MF^{φ}_K is not abelian, but there exist kernels, cokernels, image, coimage, and short exact sequences. To define tensor products and duals, in addition to the construction in MF^{φ}_K we need to define these operations on the monodromy operator: $N_{D \otimes_{K_0} D'} = N_D \otimes 1 + 1 \otimes N_{D'}$ and $N_{D^\vee} = -N_D^\vee$.

Remark 29. If $(D, \varphi_D, \mathrm{Fil}^\bullet D_K) \in \mathrm{MF}^{\varphi}_K$ then $(D, \varphi_D, N_D = 0, \mathrm{Fil}^\bullet D_K) \in \mathrm{MF}^{\varphi,N}_K$ and so many theorems need only be proven for (φ, N)-modules and the analogs will follow for φ-modules.

Lemma 5.38. *The slope decomposition $D = \oplus D(\alpha)$ in $\mathrm{Mod}^{\varphi}_{K_0}$ extends to a decomposition in MF^{φ}_K by endowing each direct summand with the subspace filtration over K. In $\mathrm{MF}^{\varphi,N}_K$ each $\oplus_{\alpha \le \alpha_0} D(\alpha)$ is stable under N.*

Proof. Consider $\Delta_{\frac{s}{r}} = \widehat{\mathbb{Q}_p^{\mathrm{ur}}}[X]/(X^r - p^s)$ on which $\phi^r = p^s$. Then by $N\phi = p\phi N$ we have

$$\begin{aligned} \phi^r N v &= p^{-r} N \phi^r v \\ &= p^{s-r} N v \end{aligned}$$

and so $Nv \in \Delta_{\frac{s}{r}-1}$. This shows that $N(\Delta_\alpha) \subset \Delta_{\alpha-1}$ and so the conclusion follows. □

Lemma 5.39. *The monodromy operator on any object in* $\mathrm{MF}_K^{\varphi,N}$ *is nilpotent.*

Proof. We have seen that $N(D(\alpha)) \subset D(\alpha - 1)$ and the result follows from the fact that an isocrystal has finitely many weights. □

Lecture 22

2012-02-29

5.4.4 Hodge polygons

Definition 5.40. For $(D, \mathrm{Fil}^\bullet D) \in \mathrm{FilVect}_K$ let $i_0 < i_1 < \ldots < i_n$ be the integers such that $\mathrm{gr}^i D \neq 0$, i.e., the indices where the jumps in the filtration occur. The Hodge polygon $P_H(D)$ of $(D, \mathrm{Fil}^\bullet D)$ is the convex polygon in the plane, starting at $(0, 0)$ and whose k-th segment has horizontal length $\dim_K \mathrm{gr}^{i_k} D$ and slope i_k. We denote by $t_H(D)$ the y-coordinate of the rightmost endpoint of $P_H(D)$.

Proposition 5.41. *We have*

1. $t_H(D \otimes_K D') = \dim_K D \cdot t_H(D') + t_H(D) \cdot \dim_K D'$,
2. $t_H(D) = t_H(\det D)$,
3. $t_H(D^\vee) = -t_H(D)$, *and*
4. *if* $0 \to D \to D' \to D'' \to 0$ *is an exact sequence in* $\mathrm{FilVect}_K$ *then* $t_H(D') = t_H(D) + t_H(D'')$.

Proof. These follow from the definitions. □

Definition 5.42. If $(D, \varphi_D, \mathrm{Fil}^\bullet D_K) \in \mathrm{MF}_K^\varphi$ then we define $t_H(D) := t_H(D_K, \mathrm{Fil}^\bullet D_K)$.

Lemma 5.43. *Let* $D \in \mathrm{FilVect}_K$.

1. *If* $D' \subset D$ *in* $\mathrm{FilVect}_K$ *then* $P_H(D')$ *lies above* $P_H(D)$.
2. *If* $f : D' \to D$ *is a morphism in* $\mathrm{FilVect}_K$ *which is an isomorphism in* Vect_K *then* $t_H(D') \leq t_H(D)$ *with equality if and only if* f *is an isomorphism in* $\mathrm{FilVect}_K$*, i.e., it respect filtrations.*

Proof.

1. This follows form the fact that the filtration on D' is the restriction of the filtration on D and the slopes of $P_H(D)$ and $P_H(D')$ are the same except the length of each segment in $P_H(D')$ is shorter so the conclusion follows.
2. The morphism f is an isomorphism in $\mathrm{FilVect}_K$ if and only if it induces an isomorphism $\det f : \det D' \to \det D$ in $\mathrm{FilVect}_K$. Now $\det f : \mathrm{Fil}^i \det D' \to \mathrm{Fil}^i \det D$ so if $i = t_H(\det D')$ is the unique index such that $\mathrm{gr}^i \det D' \neq 0$ then $\mathrm{gr}^j \det D = 0$ for $j < i$ which implies that $t_H(\det D) \geq t_H(\det D')$ and the conclusion follows from Proposition 5.41

□

5.4.5 Weakly admissible modules

The categories MF_K^{φ} and $\mathrm{MF}_K^{\varphi,N}$ will be the target categories of $\mathrm{D}_{\mathrm{cris}}$ and D_{st}, respectively, but since they are not abelian, while the category of representations is, the two Dieudonne functors (which we'll prove to be fully faithful) cannot be essentially surjective, and so we must seek abelian subcategories which are natural target categories for the two functors.

Lemma 5.44. *Let $D \in \mathrm{MF}_K^{\varphi}$ ($\mathrm{MF}_K^{\varphi,N}$). Then the following two statements are equivalent:*

1. *For all subobjects $D' \subset D$ in MF_K^{φ} ($\mathrm{MF}_K^{\varphi,N}$) the Newton polygon $P_N(D') \geq P_H(D')$.*

2. *For all subobjects $D' \subset D$ in MF_K^{φ} ($\mathrm{MF}_K^{\varphi,N}$) we have $t_N(D') \geq t_H(D')$.*

Moreover, if L/K is unramified then D satisfies the above properties in MF_K^{φ} ($\mathrm{MF}_K^{\varphi,N}$) if and only if $D \otimes_{K_0} L_0$ satisfies them in MF_L^{φ} ($\mathrm{MF}_K^{\varphi,N}$).

Proof. Note that we only need to prove the version for (φ, N)-modules as the one for φ-modules can be deduced by setting $N = 0$.

The first statement implies the second. Suppose now the second is true but for some subobject D' of D the Newton polygon $P_N(D')$ does not necessarily sit above $P_H(D')$. Since $t_N(D') \geq t_N(D')$ it follows that some vertex v of coordinates $(x, P_N(D')(x))$ of the polygon $P_N(D')$ has to lie below $P_H(D')$. Let α_0 be the slope of the segment to the left of this vertex. Let $D'' = \oplus_{\alpha \leq \alpha_0} D'(\alpha)$, which is then a subobject of D (by Lemma 5.38) so $t_N(D'') \geq t_H(D'')$. But $t_N(D'') = P_N(D')(x)$ and since the filtration on D'' is inherited from that on D' it follows that $P_H(D'')$ lies above $P_H(D')$ and so $t_H(D'') = P_H(D'')(x) \geq P_H(D')(x) > P_N(D')(x) = t_N(D'')$ giving a contradiction.

For the second statement note that if D' is a subobject of D over K_0 then $\widehat{D'}$ is a subobject of $\widehat{D}$ over $\widehat{\mathbb{Q}_p^{\mathrm{ur}}}$ and $P_N(D') = P_N(\widehat{D'})$ and $P_H(D') = P_H(\widehat{D'})$ so the conditions are satisfied over $\widehat{\mathbb{Q}_p^{\mathrm{ur}}}$. Now suppose that $\widehat{D'}$ is a subobject of $\widehat{D}$ over $\widehat{\mathbb{Q}_p^{\mathrm{ur}}}$ such that $t_N(\widehat{D'}) < t_H(\widehat{D'})$. Again Galois descent produces a subobject D' of D such that $\widehat{D'} = D' \otimes_{K_0} \widehat{\mathbb{Q}_p^{\mathrm{ur}}}$ and the conclusion follows. $\square$

Definition 5.45. An object $D \in \mathrm{MF}_K^{\varphi}$ ($\mathrm{MF}_K^{\varphi,N}$) is weakly admissible if for all subobjects $D' \subset D$ in MF_K^{φ} ($\mathrm{MF}_K^{\varphi,N}$) we have $t_N(D') \geq t_H(D')$ with equality if and only if $D = D'$. Let $\mathrm{MF}_K^{\varphi,\mathrm{wa}}$ ($\mathrm{MF}_K^{\varphi,N,\mathrm{wa}}$) be the full subcategory of MF_K^{φ} ($\mathrm{MF}_K^{\varphi,N}$) consisting of weakly admissible objects.

