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Some Two-Cardinal Results for O-Minimal Theories

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Abstract

We examine two-cardinal problems for the class of O-minimal theories. We prove that an O-minimal theory which admits some (κ, λ) must admit every (κ', λ'). We also prove that every "reasonable" variant of Chang's Conjecture is true for O-minimal structures. Finally, we generalize these results from the two-cardinal case to the δ-cardinal case for arbitrary ordinals δ.

1 Introduction

In their most general form, two-cardinal problems depend heavily on assumptions about our background set theory. Admitting cardinals conjectures, for instance, are often true in L but false in relatively straightforward extensions of L. Similarly, most variants of Chang's Conjecture have the consistency strength of quite large cardinals.

If we restrict ourselves to stable theories, two-cardinal problems become both more tractable and more amenable to model-theoretic, as opposed to set-theoretic, investigation. Lachlan has shown that any stable theory which admits some (κ, λ) must admit every (κ', λ') (see [4]). Similarly, Shelah has shown that most variants of Chang's Conjecture are true for superstable theories, and the present author has shown that this result generalizes to (almost all) $|T|^+$-saturated models of stable theories (see [6] and [2] respectively).

Two features of stable theories make these results possible. First, stable theories have a nice notion of independence (non-forking) which "explains" differences in cardinality between separate parts of a model. Second, stable theories admit an assortment of "prime model" constructions which allow us to build models respecting this independence.

Both of these features are present in O-minimal theories as well (in many ways, O-minimal theories are just ω-stable theories, but without the stability). Like stable theories, O-minimal theories have a nice notion of independence (simple algebraic independence). Like ω-stable theories, O-minimal theories admit prime models over arbitrary sets (see [5]).

In the present paper, we show that O-minimal theories are well-behaved with respect to two-cardinal problems. We begin by proving an O-minimal analog of Lachlan's theorem from [4]. We prove that if T is O-minimal and $P(N) \subset M \prec N$, then there exists N' such that $P(N') \subset M \prec N \prec N'$. From this, we conclude that an O-minimal theory which admits some (κ, λ) must admit every (κ', λ').

On the Chang's Conjecture side, we prove that if M is an O-minimal structure of type (κ, λ) and if κ' and λ' are such that $\omega \leq \lambda' \leq \kappa'$, $\lambda' \leq \lambda$, and $\kappa' \leq \kappa$, then there exists $N \prec M$ such that N is of type

(κ', λ'). In section 4, we generalize these results from the two-cardinal case to the δ cardinal case, where δ is some arbitrary ordinal.

Throughout the paper, T is countable and O-minimal and $\mathbb{M}$ is a monster model for T. We assume basic facts about O-minimal theories. These can be found in [5] and [3]. We always let "$<$" pick out the order of $\mathbb{M}$, and we assume that the "small" portions of our two-cardinal models are picked out by the predicate "P". Using the fact that T is O-minimal, we fix $\hat{c}_1, \dots, \hat{c}_{2n} \in \mathbb{M} \cup \{\pm\infty\}$ and $\hat{d}_1, \dots, \hat{d}_m \in \mathbb{M}$ such that $P(\mathbb{M}) = \bigcup_{i=1}^{n} \{x \,|\hat{c}_1 < x < \hat{c}_{i+1}\} \cup \bigcup_{i=1}^{m} \{\hat{d}_i\}$.

Notationally, we use $M, N, \dots$ to denote models and $A, B, \dots$ to denote subsets of models. We use $\alpha, \beta, \gamma, \dots$ to denote ordinals; $\kappa, \lambda, \mu, \dots$ to denote infinite cardinals; m and n to denote natural numbers; and i, j, k and l to denote either ordinals or natural numbers depending on the context. We use "$\prec$" to mean $\precneqq$.

2 Admitting Cardinals

The next lemma is, in some sense, the key to the entire paper. It shows that prime model constructions respect algebraic independence vis-a-vis definable subsets of $\mathbb{M}$.

