Lecture 8: Fast Linear Solvers (Part 2)

Naive Parallel Backward Substitution Algorithm

After elimination, we obtain upper triangular $U\boldsymbol{x} = \boldsymbol{b}^{(n-1)}$.
Assume that U is stored by rows.

$$x_n = \frac{b_n^{(n-1)}}{u_{nn}}$$

for $i = n - 1$ **to** 1

$$x_i = \frac{b_i^{(n-1)} - u_{i,i+1}x_{i+1} - u_{i,i+2}x_{i+2} - \cdots - u_{i,n}x_n}{u_{ii}}$$

for $k = n$ **to** 1
  $x_k = b_k$
  for $i = k + 1$ **to** n
    $x_k = x_k - u_{ki}x_i$
  end;
  $x_k = x_k / u_{kk}$
  broadcast x_k to all rows
end;

Naive Parallel Forward Substitution Algorithm

Consider to solve lower triangular L$\boldsymbol{x} = \boldsymbol{b}$.

for $i = 1$ **to** n

$$x_i = \frac{b_i - \sum_{j=1}^{i-1} l_{ij} x_j}{l_{ii}}$$

```
for i = 1 to n
   for j = 1 to i − 1
        b_i = b_i − l_ij x_j
   end;
   x_i = b_i / l_ii
    broadcast x_i to all rows
end;
```

Revised Forward Substitution Algorithm

// immediate-update of right hand side

for $j = 1$ **to** n

  $x_j = b_j / l_{jj}$ // compute solution

  for $i = j + 1$ **to** n

    $b_i = b_i - l_{ij} x_j$ //update right hand side

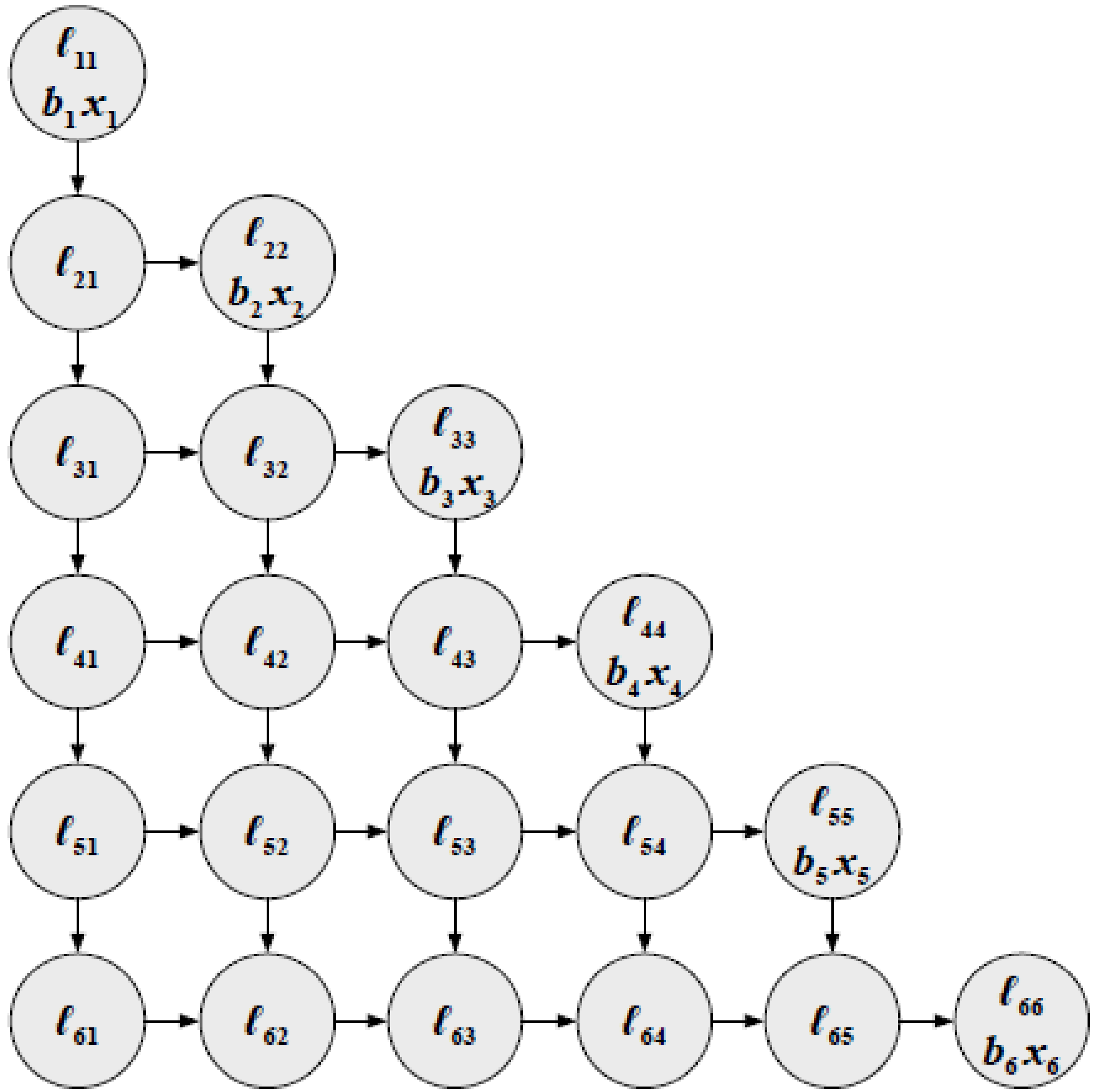
  end;