Lemma 5.46. *An object $D \in \mathrm{MF}_K^{\varphi}$ ($\mathrm{MF}_K^{\varphi,N}$) is weakly admissible if and only if for every quotient $D \twoheadrightarrow D'$ we have $t_N(D') \leq t_H(D')$.*

Proof. Let $D'' := \ker(D \twoheadrightarrow D')$ so we get an exact sequence $0 \to D'' \to D \to D' \to 0$. Assuming D is weakly admissible we get $t_N(D) = t_H(D)$ and $t_N(D'') \geq t_H(D'')$ thus $t_N(D') \leq t_H(D')$. The converse is obtained by following the above going in reverse. $\square$

Proposition 5.47. *If $D \in \mathrm{MF}_K^{\varphi}$ ($\mathrm{MF}_K^{\varphi,N}$) then D is weakly admissible if and only if $D^\vee$ is.*

Proof. This is not vacuous as slopes swap signs under duality. Again, we will show this for (φ, N)-modules and set $N = 0$ to get the result for φ-modules. Suppose $(D^\vee)' \subset D^\vee$, then $D \twoheadrightarrow D'' := ((D^\vee)')^\vee$ is a surjection. Lemma 5.46 implies that $t_N(D'') \leq t_H(D'')$. But $t_N(D'') = -t_N((D^\vee)')$ and $t_H(D'') = -t_H((D^\vee)')$ and so $t_N((D^\vee)') \geq t_H((D^\vee)')$ with equality occuring if and only if $(D^\vee)' = D^\vee$. Thus $D^\vee$ is weakly admissible. $\square$

Lecture 23

2012-03-02

Proposition 5.48. *If $0 \to D' \to D \to D'' \to 0$ is an exact sequence in MF_K^{φ} ($\mathrm{MF}_K^{\varphi,N}$) and any two of the objects are weakly admissible then the third one is also weakly admissible.*

Proof. Assume D and D'' are weakly admissible. If $D_1 \subset D'$ then it is also a subobject of D and so $t_N(D_1) \geq t_H(D_1)$ and if $D_1 = D'$ then $t_N(D') = t_N(D) - t_N(D'') = t_H(D) - t_H(D'') = t_H(D')$ and so D' is weakly admissible.

Assume D and D' are weakly admissible. Then $0 \to (D'')^\vee \to D^\vee \to (D')^\vee \to 0$ is an exact sequence with $D^\vee$ and $(D')^\vee$ weakly admissible by Proposition 5.47 and so $(D'')^\vee$ is weakly admissible which implies that D'' also is.

Assume D' and D'' are weakly admissible. Then $t_N(D) = t_N(D') + t_N(D'') = t_H(D') + t_H(D'') = t_H(D)$. Let $D_1 \subset D$. We need to check that $t_N(D_1) \geq t_H(D_1)$. Let $D_1' = D_1 \cap D$ with the subobject filtration on $(D_1')_K$ coming from D_K' (which is the same as the one coming from $(D_1)_K$), and let $D_1'' = D_1/D_1'$ endowed with the quotient filtration from D_1 (which need not be the same as the subspace filtration from D''). Also let $\overline{D}_1''$ be the image of D_1'' in D'' together with the subspace filtration from D''. Then $D_1'' \to \overline{D}_1''$ is a morphism in $\mathrm{FilVect}_K$ which is an isomorphism in Vect_K and so by Lemma 5.43 we have $t_H(D_1'') \leq t_H(\overline{D}_1'')$. Moreover, $t_N(D_1'') = t_N(\overline{D}_1'')$ since the vector space isomorphism respects φ.

Since $D_1' \subset D'$ which is weakly admissible we get that $t_N(D_1') \geq t_H(D_1')$. Since $\overline{D}_1'' \subset D''$ which is weakly admissible we get that $t_N(D_1'') = t_N(\overline{D}_1'') \geq t_H(\overline{D}_1'') \geq t_H(D_1'')$. But $0 \to D_1' \to D_1 \to D_1'' \to 0$ is exact so combining the two we get the desired conclusion. ☐

Theorem 5.49. *The categories $\mathrm{MF}_K^{\varphi,N,\mathrm{wa}}$ and $\mathrm{MF}_K^{\varphi,\mathrm{wa}}$ are abelian.*

Proof. We only need to prove that $\mathrm{MF}_K^{\varphi,N,\mathrm{wa}}$ is abelian as then $\mathrm{MF}_K^{\varphi,\mathrm{wa}}$ is automatically also abelian. The category is clearly additive (hom sets are abelian groups and finite direct sums and products exist) so we only need to check that kernels and cokernels exist and the natural map from coimage to image is an isomorphism. Kernels, cokernels, images and coimages exist in $\mathrm{MF}_K^{\varphi,N}$ so if $f : D \to D'$ is a morphism in $\mathrm{MF}_K^{\varphi,N,\mathrm{wa}}$ then in $\mathrm{MF}_K^{\varphi,M}$ we have

$$\begin{tikzcd}
0 \arrow[r] & \ker f \arrow[r] & D \arrow[r] & \operatorname{coim} f \arrow[r] \arrow[dll] & 0 \\
0 \arrow[r] & \operatorname{im} f \arrow[r] & D' \arrow[r] & \operatorname{coker} f \arrow[r] & 0
\end{tikzcd}$$

Now D and D' are weakly admissible so using the definition and Lemma 5.46 we get $t_N(\ker f) \geq t_H(\ker f)$, $t_N(\operatorname{coim} f) \leq t_H(\operatorname{coim} f)$, $t_N(\operatorname{im} f) \geq t_H(\operatorname{im} f)$ and $t_N(\operatorname{coker} f) \leq t_H(\operatorname{coker} f)$. Now the morphism $\operatorname{coim} f \to \operatorname{im} f$ is an isomorphism in Vect_K (which is abelian) so Lemma 5.43 implies that $t_H(\operatorname{coim} f) \leq t_H(\operatorname{im} f)$ and combining we get $t_N(\operatorname{coim} f) \leq t_H(\operatorname{coim} f) \leq t_H(\operatorname{im} f) \leq t_N(\operatorname{im} f)$. But the category $\mathrm{Mod}_{K_0}^{\varphi}$ is abelian and so the map $\operatorname{coim} f \to \operatorname{im} f$ is an isomorphism in $\mathrm{Mod}_{K_0}^{\varphi}$ and so $t_N(\operatorname{coim} f) = t_N(\operatorname{im} f)$ implying that $t_H(\operatorname{coim} f) = t_H(\operatorname{im} f)$ and so $\operatorname{coim} f$ and $\operatorname{im} f$ are weakly admissible. Another application of Lemma 5.43 now gives that $\operatorname{coim} f \to \operatorname{im} f$ is an isomorphism in $\mathrm{FilVect}_K$ as well, and so in $\mathrm{MF}_K^{\varphi,N}$. Finally Proposition 5.48 gives that $\ker f$ and $\operatorname{coker} f$ are weakly admissible as well. ☐

5.5 Crystalline and semistable representations

Definition 5.50. A representation is said to be crystalline if it is $\mathrm{B}_{\mathrm{cris}}$-admissible and semistable if it is B_{st}-admissible. We denote the categories $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_K)$ and $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{st}}(G_K)$.

5.5.1 $\mathrm{D}_{\mathrm{cris}}$ and D_{st}

Definition 5.51. If $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ let $\varphi := \varphi \otimes \sigma$ act on $\mathrm{D}_{\mathrm{cris}}(V)$ and φ and $N := N \otimes 1$ act on $\mathrm{D}_{\mathrm{st}}(V)$.

Lemma 5.52. *Let $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{st}}$.*

1. *$\mathrm{D}_{\mathrm{cris}}(V) = \mathrm{D}_{\mathrm{st}}(V)^{N=0}$ and so crystalline representations are semistable;*
2. *$\mathrm{D}_{\mathrm{cris}}$ goes into MF_K^{φ} and D_{st} goes into $\mathrm{MF}_K^{\varphi,N}$, they are faithful and exact;*

3. $\mathrm{D_{cris}}$ *on* $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_K)$ *and* $\mathrm{D_{st}}$ *on* $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_K)$ *respect tensor products, duals, symmetric and exterior powers;*

4. *if* V *is semistable then* $K \otimes_{K_0} \mathrm{D_{st}}(V) \cong \mathrm{D_{dR}}(V)$ *and so* V *is de Rham;*

5. *if* L/K *is complete and unramified then* $L_0 \otimes_{K_0} \mathrm{D}_{\mathrm{st},K}(V) \cong \mathrm{D}_{\mathrm{st},L}(V)$.

Proof. 1. $\mathrm{D_{st}}(V)^{N=0} = (\mathrm{B_{st}} \otimes_{\mathbb{Q}_p} V)^{G_K, N=0} = (\mathrm{B}_{\mathrm{st}}^{N=0} \otimes_{\mathbb{Q}_p} V)^{G_K} = (\mathrm{B_{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K} = \mathrm{D_{cris}}(V)$. Now $\mathrm{B_{st}}$ and $\mathrm{B_{cris}}$ are regular so if V is crystalline then $\dim_{\mathbb{Q}_p} V = \dim_{K_0} \mathrm{D_{cris}}(V) \leq \dim_{K_0} \mathrm{D_{st}}(V) \leq \dim_{\mathbb{Q}_p} V$ so V is also semistable.

2. We need to check only that φ is an isomorphism, which follows from Theorem 5.12.

3. This follows from Proposition 4.36.

4. Since $K \otimes_{K_0} \mathrm{B_{st}} \hookrightarrow \mathrm{B_{dR}}$ it follows that $K \otimes_{K_0} \mathrm{D_{st}}(V) \hookrightarrow \mathrm{D_{dR}}(V)$ as K-vector spaces. Now a dimension count in the case when V is semistable gives an isomorphism as vector spaces and so V is de Rham. Finally, the isomorphism is also one of filtered vector spaces since $K \otimes_{K_0} \mathrm{B_{st}}$ carries the subspace filtration from $\mathrm{B_{dR}}$.

5. $L_0 \otimes_{K_0} \mathrm{D}_{\mathrm{st},K}(V) \to \mathrm{D}_{\mathrm{st},L}(V)$ is a morphism in $\mathrm{MF}_L^{\varphi,N}$. Now using Proposition 4.40 we get $L \otimes_{L_0} L_0 \otimes_{K_0} \mathrm{D}_{\mathrm{st},K}(V) \cong L \otimes_K \mathrm{D}_{\mathrm{dR},K}(V) \cong \mathrm{D}_{\mathrm{dR},L}(V) \cong L \otimes_{L_0} \mathrm{D}_{\mathrm{st},L}(V)$ and so $L_0 \otimes_{K_0} \mathrm{D}_{\mathrm{st},K}(V) \to \mathrm{D}_{\mathrm{st},L}(V)$ is an isomorphism in $\mathrm{MF}_L^{\varphi,N}$.

