

Koszul Complexes

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1 Mapping Cone Construction

We begin this talk by discussing how to obtain a free resolution inductively by adding regular elements to the ring R/I . For simplicity, at the moment, we assume that R is a polynomial ring in some number of variables and that I is a homogeneous ideal. We begin with a lemma.

Lemma: Let I be an R -ideal, f a homogeneous polynomial of degree d , then the following is a graded exact sequence.

$$0 \rightarrow R(-d)/(I : f) \rightarrow R/I \rightarrow R/(I, f) \rightarrow 0.$$

Note that if f is a regular element on R/I then $I : f = I$ so that we actually have a graded exact sequence

$$0 \rightarrow R(-d)/I \rightarrow R/I \rightarrow R/(I, f) \rightarrow 0.$$

Let G_\bullet be a free resolution for the module R/I . Also suppose that F_\bullet is a free resolution for $R(-d)/I$ (which can be obtained just by shifting the grading). Thus we have a pair of resolutions

$$\begin{array}{ccccccc} \cdots & \longrightarrow & F_2 & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & R(-d)/I & \longrightarrow & 0 \\ & & \downarrow \cdot f & & \downarrow \cdot f & & \downarrow \cdot f & & \downarrow \cdot f & & \\ \cdots & \longrightarrow & G_2 & \longrightarrow & G_1 & \longrightarrow & G_0 & \longrightarrow & R/I & \longrightarrow & 0 \end{array}$$

where the vertical maps are just multiplication by f . If f is a regular element of degree d on R/I we can of course complete this diagram a bit by using the exact sequence from the lemma. To arrive at

$$\begin{array}{ccccccc} & & & & & & 0 & & & & \\ & & & & & & \downarrow & & & & \\ \cdots & \longrightarrow & F_2 & \xrightarrow{\psi_2} & F_1 & \xrightarrow{\psi_1} & F_0 & \xrightarrow{\psi_0} & R(-d)/I & \longrightarrow & 0 \\ & & \downarrow \phi_2 & & \downarrow \phi_1 & & \downarrow \phi_0 & & \downarrow & & \\ \cdots & \longrightarrow & G_2 & \xrightarrow{\xi_2} & G_1 & \xrightarrow{\xi_1} & G_0 & \xrightarrow{\xi_0} & R/I & \longrightarrow & 0 \\ & & & & & & \downarrow \pi & & & & \\ & & & & & & R/(I, f) & & & & \\ & & & & & & \downarrow & & & & \\ & & & & & & 0 & & & & \end{array}$$

Note that the map $\alpha_0 = \pi\xi_0$ is a map from $G_0 \cong R \rightarrow R/(I, f)$ a first step toward a free resolution of $R/(I, f)$.

Claim: The kernel of α_0 is generated by the images of ϕ_0 and ξ_1 .

Proof. Chasing diagrams it is clear that $\phi_0(F_0)$ and $\xi_1(G_1)$ are mapped to 0 by α_0 . The other direction involves similar chasing, which ends by showing that if $\alpha(x) = 0$ then we can write $x = \xi_1(z) + \phi_0(y)$ for z, y in the appropriate sets. \square

Thus we can define a map from $F_0 \oplus G_1 \rightarrow R/(I, f)$ and get a start to our free resolution. We can continue this process in general to obtain a chain complex

$$\cdots \longrightarrow F_{i-1} \oplus G_i \xrightarrow{\partial_i} F_{i-2} \oplus G_{i-1} \longrightarrow \cdots \longrightarrow F_0 \oplus G_1 \longrightarrow R/(I, f) \longrightarrow 0.$$

Where the map ∂_i is given by the matrix

$$\begin{bmatrix} \psi_{i-1} & 0 \\ (-1)^{i-1}\phi_{i-1} & \xi_i \end{bmatrix}$$

It is easy to check that this is indeed a chain complex as $\partial^2 = 0$. Further, it is also straightforward to check in the same way that this sequence is exact. This is because f was a regular element on R/I . If this were not the case, then we would not have an exact sequence but would still have a complex. This is called the Koszul Complex. Now we compute an example.

