A Proof of the Tietze Extension Theorem Using Urysohn's Lemma

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1 Motivation

Mathematics could be described as an ultimate quest for generalization. Everyone knows that in a (3-4-5) right triangle,

$$3^2 + 4^2 = 5^2.$$

As Pythagoras knew, however, this is the case for any right triangle. That is, the sum of the squares of the legs equals the square on the hypotenuse. Or as an equation,

$$a^2 + b^2 = c^2.$$

This of course can handle a much broader range of triangles, but it is still not satisfactory enough. This formula gives no information about triangles that are not right, so this formula was later generalized to the so-called "Law of Cosines"

$$a^2 = b^2 + c^2 - 2bc \cos A$$

This formula can be applied to any triangle that you can draw. This quest towards generalization is a good way to think of mathematics, and a good way to lead into this short topological paper.

Although the above example is quite elementary, in a way, so is the question behind the Tietze Extension Theorem. Take for instance the curve defined by $f(x) = x$ on the closed interval $[0, 1]$. The graph of this curve is a line segment. Indeed, what we really have is a continuous function (hereafter called a map):

$$f : [0, 1] \longrightarrow R$$

It is now natural to ask if there exists a continuous function g defined over the whole space R such that $f = g$ on $[0, 1]$. For this particular example, it is easy, just take $g(x) = x$.

In general, however, it might not be possible to find such a function. For example, consider $f(x) = 1/x$ on the open interval $(0, \infty)$. This is certainly continuous, but it is not possible to extend this to a continuous function g over the whole real line. Thus, as we sought above, we wish to find the general case when we can extend a function from a subset to one on the whole space.

I now state the question that we seek to answer with this paper. The answer will be the Tietze Extension Theorem presented at this paper's conclusion. For brevity, we take the word "map" to mean

Question 1. *Let S be a subset of a topological space X and let f be a map*

$$f : S \longrightarrow R.$$

What conditions must be placed on S and X such that there exists a map

$$g : X \longrightarrow R$$

and $f = g$ on S?

2 Basic Preliminaries

A background in topology will undoubtedly be needed to get the most out of this paper, but in an attempt to make this paper accessible to all readers I will briefly define all pertinent terms.

A *topology* on a set X is a family of subsets $\mathcal{T}$ such that the following properties hold:
1 Both the empty set and the whole set X belong to $\mathcal{T}$
2 The union of *any* collection of sets in $\mathcal{T}$ belongs to $\mathcal{T}$.
3 The intersection of a *finite* collection of sets in $\mathcal{T}$ belongs to $\mathcal{T}$

The set X along with the topology $\mathcal{T}$ is called a *topological space.* The elements of $\mathcal{T}$ are called *open sets.* A subset $S \subset X$ is called *closed* if $X \backslash S$ is open.

A subset S of X is a *neighborhood* of a point x if there is an open set U such that $x \in U$ and $U \subset S$. A point $x \in X$ is an interior point of S if S is a neighborhood of x. The collection of all interior points of S is called the *interior of S.*

The above is basic topology, and the reader should consult a good text on introductory topology for a more complete treatment. [2],[1] The concept of a continuous function is more subtle, and accordingly, I will spend a bit more time on it.

Definition 2.1. *Let X and Y be topological spaces, and let $f : X \longrightarrow Y$ be a function. f is continuous if $f^{-1}(V)$ is open in X whenever V is open in Y.*

Note that we are not claiming here that f has an inverse. Here, the notation $f^{-1}(V)$ should be interpreted as the pre-image V under f. Similarly, we define continuity at a single point. f is continuous at a point $x \in X$ if for every open set V in Y such that $f(x) \in V$, there exists an open set U in X such that $x \in U$ and $f(U) \subseteq V$.

What is going on here, is that for any neighborhood of the image of a point, we can find a neighborhood around the point itself that is mapped inside of this original neighborhood.

Now let us consider what happens if the image space is R. We must give R a topology, and for the purposes of this paper, we will give it what is referred to as the "standard topology". The open sets of R in this topology are all open intervals, their unions, and finite intersections.

Now we consider what is meant by saying that $f : X \longrightarrow R$ is continuous at a point x. The definition says that for any neighborhood V of $f(x)$, there must be a neighborhood U of x such that $f(U) \subset V$. Without a loss of generality, we can take V to be an interval of length ε centered at $f(x)$. Then we get the following definition.