□

Corollary 5.53. *Every semistable continuous character* $\eta : G_K \to \mathbb{Q}_p^\times$ *is also crystalline.*

Proof. Since N is nilpotent it follows $N = 0$. □

Example 5.54. $\mathrm{D_{cris}}(\mathbb{Q}_p(n)) = K_0 \cdot t^{-n}$, $\varphi(xt^{-n}) = \sigma_{K_0/\mathbb{Q}_p}(x)p^{-n}t^{-n}$ and $\mathrm{Fil}^i\, K \cdot t^{-n}$ is $K \cdot t^{-n}$ for $i \leq -n$ and 0 for $i > -n$.

Lemma 5.55. *The homomorphism* $x \mapsto \sigma(x)/x$ *on* $\mathrm{W}(\overline{\mathbb{F}_p}^\times)$ *is surjective.*

Proof. The map $x \mapsto x^{p-1}$ is surjective on $\overline{\mathbb{F}_p}$ and so on $\mathrm{W}(\overline{\mathbb{F}_p})^\times/(1 + p\,\mathrm{W}(\overline{\mathbb{F}_p}))$. Therefore it is enough to show that $x \mapsto \sigma(x)/x$ is surjective on $1 + p\,\mathrm{W}(\overline{\mathbb{F}_p})$. If $u = 1 + p^n v$ with $v \in \mathrm{W}(\overline{\mathbb{F}_p}$ let $\overline{z} \in \overline{\mathbb{F}_p}$ such that $z^p - z = v \mod p$. Let $z = [\overline{z}]$ in which case $\sigma(z) - z \equiv v \pmod p$. Letting $w = 1 + p^n z$ we have $\sigma(w)/w \equiv 1 + p^n(\sigma(z) - z) \equiv u \pmod{p^{n+1}}$. We denote $f_n(u) := w$.

Suppose $u_1 = u \in 1 + p\,\mathrm{W}(\overline{\mathbb{F}_p})$ and $w_1 = 1$. We recursively construct $u_n, w_n \in 1 + p^n\,\mathrm{W}(\overline{\mathbb{F}_p})$ by setting $w_n = f_n(u_n)$ and $u_{n+1} = u_n(\sigma(w_n)/w_n)^{-1}$. By construction $w_n \in 1 + p^n\,\mathrm{W}(\overline{\mathbb{F}_p})$ and $u_{n+1} \in 1 + p^{n+1}\,\mathrm{W}(\overline{\mathbb{F}_p})$. Then $w = \prod w_n$ converges to an element of $1 + p\,\mathrm{W}(\overline{\mathbb{F}_p})$ and $u = \sigma(w)/w$. □

Theorem 5.56. *We get functors* $\mathrm{D_{cris}} : \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_K) \to \mathrm{MF}_K^{\varphi,\mathrm{wa}}$ *and* $\mathrm{D_{st}} : \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{st}}(G_K) \to \mathrm{MF}_K^{\varphi,\mathrm{st},\mathrm{wa}}$.

Proof. We only need to show that if $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{st}}(G_K)$ then $D := \mathrm{D_{st}}(V)$ is weakly admissible.

(a) First, assume that $\dim_{\mathbb{Q}_p} V = 1$ and so by Corollary 5.53 and Proposition 5.8 there exist an integer n and an unramified character μ such that $V = \mathbb{Q}_p(\chi_{\mathrm{cycl}}^n \mu)$. Then $\widehat{D} = \mathrm{D}_{\mathrm{st},\widehat{K^{\mathrm{ur}}}}(V) = \mathrm{D}_{\mathrm{st},\widehat{K^{\mathrm{ur}}}}(\mathbb{Q}_p(n)) = \widehat{\mathbb{Q}_p^{\mathrm{ur}}} \cdot t^{-n}$. To show that $\widehat{D}$ is weakly admissible note that $\widehat{D}$ is isoclinic with slope $-n$ and $\mathrm{gr}^i\,\widehat{D} = 0$ unless $i = -n$ in which case it is one dimensional, which implies that $P_H(\widehat{D}) = P_N(\widehat{D})$. Now Lemma 5.44 shows that D is weakly admissible.

(b) If D is any dimensional but $\dim_{K_0} D' = 1$ let e' be a choice of basis with respect to which $\varphi(e') = \lambda e'$, in which case $t_N(D') = v_p(\lambda)$. Let $v_1, \ldots, v_n$ be a basis of V and consider $e' = \sum b_i \otimes v_i \in (\mathrm{B}_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_K}$. But $\varphi(e') = \lambda e'$ and $N(e') = 0$ implies that $\varphi(b_i) = \lambda b_i$ and $N(b_i) = 0$. Therefore $b_i \in B_{\mathrm{st}}^{N=0} = \mathrm{B}_{\mathrm{cris}}$. Let s be such that $e' \in \mathrm{Fil}^s \mathrm{D}_{\mathrm{dR}}(V) - \mathrm{Fil}^{s+1} \mathrm{D}_{\mathrm{dR}}(V)$, in which case $t_H(D') = s$. This implies that all $b_i \in \mathrm{Fil}^s \mathrm{B}_{\mathrm{cris}}$ but not all are in Fil^{s+1}. Let $b_j \notin \mathrm{Fil}^{s+1} \mathrm{B}_{\mathrm{cris}}$. We need to show that $s \leq v_p(\lambda)$.

Assume, for the sake of contradiction, that $s \geq v_p(\lambda) + 1$. Let $b = b_j t^{-v_p(\lambda)} \in \mathrm{Fil}^{s-v_p(\lambda)} \mathrm{B}_{\mathrm{cris}} \subset \mathrm{Fil}^1 \mathrm{B}_{\mathrm{cris}}$ in which case $\varphi(b) = ub$ where $u \in \mathrm{W}(\overline{\mathbb{F}_p})^\times$. Lemma 5.55 gives $w \in \mathrm{W}(\overline{\mathbb{F}_p})^\times$ such that $\sigma(w)/w = u$ which gives $b/w \in (\mathrm{Fil}^1 \mathrm{B}_{\mathrm{cris}})^{\varphi=1}$. But $(\mathrm{Fil}^1 \mathrm{B}_{\mathrm{cris}})^{\varphi=1} = \mathrm{Fil}^1 \mathrm{B}_{\mathrm{cris}} \cap (\mathrm{Fil}^0 \mathrm{B}_{\mathrm{cris}})^{\varphi=1} = \mathrm{Fil}^1 \mathrm{B}_{\mathrm{cris}} \cap \mathbb{Q}_p = 0$ where we used $(\mathrm{Fil}^0 \mathrm{B}_{\mathrm{cris}})^{\varphi=1} = \mathbb{Q}_p$ from the fundamental sequence in Theorem 5.13.

(c) If D is any dimensional, $\det D$ is one dimensional and we deduce $t_N(D) = t_N(\det D) = t_H(\det D) = t_H(D)$. Moreover, for every subobject $D' \subset D$ of dimension d' we need to show that $t_N(D') \geq t_H(D')$. But $t_H(D') = t_H(\det D')$ and $t_N(D') = t_N(\det D')$. Lemma 5.52 implies that $\wedge^{d'} D$ is semistable and $\det D'$ is a one dimensional subobject of a semistable representation and the conclusion follows from the previous cases. $\square$

Lecture 24

2012-03-05

5.5.2 $\mathrm{V}_{\mathrm{cris}}$ and V_{st}

Definition 5.57. For $D \in \mathrm{MF}_K^{\varphi}$ define $\mathrm{V}_{\mathrm{cris}}(D) := \mathrm{Fil}^0(\mathrm{B}_{\mathrm{cris}} \otimes_{K_0} D)^{\varphi=1}$ and for $D \in \mathrm{MF}_K^{\varphi,N}$ let $\mathrm{V}_{\mathrm{st}}(D) := \mathrm{Fil}^0(\mathrm{B}_{\mathrm{st}} \otimes_{K_0} D)^{\varphi=1,N=0}$.

