JSL 66 (Dec. 2001): 1899–1908

Partitioning Subsets of Stable Models *

Timothy Bays

Abstract

This paper discusses two combinatorial problems in stability theory. First we prove a partition result for subsets of stable models: for any A and B, we can partition A into $|B|^{<\kappa(T)}$ pieces, $\langle A_i \mid i < |B|^{<\kappa(T)}\rangle$, such that for each A_i there is a $B_i \subseteq B$ where $|B_i| < \kappa(T)$ and $A_i \underset{B_i}{\downarrow} B$. Second, if A and B are as above and $|A| > |B|$, then we try to find $A' \subset A$ and $B' \subset B$ such that $|A'|$ is as large as possible, $|B'|$ is as small as possible, and $A' \underset{B'}{\downarrow} B$. We prove some positive results in this direction, and we then discuss the optimality of these results under ZFC + GCH.

1 Introduction

The problems discussed in this paper arise from the study of Chang's Conjecture in the context of stable theories. Suppose that M is a stable model which lives in some class of models for which a nice prime model theory exists: e.g., the class of all models of an ω-stable theory, the class of $F^a_{\kappa(T)}$-saturated models of a superstable theory, or the class of $|T|^+$-saturated models of a stable theory. Suppose also that M is a two-cardinal model: i.e., for some predicate P in the language of M, $\omega \leq |P(M)| < |M|$. Given such an M, it is natural to ask Chang's Conjecture style questions: for what cardinals $\kappa \leq |M|$ and $\lambda < |P(M)|$ can we find $N \prec M$ such that $|N| = \kappa$ and $|P(N)| = \lambda$?

In [2], we show that this type of question typically reduces to a straightforward combinatorial problem. Let A and B be subsets of a stable model such that $|A| > |B|$; then we want to find A' and B' such that,

- $A' \subseteq A$ and $|A'|$ is as large as possible (ideally, $|A'| = |A|$).
- $B' \subseteq B$ and $|B'|$ is as small as possible (ideally, $|B'| < \kappa(T)$).
- $A' \underset{B'}{\downarrow} B$.

Suppose that we have some general techniques for solving this kind of problem. To obtain Chang's Conjecture style results, we argue as follows. First we find $A \subset M \setminus P(M)$ and $B \subset P(M)$ such that $|A| = |M|$, $|B| < \kappa(T)$ and $A \underset{B}{\downarrow} P(M)$. With a little bit of work, we can replace B with some N such that $N \prec M$, $|P(N)| < |P(M)|$ and N is in our class of "nice models". From here, prime model theory allows us to create a new two-cardinal model $N[A] \prec M$ as desired (note that in any such model $|P(N[A])| = |P(N)|$ while $|N[A]| = |A| = |M|$).

*This paper derives from chapter three of the author's dissertation, written under the direction of Professor Tony Martin at the University of California, Los Angeles. The dissertation was partially supported by a U.S. Department of Education graduate fellowship.

This brings us to the main topic of this paper. Let T be a complete, stable theory; let κ, λ, κ', and λ' be cardinals; and let M be any saturated model of T such that $|M| > \max\{\kappa, \lambda, \kappa', \lambda'\}$. We will say that $\underline{\dagger(\kappa, \lambda, \kappa', \lambda')}$ holds for T if whenever A and B are subsets of M such that $|A| = \kappa$ and $|B| = \lambda$, then there exists $A' \subset A$ and $B' \subset B$ such that $|A'| = \kappa'$, $|B'| = \lambda'$, and $A' \underset{B'}{\downarrow} B$. Our goal is to classify the cardinals for which $\dagger(\kappa, \lambda, \kappa', \lambda')$ holds, given only some very general information about cardinal arithmetic and T (e.g. does GCH hold? what is $\kappa(T)$? etc.).

This classification will be carried out in Sections 3–4. In Section 3, we prove several "positive" theorems which give general conditions under which $\dagger(\kappa, \lambda, \kappa', \lambda')$ holds. In Section 4, we discuss the optimality of these theorems. We show that the theorems are optimal for superstable theories and that, for unsuperstable theories, the theorems are the best one can prove using only ZFC+GCH. Finally, we note some stronger results which follow from the existence of large cardinals.

For technical reasons, we find it useful to approach questions about $\dagger$ by first considering questions about partitions. Let A and B be subsets of some stable model; then we would like to find a partition of A into $< |A|$ pieces, $\langle A_i \mid i < \lambda < |A| \rangle$, such that for each A_i there exists $B_i \subset B$ where $|B_i| < |B|$ and $A_i \underset{B_i}{\downarrow} B$. The relationship between this problem and $\dagger$ should be clear. As A is partitioned into $< |A|$ pieces, some of these pieces have to be large; and as each B_i has cardinality $< |B|$, we obtain an interesting instance of $\dagger$.