Definition 2.2. *Let X be a topological space. The function $f : X \longrightarrow R$ is continuous at $x \in X$ if for any $\varepsilon > 0$, there exists an open set $U \subset X$ containing x such that $f(U) \subseteq (f(x) - \varepsilon, f(x) + \varepsilon)$.*

3 Separation Axioms

This section discusses the idea of separating points and sets in a topological space. I begin the discussion with an example. Suppose you are given two distinct points a, b on the real line ($a < b$). Also suppose that R has the standard topology. Are there two distinct open sets, one containing a and the other containing b? The answer is of course yes, no matter how close the point are, we can also "zoom" in on the space in between them, and split it into two intervals. More rigorously, let $r = b - a$. Then the intervals $(a - \frac{r}{2}, a + \frac{r}{2})$ and $(b - \frac{r}{2}, b + \frac{r}{2})$ are disjoint and contain a and b respectively.

Naively, we think that we can always do this sort of thing with two points; that we should always be able to separate them no matter how close. As is the case often in mathematics, however, we are often led astray by our intuitions. The following example shows why.

Example 3.1.

Let X denote the set of all integers. We give X the following topology: A set $S \subset X$ is open if $X \backslash S$ is finite. An example of an open set in this so-called "co-finite" topology is the set $(..., -2, -1, 0, 4, 5, 6, ...)$ because its complement is $(1, 2, 3)$ which is finite. The reader should check that this definition coincides with the three axioms that were required of a topology.

It is now natural to ask if one can "separate" two elements as we had done above. That is, for any distinct integers a, b can we find open sets U, V such that $a \in U$, $b \in V$ and $U \cap V = \emptyset$? The answer is no. Indeed, suppose that these sets did exist. Then since U and V are open, so is $U \cap V$. But this means that $X \backslash (U \cap V)$ is finite. Which means that $U \cap V$ itself must be infinite, (hardly the empty set we were looking for!)

As the above example illustrates, we should not assume too much about the separation of points and sets from one another because sometimes this may not be possible. Below I will define four types of topological spaces that are somehow "separable" in their own ways.

Definition 3.2. *: A topological space X is a* T_1-space *if for each pair of distinct points $x, y \in X$, there exists an open set U containing y such that $x \notin U$.*

To show that these definitions are actually meaningful, we show a fact about T_1 spaces:

Lemma 3.3. *A topological space X is a T_1-space if and only if its points are closed.*

Proof. Suppose that X is a T_1-*space* and fix a point x. Each point in the complement of $\{x\}$ has an open neighborhood disjoint from $\{x\}$. Taking the union of these, we get that $X \backslash \{x\}$ is open as a union of open sets. Thus $\{x\}$ is closed. Conversely, suppose that each point of X is closed. Then clearly for any point $y \neq x$, we have $X \backslash \{x\}$ is an open neighborhood of y and X is a T_1-*space*. □

Definition 3.4. *(Hausdorff) A topological space X is a T_2-space if for each pair of distinct points $x, y \in X$, there exists disjoint open sets U and V such that $x \in U$ and $y \in V$. Obviously, every T_2-space is a T_1-space.*

Definition 3.5. *The space X is* regular *if for each closed subset E of X and each point $x \in X \backslash E$, there exist disjoint open sets U and V such that $E \subset U$ and $x \in V$. A* T_3-space *is a regular T_1-space. Again, it is easy to see that a T_3-space is a T_2-space.*

Definition 3.6. *The space X is normal if for each pair E and F of disjoint closed subsets of X, there exist disjoint open sets U and V such that $E \subseteq U$ and $F \subseteq V$. A T_4-space is a normal T_1-space.*

Finally, we see that every T_4-*space* is a T_3-*space*, and we note that

$$T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1$$

We now prove the key lemma that will be a major part of our proof of Theorem 4.2.

Lemma 3.7. *A topological space X is normal if and only if for each closed subset E of X and each open set W containing E, there exists an open set U containing E such that $\overline{U} \subset W$.*

Proof. First suppose that X is normal. Then suppose that we are given E and W as above. Then E and $X \backslash W$ are disjoint closed subsets of X. Since X is normal, there exist disjoint open sets U, V such that $E \subseteq U$ and $X \backslash W \subseteq V$. Then $\overline{U}$ is contained within $X \backslash V$, which is contained within W. Thus U is the required open set.

