2.2 Fixed-Point Iteration

Definition 2.2. The number p is a **fixed point** for a given function $g(x)$ if $g(p) = p$.

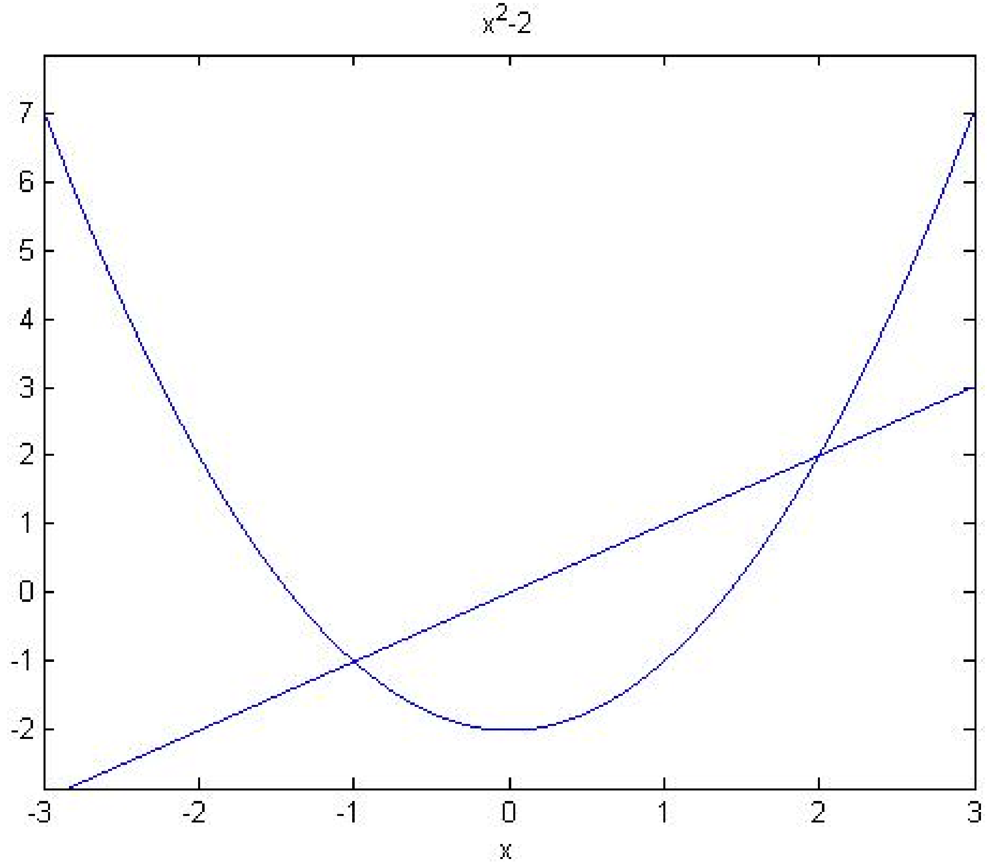
Geometric interpretation of **fixed point.**

- Consider the graph of function $g(x)$, and the graph of equation $y = x$.
- If they intersect, what are the coordinates of the intersection point?

Key: Fixed point is the solution of eq. $x = g(x)$.

Example 2.2.1.

Determine the fixed points of the function $g(x) = x^2 - 2$.



Connection between Fixed-point Problem and Root-Finding Problem

1. Given a root-finding problem, i.e., to solve $f(x) = 0$. Suppose a root is p, so that $f(p) = 0$.

   There are many ways to define $g(x)$ with fixed-point at p.

   For example, define $g(x) = x - f(x)$,

   or define $g(x) = x + 3f(x)$,

   …

2. If $g(x)$ has a fixed-point at p, then $f(x)$ defined by $f(x) = x - g(x)$ has a zero at p.

Sufficient Conditions for Existence and Uniqueness of a Fix Point

Theorem 2.3. Existence and Uniqueness Theorem

i. If $g \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$, then g has at least one **fixed-point** in $[a,b]$

ii. If, in addition, $g'(x)$ exists on (a,b) and a positive constant $k < 1$ exists with

$$|g'(x)| \leq k, \qquad \text{for all } x \in (a,b),$$

then there is **exactly one** **fixed-point** in $[a,b]$.