In Section 2, we address this problem. The main theorem in the section is the following:

Theorem: *Let A and B be arbitrary subsets of a stable model. Then we can partition A into $|B|^{<\kappa(T)}$ pieces, $\langle A_i \mid i < |B|^{<\kappa(T)} \rangle$, such that for each A_i there is a $B_i \subseteq B$ where $|B_i| < \kappa(T)$ and $A_i \underset{B_i}{\downarrow} B$.*

We also show that in a few cases, say when $|B|$ is particularly small or when we are willing to let $|B_i|$ be large, we can get slightly better partitions. Finally, and with an eye toward the project of Section 4, we show that these results are optimal under ZFC+GCH.

Throughout the paper, T is stable and countable, and $\mathbb{M}$ is a monster model for T. We assume basic facts about stable theories. These can be found in [1], [4], or [5]. Notationally, we use $M, N, \ldots$ to denote models and $A, B, \ldots$ to denote subsets of models. We use $\alpha, \beta, \gamma, \ldots$ to denote ordinals; $\kappa, \lambda, \mu, \ldots$ to denote infinite cardinals; m and n to denote natural numbers; and i, j, k and l to denote either ordinals or natural numbers depending on the context. We use $\subset$ to mean $\subseteq$.

2 Partitions

Our main goal in this section is to prove a partition theorem for subsets of stable models. We begin with the following lemma. Its proof is due to Shelah and is contained within his proof of IX 1.4 in [5].

Lemma 1 *Let A and B be subsets of $\mathbb{M}$ such that $cf(|B|) \geq \kappa(T)$. Then we can partition A into $cf(|B|)$ pieces, $\langle A_i \mid i < cf(|B|) \rangle$, such that for each A_i there is a $B_i \subseteq B$ where $|B_i| < |B|$ and $A_i \underset{B_i}{\downarrow} B$.*

Proof. Let $\nu = cf(|B|)$ and let $\langle B_i |\ i \leq \nu \rangle$ be an increasing, continuous sequence of subsets of B such that $|B_i| < |B|$ for $i < \nu$, and $B_\nu = B$. We define an increasing sequence of subsets of A, $\langle A_i \mid i < \nu \rangle$, by induction: A_i is a maximal subset of A such that $\bigcup_{j<i} A_j \subseteq A_i$ and $A_i \underset{B_i}{\downarrow} B$.

Now I claim that $\bigcup_{i<\nu} A_i = A$. For suppose $a \in A \setminus (\bigcup\limits_{i<\nu} A_i)$. Then for each i, $A_i \cup \{a\} \underset{B_i}{\not\downarrow} B$.

$$\begin{aligned} &\Rightarrow \quad a \underset{A_i \cup B_i}{\not\downarrow} A_i \cup B \\ &\Rightarrow \quad \text{for some } j > i, \ a \underset{A_i \cup B_i}{\not\downarrow} A_i \cup B_j . \\ &\Rightarrow \quad \text{for some } j > i, \ a \underset{A_i \cup B_i}{\not\downarrow} A_j \cup B_j . \end{aligned}$$

So, letting $p_i = tp(a, A_i \cup B_i)$, we get a cofinal subsequence of $\langle p_i \mid i < \nu \rangle$ which is a forking sequence. As ν is regular and $\kappa(T) \leq \nu$, this is a contradiction.

Now, replace each A_i with $A_i \setminus \bigcup_{j<i} A_j$. Then $\mathcal{P} = \{A_i \ : \ i < \nu\}$ is a partition of A, and for each $i < \nu$, $A_i \underset{B_i}{\downarrow} B$. As $|B_i| < |B|$ for each i, we are done. □

Theorem 2 *Let A and B be subsets of $\mathbb{M}$. Then we can partition A into $|B|^{<\kappa(T)}$ pieces, $\langle A_i \mid i < |B|^{<\kappa(T)} \rangle$, such that for each A_i there is a $B_i \subseteq B$ where $|B_i| < \kappa(T)$ and $A_i \underset{B_i}{\downarrow} B$.*

Proof. The proof is by induction on $|B|$. For $|B| < \kappa(T)$, there is nothing to prove; so, we assume that $|B| \geq \kappa(T)$. We take cases on $cf(|B|)$.

Case 1 ($cf(|B|) \geq \kappa(T)$) Using Lemma 1, we partition A into $cf(|B|)$ pieces, $\langle A_i \mid i < cf(|B|) \rangle$, such that for each A_i there is a $B_i \subseteq B$ where $|B_i| < |B|$ and $A_i \underset{B_i}{\downarrow} B$.

Next, we apply the induction hypothesis and partition each A_i into $|B_i|^{<\kappa(T)}$ pieces, $\langle A_{i,j} \mid j < |B_i|^{<\kappa(T)} \rangle$, such that each $A_{i,j}$ has an associated $B_{i,j} \subset B_i$ where $|B_{i,j}| < \kappa(T)$ and $A_{i,j} \underset{B_{i,j}}{\downarrow} B_i$. By the transitivity of forking, $A_{i,j} \underset{B_{i,j}}{\downarrow} B$. Note that the number of such $A_{i,j}$ is at most $|B| \cdot |B|^{<\kappa(T)} = |B|^{<\kappa(T)}$. So, if we let $\mathcal{P} = \{A_{i,j} \ : \ i < cf(|B|) \text{ and } j < |B|^{<\kappa(T)}\}$ be our partition, we are done.

