Name:________________________

107 Activity 5: Pascal's Triangle

In §5.5, we used the notation $C(m,r)$ for the number of combinations of m things taken r at a time. One way of thinking about it was as if we took all possible permutations of m taken r at a time, $P(m,r)$, and divided out by the number of ways of listing out r things, $P(r,r) = r!$. We got the following general formula for $C(m,r)$:

$$C(m,r) = \frac{P(m,r)}{r!} = \frac{m\cdot(m-1)\cdot(m-2)\cdot\cdots\cdot(m-r+1)}{r\cdot(r-1)\cdot(r-2)\cdot\cdots\cdot 2\cdot 1}$$

Together, we often call the $C(m,r)$'s *binomial coefficients* after the Binomial Theorem that they appear in. From now on, I'll often call them that.

Pascal's Triangle

arranges the binomial coefficients together so we can see patterns.

1	0th row
1 1	1st row
1 2 1	2nd row
1 3 3 1	3rd row
1 4 6 4 1	4th row
1 5 10 10 5 1	5th row
1 6 15 20 15 6 1	6th row

The mth row has the numbers $C(m,r)$ for all possible r between 0 and m. For instance, the 2nd row lists $C(2,0)$, $C(2,1)$, and $C(2,2)$ from left to right. We officially define $C(0,0) = 1$ and $C(m,0) = 1$ for all numbers m, for completeness and to allow the symmetry to come through.

Notice that every row begins and ends with 1. Notice the second entry in every row is the number of that row (ie, the 3rd row is the one where the second entry is 3.) Also, each row has exactly one more entry than the previous one, and the number of entries is closely related to the number in the $C(m,r)$'s for that row. For instance, row 5 has $6 = 5+1$ numbers in it.

1. **“The power of 2”**

(a) Add up the entries in each row.

							1							0th row sum=
						1		1						1st row sum =
					1		2		1					2nd row sum=
				1		3		3		1				3rd row sum=
			1		4		6		4		1			4th row sum=
		1		5		10		10		5		1		5th row sum=
	1		6		15		20		15		6		1	6th row sum=

(b) What is the pattern in these sums?

(c) Imagine you flip a coin 4 times and record the sequences of heads and tails. How many different sequences are possible?

(d) How many of those sequences will have no heads?

(e) How many will have exactly one head in them?

(f) How many will have exactly two heads in them?

(g) How many will have exactly three heads?

(h) How many will have all heads?

(i) If I flip a coin 12 times, how many sequences have exactly 4 heads?

2. Symmetry

$$
\begin{array}{ccccccccccccc}
 & & & & & & C(0,0) & & & & & & \\
 & & & & & C(1,0) & & C(1,1) & & & & & \\
 & & & & C(2,0) & & C(2,1) & & C(2,2) & & & & \\
 & & & C(3,0) & & C(3,1) & & C(3,2) & & C(3,3) & & & \\
 & & C(4,0) & & C(4,1) & & C(4,2) & & C(4,3) & & C(4,4) & & \\
 & C(5,0) & & C(5,1) & & C(5,2) & & C(5,3) & & C(5,4) & & C(5,5) & \\
C(6,0) & & C(6,1) & & C(6,2) & & C(6,3) & & C(6,4) & & C(6,5) & & C(6,6)
\end{array}
$$

Above are the first 6 rows of Pascal's triangle, this time listed also in their binomial coefficient form. The first pattern we want to investigate is *symmetry*, or the way the triangle appears to repeat itself as if the right side were a mirror image of the left side.

(a) Which coefficients in row 3 and row 5 have the same value? Which coefficients in row 4 and row 6 have the same value?

(b) What sort of pattern do you see? Can you find a relationship between the k and n when $C(5,k) = C(5,n)$ or $C(6,k) = C(6,n)$?

(c) Use this pattern to guess which other number in row 7 is equal to $C(7,6)$ and what other number in row 8 will have the same value as $C(8,3)$.

(d) Can I use the rule I figured out to determine $C(8,4)$? Why or why not?

(e) Suppose we are in the mth row of Pascal's triangle. Can you figure out which numbers in that row are equal?

3. Other sums in the triangle

(a) Look at the 2nd row of Pascal's triangle. Starting from the left, add the first two entries in the row together. Do you get a number that appears anywhere else in the triangle?

(b) This time add the second and third numbers in the row together. Do you get a number that appears anywhere else in the triangle?

(c) Now look at the third row. Calculate the sums of numbers that are next to each other (the 1st and 2nd numbers, the 2nd and 3rd numbers, etc). Do you get a number that appears anywhere else in the triangle?

(d) What pattern do you see?

(e) Use this pattern to calculate the numbers of the 7th row of Pascal's triangle. Remember you know the row will begin and end with 1.

(f) Figure out how to write $C(8, 3)$ as a sum of two numbers of the form $C(7, k) + C(7, n)$.