end;

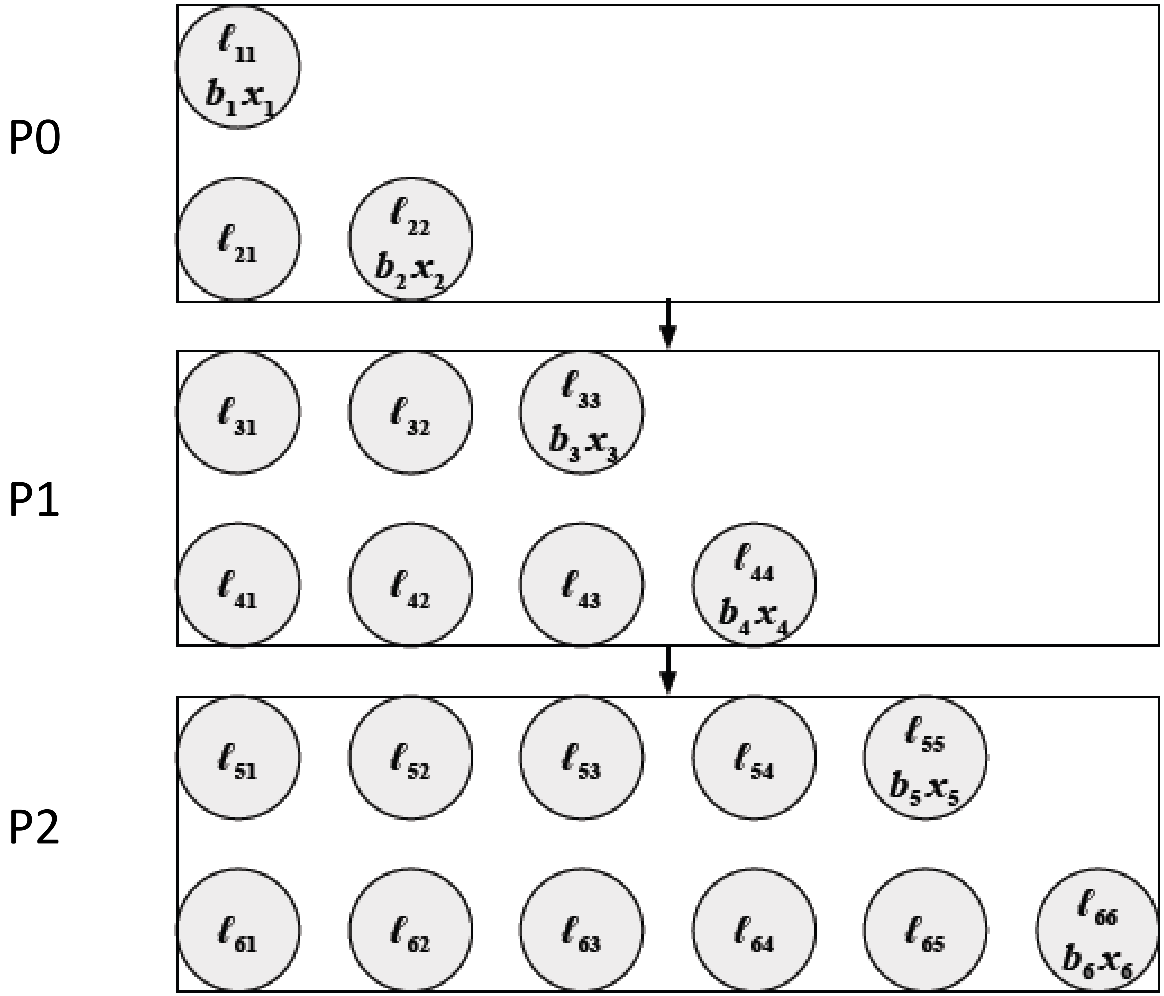
Parallel Forward Substitution Algorithm

- Assume a fine-grained decomposition in which process p_{ij} stores l_{ij} and compute $l_{ij}x_j$ for $i = 2, \ldots, n, j = 1, \ldots, i-1$
- Assume p_{ii} stores l_{ii} and b_i, collects $\sum_{j=1}^{i-1} l_{ij}x_j$ and computes $x_i = \frac{b_i - \sum_{j=1}^{i-1} l_{ij}x_j}{l_{ii}}$ for $i = 1, \ldots, n$