Theorem 5.58. *The functors $\mathrm{D}_{\mathrm{cris}}$ (D_{st}) is fully faithful and the inverse on its essential image is $\mathrm{V}_{\mathrm{cris}}$ (V_{st}).*

Proof. We show this for D_{st}. That $V \cong \mathrm{V}_{\mathrm{st}}(\mathrm{D}_{\mathrm{st}}(V))$ follows from the fundamental exact sequence and before. Let $V, V' \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{st}}(G_K)$ and let $D = \mathrm{D}_{\mathrm{st}}(V)$ and $D' = \mathrm{D}_{\mathrm{st}}(V')$. We already know that D_{st} is faithful, as B_{st} is regular, so we only need to show that $\mathrm{Hom}_{\mathbb{Q}_p[G_K]}(V', V) \to \mathrm{Hom}_{\mathrm{MF}_K^{\varphi,N}}(D', D)$ is surjective. Let $T : D' \to D$ respecting φ, N, as well as filtrations over K. Then $1 \otimes T : \mathrm{B}_{\mathrm{st}} \otimes_{K_0} D' \to \mathrm{B}_{\mathrm{st}} \otimes_{K_0} D$ and using the comparison isomorphisms $\alpha_{\mathrm{st},D} : \mathrm{B}_{\mathrm{st}} \otimes_{K_0} D \cong \mathrm{B}_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V$ and $\alpha_{\mathrm{st},D'} : \mathrm{B}_{\mathrm{st}} \otimes_{K_0} D' \cong \mathrm{B}_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V'$ we get $\widetilde{T} := \alpha_{\mathrm{st},D} T \alpha_{\mathrm{st},D'}^{-1} : \mathrm{B}_{\mathrm{st}} \otimes_{K_0} V' \to \mathrm{B}_{\mathrm{st}} \otimes_{K_0} V$. We already know that the comparison isomorphism is an isomorphism of filtered vector spaces over K by Proposition 4.36, and it respect φ and N so $\widetilde{T}$ respects φ, N and filtrations over K as well. Taking φ and N invariants we get $(\mathrm{B}_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V')^{\varphi=1,N=0} \to (\mathrm{B}_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{\varphi=1,N=0}$ which gives a map $\mathrm{B}_{\mathrm{st}}^{\varphi=1,N=0} \otimes_{\mathbb{Q}_p} V' \to \mathrm{B}_{\mathrm{st}}^{\varphi=1,N=0} \otimes_{\mathbb{Q}_p} V$. Tensoring with K and restricting to Fil^0 gives (since $(\mathrm{Fil}^0 \mathrm{B}_{\mathrm{st}})^{\varphi=1,N=0} = (\mathrm{Fil}^0 \mathrm{B}_{\mathrm{cris}})^{\varphi=1} = \mathbb{Q}_p$) a map $V' \to V$. $\square$

5.5.3 The main theorems: admissibility and the p-adic monodromy conjecture

The main results are the following:

Theorem 5.59. *The functors $\mathrm{D}_{\mathrm{cris}} : \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_K) \to \mathrm{MF}_K^{\varphi,\mathrm{wa}}$ and $\mathrm{D}_{\mathrm{st}} : \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{st}}(G_K) \to \mathrm{MF}_K^{\varphi,N,\mathrm{wa}}$ are equivalences of categories.*

Theorem 5.60. *If $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(G_K)$ there exists a finite extension L/K such that $V|_{G_L} \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{st}}(G_L)$, in other words, de Rham representations are potentially semistable.*

5.6 Bloch-Kato

The reference for this section is [2].

5.6.1 H^1_e, H^1_f and H^1_g

Definition 5.61. For $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ define

$$\begin{aligned}
H^1(K,V) &= H^1(G_K,V) \\
H^1_g(K,V) &= \ker(H^1(G_K,V) \to H^1(G_K, \mathrm{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V) && \text{“geometric”} \\
H^1_f(K,V) &= \ker(H^1(G_K,V) \to H^1(G_K, \mathrm{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V) && \text{“finite”} \\
H^1_e(K,V) &= \ker(H^1(G_K,V) \to H^1(G_K, \mathrm{B}_{\mathrm{cris}}^{\varphi=1} \otimes_{\mathbb{Q}_p} V) && \text{“exponential”}
\end{aligned}$$

Note that $H^1_e(K,V) \subset H^1_f(K,V) \subset H^1_g(K,V) \subset H^1(K,V)$.

Lemma 5.62. *If $V \in \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_K)$ then $H^1(K, \mathrm{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V) \hookrightarrow H^1(K, \mathrm{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)$.*

Proof. The exact sequence $0 \to \mathrm{B}^+_{\mathrm{dR}} \to \mathrm{B}_{\mathrm{dR}} \to \mathrm{B}_{\mathrm{dR}} / \mathrm{B}^+_{\mathrm{dR}} \to 0$ gives the exact sequence $0 \to \mathrm{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V \to \mathrm{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V \to \mathrm{B}_{\mathrm{dR}} / \mathrm{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V \to 0$ which in turn gives the exact sequence

$$0 \to H^0(K, \mathrm{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V) \to H^0(K, \mathrm{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V) \to H^0(K, \mathrm{B}_{\mathrm{dR}} / \mathrm{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)$$

But then

$$\begin{aligned}
\dim_K \mathrm{D}_{\mathrm{dR}}(V) &= \dim_K H^0(K, \mathrm{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V) \\
&\leq \dim_K H^0(K, \mathrm{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V) + \dim_K H^0(K, \mathrm{B}_{\mathrm{dR}} / \mathrm{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)
\end{aligned}$$

Now $0 \to t^{i+1} \mathrm{B}^+_{\mathrm{dR}} \to t^i \mathrm{B}^+_{\mathrm{dR}} \to \mathbb{C}_p(i) \to 0$ gives

$$0 \to H^0(K, t^{i+1} \mathrm{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V) \to H^0(K, t^i \mathrm{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V) \to H^0(K, \mathbb{C}_p(i) \otimes_{\mathbb{Q}_p} V)$$

and we deduce that

$$\dim_K H^0(K, t^i \mathrm{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V) \leq \dim_K H^0(K, t^{i+1} \mathrm{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V) + \dim_K H^0(K, \mathbb{C}_p(i) \otimes_{\mathbb{Q}_p} V)$$

Similarly for $i < 0$

$$\dim_K H^0(K, t^i \mathrm{B}^+_{\mathrm{dR}} / \mathrm{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V) \leq \dim_K H^0(K, t^{i+1} \mathrm{B}^+_{\mathrm{dR}} / \mathrm{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V) + \dim_K H^0(K, \mathbb{C}_p(i) \otimes_{\mathbb{Q}_p} V)$$

Since $H^0(K, \mathrm{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V) = \varinjlim H^0(K, \mathrm{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)$ such that for $i \geq i_\infty$ we have $\mathrm{Fil}^i \mathrm{D}_{\mathrm{dR}}(V) = (t^i \mathrm{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K} = 0$. Moreover, since $H^0(K, \mathrm{B}_{\mathrm{dR}} / \mathrm{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V) = \varinjlim H^0(K, t^i \mathrm{B}^+_{\mathrm{dR}} / \mathrm{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)$ is finite dimensional it follows that there exists an integer $i_{-\infty} \leq 0$ such that for $i \leq i_{-\infty}$ we have $H^0(K, \mathrm{B}_{\mathrm{dR}} / \mathrm{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V) = H^0(K, t^i \mathrm{B}^+_{\mathrm{dR}} / \mathrm{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)$. Inductively we have

$$\begin{aligned}
\dim_K H^0(K, \mathrm{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V) &\leq \dim_K H^0(K, t^{i_\infty} \mathrm{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V) + \sum_{i=0}^{i_\infty} \dim_K H^0(K, \mathbb{C}_p(i) \otimes_{\mathbb{Q}_p} V) \\
&\leq \sum_{i=0}^{\infty} \dim_K H^0(K, \mathbb{C}_p(i) \otimes_{\mathbb{Q}_p} V)
\end{aligned}$$

and

$$\begin{aligned}
\dim_K H^0(K, \mathrm{B}_{\mathrm{dR}} / \mathrm{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V) &= \dim_K H^0(K, t^{i_{-\infty}} \mathrm{B}^+_{\mathrm{dR}} / \mathrm{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V) \\
&\leq \sum_{i=i_{-\infty}}^{-1} \dim_K H^0(K, \mathbb{C}_p(i) \otimes_{\mathbb{Q}_p} V) \\
&\leq \sum_{i=-\infty}^{-1} \dim_K H^0(K, \mathbb{C}_p(i) \otimes_{\mathbb{Q}_p} V)
\end{aligned}$$

Combining the two we have

$$\begin{aligned}\dim_K \mathrm{D}_{\mathrm{dR}}(V) &\leq \sum_{-\infty}^{\infty} \dim_K H^0(K, \mathbb{C}_p(i) \otimes_{\mathbb{Q}_p} V) \\ &= \dim_K \mathrm{D}_{\mathrm{HT}}(V)\end{aligned}$$

and the two are equal since V is assumed to be de Rham. Therefore the map $H^0(K, \mathrm{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V) \to H^0(K, \mathrm{B}_{\mathrm{dR}} / \mathrm{B}_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} V)$ is surjective which in the long exact sequence gives that the map $H^1(K, \mathrm{B}_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} V) \to H^1(K, \mathrm{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)$ is injective as desired. ☐

Proposition 5.63. *If V is de Rham then the following diagram is commutative and the two rows are exact:*

$$\begin{tikzcd}
0 \arrow[r] & H^0(K,V) \arrow[r, "\alpha"] \arrow[d, equal] & \mathrm{D}_{\mathrm{cris}}(V)^{\varphi=1} \oplus \mathrm{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V) \arrow[r, "\beta"] \arrow[d, hook] & \mathrm{D}_{\mathrm{dR}}(V) \arrow[r] \arrow[d, "{(0,\mathrm{id})}"] & H^1_e(K,V) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & H^0(K,V) \arrow[r, "\alpha"] & \mathrm{D}_{\mathrm{cris}}(V) \oplus \mathrm{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V) \arrow[r, "\gamma"] & \mathrm{D}_{\mathrm{cris}}(V) \oplus \mathrm{D}_{\mathrm{dR}}(V) \arrow[r] & H^1_f(K,V) \arrow[r] & 0
\end{tikzcd}$$

where

$$\begin{aligned}\alpha(x) &= (x,x) \\ \beta(x,y) &= x-y \\ \gamma(x,y) &= (x-\varphi(x), x-y)\end{aligned}$$

Proof. We may rewrite the exact sequences in Theorem 5.13 as

$$0 \longrightarrow \mathbb{Q}_p \xrightarrow{\alpha} \mathrm{B}_{\mathrm{cris}}^{\varphi=1} \oplus \mathrm{B}_{\mathrm{dR}}^+ \xrightarrow{\beta} \mathrm{B}_{\mathrm{dR}} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Q}_p \xrightarrow{\alpha} \mathrm{B}_{\mathrm{cris}} \oplus \mathrm{B}_{\mathrm{dR}}^+ \xrightarrow{\gamma} \mathrm{B}_{\mathrm{cris}} \oplus \mathrm{B}_{\mathrm{dR}} \longrightarrow 0$$