Lemma 1 *Let M be a model and let $A \supset M$ be algebraically closed and such that $P(A) \subset M$. Let $M[A]$ be prime over A. Then $P(M[A]) \subset M$.*

Proof. Suppose not, and let $b \in P(M[A]) \setminus M$. As $M[A]$ is atomic over A, there is some formula $\psi(x, \bar{a})$ over A such that $\psi(x, \bar{a})$ isolates $\operatorname{tp}(b,\, A)$. Since A is algebraically closed, we may assume that ψ is of the form "$a_1 < x < a_2$" where $a_1, a_2 \in A \cup \{\pm\infty\}$. Further, since ψ both isolates $\operatorname{tp}(b,\, A)$ and entails that b is in $P(\mathbb{M})$, there must be some i such that $\psi(x) \vdash \hat{c}_i < x < \hat{c}_{i+1}$. Hence, $\hat{c}_i \leq a_1 < a_2 \leq \hat{c}_{i+1}$.

Because of this, and because $P(A) \subset M$, both a_1 and a_2 must live in M. Thus, since $a_1 < b < a_2$ and $M \prec \mathbb{M}$, M must satisfy "$\exists x(a_1 < x < a_2)$". Hence, for some $n_1 \in M \cap (a_1, a_2)$, ψ does not decide either "$a_1 < x < n_1$" or "$n_1 < x < a_2$" . So, ψ fails to isolate $\operatorname{tp}(b,\, A)$ for a contradiction. □

Lemma 2 *Let $P(N) \subset M \prec N$ and let $a \in N \setminus M$. Let $A \supset N$ be algebraically closed, let $p \in S(A)$ be an heir of tp(a, M), and let $b \models p$. Then, $P(acl(Ab)) \subset A$.*

Proof. Suppose not, and let $c \in P(\operatorname{acl}(Ab)) \setminus A$. Since $c \in \operatorname{acl}(Ab)$ and $c \notin A$, $b \in \operatorname{acl}(Ac)$. Let $\psi(x, \bar{m}, \bar{a}, c)$ witness this. Then,

$$\models \exists y[P(y) \wedge \exists^{=n} x\, \psi(x, \bar{m}, \bar{a}, y) \wedge \psi(b, \bar{a}, \bar{n}, y)].$$

Since $\operatorname{tp}(b,\, A)$ is an heir of $\operatorname{tp}(a,\, M)$, there is some $\bar{m}'$ such that,

$$\models \exists y[P(y) \wedge \exists^{=n} x\, \psi(x, \bar{m}, \bar{m}', y) \wedge \psi(a, \bar{m}, \bar{m}', y)].$$

Since N must satisfy this formula, and since $P(N) \subset M$, we conclude that a must be algebraic over M. But, this contradicts $a \in N \setminus M$. □

Theorem 3 *Suppose $M \prec N$ and $P(N) = P(M)$. Then there exists N' such that $N \prec N'$ and $P(N') = P(M)$.*

Proof. Choose $a \in N \setminus M$, and let $p \in S(N)$ be an heir of $\mathrm{tp}(a,\, M)$. Let $b \models p$. Then by lemma 2, $P(\mathrm{acl}(Nb)) \subset N$. So, $P(\mathrm{acl}(Nb)) \subset M$. Let N' be prime over $\mathrm{acl}(Nb)$. By lemma 1, $P(N') \subset M$ as desired. □

Theorem 4 *If T admits some (κ, λ) where $\kappa > \lambda$, then T admits every (κ', λ').*

Proof. Let M witness the fact that T admits (κ, λ). By the downward Löwenheim-Skolem theorem, there exists $M' \prec M$ such that $P(M) \subset M'$. Let "U" be a fresh predicate and expand M by letting $U(M) = M'$. By compactness, we can obtain a model N such that $N \models \mathrm{Th}(\langle M, U\rangle)$ and $|N| = |U(N)| = \lambda'$. Note that since N satisfies $\mathrm{Th}(\langle M, U\rangle)$, $P(N) \subset U(N)$ and, vis-a-vis our original language, $U(N) \prec N$.