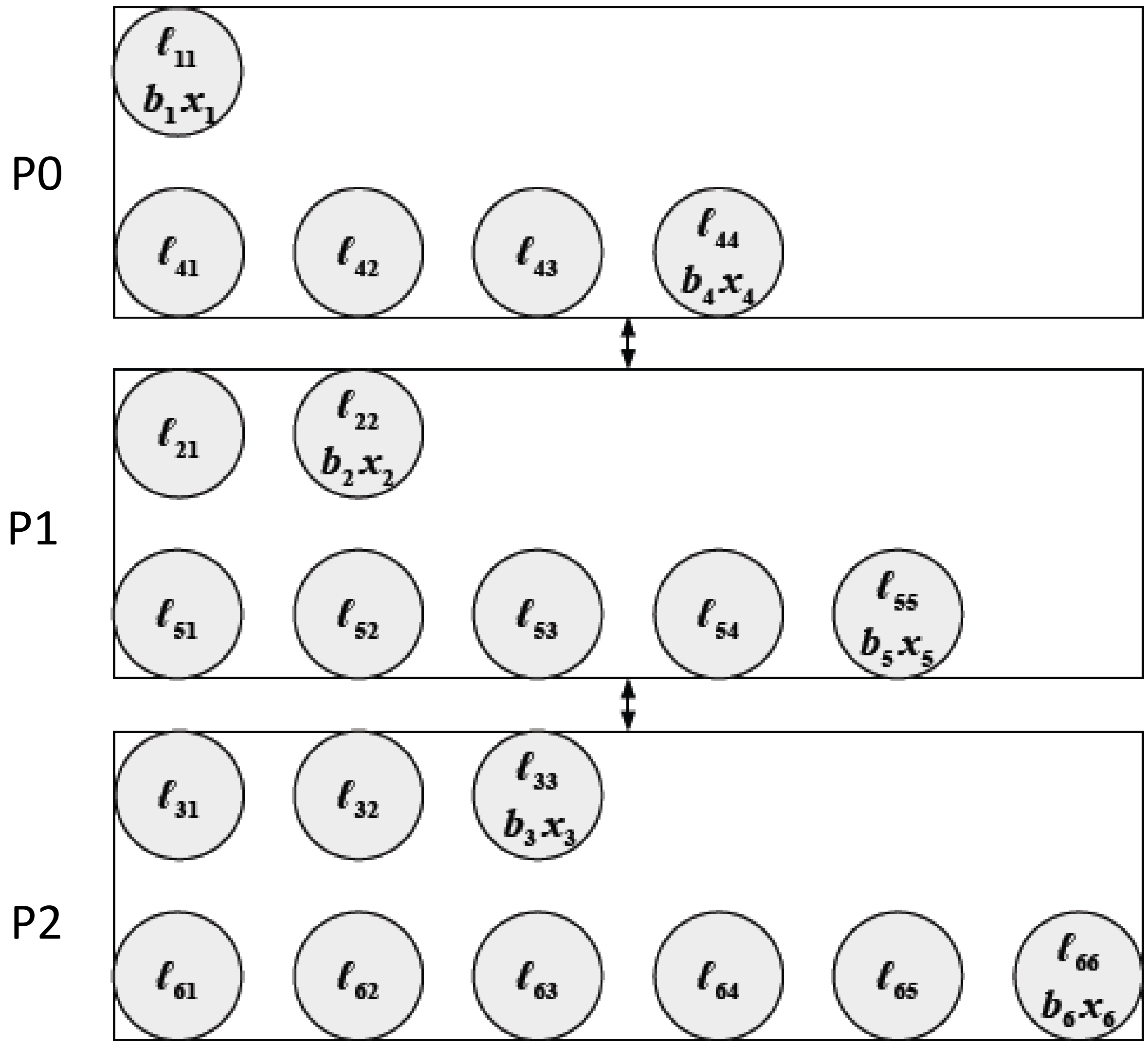
Primary Tasks and Communication



1D Row Block Mapping



1D Row Cyclic Mapping



Forward Substitution Parallel Algorithm Based on 1D Row Mapping

```
// immediate-update of right hand side
for j = 1 to n
   for process holding jth row
   x_j = b_j / l_jj        // compute solution
   end;
   broadcast x_j to all processes
   for process holding ith row i > j
      b_i = b_i - l_ij x_j  //update right hand side
   end;
end;
```

References

- G. Li and T. F. Coleman. A new method for solving triangular systems on distributed-memory message-passing multiprocessors, SIAM *J. Sci. Stat. Comput.* 10:382-396, 1989
- E. E. Santos. On designing optimal parallel triangular solvers,
- *Information and Computation* 161:72-210, 2000

Gaussian Elimination and Sparse System

Consider to solve the tridiagonal system

$$a_i x_{i-1} + b_i x_i + c_i x_{i+1} = F_i, \qquad i = 1, \dots, n$$

for unknowns $x_1, \dots, x_i, \dots, x_n$ (So $x_0 = x_{n+1} = 0$).
Here a_i, b_i, c_i, and F_i are given.

- *Let m be the band width.*

for $k = 1$ **to** $n - 1$ // loop over columns
  for $i = k + 1$ **to** $\min(k + m, n)$
    $l_{ik} = a_{ik}/a_{ii}$ // multipliers for kth column
    $b_i = b_i - l_{ik} b_k$
  end;
  for j $= k + 1$ **to** $\min(k + m, n)$
    for $i = k + 1$ **to** $\min(k + m, n)$
      $a_{ij} = a_{ij} - l_{ik} a_{kj}$ // elimination step
    end;
  end;
end;

Parallel Cyclic Reduction for Tridiagonal System