From the first one we get the exact sequence

$$0 \to \mathbb{Q}_p \to \mathrm{D}_{\mathrm{cris}}(V)^{\varphi=1} \oplus \mathrm{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V) \to \mathrm{D}_{\mathrm{dR}}(V) \to H^1(K,V) \to$$

$$\to H^1(K, \mathrm{B}_{\mathrm{cris}}^{\varphi=1} \otimes V) \oplus H^1(K, \mathrm{B}_{\mathrm{dR}}^+ \otimes V) \to H^1(K, \mathrm{B}_{\mathrm{dR}} \otimes V)$$

Lemma 5.62 shows that $H^1(K, \mathrm{B}_{\mathrm{dR}}^+ \otimes V) \to H^1(K, \mathrm{B}_{\mathrm{dR}} \otimes V)$ is injective so the kernel of the map $H^1(K, \mathrm{B}_{\mathrm{cris}}^{\varphi=1} \otimes V) \oplus H^1(K, \mathrm{B}_{\mathrm{dR}}^+ \otimes V) \to H^1(K, \mathrm{B}_{\mathrm{dR}} \otimes V)$ is the same as the kernel of the map $H^1(K, \mathrm{B}_{\mathrm{cris}}^{\varphi=1} \otimes V) \to H^1(K, \mathrm{B}_{\mathrm{dR}} \otimes V)$. But the image of $H^1(K,V) \to H^1(K, \mathrm{B}_{\mathrm{cris}}^{\varphi=1} \otimes V) \oplus H^1(K, \mathrm{B}_{\mathrm{dR}}^+ \otimes V) \to H^1(K, \mathrm{B}_{\mathrm{dR}} \otimes V)$ is in this kernel and so the map factors through $H^1(K,V) \to H^1(K, \mathrm{B}_{\mathrm{cris}}^{\varphi=1} \otimes V)$. But then the map $\mathrm{D}_{\mathrm{dR}}(V) \to H^1(K,V)$ has image in $H^1_e(K,V) = \ker(H^1(K,V) \to H^1(K, \mathrm{B}_{\mathrm{cris}}^{\varphi=1} \otimes V))$.

The second exact sequence in the proposition follows analogously from the second fundamental sequence. Indeed, we get

$$0 \to H^0(K,V) \to \mathrm{D}_{\mathrm{cris}}(V) \oplus \mathrm{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V) \to \mathrm{D}_{\mathrm{cris}}(V) \oplus \mathrm{D}_{\mathrm{dR}}(V) \to$$

$$\to H^1(K,V) \to H^1(K, \mathrm{B}_{\mathrm{cris}} \otimes V) \oplus H^1(K, \mathrm{B}_{\mathrm{dR}}^+ \otimes V) \to H^1(K, \mathrm{B}_{\mathrm{cris}} \otimes V) \oplus H^1(K, \mathrm{B}_{\mathrm{dR}} \otimes V)$$

Again, since $H^1(K, \mathrm{B}_{\mathrm{dR}}^+ \otimes V) \hookrightarrow H^1(K, \mathrm{B}_{\mathrm{dR}} \otimes V)$ we deduce that the kernel of the last map lies in $H^1(K, \mathrm{B}_{\mathrm{cris}} \otimes V)$ and so the image of $H^1(K,V)$ lies in $H^1(K, \mathrm{B}_{\mathrm{cris}} \otimes V)$. But that implies that the image of $\mathrm{D}_{\mathrm{cris}}(V) \oplus \mathrm{D}_{\mathrm{dR}}(V)$ lies in the kernel $H^1_f(K,V)$ of $H^1(K,V) \to H^1(K, \mathrm{B}_{\mathrm{cris}} \otimes V)$. ☐

Corollary 5.64. *If V is de Rham then*

1. $\dim_{\mathbb{Q}_p} H^1_f(K,V) = \dim_{\mathbb{Q}_p}(\mathrm{D}_{\mathrm{dR}}(V)/\operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V)) + \dim_{\mathbb{Q}_p} H^0(K,V)$ *and*
2. $H^1_f(K,V)/H^1_e(K,V) \cong \mathrm{D}_{\mathrm{cris}}(V)/(1-\varphi)\,\mathrm{D}_{\mathrm{cris}}(V)$.

Proof. 1. From the second exact sequence in Proposition 5.63 we compute

$$\begin{aligned}\dim_{\mathbb{Q}_p} H^1_f(K,V) &= \dim_{\mathbb{Q}_p} H^0(K,V) - \dim_{\mathbb{Q}_p} \mathrm{D}_{\mathrm{cris}}(V) \oplus \operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V) + \dim_{\mathbb{Q}_p} \mathrm{D}_{\mathrm{cris}}(V) \oplus \mathrm{D}_{\mathrm{dR}}(V)\\ &= \dim_{\mathbb{Q}_p}(\mathrm{D}_{\mathrm{dR}}(V)/\operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V)) + \dim_{\mathbb{Q}_p} H^0(K,V)\end{aligned}$$

2. In the diagram in Proposition 5.63 we rewrite the map $\beta : \mathrm{D}_{\mathrm{cris}}(V)^{\varphi=1} \oplus \operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V) \to \mathrm{D}_{\mathrm{dR}}(V)$ as $\gamma : \mathrm{D}_{\mathrm{cris}}(V)^{\varphi=1} \oplus \operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V) \to 0 \oplus \mathrm{D}_{\mathrm{dR}}(V)$. We have

$$\begin{tikzcd}
0 \arrow[r] & H^0(K,V) \arrow[r] \arrow[d, equal] & \mathrm{D}_{\mathrm{cris}}(V)^{\varphi=1} \oplus \operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V) \arrow[r] \arrow[d, hook] & \gamma(\mathrm{D}_{\mathrm{cris}}(V)^{\varphi=1} \oplus \operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V)) \arrow[r] \arrow[d] & 0\\
0 \arrow[r] & H^0(K,V) \arrow[r] & \mathrm{D}_{\mathrm{cris}}(V) \oplus \operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V) \arrow[r] & \gamma(\mathrm{D}_{\mathrm{cris}}(V) \oplus \operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V)) \arrow[r] & 0
\end{tikzcd}$$

and the snake lemma gives

$$\begin{aligned}\frac{\gamma(\mathrm{D}_{\mathrm{cris}}(V) \oplus \operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V))}{\gamma(\mathrm{D}_{\mathrm{cris}}(V)^{\varphi=1} \oplus \operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V))} &\cong \gamma\left(\frac{\mathrm{D}_{\mathrm{cris}}(V) \oplus \operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V)}{\mathrm{D}_{\mathrm{cris}}(V)^{\varphi=1} \oplus \operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V)}\right)\\ &\cong \gamma(\mathrm{D}_{\mathrm{cris}}(V)/\mathrm{D}_{\mathrm{cris}}(V)^{\varphi=1})\\ &\cong (1-\varphi)\,\mathrm{D}_{\mathrm{cris}}(V)\end{aligned}$$

The snake lemma applied to the diagram

$$\begin{tikzcd}
0 \arrow[r] & \gamma(\mathrm{D}_{\mathrm{cris}}(V)^{\varphi=1} \oplus \operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V)) \arrow[r] \arrow[d] & 0 \oplus \mathrm{D}_{\mathrm{dR}}(V) \arrow[r] \arrow[d] & H^1_e(K,V) \arrow[r] \arrow[d, hook] & 0\\
0 \arrow[r] & \gamma(\mathrm{D}_{\mathrm{cris}}(V) \oplus \operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V)) \arrow[r] & \mathrm{D}_{\mathrm{cris}}(V) \oplus \mathrm{D}_{\mathrm{dR}}(V) \arrow[r] & H^1_f(K,V) \arrow[r] & 0
\end{tikzcd}$$

gives

$$0 \to \underbrace{\frac{\gamma(\mathrm{D}_{\mathrm{cris}}(V) \oplus \operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V))}{\gamma(\mathrm{D}_{\mathrm{cris}}(V)^{\varphi=1} \oplus \operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V))}}_{(1-\varphi)\,\mathrm{D}_{\mathrm{cris}}(V)} \to \mathrm{D}_{\mathrm{cris}}(V) \to H^1_f(K,V)/H^1_e(K,V) \to 0$$

which implies the statement.