Case 2 ($cf(|B|) < \kappa(T)$) Because T is countable and $cf(|B|) < \kappa(T) \leq |B|$, we know that $\kappa(T) = \omega_1$ and $cf(|B|) = \omega$. Hence, we let $\langle B_i \mid i < \omega \rangle$ be an increasing sequence of subsets of B such that $|B_i| < |B|$ for $i < \omega$, and $\bigcup_{i<\omega} B_i = B$. By induction, we construct a sequence of partitions, $\langle \mathcal{P}_i \mid i < \omega \rangle$, satisfying the following conditions:

1. Each $\mathcal{P}_i$ is a partition of A into $|B_i|^{<\kappa(T)}$ pieces.
2. For each $A_{i,j}$ in $\mathcal{P}_i$ there exists $B_{i,j} \subset B_i$ such that $|B_{i,j}| < \kappa(T)$ and $A_{i,j} \underset{B_{i,j}}{\downarrow} B_i$.
3. If $j > i$, then $\mathcal{P}_j$ refines $\mathcal{P}_i$

We begin by applying the original induction hypothesis to partition A over B_0 so as to satisfy conditions 1 and 2.

Suppose we have $\mathcal{P}_i = \langle A_{i,j} \mid j < |B_i|^{<\kappa(T)} \rangle$. As $|B_{i+1}| < |B|$, we can again apply our original induction hypothesis and partition each $A_{i,j}$ into $|B_{i+1}|^{<\kappa(T)}$ pieces, $\langle A_{i,j,k} \mid k < |B_{i+1}|^{<\kappa(T)} \rangle$, such that each $A_{i,j,k}$ has an associated $B_{i,j,k} \subseteq B_{i+1}$ where $|B_{i,j,k}| < \kappa(T)$ and $A_{i,j,k} \underset{B_{i,j,k}}{\downarrow} B_{i+1}$. We then set,

$$\mathcal{P}_{i+1} = \{A_{i,j,k} : \ j < |B_i|^{<\kappa(T)} \text{ and } k < |B_{i+1}|^{<\kappa(T)}\}.$$

Clearly, $\mathcal{P}_{i+1}$ satisfies conditions 2 and 3. Since $(|B_i|^{<\kappa(T)}) \cdot (|B_{i+1}|^{<\kappa(T)}) = |B_{i+1}|^{<\kappa(T)}$, condition 1 is satisfied as well.

Given the sequence $\langle \mathcal{P}_i \mid i < \omega \rangle$, we define our final $\mathcal{P}$ through its associated equivalence relation. Using the obvious notation, we set:

$$a \sim c \Longleftrightarrow \text{ for every } i < \omega, \ a \sim_i c.$$

Note that for every $\hat{A} \in \mathcal{P}$ and every $i < \omega$, there is some $\hat{A}_i \in \mathcal{P}_i$ and an associated $\hat{B}_i \subset B_i$ such that $\hat{A} \subset \hat{A}_i$, $|\hat{B}_i| < \kappa(T)$ and $\hat{A}_i \underset{\hat{B}_i}{\downarrow} B_i$. Hence, by forking continuity, $\hat{A} \underset{\bigcup_{i<\omega} \hat{B}_i}{\downarrow} B$.

As to our cardinality conditions, notice that since $|\hat{B}_i| < \kappa(T)$ for every $i < \omega$, $|\bigcup_{i<\omega} \hat{B}_i| < \kappa(T)$ (remember that $\kappa(T) = \omega_1$ in the "case 2" situation). Similarly, the following computation shows that $|\mathcal{P}| \leq |B|^{<\kappa(T)}$:

$$|\mathcal{P}| \leq \prod_{i<\omega} |\mathcal{P}_i| \leq \prod_{i<\omega} |B_i|^{<\kappa(T)} \leq \prod_{i<\omega} |B|^{<\kappa(T)} = (|B|^{<\kappa(T)})^{\omega} = |B|^{<\kappa(T)}.$$

Again, the last step in this computation depends on the fact that $\kappa(T) = \omega_1$ in the "case 2" situation. □

Remark: Notice that for superstable theories, Theorem 2 lets us partition A into only $|B|$ pieces (since $|B|^{<\kappa(T)} = |B|$ for superstable theories). Even for stable theories, the factor of $|B|^{<\kappa(T)}$ only comes into play when we try to partition A "over" some B such that $cf(|B|) < \kappa(T)$. If our induction does not pass through such a B, then even unsuperstable theories will admit partitions of size $|B| < |B|^{<\kappa(T)}$. We have:

Corollary 3 *Let A and B be as in Theorem 2. Let $\mu \leq |B|$ be such that $cf(\mu) \geq \kappa(T)$ and there are no singular cardinals κ such that $\mu < \kappa \leq |B|$. Then for arbitrary A, we can partition A into $|B|$ pieces, $\langle A_i \mid i < |B| \rangle$, such that for each A_i there is a $B_i \subseteq B$ where $|B_i| < \mu$ and $A_i \underset{B_i}{\downarrow} B$.*

Proof. We follow the proof of Theorem 2, limiting our induction to cardinals κ such that $\mu \leq \kappa \leq |B|$. Since $cf(\mu) \geq \kappa(T)$ and there are no singular cardinals between μ and $|B|$, our induction never enters a "case 2" situation. Hence, we can preserve partitions of size $|B|$ throughout the induction. □

Corollary 4 *Let A and B be arbitrary subsets of $\mathbb{M}$ such that $|B| < \aleph_\omega$. Then we can partition A into $|B|$ pieces, $\langle A_i \mid i < |B| \rangle$, such that for each A_i there is a $B_i \subseteq B$ where $|B_i| < \kappa(T)$ and $A_i \underset{B_i}{\downarrow} B$.*

Proof. Apply Corollary 3. □

Remark: Given some initial instance of non-forking, it is easy to check that this non-forking can be preserved through the proofs of Lemma 1 and Theorem 2. This gives us a usefull strengthening of Theorem 2.

Theorem 5 *Let B, C and B_C be subsets of $\mathbb{M}$ such that $C \underset{B_C}{\downarrow} B$. Then we can partition A into $|B|^{<\kappa(T)}$ pieces, $\langle A_i \mid i < |B|^{<\kappa(T)} \rangle$, such that for each A_i there is a $B_i \subseteq B$ where $|B_i| < \kappa(T)$ and $A_i \cup C \underset{B_i \cup B_C}{\downarrow} B$.*

Proof. Just like the proofs of Lemma 1 and Theorem 2. □

At this point, we turn to examining the optimality of Theorem 2. To keep our cardinal arithmetic manageable, we assume GCH throughout this discussion. Under GCH, Theorem 2 shows two things:

- If $cf(|B|) \geq \kappa(T)$, then we can partition any A into $|B|$ pieces, $\langle A_i \mid i < |B| \rangle$, such that for each A_i there is a $B_i \subseteq B$ where $|B_i| < \kappa(T)$ and $A_i \underset{B_i}{\downarrow} B$.
- If $cf(|B|) < \kappa(T)$, then we can partition any A into $|B|^+$ pieces, $\langle A_i \mid i < |B|^+ \rangle$, such that for each A_i there is a $B_i \subseteq B$ where $|B_i| < \kappa(T)$ and $A_i \underset{B_i}{\downarrow} B$.

The first of these points shows that Theorem 2 is optimal when $cf(|B|) \geq \kappa(T)$. To see this, let $\kappa \geq \kappa(T)$ and let $B \subset \mathbb{M}$ be such that B is algebraically independent, $|B| = \kappa$, and $B \cap \mathrm{acl}(\emptyset) = \emptyset$. Then we cannot find $B' \subset B$ and $B'' \subset B$ such that $|B''| < \kappa(T) \leq |B'|$ and $B' \underset{B''}{\downarrow} B$ (as $B' \underset{B''}{\downarrow} B$ entails $B'' \supset B'$). Hence, by setting $A = B$ or even just $A \supset B$, we witness the impossibility of partitioning A into less than κ reasonably homogenous pieces.

Next, we consider the case in which $cf(|B|) < \kappa(T)$, and we show that one cannot, in general, construct partitions of size $< |B|^+$. We note that since $cf(|B|) < \kappa(T)$, $cf(|B|) = \omega$ and $\kappa(T) = \omega_1$. So, we fix a cardinal, κ, such that $\mathrm{cf}(\kappa) = \omega$, and we consider the following example.

Example 6 Let $\mathcal{L} = \{P, Q, \langle F_i \mid i < \omega \rangle\}$ where P and Q are unary predicates and each F_i is a binary relation. Let a model $N = N_\kappa$ for $\mathcal{L}$ be given as follows:

- $P(N) = \kappa$; $Q(N) = {}^{\omega}\kappa$; $N = P(N) \cup Q(N)$.
- $F_i : Q(N) \to P(N)$ by $F_i(\eta) = \eta(i)$.

Here is the intuitive idea. $P(N)$ and $Q(N)$ are disjoint sets. $P(N)$ has no intrinsic structure, while elements of $Q(N)$ "code up" countable subsets of $P(N)$ via the sequence $\langle F_i \mid i < \omega \rangle$. Let $T = Th(N)$. It is easy to check that T is stable and quantifier eliminable and that $\kappa(T) = \omega_1$. Similarly, it is clear that $|P(N)| = \kappa$ and $|Q(N)| = \kappa^+$.