- When $m < p$, neither row-cyclic nor column-cyclic decomposition is efficient. Because only m processors are actively used.
- Assume that $n = 2^p - 1$, where p is the number of processors. If $n \neq 2^p - 1$, then add a trivial equation $x_i = 0, i = n + 1, \dots, 2^p - 1$.

- Consider the case when $n = 7 = 2^3 - 1$.
- Key idea:
  - Combine linearly equations to eliminate the odd-numbered unknowns $x_1, x_3, x_5, \ldots$ in the first stage.
    - Adding a multiple of $(i-1)th$ equation and a multiple of $(i+1)th$ equation to ith equation to eliminate x_{i-1} and x_{i+1} from the ith equation for $i = 2,4, \ldots$.
  - Then renumber unknowns and repeat this process till there is a single equation with one unknown.
  - Solve backward to obtain the rest of the unknowns.

Remark: Cyclic Reduction is a divide-and-conquer method

- Multiply parameters $\alpha_2, \beta_2, \gamma_2$ to the first three equations respectively to get:

$$\alpha_2 b_1 x_1 + \alpha_2 c_1 x_2 = \alpha_2 F_1$$
$$\beta_2 a_2 x_1 + \beta_2 b_2 x_2 + \beta_2 c_2 x_3 = \beta_2 F_2$$
$$\gamma_2 a_3 x_2 + \gamma_2 b_3 x_3 + \gamma_2 c_3 x_4 = \gamma_2 F_3$$

To eliminate x_1 and x_3, add above three equations and let

$$\beta_2 = 1$$
$$\alpha_2 b_1 + \beta_2 a_2 = 0$$
$$\beta_2 c_2 + \gamma_2 b_3 = 0$$
$$\Rightarrow \hat{b}_2 x_2 + \hat{c}_2 x_4 = \hat{F}_2$$

Where

$$\hat{b}_2 = \alpha_2 c_1 + \beta_2 b_2 + \gamma_2 a_3$$
$$\hat{c}_2 = \gamma_2 c_3$$
$$\hat{F}_2 = \alpha_2 F_1 + \beta_2 F_2 + \gamma_2 F_3$$