Let $I = (f_1, \dots, f_n)$ be a complete intersection with f_1, \dots, f_n forming a regular sequence with f_i of degree d_i . Let $I_0 = 0$. Then we have a free resolution for R/I_0 easily

$$0 \longrightarrow R \longrightarrow R/I_0 \longrightarrow 0.$$

Then from our previous work we know that we have a free resolution for $R/(f_1)$ from the Koszul complex by

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & R(-d_1) & \longrightarrow & R(-d_1)/I_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & R & \longrightarrow & R/I_0 & \longrightarrow & 0 \\ & & & & \downarrow & & \\ & & & & R/(f_1) & & \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

Tracing diagonals we get the resolution,

$$0 \longrightarrow R(-d_1) \longrightarrow R \longrightarrow R/I \longrightarrow 0.$$

Of course, this is no surprise and isn't even that impressive since it is simply the vertical column. But now we replace this sequence with the bottom row in the previous diagram to compute a free resolution of $R/(f_1, f_2)$.

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 0 & \longrightarrow & R(-d_1 - d_2) & \longrightarrow & R(-d_2) & \longrightarrow & R(-d_2)/(f_1) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & R(-d_1) & \longrightarrow & R & \longrightarrow & R/(f_1) \longrightarrow 0 \\
 & & & & & & \downarrow \\
 & & & & & & R/(f_1, f_2) \\
 & & & & & & \downarrow \\
 & & & & & & 0
 \end{array}$$

Yielding the resolution

$$0 \longrightarrow R(-d_1 - d_2) \longrightarrow R(-d_1) \oplus R(-d_2) \longrightarrow R \longrightarrow R/(f_1, f_2) \longrightarrow 0$$

Ignoring the grading for a moment we can simply look at the betti numbers of the resolution as we add f_3 to our ideal.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & R & \longrightarrow & R^2 & \longrightarrow & R \longrightarrow R/(f_1, f_2) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & R & \longrightarrow & R^2 & \longrightarrow & R \longrightarrow R/(f_1, f_2) \longrightarrow 0
 \end{array}$$

We see that the next set of Betti numbers will be 1, 3, 3, 1 and in general it is easy to see the following fact

Proposition If f_1, \dots, f_n is a regular sequence then ignoring the grading, the free resolution for $R/(f_1, \dots, f_n)$ is given by

$$0 \longrightarrow R^{b_0} \longrightarrow R^{b_1} \longrightarrow \dots \longrightarrow R^{b_n} \longrightarrow R/(f_1, \dots, f_n) \longrightarrow 0$$

where b_i is given by $\binom{n}{i}$.

Example: This gives a concrete free resolution for a field k when considered a quotient $k[x_1, \dots, x_n]/(x_1, \dots, x_n)$.

Before presenting our next example, we state a lemma concerning Tor.

Lemma: Suppose F is a finitely generated graded $S = k[x_1, \dots, x_n]$ module. If $\text{Tor}_1(F, k) = 0$ then F is free.

Proof. Note that $F \otimes k = F/IF$ where $I = (x_1, \dots, x_n)$, a maximal ideal of S . Thus $F \otimes k$ is naturally an $A/I = k$ -module, or a k -vector space. Choose homogeneous elements v_1, \dots, v_r in F so that their images in F/IF form a basis for this vector space. Let L be a free module with basis e_1, \dots, e_r . Let

$$L \rightarrow F$$

be the graded homomorphism sending $e_i \mapsto v_i$ for $i = 1, \dots, n$. We will show that this is an isomorphism. Let C be the cokernel, so

$$L \rightarrow F \rightarrow C \rightarrow 0$$

is exact. Tensoring with k yields an exact sequence

$$L \otimes k \rightarrow F \otimes k \rightarrow C \otimes k \rightarrow 0.$$

Since by construction, the map $L \otimes k \rightarrow F \otimes k$ is surjective, it follows that $C \otimes k = 0$. We now prove a Nakayama like lemma to show that $C = 0$.