Now suppose that $\langle A_j \mid j < \lambda \rangle$ is a partition of $Q(N)$ into fewer than κ^+ pieces. As $|Q(N)| = \kappa^+$, there must be some $j < \lambda$ such that $|A_j| = \kappa^+$. And as no set of size $< \kappa$ can have κ^+ distinct subsets (by GCH), $|\bigcup_{i<\omega} F_i[A_j]| = \kappa$. Further, it is easy to see that if $B \subset P(N)$ such that $\bigcup_{i<\omega} F_i[A_j] \not\subset B$, then $A_j \underset{B}{\not\downarrow} P(N)$ (as witnessed, for instance, by some formula of the form "$x = F_i(y)$"). So, there is no $B \subset P(N)$ such that $|B| < \kappa(T)$ and $A_j \underset{B}{\downarrow} P(N)$. □

Remarks: (1.) These examples show that Theorem 2 is optimal with respect to partitioning A into a small number of A_i pieces such that the associated B_i pieces have cardinality $< \kappa(T)$. For fixed $\mu < |B|$, the same examples show that we cannot get better partitions by letting the B_i pieces have cardinality $< \mu$.

(2.) Suppose $\kappa(T) \leq cf(|B|) < |B|$. Then by Lemma 1, we can partition A into only $cf(|B|)$ pieces, $\langle A_i \mid i < cf(|B|)\rangle$, such that the associated B_i pieces have cardinality $< |B|$ (so, there is no *fixed* $\mu < |B|$ such that $|B_i| < \mu$ for all i). By the example from the $cf(|B|) \geq \kappa(T)$ case above, this is also optimal.

(3.) With minor modifications, everything in this section generalizes to uncountable theories (though the proofs become notational bogs rather rapidly). In Lemma 1 (hence in Theorem 2), we need to replace $\kappa(T)$ with $\kappa_r(T)$ to make our computations come out right. In the proof of Theorem 2 we need a transfinite sequence of partitions $\langle \mathcal{P}_i \mid i < cf(|B|)\rangle$; but, if we simply apply the direct limit construction from Theorem 2 at all limit ordinals, this construction goes through exactly as before. In Example 6, we need to consider structures with uncountably many functions, $\langle F_i \mid i < \lambda\rangle$.

3 Combinatorics

In this section we prove several positive results concerning the $\dagger(\kappa, \lambda, \kappa', \lambda')$ relation. We begin by noting that questions about † are only interesting when $\kappa' \leq \kappa$, $\lambda' < \lambda$, $\lambda < \kappa$, and $\lambda' < \kappa'$; if any of these conditions fail, then questions about † become trivial. For notational convenience, we define a function,

$$\Phi(\mu, \nu) = \begin{cases} \nu & \text{if } T \text{ is superstable} \\ \nu & \text{if } T \text{ is stable and there is no cardinal } \xi \\ & \text{such that } \mu < \xi \leq \nu \text{ and } cf(\xi) < \kappa(T) \\ \nu^\omega & \text{otherwise.} \end{cases}$$

The point of this notation is as follows. Let $\mu < \nu$ and let B be a set of cardinality ν. Then Theorem 2 and Corollary 3 allow us to partition an arbitrary A into $\Phi(\mu, \nu)$ pieces, $\langle A_i \mid i < \Phi(\mu, \nu)\rangle$, such that such that each A_i has an associated $B_i \subseteq B$ where and $A_i \underset{B_i}{\downarrow} B$ and $|B_i| = \mu$. Given this, our main theorem is:

Theorem 7 *Let κ, λ, κ' and λ' be cardinals such that $\lambda < \kappa$, $\kappa' \leq \kappa$, $\lambda' < \lambda$. Suppose that one of the following conditions holds:*

1. *$\kappa' < \kappa$, and $\Phi(\lambda', \lambda) < \kappa$.*
2. *$\kappa' = \kappa$, and $\Phi(\lambda', \lambda) < cf(\kappa)$.*
3. *$\kappa' = \kappa$, $\Phi(\lambda', \lambda) < \kappa$, and $cf(\kappa) \leq \lambda'$.*

Then $\dagger(\kappa, \lambda, \kappa', \lambda')$ holds.

Proof. Fix A and B such that $|A| = \kappa$ and $|B| = \lambda$; we take cases on the conditions in the theorem.

Suppose first that condition 1 holds. Then we partition A into $\Phi(\lambda', \lambda)$ pieces $\langle A_i \mid i < \Phi(\lambda', \lambda)\rangle$, such that for each A_i there is a $B_i \subseteq B$ where $|B_i| = \lambda'$ and $A_i \underset{B_i}{\downarrow} B$. Since $\Phi(\lambda', \lambda) < \kappa$, one of these A_i sets has size at least κ'. Making this set smaller as necessary, we are done.

Similarly, suppose condition 2 holds. Once again we partition A into $\Phi(\lambda', \lambda)$ pieces $\langle A_i \mid i < \Phi(\lambda', \lambda)\rangle$ and note that, since $\Phi(\lambda', \lambda) < cf(\kappa)$, one of the A_i sets must have size $\kappa = \kappa'$.

