3. The proof of Theorem 8 on p. 331 of Lay shows that if A is the matrix for a transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ and $\mathcal{B}$ is a basis of $\mathbb{R}^n$, then the $\mathcal{B}$-matrix for T is given by $P^{-1}AP$ where P has the elements of $\mathcal{B}$ as columns.

Here then $P = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}$

so the $\mathcal{B}$-matrix is

$$\begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} -2 & 2 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 4 & 2 \end{bmatrix}$$

So the answer is

$$\boxed{-3}$$

4. The characteristic polynomial of A is $\det(A-\lambda I)$ so here it is

$$\det\begin{pmatrix} 2-\lambda & 0 & 1 \\ 0 & 3-\lambda & -2 \\ 1 & 1 & -\lambda \end{pmatrix} = (2-\lambda)\big((3-\lambda)\cdot-\lambda - -2\big) - 0(0\cdot-\lambda-1)$$
$$+ 1\cdot\big(0-(3-\lambda)\big)$$
$$= (2-\lambda)(-3\lambda+\lambda^2+2) - (3-\lambda)$$
$$= -6\lambda+2\lambda^2+4+3\lambda^2-\lambda^3-2\lambda-3+\lambda$$
$$= -\lambda^3+5\lambda^2-7\lambda+1$$

5. The first statement is False.
See example 3 in section 5.3 for a 3×3 matrix with only 2 eigenvalues that is diagonalizable

5. The second statement is True

Since v is an eigenvector, $Av = \lambda v$ for some λ.

Then $A^2 v = A \cdot Av = A \cdot \lambda v = \lambda \cdot Av = \lambda \cdot \lambda v = \lambda^2 v$

So $A^2 v = \lambda^2 v$ and v is an eigenvector of A^2 with eigenvalue λ^2

The third statement is True

Since a linear ~~ta~~ transformation satisfies

$T(cu) = cT(u)$ where u is a vector and c a scalar,

$T(0) = T(0 \cdot u) = 0 \cdot T(u) = 0$.

6. As in Example 2 on p. 274 of Lay,

we see

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & -1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & -1 \end{bmatrix} \text{ so}$$

$\begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix}$ is the change of coordinates matrix from basis $\mathcal{B}$ to basis $\mathcal{C}$.

7. The characteristic equation of our matrix is

$$0 = \det\begin{pmatrix} 3-\lambda & 1 \\ -2 & 1-\lambda \end{pmatrix} = (3-\lambda)(1-\lambda) - -2 = 3-4\lambda+\lambda^2+2 = \lambda^2-4\lambda+5$$

By the quadratic equation, the solutions are

$$\frac{4 \mp \sqrt{16-4\cdot5}}{2} = \frac{4 \mp \sqrt{-4}}{2} = 2 \mp i$$

The eigenspace for $2+i$ is the null space of

$$\begin{pmatrix} 3-(2+i) & 1 \\ -2 & 1-(2+i) \end{pmatrix} = \begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix}$$

$$\begin{pmatrix} 1-i & 1 & 0 \\ -2 & -1-i & 0 \end{pmatrix} \sim \begin{pmatrix} 1-i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$1+i$ times row 1
added to row 2

Then we have x_2 free and $x_1(1-i) + x_2 = 0$ so

$$x_1 = \frac{-1}{1-i} x_2 = \frac{-i-1}{2} x_2$$ so the eigenspace

is the space spanned by $\begin{bmatrix} -i-1 \\ 2 \end{bmatrix}$.

The eigenspace for $2-i$ will ~~be the complex~~ have a basis that is the complex conjugate to that of $2+i$, so the space is that spanned by $\begin{bmatrix} i-1 \\ 2 \end{bmatrix}$