- Multiply parameters $\alpha_4, \beta_4, \gamma_4$ to the third, fourth and fifth equations respectively and add to eliminate x_3 and x_5:

$$\Rightarrow \hat{a}_4 x_2 + \hat{b}_4 x_4 + \hat{c}_4 x_6 = \hat{F}_4$$

Where

$$\hat{a}_4 = \alpha_4 a_3$$
$$\hat{b}_4 = \alpha_4 c_3 + \beta_4 b_4 + \gamma_4 a_5$$
$$\hat{c}_4 = \gamma_4 c_5$$
$$\hat{F}_4 = \alpha_4 F_3 + \beta_4 F_4 + \gamma_4 F_5$$

$\alpha_4, \beta_4, \gamma_4$ are determined by :

$$\beta_4 = 1$$
$$\alpha_4 b_3 + \beta_4 a_4 = 0$$
$$\beta_4 c_4 + \gamma_4 b_5 = 0$$

- Finally, multiply parameters $\alpha_6, \beta_6, \gamma_6$ to the fifth, sixth and seventh equations respectively and add to eliminate x_5 and x_7:

$$\Rightarrow \hat{a}_6 x_4 + \hat{b}_6 x_6 = \hat{F}_6$$

Where

$\alpha_6, \beta_6, \gamma_6$ are determined by :

$$\beta_6 = 1$$
$$\alpha_6 b_5 + \beta_6 a_6 = 0$$
$$\beta_6 c_6 + \gamma_6 b_7 = 0$$

- In stage two:

$$\hat{b}_2 x_2 + \hat{c}_2 x_4 = \hat{F}_2$$
$$\hat{a}_4 x_2 + \hat{b}_4 x_4 + \hat{c}_4 x_6 = \hat{F}_4$$
$$\hat{a}_6 x_4 + \hat{b}_6 x_6 = \hat{F}_6$$

  - Repeat the same elimination process, which leads to only one equation

$$\alpha_4^* x_4 = F_4^*$$

  - Use backward substitution, x_2 and x_6 are solved by $\hat{b}_2 x_2 + \hat{c}_2 x_4 = \hat{F}_2$ and $\hat{a}_6 x_4 + \hat{b}_6 x_6 = \hat{F}_6$
  - Use the original equations to solve for x_1, x_3, x_5, x_7.

Cyclic Reduction Algorithm

```
for(i=0; i < log₂(size+1)-1;i++)          // levels of reduction
{
    for(j=2^(i+1) − 1; j <size; j=j+ 2^(i+1))   // rows that are reduced
    {
        offset = 2^i;
        index1 = j – offset;   // index of row before the jth row
        index2 = j + offset;   // index of row after the jth row
        α = A[j][index1]/A[index1][index1];
        γ = A[j][index2]/A[index2][index2];

        for(k=0; k < size; k++)
            A[j][k] -= αA[index1][k] + γA[index2][k];    // do the reduction to  have only
                                                          // jth row being active
        F[j] -= αF[index1] + γF[index2];
    }
}
```

Backward Substitution

```
int  index = (size-1)/2;
x[index]  = F[index]/A[index][index];

for(i=log2(size+1)+2;i>=0; i--)
{
    for(j=2^(i+1) - 1; j <size; j=j+ 2^(i+1))
    {
        offset = 2^i;
        index1 = j – offset;
        index2 = j + offset;

         x[index1] = F[index1];
         x[index2] = F[index2];
        for(k=0; k < size; k++)
        {
            if(k! = index1)
                x[index1] -= A[index1]*x[k];
            if(k != index2)
                x[index2] -= A[index2][k]*x[k];
        }
        x[index1] = x[index1]/A[index1][index1];
        x[index2] = x[index2]/A[index2][index2];
    }
}
```

Source of Parallelism

- Simultaneous reduction of equations in the system
- Simultaneous backward substitution to solve for the solution

Row Decomposition

For $i = 1, \ldots, n$ of equations

Process i stores ith equation.

Computation

when $modulus(i, 2) = 0$, do row reduction to yield updated ith equation

Communication

ith equation receive $(i-1)th$ and $(i+1)th$ equations for $modulus(i, 2) = 0$

Reference

- B. Buzbbe, G. Golub, and C. Nielsen.
  On direct methods for solving Poisson's equation. SIAM *J. Numer. Anal.,* 7:627-656, 1970
- J. Dongarra and S. Johnsson. Solving banded systems on a parallel processor, *Parallel Computing* 5:219-246, 1987
- M. Hegland. On the parallel solution of tridiagonal systems by wrap-around partitioning and incomplete LU factorization, *Numer. Math.* 59:453-472, 1991
- V. Mehrmann. Divide and conquer methods for block tridiagonal systems, Parallel Computing 19:257-280, 1993