Finally, suppose condition 3 holds. Let $\langle \kappa_i \mid i < cf(\kappa)\rangle$ be increasing such that $\kappa = \sum_{i<cf(\kappa)} \kappa_i$. We construct a sequence, $\langle (A_i, B_i) \mid i < cf(\kappa)\rangle$, such that for each i:

- $A_i \subset A$ and $|A_i| \geq \kappa_i$.
- $B_i \subset B$ and $|B_i| = \lambda'$.
- $\bigcup_{j\leq i} A_j \underset{\bigcup_{j\leq i} B_j}{\downarrow} B$.

Assume we have constructed this sequence for $j < i$. By forking continuity, we know that $\bigcup_{j<i} A_j \underset{\bigcup_{j<i} B_j}{\downarrow} B$. Using Lemma 5 (or the obvious analog of Lemma 3), we partition A into $\Phi(\lambda', \lambda)$ pieces, $\langle \hat{A}_k \mid k < \Phi(\lambda', \lambda)\rangle$, such that for each $\hat{A}_k$ there is a $\hat{B}_k \subseteq B$ where $|\hat{B}_k| = \lambda'$ and $\hat{A}_k \cup \bigcup_{j<i} A_j \underset{\hat{B}_k \cup \bigcup_{j<i} B_j}{\downarrow} B$. Since $\Phi(\lambda', \lambda) < \kappa$, one of the $\hat{A}_k$ sets must have cardinality at least κ_i. Choosing such an $\hat{A}_k$, we let $A_i = \hat{A}_k \cup \bigcup_{j<i} A_j$ and $B_i = \hat{B}_k \cup \bigcup_{j<i} B_j$ (note that since $i < cf(\kappa) \leq \lambda'$, $|\bigcup_{j<i} B_j| = \lambda'$).

Let $A' = \bigcup_{i<cf(\kappa)} A_i$ and let $B' = \bigcup_{i<cf(\kappa)} B_i$. By forking continuity, $A' \underset{B'}{\downarrow} B$. Further, $|A'| = \sum_{i<cf(\kappa)} \kappa_i = \kappa$. Finally, since $cf(\kappa) \leq \lambda', |B'| = |\bigcup_{i<cf(\kappa)} B_i| = \lambda'$. □

Under several conditions, the use of Φ in this theorem can be eliminated. This renders the theorem itself somewhat more perspicuous. The following three corollaries give the most significant simplifications of the theorem.

Corollary 8 *Let κ, λ, κ' and λ' be as in the theorem. Suppose that $\lambda^\omega < \kappa$ and that one of the following holds:*

1. $\kappa' < \kappa$.
2. $\kappa' = \kappa$ *and* $\lambda^\omega < cf(\kappa)$.
3. $\kappa' = \kappa$ *and* $cf(\kappa) \leq \lambda'$.

Then $\dagger(\kappa, \lambda, \kappa', \lambda')$ holds.

Proof. Since $\Phi(\lambda', \lambda)$ is at most λ^ω, $\lambda^\omega < \kappa \Rightarrow \Phi(\lambda', \lambda) < \kappa$. Hence, conditions 1–3 of the corollary reduce to conditions 1–3 of Theorem 7. □

Corollary 9 *Let κ, λ, κ' and λ' be as in the theorem. Suppose that T is superstable or that there are no singular cardinals between λ' and λ. Finally, suppose that one of the following holds:*

1. $\kappa' < \kappa$.
2. $\kappa' = \kappa$ *and* $\lambda < cf(\kappa)$.
3. $\kappa' = \kappa$ *and* $cf(\kappa) \leq \lambda'$.

Then $\dagger(\kappa, \lambda, \kappa', \lambda')$ holds.

Proof. The conditions of the corollary entail that $\Phi(\lambda', \lambda) = \lambda) < \kappa$. Hence, conditions 1–3 of the corollary reduce to conditions 1–3 of Theorem 7. □

Corollary 10 *Suppose that GCH holds, and let κ, λ, κ' and λ' be as in the theorem. Suppose also that one of the following holds:*

1. $cf(\lambda) \geq \kappa(T)$ *and* $\kappa' < \kappa$
2. $cf(\lambda) \geq \kappa(T)$ *and* $\lambda < cf(\kappa)$.
3. $cf(\lambda) \geq \kappa(T)$ *and* $cf(\kappa) \leq \lambda'$.
4. $cf(\lambda) < \kappa(T)$, $\lambda^+ < \kappa$, *and* $\kappa' < \kappa$.
5. $cf(\lambda) < \kappa(T)$, $\lambda^+ < \kappa$, *and* $\lambda^+ < cf(\kappa)$.
6. $cf(\lambda) < \kappa(T)$, $\lambda^+ < \kappa$, *and* $cf(\kappa) \leq \lambda'$.