Conversely, suppose that the condition above is valid, then let E and F be arbitrary disjoint closed subsets of X. Then $W = X \backslash F$ is an open subset containing E. From the conditions we assumed, there exists an open set U containing E such that $\overline{U} \subset W$. Then U and $X \backslash \overline{U}$ are disjoint open sets. To show that X is normal, we must show that $X \backslash \overline{U}$ contains F. Of course this is easy to see. Indeed, since $\overline{U} \subset W \Rightarrow X \backslash W \subset X \backslash \overline{U}$, we have:

$$F = X \backslash W \subset X \backslash \overline{U}. \quad \square$$

4 Urysohn's Lemma

In this section of the paper we will prove *Urysohn's Lemma*. Roughly speaking, this lemma allows us to create a function that is 1 near a certain point, and 0 very far from this point. These functions are generally called *Bump Functions* because their graph looks like a bump.

For the proof of the Lemma, we will need the following definition:

Definition 4.1. *A dyadic rational number is a rational number of the form $p/2^n$, where p and n are integers.*

Although it is beyond the scope of this paper to discuss the notion of density in full detail, the reader can consult the sources in the bibliography for a full treatment. It will suffice here, to say that a subset of the real numbers is dense, if any interval contains at least one element of the subset. It is clear that the dyadic rational numbers are dense in R.

Theorem 4.2. *(Urysohn's Lemma and the Tietze Extension Theorem): Let E and F be disjoint closed subsets of a normal topological space. Then there exists a continuous function f from X to the unit interval $[0, 1]$ such that $f = 0$ on E and $f = 1$ on F.*

Proof. Set $V = X \backslash F$, an open set containing E. By Lemma (3.7), there exists an open set $U_{1/2}$ such that:

$$E \subset U_{1/2} \subset \overline{U}_{1/2} \subset V.$$

Using the same lemma again on the open set $U_{1/2}$ containing E, we get an open set $U_{1/4}$ such that:

$$E \subset U_{1/4} \subset \overline{U}_{1/4} \subset U_{1/2}.$$

Similarly, we apply the lemma to the open set V containing the closed set $\overline{U}_{1/2}$ to get an open set $U_{3/4}$, so the following holds:

$$E \subset U_{1/4} \subset \overline{U}_{1/4} \subset U_{1/2} \subset \overline{U}_{1/2} \subset U_{3/4} \subset \overline{U}_{3/4} \subset V.$$

Continuing in this manner, we construct an open set U_r for each dyadic rational $r \in (0,1)$, such that:

$$\overline{U}_r \subset U_s, \qquad 0 < r < s < 1, \tag{4.1}$$
$$E \subset U_r, \qquad 0 < r < 1, \tag{4.2}$$
$$U_r \subset V, \qquad 0 < r < 1. \tag{4.3}$$

We now wish to define a function f so that the sets ∂U_r are level sets of f on which f assumes the value r. This is done by letting $f(x) = 0$ if $x \in U_r$ for all $r > 0$ and

$$f(x) = \sup\{r : x \notin U_r\}$$

Note that (4.2) implies that $f = 0$ on E, and it is obvious that $f = 1$ on F and that otherwise, $0 \leq f \leq 1$. It suffices to show that f is continuous.

Let $x \in X$. First we assume that $0 \leq f(x) \leq 1$. (The cases at 0 and 1 are much easier and the method is entirely the same.) Let $\varepsilon > 0$. Choose dyadic rational numbers r and s such that $r, s < 1$ and

$$f(x) - \varepsilon < r < f(x) < s < f(x) + \varepsilon.$$

(Note here that we use the fact that the dyadic numbers are dense in R.) Then it is clear that $x \notin U_t$ for dyadic rational numbers between r and $f(x)$, so that by (4.1) $x \notin \overline{U}_r$. On the other hand, $x \in U_s$. Hence if we let $W = U_s \backslash \overline{U}_r$, then W is an open neighborhood of x. Further, if we let $y \in W$, then from the definition of f we see that $r \leq f(y) \leq s$. This shows that

$$|f(y) - f(x)| < \varepsilon$$

for $y \in W$ so that f is continuous at x. □

Before we continue, a few more facts must be stated.