Lemma: Let N be a graded module over A . Let $I = (x_1, \dots, x_n)$. Then if $N/IN = 0$ then $N = 0$. Note that $N \otimes k = N/IN$.

Proof. The proof is clear. Just take an element f of minimum degree in N . Then since $f + IN = 0$, we must have $f \in IN$ but each element of I has degree > 0 so this is impossible. \square

Thus if E is the kernel of our map $L \rightarrow F$ then we now have an exact sequence

$$0 \rightarrow E \rightarrow L \rightarrow F \rightarrow 0$$

we must show that $E = 0$. But this is where we get to use Tor! The Tor exact sequence gives us

$$0 = \text{Tor}_1(F, k) \rightarrow E \otimes k \rightarrow L \otimes k \rightarrow F \otimes k \rightarrow 0.$$

By construction, $L \otimes k$ is isomorphic to $F \otimes k$, so $E \otimes k = 0$ and by the lemma, again, we see that $E = 0$. \square

Example: (Hilbert Syzygy Theorem) Let F be a finitely generated module over A . Then A has projective dimension less than n . In other words, if

$$0 \rightarrow K \rightarrow F_{n-1} \rightarrow \dots \rightarrow F_0 \rightarrow F \rightarrow 0 \tag{1}$$

is exact with F_i free, then K is free.

Proof. Since the Koszul complex is a free resolution for k of length n , tensoring with F we see that $\text{Tor}_{n+1}(k, F) = 0$. Tensoring (1) with k we see that $0 = \text{Tor}_{n+1}(F, k) = \text{Tor}_1(K, k)$ so K is free. \square

2 The Koszul Complex

In this section we follow Matsumura pp. 127-130 and show that we can in general construct a complex for any sequence of elements in a ring and that exactness is a more general result.

Given a ring R and $x_1, \dots, x_n \in R$, define the Koszul complex K_\bullet as follows: set $K_0 = A$, and $K_p = 0$ if $p > n$ and for $1 \leq p \leq n$ define

$$K_p = \bigoplus R \cdot e_{i_1} \wedge \dots \wedge e_{i_p}$$

to be the free module of rank $\binom{n}{p}$ with basis $\{e_{i_1} \wedge \dots \wedge e_{i_p} : 1 \leq i_1 < \dots < i_p \leq n\}$ (A standard construction)

The differential map $\partial : K_p \rightarrow K_{p-1}$ is defined by

$$\partial(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum_{r=1}^p (-1)^{r-1} x_{i_r} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_r}} \wedge \cdots \wedge e_{i_p}$$

(for $p = 1$, set $\partial(e_i) = x_i$). This is no doubt a horrendous expression, but nonetheless one that the reader has probably encountered before. In words it is simple, remove one of the e_{i_j} at a time and multiply by x_{i_j} and then sum while alternating signs.

For example,

$$\partial(e_1 \wedge e_2 \wedge e_3) = x_1(e_2 \wedge e_3) - x_2(e_1 \wedge e_3) + x_3(e_1 \wedge e_2).$$

It is clear that $\partial^2 = 0$ as is checked in any basic topology course. This is called the Koszul complex and is written $K_\bullet(x_1, \dots, x_n)$ or $K_\bullet(\underline{x})$.

Let's do some examples now, if $n = 1$ then the the complex $K_\bullet(\underline{x})$ is given by

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow A \xrightarrow{x} A \longrightarrow 0.$$

If $n = 2$ then $K_\bullet(x, y)$ is the complex

$$\cdots \longrightarrow 0 \longrightarrow A \xrightarrow{\begin{bmatrix} x \\ -y \end{bmatrix}} A^2 \xrightarrow{\begin{bmatrix} x & y \end{bmatrix}} A \longrightarrow 0$$

Finally, for $n = 3$, $K_\bullet(x, y, z)$ is the complex

$$\cdots \longrightarrow 0 \longrightarrow A \xrightarrow{\psi_3} A^3 \xrightarrow{\psi_2} A^3 \xrightarrow{\psi_1} A \longrightarrow 0$$

where

$$\psi_3 = \begin{bmatrix} x \\ -y \\ z \end{bmatrix}, \quad \psi_2 = \begin{bmatrix} 0 & -z & -y \\ -z & 0 & x \\ y & x & 0 \end{bmatrix}, \quad \psi_1 = [x \ y \ z]$$

Note that one has to be careful about choosing the order of the bases for these matrices in order for everything to make sense. It is perhaps better not to worry about these things and just use the definition using wedge products.