Then $\dagger(\kappa, \lambda, \kappa', \lambda')$ *holds.*

Proof. If T is superstable, then cases 4–6 in this corollary are impossible and cases 1–3 correspond to the three cases of Corollary 9. Similarly, if T is unsuperstable and $cf(\lambda) > \omega$, then cases 4–6 in this corollary are impossible and cases 1–3 correspond to the three cases of Corollary 8. Finally, if T is unsuperstable and $cf(\lambda) = \omega$, then cases 1–3 of this corollary become impossible and cases 4–6 correspond to the three cases of Corollary 8. □

Remarks: (1.) For superstable theories, Shelah gives an alternate proof of Corollary 9 (see [5] V, 6.16-6.17). Because Shelah's proof makes extensive use of large independent sets, it does not generalize to the non-superstable case.

(2.) The results of this section generalize to uncountable theories. The generalizations are straightforward, with no essentially new ideas required. However, the cardinal arithmetic involved in such generalizations is sufficiently complicated that it rapidly obscures all of the main ideas (especially if we do not assume GCH).

4 Some Countermodels

In the last section, we gave some general conditions under which $\dagger(\kappa, \lambda, \kappa', \lambda')$ holds. In this section, we discuss the degree to which these conditions are optimal. To keep our cardinal arithmetic manageable, we assume GCH throughout this discussion (but, see the remarks following Example 11). We also define the following "Ramsey style" modification of $\dagger$: we say that $\dagger'(\kappa, \lambda, \lambda')$ holds if $\dagger(\kappa, \lambda, \kappa, \lambda')$ holds. Given this, we show the following three things:

- The results of Section 3 are optimal for superstable theories.
- The results of Section 3 are optimal for the $\dagger'$ relation.
- The results of Section 3 are the best that can be proved using only ZFC+GCH.

As noted in the last section, questions about $\dagger(\kappa, \lambda, \kappa', \lambda')$ are only interesting if $\kappa' \leq \kappa$, $\lambda' < \lambda$, $\lambda < \kappa$, and $\lambda' < \kappa'$. Hence, we will take these conditions for granted throughout this section.

We begin by showing that Theorem 7 gives optimal results under the assumption that T is superstable (i.e. that Corollary 9 is optimal). Note that by Corollary 9 the only cases in which $\dagger(\kappa, \lambda, \kappa', \lambda')$ can fail are cases in which κ is singular, $\kappa' = \kappa$, and $\lambda' < cf(\kappa) \leq \lambda$. We fix some particular κ, κ', λ, and λ' satisfying these conditions and consider the following example.

Example 11 Let $\mathcal{L} = \{P, Q, G\}$ where P and Q are unary predicates and G is a binary relation. Let a model for $\mathcal{L}$ be given as follows:

- $P(M) = \lambda$; $Q(M) = \lambda \times \kappa$; $M = P(M) \cup Q(M)$.
- G is a function from $Q(M)$ to $P(M)$ such that $G((\alpha, \beta)) = \alpha$.

Intuitively, $P(M)$ is an infinite set with no intrinsic structure. Using G, we associate an infinite collection of (otherwise undifferentiated) elements of $Q(M)$ to each member of $P(M)$. Let $T = Th(M)$; it is easy to check that T is superstable (indeed ω-stable) and quantifier eliminable.

Let $\langle \kappa_i \mid i < cf(\kappa) \rangle$ be increasing and cofinal in κ. Let N be a submodel of M such that $P(N) = P(M)$ and

$$(\alpha, \beta) \in Q(N) \Longleftrightarrow \text{either } \ \alpha < cf(\kappa) \ \& \ \beta < \kappa_\alpha \quad \text{or} \quad \beta < \omega.$$

So, $|Q(N)| = \kappa$ and $|P(N)| = \lambda$. However, suppose $A \subset Q(N)$ such that $|A| = \kappa$. Then a trivial combinatorial argument shows that $G[A]$ must have cardinality at least $cf(\kappa)$. Further, it is easy to see that given any $B \subset P(N)$ such that $G[A] \not\subset B$, $A \not\perp_B P(N)$ (as witnessed, for instance, by the formula "x=G(y)"). So, $Q(N)$ and $P(N)$ witness the failure of $\dagger(\kappa, \lambda, \kappa', \lambda')$ as desired. □

Remark: Note that nothing in this argument depends on GCH. Hence, for superstable theories, Corollary 9 is optimal whether or not GCH holds.

Next, we turn to examine unsuperstable theories, and we consider the cases left open by Corollary 10. It is straightforward to show that Corollary 10 leaves only four cases in which $\dagger(\kappa, \lambda, \kappa', \lambda')$ can fail:

1. $\kappa' = \kappa$; and $\lambda' < cf(\kappa) \leq \lambda$.
2. $cf(\lambda) = \omega$; $\lambda^+ = \kappa$; and ${\lambda'}^\omega < \kappa'$.
3. $cf(\lambda) = \omega$; $\lambda^+ = \kappa$; and ${\lambda'}^\omega = \kappa'$.
4. $cf(\lambda) = \omega$; $\lambda^+ < \kappa$; $\kappa' = \kappa$; and $cf(\kappa) = \lambda^+$.