Theorem 4.3. *Let S be a set, and let X be a complete metric space. If $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence of functions from S to X, then there exists a function f from S to X such that $\{f_n\}$ converges uniformly to f. Note that R equipped with the standard metric is complete.*

Theorem 4.4. *The limit of a uniformly convergent sequence of continuous functions from a topological space X to a metric space Y is continuous.*

The proofs of these theorems are not difficult, but as they deal with a basic metric and point-set topology, we omit the proofs and note that the (short) proofs can be found in [2] as theorems I.2.5 and II.3.5 respectively. We are now ready to prove the main result.

Theorem 4.5. *(Tietze Extension Theorem): Let X be a normal topological space, let Y be a closed subset of X, and let f be a continuous real-valued function on Y. Then there exists a continuous real-valued function h on X such that $h = f$ on Y.*

Proof. We begin by considering the case when f is bounded. We set $c_0 = \sup\{|f(y)| : y \in Y\}$, and define two subsets of X.

$$\begin{aligned} E_0 &= \{y \in Y : f(y) \leq -c/3\}, \\ F_0 &= \{y \in Y : f(y) \geq c/3\}. \end{aligned}$$

It should be clear that E_0 and F_0 are closed disjoint subsets of X. (They are the inverse image of a closed set in R.) Taking a linear combination of a constant function and the function appearing in Urysohn's Lemma, we find a continuous real-valued function g_0 on X such that $-c/3 \leq g_0 \leq c/3$, $g_0 = -c/3$ on E_0 and $g_0 = c/3$ on F_0. In particular,

$$\begin{aligned} |g_0| &\leq c/3, \\ |f - g_0| &\leq 2c/3 \quad \text{on } Y. \end{aligned}$$

Now consider the function $f - g_0$, and let

$$\begin{aligned} E_1 &= \{y \in Y : (f - g_0)(y) \leq -2c/9\}, \\ F_1 &= \{y \in Y : (f - g_0)(y) \geq 2c/9\}. \end{aligned}$$

Now we construct another map g_1, (in the same way that we constructed g_0,) such that

$$-2c/9 \leq g_1 \leq 2c/9$$

and $g_1 = -2c/9$ on E_1 and $g_1 = 2c/9$ on F_1. As before, we have the following:

$$\begin{aligned} |g_1| &\leq 2c/9, \\ |f - g_0 - g_1| &\leq 4c/9 \quad \text{on } Y. \end{aligned}$$

We continue again, inductively generating functions $\{g_n\}_{n=0}^{\infty}$ such that

$$|g_n| \leq 2^n c/3^{n+1}, \tag{4.4}$$

$$|f(y) - g_0(y) - g_1(y) - \cdots - g_{n-1}(y)| \leq 2^{n+1} c/3^{n+1} \quad \text{on } Y. \tag{4.5}$$

Now set

$$h_n = g_0 + \cdots + g_n, \qquad n \geq 1$$

If $n \geq m$, then

$$\begin{aligned} |h_n - h_m| &= |g_{m+1} + \cdots + g_n \\ &\leq \left(\left(\frac{2}{3}\right)^{m+1} + \cdots + \left(\frac{2}{2}\right)^{n} \right) \frac{c_0}{3} \\ &\leq \left(\frac{2}{3}\right)^{m+1} c_0. \end{aligned}$$

Consequently the sequence $\{h_n\}$ is Cauchy, which, by (4.3), converges uniformly to a real-valued function h on X. Also, by (4.4), h is continuous. Thus h is the desired extension.

Now suppose that the function f is not bounded on Y. Then, choosing your favorite homeomorphism $h : (-\infty, \infty) \to (-1, 1)$, we see that the function $h \circ f$ is bounded on Y, and thus can be extended to some function g on X. Thus the function $h^{-1}g$ is continuous on X and agrees with f on Y so it is the required extension. □

References

[1] M. A. Armstrong. *Basic Topology.* Springer-Verlag New York Inc., 1983.

[2] Theodore W. Gamelin and Robert Everist Greene. *Introduction to Topology.* Dover Publications, Mineola, NY, 1999.