□

5.6.2 Tate duality

First, we start with the setup. Twisting by t in Theorem 5.13 gives

$$0 \to \mathbb{Q}_p(1) \to \mathrm{B}_{\mathrm{cris}}^{\varphi=p} \oplus t\,\mathrm{B}_{\mathrm{dR}}^+ \to \mathrm{B}_{\mathrm{dR}} \to 0$$

which when restricted to Fil^0 gives

$$0 \to \mathbb{Q}_p(1) \to (\mathrm{B}_{\mathrm{cris}}^{\varphi=p} \cap \mathrm{B}_{\mathrm{dR}}^+) \oplus t\,\mathrm{B}_{\mathrm{dR}}^+ \to \mathrm{B}_{\mathrm{dR}}^+ \to 0$$

or

$$0 \to \mathbb{Q}_p(1) \to (\mathrm{B}_{\mathrm{cris}}^{\varphi=p} \cap \mathrm{B}_{\mathrm{dR}}^{+}) \to \mathrm{B}_{\mathrm{dR}}^{+} / t\, \mathrm{B}_{\mathrm{dR}}^{+} \to 0$$

which is

$$0 \to \mathbb{Q}_p(1) \to \mathrm{Fil}^0 \mathrm{B}_{\mathrm{cris}}^{\varphi=p} \to \mathbb{C}_p \to 0$$

Lemma 5.65. [2] *Let V be a de Rham representation and $\gamma : H^1(K, \mathbb{C}_p) \to H^2(K, \mathbb{Q}_p(1)) \cong \mathbb{Q}_p$ be the connecting homomorphism obtained from the exact sequence $0 \to \mathbb{Q}_p(1) \to \mathrm{Fil}^0 \mathrm{B}_{\mathrm{cris}}^{\varphi=p} \to \mathbb{C}_p \to 0$. Then the composite $H^1(K, \mathbb{C}_p \otimes_{\mathbb{Q}_p} V) \times H^0(K, \mathbb{C}_p \otimes_{\mathbb{Q}_p} V^*) \xrightarrow{\smile} H^1(K, \mathbb{C}_p) \xrightarrow{\gamma} H^2(K, \mathbb{Q}_p(1)) \cong \mathbb{Q}_p$ is a perfect pairing.*

Proof. The representation V is Hodge-Tate so we only need to prove the statement for $V = \mathbb{Q}_p(n)$ as the general V is a direct sum of these. But when $n \neq 0$ the two cohomology groups vanish so we only need to show that $H^1(K, \mathbb{C}_p) \times H^0(K, \mathbb{C}_p) \to \mathbb{Q}_p$ is perfect. But $H^1(K, \mathbb{C}_p) = K \cdot \log \chi_{\mathrm{cycl}}$ and $H^0(K, \mathbb{C}_p) = K$ so we only need to show that $\gamma \neq 0$.

If $\delta : H^0(K, \mathbb{C}_p) \to H^1(K, \mathbb{Q}_p(1))$ is the connecting homomorphism in degree 0 then there is a commutative diagram

$$\begin{tikzcd}
H^1(K, \mathbb{Q}_p) \otimes H^0(K, \mathbb{C}_p) \arrow[r, "\mathrm{id} \otimes \delta"] \arrow[d, "\smile"] & H^1(K, \mathbb{Q}_p) \otimes H^1(K, \mathbb{Q}_p(1)) \arrow[d, "\smile"] \\
H^1(K, \mathbb{C}_p) \arrow[r, "\gamma"] & H^2(K, \mathbb{Q}_p(1))
\end{tikzcd}$$

Since the right vertical map is the (nondegenerate) Tate pairing, to show that γ does not vanish it is enough to show that δ does not vanish. Suppose $x \in K = H^0(K, \mathbb{C}_p)$ such that $\delta(x) = 0$. Then in the exact sequence

$$H^0(K, \mathrm{Fil}^0 \mathrm{B}_{\mathrm{cris}}^{\varphi=p}) \to H^0(K, \mathbb{C}_p) \xrightarrow{\delta} H^1(K, \mathbb{Q}_p(1))$$

x would have to be the image in $H^0(K, \mathbb{C}_p)$ of some $y \in H^0(K, \mathrm{Fil}^0 \mathrm{B}_{\mathrm{cris}}^{\varphi=p})$. But

$$\begin{aligned} H^0(K, \mathrm{Fil}^0 \mathrm{B}_{\mathrm{cris}}^{\varphi=p}) &= \mathrm{Fil}^0 \mathrm{B}_{\mathrm{cris}}^{G_K, \varphi=p} \\ &= \mathrm{Fil}^0 K_0^{\varphi=p} \\ &= K_0^{\varphi=p} \end{aligned}$$

but on K_0 Frobenius is $\sigma_{K_0} \in G_{K_0/\mathbb{Q}_p}$ which preserves valuation and so $K_0^{\varphi=p} = 0$. Therefore $y = 0$ so $x = 0$ and thus δ is injective. ☐

Proposition 5.66. *Let $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(G_K)$ and let $V^* = \mathrm{Hom}_{\mathbb{Q}_p[G_K]}(V, \mathbb{Q}_p)$. Via the perfect Tate pairing*

$$H^1(K, V) \times H^1(K, V^*(1)) \to H^2(K, \mathbb{Q}_p(1)) \cong \mathbb{Q}_p$$

the following are annihilators of each other:

1. *$H^1_f(K, V)$ and $H^1_f(K, V^*(1))$*
2. *$H^1_e(K, V)$ and $H^1_g(K, V^*(1))$.*

Proof. We first remark that Tate duality implies that $\dim_{\mathbb{Q}_p} H^0(K, V^*(1)) = \dim_{\mathbb{Q}_p} H^2(K, V)$ while $\dim_{\mathbb{Q}_p} H^0(K, V) - \dim_{\mathbb{Q}_p} H^1(K, V) + \dim_{\mathbb{Q}_p} H^2(K, V) = -[K : \mathbb{Q}_p] \dim_{\mathbb{Q}_p} V$ by the local Euler-Tate characteristic formula. Therefore $\dim_{\mathbb{Q}_p} H^1(K, V) = \dim_{\mathbb{Q}_p} H^0(K, V) + \dim_{\mathbb{Q}_p} H^0(K, V^*(1)) + [K : \mathbb{Q}_p] \dim_{\mathbb{Q}_p} V$. Since $(V^*(1))^*(1) \cong V$ and $\dim V = \dim V^*(1)$ it follows that $\dim_{\mathbb{Q}_p} H^1(K, V) = \dim_{\mathbb{Q}_p} H^1(K, V^*(1))$.

[2] not covered in class

1. Let $\alpha \in H^1_f(K,V)$. Note that the fundamental sequence $0 \to \mathbb{Q}_p \to \mathrm{B}_{\mathrm{cris}} \oplus \mathrm{B}^+_{\mathrm{dR}} \to \mathrm{B}_{\mathrm{cris}} \oplus \mathrm{B}_{\mathrm{dR}} \to 0$ gives a connecting homomorphism $\varepsilon : H^1(K, \mathrm{B}_{\mathrm{cris}} \otimes V \otimes V^*(1)) \oplus H^1(K, \mathrm{B}_{\mathrm{dR}} \otimes V \otimes V^*(1)) \to H^2(K, V \otimes V^*(1))$ and together with the bottom exact sequence of Proposition 5.63 we get a commutative diagram

$$\begin{tikzcd}
\mathrm{D}_{\mathrm{cris}}(V^*(1)) \oplus \mathrm{D}_{\mathrm{dR}}(V^*(1)) \arrow[r, two heads] \arrow[d, "\smile\alpha"] & H^1_f(K, V^*(1)) \arrow[d, "\smile\alpha"] \\
H^1(K, \mathrm{B}_{\mathrm{cris}} \otimes V \otimes V^*(1)) \oplus H^1(K, \mathrm{B}_{\mathrm{dR}} \otimes V \otimes V^*(1)) \arrow[r, "\varepsilon"] & H^2(K, V \otimes V^*(1))
\end{tikzcd}$$

Since the image of α in $H^1(K, \mathrm{B}_{\mathrm{dR}} \otimes V)$ is trivial the left vertical map vanishes and so $H^1_f(K, V^*(1)) \smile \alpha = 0$ which implies that $H^1_f(K,V)$ annihilates $H^1_f(K, V^*(1))$.

To show that they are exact annihilators it is enough to show that $\dim_{\mathbb{Q}_p} H^1_f(K,V) + \dim_{\mathbb{Q}_p} H^1_f(K, V^*(1)) = \dim_{\mathbb{Q}_p} H^1(K,V)$. But Corollary 5.64 implies that

$$\begin{aligned}
\dim_{\mathbb{Q}_p} H^1_f(K,V) &= \dim_{\mathbb{Q}_p} \mathrm{D}_{\mathrm{dR}}(V)/\operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V) + \dim_{\mathbb{Q}_p} H^0(K,V) \\
\dim_{\mathbb{Q}_p} H^1_f(K,V^*(1)) &= \dim_{\mathbb{Q}_p} \mathrm{D}_{\mathrm{dR}}(V^*(1))/\operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V^*(1)) + \dim_{\mathbb{Q}_p} H^0(K,V^*(1))
\end{aligned}$$

But $\dim_{\mathbb{Q}_p} \mathrm{D}_{\mathrm{dR}}(V) = \dim_{\mathbb{Q}_p} \mathrm{D}_{\mathrm{dR}}(V^*(1)) = [K : \mathbb{Q}_p] \dim_{\mathbb{Q}_p} V$ by the fact that V and $V^*(1)$ are de Rham. Moreover (cf. the proof of Lemma 5.62)

$$\begin{aligned}
\dim_{\mathbb{Q}_p} \operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V) &= \dim_{\mathbb{Q}_p} \operatorname{gr} \operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V) \\
&= \sum_{n \geq 0} \dim_{\mathbb{Q}_p} H^0(K, \mathbb{C}_p(n) \otimes V) \\
\dim_{\mathbb{Q}_p} \operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V^*(1)) &= \sum_{n \geq 0} \dim_{\mathbb{Q}_p} H^0(K, \mathbb{C}_p(-n-1) \otimes V) \\
&= \sum_{n \leq -1} \dim_{\mathbb{Q}_p} H^0(K, \mathbb{C}_p(n) \otimes V)
\end{aligned}$$

which implies that

$$\begin{aligned}
\dim_{\mathbb{Q}_p} \operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V) + \dim_{\mathbb{Q}_p} \operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V^*(1)) &= \dim_{\mathbb{Q}_p} \mathrm{D}_{\mathrm{HT}}(V) \\
&= [K : \mathbb{Q}_p] \dim_{\mathbb{Q}_p} V
\end{aligned}$$

Therefore

$$\begin{aligned}
\dim_{\mathbb{Q}_p} H^1_f(K,V) + \dim_{\mathbb{Q}_p} H^1_f(K,V^*(1)) &= [K : \mathbb{Q}_p] \dim_{\mathbb{Q}_p} V + \dim_{\mathbb{Q}_p} H^0(K,V) + \dim_{\mathbb{Q}_p} H^0(K,V^*(1)) \\
&= \dim_{\mathbb{Q}_p} H^1(K,V)
\end{aligned}$$

by the computations at the beginning of this proof.