The first of these cases is covered by the model from Example 11, and the second by the model from Example 6. The fourth is covered by a straightforward combination of these two models. Since these are the only cases in which $\kappa' = \kappa$ is possibe, we know that Theorem 7 and Corollary 10 are optimal with respect to the $\dagger'$ relation.

At this point, we turn to the third case in which $\dagger(\kappa, \lambda, \kappa', \lambda')$ can fail. This case cannot be treated via simple countermodels, as it turns out to be independent of set theory. Because the proof of independence involves several convoluted forcing constructions—and doesn't provide very much *model-theoretic* insight—we simply sketch the key ideas here. We begin with an example showing that ZFC+GCH cannot prove $\dagger(\lambda^+, \lambda, \omega_1, \omega)$ for any λ such that $cf(\lambda) = \omega$.

Example 12 We work in L and fix an appropriate λ. Let $N = N_\lambda$ be defined as in Example 6. In particular, we know that $(|P(N)| = \lambda)^L$ and that $(|Q(N)| = \lambda^+)^L$.

Force over L using the partial order $Coll(\omega, \omega_1)$. Since this forcing is ω_2-c.c., it preserves all cardinals and cofinalities $\geq \omega_2$; in particular, it doesn't affect either λ or λ^+. Also, GCH continues to hold in the generic extension. Consider N from the perspective of $L[G]$ and suppose there exist $A, B \in L[G]$ such that $A \subset Q(N)$, $B \subset P(N)$, $(|A| = \aleph_1)^{L[G]}$, $(|B| = \aleph_0)^{L[G]}$, and $A \underset{B}{\downarrow} P(N)$. Since $Coll(\omega, \omega_1)$ is ω_2-c.c, there is some $B' \in L$ such that $B \subset B' \subset P(N)$ and $(|B'| = \omega_1)^L$. Working in L, we let $A' = \{q \in Q(N) \mid \text{for every } i < \omega,\ F_i(q) \in B'\}$. By an argument like that of Example 6, we know that $(|A'| = \omega_1)^L$. Moving to $L[G]$, therefore, we know that $(|A'| = \omega)^{L[G]}$. Since $A \subset A'$, this is a contradiction. Thus, we have our result: in $L[G]$, $Q(N)$ and $P(N)$ witness the failure of $\dagger(\lambda^+, \lambda, \omega_1, \omega)$. □

Remark: This example generalizes to cases in which λ' is greater than ω, although the forcing constructions are quite a bit more complicated. Typically, for instance, they use Jensen's Covering Theorem in place of chain conditions and (so) depend on the assumption that $0^\#$ does not exist in the ground model.

Example 12 and its cousins show that we cannot do better than Corollary 10 if we limit ourselves to ZFC+GCH. If we assume large cardinals, however, we can do a little better. The following is the key result:

Proposition 13 *If Chang's Conjecture holds between* (κ, λ) *and* (κ', λ')*, then* $\dagger(\kappa, \lambda, \kappa', \lambda')$ *holds as well.*

Proof. Let A and B be arbitrary subsets of $\mathbb{M}$ such that $|A| = \kappa$ and $|B| = \lambda$. Let $\langle F^i_j \mid i < \omega \text{ and } j < \omega\rangle$ be a collection of functions such that $F^i_j : {}^iA \to B$ and for every $\bar{a} \in {}^iA$, $tp(\bar{a}, B)$ does not fork over $\bigcup_{j<\omega} F^i_j[\bar{a}]$. Since Chang's conjecture holds between (κ, λ) and (κ', λ'), we can find $A' \subset A$ and $B' \subset B$ such that $|A'| = \kappa'$, $|B'| = \lambda'$, and $\bigcup_{i,j<\omega} F^i_j[A'] \subset B'$. Hence, $A' \underset{B'}{\downarrow} B$ as desired. □

In [3], Shelah, Magidor and Levinski show how, starting with assumptions slightly stronger than a 1-huge cardinal, we can prove the consistency of many "case three" instances of Chang's Conjecture (including, for instance, all instances of the form $(\lambda^+, \lambda) \longrightarrow (\omega_1, \omega)$ where $cf(\lambda) = \omega$). This gives us a method for getting slightly stronger results than those obtained in Section 3. It also completes our proof that, modulo the consistency of large cardinals, "case three" instances of † are independent of ZFC+GCH.

References

[1] John Baldwin. *Fundamentals of Stability Theory.* Springer-Verlag, Berlin, 1988.

[2] Timothy Bays. *Multi-Cardinal Phenomena in Stable Theories.* PhD thesis, University of California, Los Angeles, 1994.

[3] Jean-Pierre Levinski, Menachem Magidor, and Saharon Shelah. Chang's Conjecture for $\aleph_\omega$. *Israel Journal of Mathematics*, 69:161–172, 1990.

[4] Anand Pillay. *An Introduction to Stability Theory.* Clarendon Press, Oxford, 1983.

[5] Saharon Shelah. *Classification Theory and the Number of Non-Isomorphic Models.* North-Holland, Amsterdam, 1990.