Returning to our original language, we let N' be the submodel of N which was picked out by "U". By induction, we construct a strictly increasing sequence of models, $\langle N_i \,|\, i \leq \kappa'\rangle$, such that for every $i \leq \kappa'$, $P(N_i) = P(N')$. We start by letting $N_0 = N$. Given N_i, we apply theorem 3 to obtain N_{i+1}. Finally, for limit i, we let $N_i = \bigcup_{j<i} N_j$. At the end of the day, $N_{\kappa'}$ is a (κ', λ')-model as desired. □

Remarks: (1.) Lemma 2 resembles a result from stability theory. In stability theory, we say that a type $p \in S(B)$ is *foreign* to some definable set P if for every $A \supset B$ and every $p' \in S(A)$ a non-forking extension of p, if $b \models p'$ and $\mathbb{M} \models P(c)$, then $b \downarrow_A c$. For stable theories, if $P(N) \subset M \prec N$ and $a \in N \setminus M$, then $\mathrm{tp}(a,\, M)$ is foreign to P (see [2]). Lemma 2 says the same thing, modulo the need to redefine "non-forking extension" and "$b \downarrow_A c$" so as to make sense in the O-minimal context.

(2.) Note that the proof of theorem 3 actually gives something slightly stronger than theorem 3 itself. Let M and N be as in the theorem and let $N' \supset N$ (here, $P(N')$ need not be a subset of M). Then our proof shows that there must exist N'' such that $N' \prec N''$ and $P(N'') = P(N')$. So, any model which *contains* a pair of models like those in theorem 3, can be extended without adding new members of $P(\mathbb{M})$. This will be important in section 4.

(3.) We have proved theorem 4 for countable languages only (since countability is a background assumption for this paper). The result extends trivially to uncountable languages, however. If an uncountable theory T admits some (κ, λ) (where $\kappa > \lambda$), then all its countable subtheories admit every (κ', λ') (by theorem 4). Hence, by a result of Vaught's, T itself admits every (κ', λ') (see [7]).

3 Chang's Conjectures

Our first lemma "relativizes" theorem 3 so as to work within a particular two-cardinal model. Given a two-cardinal model M, the lemma allows us to expand arbitrary submodels of M, without adding new members of $P(M)$ to these submodels.

Lemma 5 *Let M be a model of type (κ, λ) where $\kappa > \lambda$. Let $N \prec M$ be such that $|N| < \kappa$. Then there exists N' such that $N \prec N' \prec M$, $|N'| = |N|$, and $P(N') = P(N)$.*

Proof. Let M and N be as given. We define an equivalence relation on $M \setminus N$ by,

$$m_1 \sim_N m_2 \iff m_1 \in \operatorname{acl}(Nm_2) \iff m_2 \in \operatorname{acl}(Nm_1).$$

Since each equivalence class has cardinality at most $|N|$, this relation partitions $M \setminus N$ into κ different pieces. Since there are only λ members of $P(M)$, some equivalence class does not contain members of $P(M)$.

Let A be some such equivalence class. Then $N \cup A$ is algebraically closed, and $P(N \cup A) \subset N$. Let $N[A] \prec M$ be prime over $N \cup A$. Clearly, $|N[A]| = |N|$, and by lemma 1, $P(N[A]) = P(N)$. Hence, $N[A]$ is the desired N'. □

Theorem 6 *Let M be a model of type (κ, λ) where $\kappa > \lambda$. Suppose that $\lambda' \leq \lambda$, $\kappa' \leq \kappa$, and $\omega \leq \lambda' \leq \kappa'$. Then there exists $N \prec M$ such that N is of type (κ', λ').*

Proof. Using the downward Löwenheim-Skolem theorem, we construct $M' \prec M$ such that $|M'| = |P(M')| = \lambda'$. By induction, we construct a strictly increasing sequence of models, $\langle N_i \,|\, i \leq \kappa' \rangle$, such that for every $i \leq \kappa'$, $N_i \prec M$, $P(N_i) = P(M')$, and $|N_i| = \lambda' + |i\,|$.