2. Let $\delta : \mathrm{D}_{\mathrm{dR}}(V^*(1)) \twoheadrightarrow H^1_e(K, V^*(1))$ be the boundary map from the first row of Proposition 5.63 applied to $V^*(1)$, and let $\varepsilon : H^1(K, \mathrm{B}_{\mathrm{dR}} \otimes \mathbb{Q}_p(1)) \to H^2(K, \mathbb{Q}_p(1))$ be the boundary map obtained from the exact sequence $0 \to \mathbb{Q}_p(1) \to (\mathrm{B}^{\varphi=1}_{\mathrm{cris}} \oplus \mathrm{B}^+_{\mathrm{dR}}) \otimes \mathbb{Q}_p(1) \to \mathrm{B}_{\mathrm{dR}} \otimes \mathbb{Q}_p(1) \to 0$. Then the following diagram is commutative

$$\begin{tikzcd}
H^1(K,V) \otimes \mathrm{D}_{\mathrm{dR}}(V^*(1)) \arrow[r, two heads, "(\mathrm{id},\delta)"] \arrow[d] & H^1(K,V) \otimes H^1_e(K,V^*(1)) \arrow[r, hook] & H^1(K,V) \otimes H^1(K,V^*(1)) \arrow[d, "\smile"] \\
H^1(K, \mathrm{B}_{\mathrm{dR}} \otimes V) \otimes \mathrm{D}_{\mathrm{dR}}(V^*(1)) \arrow[r, "\smile"] & H^1(K, \mathrm{B}_{\mathrm{dR}} \otimes \mathbb{Q}_p(1)) \arrow[r, "\varepsilon"] & H^2(K, \mathbb{Q}_p(1)) \cong \mathbb{Q}_p
\end{tikzcd}$$

Let $\alpha \in H^1(K,V)$ and let α_{dR} be its image in $H^1(K, \mathrm{B}_{\mathrm{dR}} \otimes V)$. Since the right vertical map is the perfect Tate pairing, it follows that α annihilates $H^1_e(K, V^*(1))$ if and only if α annihilates $\mathrm{D}_{\mathrm{dR}}(V^*(1))$ under the composite pairing, if and only if α_{dR} annihilates $\mathrm{D}_{\mathrm{dR}}(V^*(1))$ under $\varepsilon\circ \smile$. We will show that $\varepsilon\circ \smile$ is a perfect pairing, which implies that $\alpha_{\mathrm{dR}} = 0$ and so α annihilates $H^1_e(K, V^*(1))$ if and only if $\alpha \in H^1_g(K, V^*(1))$.

We now show that $\varepsilon\circ \smile$ is a perfect pairing[3]. Suppose that $\alpha \in H^1(K, t^n \mathrm{B}^+_{\mathrm{dR}} \otimes V)$ annihilates $\mathrm{D}_{\mathrm{dR}}(V^*(1))$. There exists a commutative diagram (the reason the diagram is commutative is that to obtain the bottom row we multiplied by t the fundamental sequence, whereas for the top row we did not)

$$\begin{tikzcd}
H^1(K, t^n \mathrm{B}^+_{\mathrm{dR}} \otimes V) \arrow[d] & \otimes & \mathrm{D}_{\mathrm{dR}}(V^*(1)) \arrow[r] & H^2(K, \mathbb{Q}_p(1)) \cong \mathbb{Q}_p \arrow[dd, equal] \\
H^1(K, \mathbb{C}_p(n) \otimes V) \arrow[d, equal] & & \mathrm{Fil}^{-n-1} \mathrm{D}_{\mathrm{dR}}(V^*(1)) = \mathrm{Fil}^{-n} \mathrm{D}_{\mathrm{dR}}(V^*)t^{-1} \arrow[u, hook] \arrow[d] & \\
H^1(K, \mathbb{C}_p \otimes V(n)) & \otimes & H^0(K, \mathbb{C}_p \otimes V(n)^*) \arrow[r] & \mathbb{Q}_p
\end{tikzcd}$$

where the surjection $\mathrm{Fil}^{-n} \mathrm{D}_{\mathrm{dR}}(V^*) = H^0(K, \mathrm{B}^+_{\mathrm{dR}} \otimes V(n)^*) \twoheadrightarrow H^0(K, \mathbb{C}_p \otimes V(n))$ follows from the proof of Lemma 5.62 and the surjection $H^1(K, t^n \mathrm{B}^+_{\mathrm{dR}} \otimes V) \twoheadrightarrow H^1(K, \mathbb{C}_p(n) \otimes V)$ from Proposition 4.34. Since α annihilates $\mathrm{D}_{\mathrm{dR}}(V^*(1))$, its image in the bottom row annihilates $H^0(K, \mathbb{C}_p \otimes V^*)$. But the bottom row is perfect by Lemma 5.65 applied to $V(n)$, so it follows that the image of α is zero and so α is in fact in $H^1(K, t^{n+1} \mathrm{B}^+_{\mathrm{dR}} \otimes V)$. Now let $\alpha \in H^1(K, \mathrm{B}_{\mathrm{dR}} \otimes V)$ annihilate $\mathrm{D}_{\mathrm{dR}}(V^*(1))$. Let $i \in \mathbb{Z}$ such that $\alpha \in H^1(K, t^i \mathrm{B}^+_{\mathrm{dR}} \otimes V)$. Inductively we get that $\alpha \in H^1(K, t^n \mathrm{B}^+_{\mathrm{dR}} \otimes V)$ for all $n \geq i$. But for $n >> 0$ we have $H^1(K, t^n \mathrm{B}^+_{\mathrm{dR}} \otimes V) = 0$ so $\alpha = 0$.

Now suppose that $\beta \in \mathrm{D}_{\mathrm{dR}}(V^*(1))$ annihilates $H^1(K, t^n \mathrm{B}^+_{\mathrm{dR}} \otimes V)$. Then since the top left map is surjective we would get that the image of β in $H^0(K, \mathbb{C}_p \otimes V(n)^*)$ annihilates $H^1(K, \mathbb{C}_p \otimes V(n))$ and by perfectness of the bottom pairing we deduce that the image of β is trivial. But that would imply that $\beta \in (t^{n+1} \mathrm{B}^+_{\mathrm{dR}} \otimes V)^{G_K}$ and inductively we again deduce that $\beta = 0$.

□

Lecture 26
2012-03-09

5.6.3 Computations

Proposition 5.67. *Let $r \in \mathbb{Z}$. We have the following table of dimensions*

r	$\dim H^1_e(K, \mathbb{Q}_p(r))$	$\dim H^1_f(K, \mathbb{Q}_p(r))$	$\dim H^1_g(K, \mathbb{Q}_p(r))$	$\dim H^1(K, \mathbb{Q}_p(r))$
$r < 0$	0	0	0	$[K : \mathbb{Q}_p]$
$r = 0$	0	1	1	$[K : \mathbb{Q}_p] + 1$
$r = 1$	$[K : \mathbb{Q}_p]$	$[K : \mathbb{Q}_p]$	$[K : \mathbb{Q}_p] + 1$	$[K : \mathbb{Q}_p + 1$
$r > 1$	$[K : \mathbb{Q}_p]$	$[K : \mathbb{Q}_p]$	$[K : \mathbb{Q}_p]$	$[K : \mathbb{Q}_p]$

Proof. We denote $h^\bullet = \dim_{\mathbb{Q}_p} H^\bullet$. First, we have that $h^0(K, \mathbb{Q}_p(r)) = \delta_{r=0}$ and $h^0(K, \mathbb{Q}_p(r)^*(1)) = h^0(K, \mathbb{Q}_p(-r+1)) = \delta_{r=1}$ and so by the Tate characteristic formula

$$\begin{aligned}
h^1(K, \mathbb{Q}_p(r)) &= [K : \mathbb{Q}_p] + h^0(K, \mathbb{Q}_p(r)) + h^0(K, \mathbb{Q}_p(r)^*(1)) \\
&= [K : \mathbb{Q}_p] + \delta_{r=0} + \delta_{r=1}
\end{aligned}$$

[3] not covered in class

This gives the fourth column. Moreover,

$$h^1_f(K, \mathbb{Q}_p(r)) = \dim_{\mathbb{Q}_p} \mathrm{D}_{\mathrm{dR}}(\mathbb{Q}_p(r))/\operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(\mathbb{Q}_p(r)) + h^0(K, \mathbb{Q}_p(r))$$

and since $\mathrm{gr}^{-r} \mathrm{D}_{\mathrm{dR}}(\mathbb{Q}_p(r)) = K$ we deduce that $\mathrm{D}_{\mathrm{dR}}(\mathbb{Q}_p(r)) = \operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(\mathbb{Q}_p(r))$ when $r \leq 0$ and otherwise $\operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(\mathbb{Q}_p(r)) = 0$ giving $\mathrm{D}_{\mathrm{dR}}(\mathbb{Q}_p(r))/\operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(\mathbb{Q}_p(r)) \cong K$ having $\mathbb{Q}_p$ dimension $[K : \mathbb{Q}_p]$. This gives the second column.