Note:

1. $g \in C[a,b] \rightarrow$ g is continuous in $[a,b]$
2. $g(x) \in [a,b] \rightarrow$ range of g is in $[a,b]$

Example 2. Show $g(x) = \frac{x^2-1}{3}$ has a unique fixed point on [-1, 1].

Example 3. Show that **Theorem 2.3** does not ensure a unique fixed point of $g(x) = 3^{-x}$ on the interval [0, 1], even through a unique fixed point on this interval does exist.

Solution: $g'(x) = -3^{-x}\ln(3)$.

$g'(x) < 0$ on [0,1]. So g is strictly decreasing on [0,1].

$g(1) = \frac{1}{3} \le g(x) \le g(0) = 1$, for $0 \le x \le 1$.

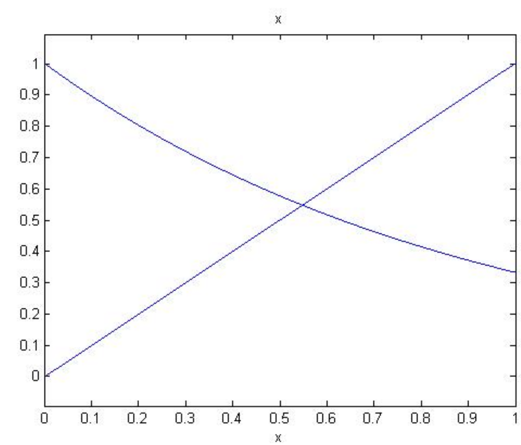
Condition (i) of **Theorem 2.3** ensures there is at least one fixed point.

Since $|g'(0.01)| = |-3^{-0.01}\ln(3)| \approx 1.0866$,

$|g'(x)| \nleq 1$ on (0,1).

Since condition in (ii) of **Theorem 2.3** is NOT satisfied, **Theorem 2.3** can not determine uniqueness.

Graphs of 3^{-x} and $y = x$:



Fixed-Point Iteration Algorithm

- Choose an initial approximation p_0, generate sequence $\{p_n\}_{n=0}^{\infty}$ by $p_n = g(p_{n-1})$.
- If the sequence converges to p, then

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} g(p_{n-1}) = g\left(\lim_{n\to\infty} p_{n-1}\right) = g(p)$$

Example.

Determine the fixed point of the function $g(x) = \cos(x)$ for $x \in [-0.1, 1.8]$.

Soln: choose $p_0 = 0.3$, $p_1 = \cos(0.3) = 0.955336$,

$p_2 = \cos(0.955336)$=0.577334,

$p_3 = \cos(0.577334)$=0.837921,

…

Remark: See also the Matlab code.

INPUT **p0**; tolerance **TOL**; maximum number of iteration **N0**.

OUTPUT solution **p** or message of failure

STEP1 Set i = 1. // init. counter

STEP2 While i ≤ N0 do Steps 3-6

    STEP3 Set **p** = g(**p0**).

    STEP4 If |**p**-**p0**| < **TOL** then

        OUTPUT(p); // successfully found the solution

        STOP.

    STEP5 Set i = i + 1.

    STEP6 Set **p0** = **p**. // update **p0**

STEP7 OUTPUT(“The method failed after **N0** iterations”);

STOP.

Convergence

Fixed-Point Theorem 2.4

Let $g \in C[a,b]$ be such that $g(x) \in [a,b]$, for all $x \in [a,b]$. Suppose, in addition, that g' exists on (a,b) and that a constant $0 < k < 1$ exists with

$$|g'(x)| \leq k, \qquad \text{for all } x \in (a,b)$$

Then, for any number p_0 in $[a,b]$, the sequence defined by

$$p_n = g(p_{n-1})$$

converges to the unique fixed point p in $[a,b]$.

Corollary 2.5

If g satisfies the above hypotheses, then bounds for the error involved using p_n to approximating p are given by

$$|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\}$$

$$|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0|$$

Illustration Equation $x^3 + 4x^2 - 10 = 0$ has a unique root in [1,2]. Use algebraic manipulation to obtain fixed-point iteration function g to solve this root-finding problem.