We now give a third definition of the Koszul complex which is given by forming the tensor product of two complexes.

Definition: Let K_\bullet and L_\bullet be two complexes. Then we form the complex $K \otimes L$ by

$$(K \otimes L)_n = \bigoplus_{p+q=n} K_p \otimes L_q$$

and the boundary map given by

$$\partial(x \otimes y) = \partial x \otimes y + (-1)^p x \otimes \partial y$$

for $x \in K_p$ and $y \in L_q$.

Notice that if $K = L$ are the complex $0 \rightarrow R \rightarrow R \rightarrow 0$ then their tensor product is

$$0 \rightarrow R \rightarrow R^2 \rightarrow R \rightarrow 0.$$

If we tensor this again with K we obtain

$$0 \rightarrow R \rightarrow R^3 \rightarrow R^3 \rightarrow 0.$$

and can define the Koszul complex in the following way: Let K_i $1 \leq i \leq n$ be the complex $0 \rightarrow R \rightarrow R \rightarrow 0$ where the map $m_i : R \rightarrow R$ is given by multiplication by x_i . Then the Koszul complex is given by $K = \bigotimes K_i$ the reader can verify that the maps given in this construction match up with those in the first two examples. The second definition in terms of wedge products is the one that Koszul proposed.

Definition: For an R -module M we can define $K_\bullet(\underline{x}, M)$ as the complex $K_\bullet(\underline{x}) \otimes_R M$. Moreover, if C_\bullet is an arbitrary complex of R modules then we define $C_\bullet(\underline{x}) = C_\bullet \otimes K_\bullet(\underline{x})$.

Remark: Note that since $K_\bullet(x_1, \dots, x_n) = K_\bullet(x_1) \otimes \dots \otimes K_\bullet(x_n)$ and tensor products of complexes are commutative, we have that the Koszul complex is invariant under permutation of x_1, \dots, x_n .

The Koszul complex has homology groups $H_p(K(\underline{x}, M))$, which we abbreviate by $H_p(\underline{x}, M)$. It is easy to see that

$$H_0(\underline{x}, M) \cong M/\underline{x}M = M/(x_1M + \dots + x_nM)$$

and

$$H_n(\underline{x}, M) \cong \{m \in M : x_i m = 0 \text{ for all } i\}.$$

Theorem: Let C be a complex of R modules and $x \in R$. Then we get an exact sequence of complexes

$$0 \rightarrow C \rightarrow C(x) \rightarrow C' \rightarrow 0$$

where C' is the complex obtained by shifting the degrees in C up by 1. (That is $C'_{p+1} = C_p$ and they share the same differential.) The homology long sequence obtained by this has the form

$$\dots \longrightarrow H_p(C) \longrightarrow H_p(C(x)) \longrightarrow H_{p-1}(C) \xrightarrow{(-1)^{p-1}x} H_{p-1}(C) \longrightarrow \dots$$

Finally, we have $x \cdot H_p(C(x)) = 0$ for all p .

Proof. Since $K(x)$ is given by $0 \rightarrow R \rightarrow R \rightarrow 0$, the complex $C(x)$ is given by

$$\dots \longrightarrow C_n \oplus C_{n-1} \longrightarrow \dots \longrightarrow C_1 \oplus C_0 \longrightarrow C_0 \longrightarrow 0.$$

The boundary map for tensor products of complexes gives that

$$\partial(\xi, \eta) = (\partial\xi + (-1)^{p-1}x\eta, \partial\eta).$$

The exactness follows easily by considering the following diagram

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & C_n & \longrightarrow & \cdots & \longrightarrow & C_1 & \longrightarrow & C_0 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & C_n \oplus C_{n-1} & \longrightarrow & \cdots & \longrightarrow & C_1 \oplus C_0 & \longrightarrow & C_0 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & C_{n-1} & \longrightarrow & \cdots & \longrightarrow & C_0 & \longrightarrow & 0 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & & & \\
& & 0 & & 0 & & & & & & .
\end{array}$$