We know from Corollary 5.64 that $h^1_f(K, \mathbb{Q}_p(r)) - h^1_e(K, \mathbb{Q}_p(r)) = \dim_{\mathbb{Q}_p} \mathrm{D}_{\mathrm{cris}}(\mathbb{Q}_p(r))/(1-\varphi) \mathrm{D}_{\mathrm{cris}}(\mathbb{Q}_p(r))$. But the exact sequence $0 \to \mathrm{D}_{\mathrm{cris}}(V)^{\varphi=1} \to \mathrm{D}_{\mathrm{cris}}(V) \to \mathrm{D}_{\mathrm{cris}}(V) \to \mathrm{D}_{\mathrm{cris}}(V)/(1-\varphi)\mathrm{D}_{\mathrm{cris}}(V) \to 0$ show that

$$h^1_f(K, \mathbb{Q}_p(r)) - h^1_e(K, \mathbb{Q}_p(r)) = \dim_{\mathbb{Q}_p} \mathrm{D}_{\mathrm{cris}}(\mathbb{Q}_p(r))^{\varphi=1}$$

But $\mathrm{D}_{\mathrm{cris}}(\mathbb{Q}_p(r))^{\varphi=1} = (K_0 t^{-r})^{\varphi=1} = K_0^{\sigma_{K_0}=p^r}$ which is 0 when $r \neq 0$ as σ_{K_0} preserves v_p on K_0. If $r = 0$ then $K_0^{\sigma_{K_0}=1} = \mathbb{Q}_p$ and so $h^1_f(K, \mathbb{Q}_p) - h^1_e(K, \mathbb{Q}_p) = 1$, giving the first column from the second one.

Finally, Proposition 5.66 gives that

$$h^1_g(K, V) = h^1(K, V^*(1)) - h^1_e(K, V^*(1))$$

giving the third column. ☐

5.6.4 Extensions

Remark 30. Let $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$. Extensions $0 \to V \to W \to \mathbb{Q}_p \to 0$ are in bijection with elements of $c_W \in H^1(K, V)$. The representation W is de Rham (crystalline) if and only if $c_W \in H_g(K, V)$ ($c_W \in H^1_f(K, V)$).

Corollary 5.68. *1. There exists a (necessarily Hodge-Tate) extension* $0 \to \mathbb{Q}_p(-1) \to V \to \mathbb{Q}_p \to 0$ *which is not de Rham.*

2. *When* $r \geq 1$ *all extensions* $0 \to \mathbb{Q}_p(r) \to V \to \mathbb{Q}_p \to 0$ *are de Rham.*

3. *When* $r \geq 2$ *all extensions* $0 \to \mathbb{Q}_p(r) \to V \to \mathbb{Q}_p \to 0$ *are crystalline.*

4. *There exists a de Rham but not crystalline extension* $0 \to \mathbb{Q}_p(1) \to V \to \mathbb{Q}_p \to 0$.

Proof. 1. Follows from Remark 30 since $\dim H^1(K, \mathbb{Q}_p(-1)) = [K : \mathbb{Q}_p] > 0$ and $H_g(K, \mathbb{Q}_p(-1)) = 0$.

2. Follows from the fact that $\dim H^1(K, \mathbb{Q}_p(r)) = \dim H^1_g(K, \mathbb{Q}_p(r))$.

3. Follows from the fact that $\dim H^1(K, \mathbb{Q}_p(r)) = \dim H^1_f(K, \mathbb{Q}_p(r))$.

4. Follows from the fact that $\dim H^1_g(K, \mathbb{Q}_p(1)) > \dim H^1_f(K, \mathbb{Q}_p(1))$.

☐

Example 5.69. If $E/\mathbb{Q}_p$ is an elliptic curve with multiplicative reduction then (using Tate curves) we get an extension $0 \to \mathbb{Q}_p(1) \to V_pE \to \mathbb{Q}_p \to 0$, and so V_pE is necessarily de Rham (even semistable using Perrin-Riou).

5.7 Ordinary representations

Corollary 5.68 implies that all extensions $0 \to \mathbb{Q}_p(1) \to V \to \mathbb{Q}_p \to 0$ are de Rham. Perrin-Riou computed extensions in $\mathrm{MF}_K^{\varphi,N,\mathrm{wa}}$ to show that in fact all such extensions are semistable (cf. also [3, Lemma 8.3.9]). The main complication in [5] is the lack of availability at the time of the equivalence of categories $\mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(G_K) \cong \mathrm{MF}_K^{\varphi,N,\mathrm{wa}}$. Assuming this, showing that such extensions are semistable is straightforward.

Lemma 5.70. *We have*

$$\mathrm{Ext}^1_{\mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(G_K)}(\mathbb{Q}_p, \mathbb{Q}_p(1)) \cong K \times \mathbb{Q}_p$$

Proof. From the equivalence of categories we deduce:

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(G_K)}(\mathbb{Q}_p,\mathbb{Q}_p(1)) &\cong \mathrm{Ext}^1_{\mathrm{MF}_K^{\varphi,N,\mathrm{wa}}}(\mathrm{D}_{\mathrm{st}}(\mathbb{Q}_p),\mathrm{D}_{\mathrm{st}}(\mathbb{Q}_p(1)))\\ &\cong \mathrm{Ext}^1_{\mathrm{MF}_K^{\varphi,N,\mathrm{wa}}}(K_0,K_0\langle -1\rangle)\\ &\cong \mathrm{Ext}^1_{\mathrm{MF}_K^{\varphi,N,\mathrm{wa}}}(K_0\langle 1\rangle,K_0)\end{aligned}$$

Extensions $D\in \mathrm{Ext}^1_{\mathrm{MF}_K^{\varphi,N,\mathrm{wa}}}(K_0\langle 1\rangle,K_0)$ have a basis e_0,e_1 such that "$K_0'' = K_0e_0$ and "$K_0\langle 1\rangle'' = K_0e_1$, i.e., $\varphi(e_0)=e_0$ with slope 0 ($D(0)=K_0e_0$) and $\varphi(e_1)=pe_1$ with slope 1 ($D(1)=K_0e_1$). Since N decreases slope by 1 we deduce that there exists $\alpha\in K_0$ such that $Ne_1=\alpha e_0$ and $Ne_0=0$. But $\varphi Ne_1=\varphi(\alpha e_0)=\sigma_{K_0}(\alpha)e_0$ and $N\varphi e_1 = Npe_1 = p\alpha e_0$. Since $N\varphi = p\varphi N$ we deduce that $\alpha=\sigma_{K_0}(\alpha)$ and so $\alpha\in\mathbb{Q}_p$.

The Hodge-Tate weights are 0 and 1 and so $\mathrm{Fil}^0 D_K = D_K$, $\mathrm{Fil}^2 D_K = 0$ and $\mathrm{Fil}^1 D_K \subset Ke_0\oplus Ke_1$ is a K-line. If $\mathrm{Fil}^1 D_K = Ke_0$ then K_0e_0 is a subobject of D in $\mathrm{MF}_K^{\varphi,N}$ with $t_H(K_0e_0)=1>0=t_N(K_0e_0)$. Therefore $\mathrm{Fil}^1 D_K = K(e_1-\mathcal{L}e_0)$ for some $\mathcal{L}\in K$. So to D we attached $(\mathcal{L},\alpha)\in K\times\mathbb{Q}_p$ and one can check that all $\mathcal{L}\in K$ and $\alpha\in\mathbb{Q}_p$ give weakly admissible, and hence admissible D. $\square$

Proposition 5.71. *If* $0\to\mathbb{Q}_p(1)\to V\to\mathbb{Q}_p\to 0$ *then* V *is semistable.*

Proof. It is enough, by the previous lemma, to give an isomorphism $H^1(G_K,\mathbb{Q}_p(1))\cong K\times\mathbb{Q}_p$. But Kummer theory gives $H^1(G_K,\mathbb{Z}_p(1))\cong \varprojlim K^\times/(K^\times)^{p^n}$ and so $H^1(G_K,\mathbb{Q}_p(1))\cong\varprojlim K^\times/(K^\times)^{p^n}\otimes\mathbb{Q}_p$.

If $v_p(x)>\dfrac{1}{p-1}$ then $\exp(x)$ converges and in that case $\exp(px)=\exp(x)^p$. Consider the map $K\times\mathbb{Q}_p\to\varprojlim K^\times/(K^\times)^{p^n}\otimes\mathbb{Q}_p$ taking (x,q) to $\varpi_K^{p^nq}\exp(p^nx)\otimes p^{-n}$ for n large enough to make the exponential convergent and $p^nq\in\mathbb{Z}_p$. This map is clearly injective so we only need surjectivity. Note that $K^\times = \varpi_K^{\mathbb{Z}}\times k_K^\times\times(1+\mathfrak{m}_K)$ and so (since $p\nmid \#k_K^\times$)

$$\varprojlim K^\times/(K^\times)^{p^n}\cong\varpi_K^{\mathbb{Z}_p}\times\varprojlim(1+\mathfrak{m}_K)/(1+\mathfrak{m}_K)^{p^n}$$

which via the log map (normalized such that $\log\varpi_K=0$) goes to $\varprojlim\mathfrak{m}_K/p^n\mathfrak{m}_K=\mathfrak{m}_K$ which is complete. The map $\varprojlim K^\times/(K^\times)^{p^n}\otimes\mathbb{Q}_p\to K\times\mathbb{Q}_p$ given by $x\otimes q\mapsto(q\log x, qv_p(x))$ is an inverse to $(x,q)\mapsto\varpi_K^{p^nq}\exp(p^nx)\otimes p^{-n}$. $\square$

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