We begin by letting $N_0 = M'$. Given N_i where $i < \kappa'$, we apply lemma 5 to obtain N_{i+1}. Finally, for limit i, we let $N_i = \bigcup_{j<i} N_j$. At the end of the day, $|N_{\kappa'}| = \kappa'$ and $|P(N_{\kappa'})| = |P(M')| = \lambda'$ as desired. □

Remark: Note that this proof of theorem 6 allows us a great deal of freedom in choosing the "bottom" portion of our (κ', λ')-models. In particular, for any $M' \prec M$, there exist arbitrarily large N such that $M' \prec N \prec M$ and $P(N) = P(M')$.

4 δ-Cardinal Theorems

In this section we generalize theorems 4 and 6 to the δ-cardinal case. For notational convenience, we choose a fixed sequence of formulas over $\mathbb{M}$, $\langle \psi_i(x, \bar{m}_i) \,|\, i < \delta \rangle$, and we let $A \subset \mathbb{M}$ be minimal such that every ψ_i is also over A.

Theorem 7 *Suppose that for every $i < \delta$, there exists M_i such that $A \subset M_i$ and $|\psi_i(M_i)| > |\bigcup_{j<i} \psi_j(M_i)|$. Let $F : \delta \to CARD$ be increasing and such that $F(0) \geq |\delta| + \omega$. Then there exists a model M_F such that $A \subset M_F$ and for every $i < \delta$, $|\psi_i(M_F)| = F(i)$.*

Proof. We construct by induction a strictly increasing sequence of models, $\langle N_i \,|\, i < \delta \rangle$, such that:

1. for every i, $A \subset N_i$,
2. for every i, $|\psi_i(N_i)| = |N_i| = F(i)$, and
3. for every $j < i < \delta$, $\psi_j(N_i) = \psi_j(N_j)$.

At the end of the day, $N_\delta = \bigcup_{i<\delta} N_i$ will be the desired M_F.

For $i = 0$, we argue as in the proof of theorem 4. Using compactness, we obtain a sequence of pairs, $\langle M_i'' \prec M_i' \,|\, i < \delta\rangle$, such that for every $i, j < \delta$,

- $A \cup \bigcup_{j<i} \psi_j(M_i') \subset M_i''$
- $\psi_i(M_i') \not\subset M_i''$, and
- $|\psi_j(M_i')| = |M_i'| = F(0)$.

We then let N_0 be arbitrary such that $|N_0| = F(0)$ and $\bigcup_{i<\delta} M_i' \subset N_0$.

For $i > 0$, we assume that N_j has been defined for all $j < i$. By induction, we construct a strictly increasing sequence, $\langle N_k^i \,|\, k < F(i)\rangle$, such that for every $k < F(i)$ and $j < i$, $\psi_j(N_k^i) = \psi_j(N_j)$. We start by letting $N_0^i = \bigcup_{j<i} N_j$. Note that by our induction hypothesis on the N_j's (clause 3, in particular), $\psi_j(N_0^i) = \psi_j(N_j)$ as desired. Similarly, for k limit, we simply let $N_k^i = \bigcup_{l<k} N_l^i$.

For k a successor, we we choose $a \in \psi_i(M_i') \setminus M_i''$, we let $p \in S(N_{k-1}^i)$ be an heir of $\mathrm{tp}(a, M'')$, and we let $b \models p$. By lemma 2, $\psi_j(\mathrm{acl}(bN_{k-1}^i)) \subset N_{k-1}^i$ for every $j < i$. We let N_k^i be prime over $\mathrm{acl}(bN_{k-1}^i)$. Then by lemma 1, $\psi_j(N_k^i) \subset N_{k-1}^i$ for every $j < i$. So, by the induction hypothesis on N_{k-1}^i, $\psi_j(N_k^i) = \psi_j(N_j)$ as desired.