We always have a long homology sequence. What remains is to show that the induced map from $H_{p-1}(C) \rightarrow H_{p-1}(C)$ is given by $(-1)^{p-1}x$. So suppose we have an element $\eta \in C_{p-1} = C'_p$ that is in the kernel of ∂ . Then in $C(x)$ we have $\partial(0, \eta) = ((-1)^{p-1}x\eta, 0)$ so in C_{n-1} (the top line) we see that η is sent to $(-1)^{p-1}x\eta$.

Finally, if $d(\xi, \eta) = 0$ then $\partial(\eta) = 0$ and $\partial\xi = (-1)^p x\eta$, so that we have

$$x(\xi, \eta) = \partial(0, (-1)^p \xi) \in \partial C_{p+1}(x),$$

and therefore $x \cdot H_p(C(x)) = 0$. □

Remark: Applying this theorem to $K(\underline{x}, M)$ we see that the ideal $(\underline{x}) = (x_1, \dots, x_n)$ annihilates the homology groups.

Theorem(i) Let R be a ring, M an R -module, and x_1, \dots, x_n a regular sequence for M . Then

$$H_p(\underline{x}, M) = 0 \quad \text{for } p > 0 \quad \text{and} \quad H_0(\underline{x}, M) = M/\underline{x}M.$$

(ii) Suppose that one of the following two conditions holds:

(α) (R, m) is a local ring, $x_1, \dots, x_n \in m$ and M is a finite R -module.

(β) R is an \mathbb{N} -graded ring, M is an \mathbb{N} -graded R -module, and x_1, \dots, x_n are homogeneous elements of degree > 0 . Then the converse of (i) holds in the following strong form: if $H_1(\underline{x}, M) = 0$ and $M \neq 0$ then x_1, \dots, x_n is a regular sequence for M .

Proof. (i) We induct on n . When $n = 1$ we have $H_1(x, M) = \{m \in M : xm = 0\} = 0$, so that the first step is done. When $n > 1$ and $p > 1$ the previous theorem gives an exact sequence

$$0 = H_p(x_1, \dots, x_{n-1}, M) \rightarrow H_p(x_1, \dots, x_n, M) \rightarrow H_{p-1}(x_1, \dots, x_{n-1}, M) = 0.$$

So that $H_p(x_1, \dots, x_n, M) = 0$. For $p = 1$, we have

$$0 \rightarrow H_1(\underline{x}, M) \rightarrow H_0(x_1, \dots, x_{n-1}, M) = M/(x_1, \dots, x_{n-1})M \rightarrow M/(x_1, \dots, x_{n-1})M$$

where the last map is just multiplication by $\pm x_n$. Since x_n is a regular element, This shows that $H_1 = 0$.

(ii) Assuming either (α) or (β) if $M \neq 0$ then $M_i = M/(x_1, \dots, x_i)M \neq 0$. By hypothesis and by the previous theorem, we have

$$H_1(x_1, \dots, x_{n-1}, M) \xrightarrow{\pm x_n} H_1(x_1, \dots, x_{n-1}, M) \longrightarrow H_1(\underline{x}, M) = 0;$$

but by the hypotheses, these homology groups are finite R modules or \mathbb{N} graded modules, so by NAK we have $H_1(x_1, \dots, x_{n-1}, M) = 0$. Thus by induction we know that x_1, \dots, x_n is exact. Then from the exact sequence

$$0 \rightarrow H_1(\underline{x}, M) \rightarrow H_0(x_1, \dots, x_{n-1}, M) = M/(x_1, \dots, x_{n-1})M \rightarrow M/(x_1, \dots, x_{n-1})M,$$

since the last map is multiplication by x_n and exactness implies injectivity, we know that x_n is a regular element as well. \square