When the construction of $\langle N_k^i \,|\, k < F(i)\rangle$ is finished, we set $N_i = \bigcup_{k<F(i)} N_k^i$. Note that conditions 1 and 2 on our induction are satisfied trivially (as $A \subset N_0 \subset N_i$, and the N_k^i-construction added exactly $F(i)$ pieces of $\psi_i(\mathbb{M})$ to N_i). Condition 3 is satisfied because of the induction hypothesis on the subsidiary N_k^i-construction. □

Remark: Note that theorem 7 does not require us to start with an initial δ-cardinal model which witnesses all of the relevant cardinality splits simultaneously. We can start with a collection of models, each witnessing a different cardinality split, and then parlay these models into a single δ-cardinal model which exhibits some desired sequence of cardinality splits.

Theorem 8 *Let $M \supset A$ be such that for every $i < \delta$, $|\psi_i(M)| > |\bigcup_{j<i} \psi_j(M)|$. Let $F : \delta \to CARD$ be increasing and such that for every $i < \delta$, $\delta + \omega \le F(i) \le |\psi_i(M)|$. Then there exists $N_F \prec M$ such that $A \subset N_F$ and for every $i < \delta$, $|\psi_i(N_F)| = F(i)$.*

Proof. We construct by induction a strictly increasing sequence of models $\langle N_i \,|\, i < \delta\rangle$, such that:

1. for every i, $A \subset N_i \prec M$,
2. for every i, $|\psi_i(N_i)| = |N_i| = F(i)$, and
3. for every $j < i < \delta$, $\psi_j(N_i) = \psi_j(N_j)$.

At the end of the day, we simply let $N_F = \bigcup_{i<\delta} N_i$.

For $i = 0$, we apply the downward Löwenheim-Skolem theorem to obtain $N_0 \prec M$ such that $A \subset N_0$ and for every $i < \delta$, $|\psi_i(N_0)| = |N_0| = F(0)$. For $i > 0$, we assume that N_j has been defined for every $j < i$, and we let $\hat{N}_i = \bigcup_{j<i} N_j$. Note that for $j < i$, $\psi_j(\hat{N}_i) = \psi_j(N_j)$. Next, we let $B_i \subset \psi_i(M)$ be maximal such that $\hat{N}_i \cup B_i$ is algebraically closed and for every $j < i$, $\psi_j(\hat{N} \cup B_i) \subset \hat{N}_i$.

Claim: $|B_i| = |\psi_i(M)|$

Pf. Suppose $|B_i| < |\psi_i(M)|$. As in the proof of theorem 5, we partition $\psi_i(M) \setminus B_i$ via the equivalence relation: $a \sim b \iff a \in \mathrm{acl}(\hat{N}_i \cup B_i \cup \{b\})$. Since there are $|\psi_i(M)|$ equivalence classes and $< |\psi_i(M)|$ elements in $|\bigcup_{j<i} \psi_j(M)|$, some equivalence class does not intersect $|\bigcup_{j<i} \psi_j(M)|$. Adding this class to B_i, we contradict the maximality of B_i. $\square$ (claim)

Let $B_i' \subset B_i$ be such that $\hat{N}_i \cup B_i'$ is algebraically closed and $|B_i'| = F(i)$. Let $N_i \prec M$ be prime over $\hat{N}_i \cup B_i'$. Clearly, $A \subset N_i$ and $|\psi_i(N_i)| = |N_i| = F(i)$. Further, by lemma 1, $\psi_j(N_i) \subset \hat{N}_i$ for every $j < i$. Hence, by the construction of $\hat{N}_i$, $\psi_j(N_i) = \psi_j(N_j)$ as desired. $\square$

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