Multi-Cardinal Phenomena In Stable Theories

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1994

Contents

Abstract

In this dissertation we study two-cardinal phenomena—both of the admitting cardinals variety and of the Chang's Conjecture variety—under the assumption that all our models have stable theories. All our results involve two, relatively widely accepted generalizations of the traditional definitions in this area. First, we allow the relevant subsets of our models to be picked out by (perhaps infinitary) partial types; second we consider δ-cardinal problems as well as two-cardinal problems.

We begin by examining phenomena related to admitting cardinals, and we provide two separate methods of obtaining cardinal transfer results. Suppose first that we are working with a countable language, and that all the relevant subsets of our models are picked out by single predicates. In this context, it is well known that certain types of two-cardinal phenomena are essentially equivalent to a strong type of forking independence (Hrushovski's foreignness). We generalize this result to obtain the following:

Theorem: *Let $N \prec M$, let $\Gamma(x)$ be a partial type over N such that $\Gamma(M) = \Gamma(N)$. Let $a \in M \setminus N$, let $p = tp(a, N)$, and suppose that one of the following conditions holds:*

1. *M and N are λ-compact and $|\Gamma| < \lambda$.*
2. *M and N are $\mathbf{F}^a_\lambda$-saturated for some $\lambda \geq \kappa(T)$ and Γ is over some A such that $|A| < \lambda$.*

Then p is foreign to Γ.

Exploiting this theorem allows us to prove an initial collection of δ-cardinal transfer results. Roughly, we use the initial existence of a collection of multi-cardinal models to obtain a sequence of types foreign to the ∞-definable sets whose sizes we wish to manipulate. We then obtain new multi-cardinal models by building appropriately sized Morley sequences for these types and closing under some prime model constructions.

Next, we show that a second type of two-cardinal phenomenon implies the existence of large sets which are indiscernible over specified parts of our models—i.e. those parts whose sizes we wish to manipulate. By stretching and/or shrinking these sets of indiscernibles, and then applying some prime model theory, we obtain a second collection of δ-cardinal transfer results.

Finally, we consider some variants of Chang's Conjecture in the context of stable theories. Many of the basic prime model constructions in this part of the dissertation are variants of the constructions used in the admitting cardinals sections, relativized to work inside particular models. A central technical problem involves splitting a model into various "large" sets which have relatively little to do with certain specified

smaller sets. To solve this problem, we examine various methods for splitting up stable models; we obtain the following result:

Theorem: *Let A and B be arbitrary subsets of $\mathbb{M}$. Then we can partition A into $|B|^{<\kappa_r(T)}$ pieces $\langle A_i \mid i < |B|^{<\kappa_r(T)} \rangle$, such that for each A_i there is a $B_i \subseteq B$ where $|B_i| < \kappa_r(T)$ and $A_i \downarrow_{B_i} B$.*

Using this result, we prove a series of δ-cardinal variants of Chang's Conjecture.

In the course of our analysis, we consider the degree to which the result mentioned above provides optimal methods for building "large" sets which are independent from certain specified smaller sets. We show that in many cases—in particular, in cases in which the sets in question are contained within models of superstable or totally transcendental theories—the result is optimal. In the general case in which the sets in question are contained within models of arbitrary stable theories, the result is very close to optimal. In these cases, we investigate the result's optimality, both under the assumption that GCH holds and under the assumption that the universe is "suitably generic" over some inner model in which GCH holds (though GCH and even SCH may fail in the generic extension).

Chapter 1

Classical Multi-Cardinal Phenomena

In this chapter, we develop a portion of the classical theory of multi-cardinal phenomena. Everything in the chapter is either known or is an easy generalization of known results. The chapter introduces several notational devices which are useful in later chapters; otherwise, notation throughout the dissertation is standard (see section 1.2 for more on notation). Throughout the dissertation, we use basic facts about model theory, including the theory of forking, freely. Facts unrelated to forking can be found in [CK] or [H]; facts related to forking can be found in [Ba], [Sh] or [Mk].

1.1 Introduction

The Upward Löwenheim-Skolem theorem says that every infinite model for a language $\mathfrak{L}$ has an elementarily equivalent model in every cardinality greater than $|\mathfrak{L}|$. The Downward Löwenheim-Skolem theorem says that every infinite model M has an elementary submodel N in every cardinality such that $|\mathfrak{L}| \leq |N| \leq |M|$. These theorems involve the cardinalities of a single definable subset of a given model (i.e. that defined by "x=x"): they say that first order logic cannot pin down such cardinalities.

It is natural to conjecture that these theorems generalize from one definable subset of a model to two or more such subsets. To formulate such conjectures, we employ a notation introduced by Vaught in [V]. Let M be a model, and let $P_0, \ldots, P_n$ be a series of predicates such that $P_0(M) \supseteq P_1(M) \supseteq \ldots \supseteq P_n(M)$. We say that M is of **type** $(\kappa\,; \lambda_0, \ldots, \lambda_n)$ if $|M| = \kappa$ and for each $i < n$, $|P_i(M)| = \lambda_i$. We say that a theory T **admits** $(\kappa\,; \lambda_0, \ldots, \lambda_n)$ if there is a model M such that M is of type $(\kappa\,; \lambda_0, \ldots, \lambda_n)$ and $M \models T$.

Using this notation, the two cardinal version of the Upward Löwenheim-Skolem theorem says that if $\kappa > \lambda \geq |\mathfrak{L}|$, then any theory which admits $(\kappa\,; \lambda)$, admits every $(\kappa'; \lambda')$. The two cardinal version of the Downward Löwenheim-Skolem theorem says that if $\kappa > \lambda$, $\kappa' \geq \lambda'$, $\kappa \geq \kappa'$ and $\lambda \geq \lambda' \geq |\mathfrak{L}|$, then any M of type $(\kappa\,; \lambda)$ has an elementary submodel of type $(\kappa'; \lambda')$. Natural generalizations to three or more cardinals can be formed in the obvious way.

Unfortunately, all of these natural generalizations are false. Robinson has produced a countable theory which admits $(\kappa\,;\lambda)$ exactly when $\kappa \leq 2^{\lambda}$. This is clearly enough to refute all of the generalizations in question. Further, many special cases of the downwards two-cardinal conjecture appear to be independent of ZFC. For the moment, let us consider only countable languages. Then we say that **Chang's Conjecture** holds between $(\kappa\,;\lambda)$ and $(\kappa'; \lambda')$ just in case any model of type $(\kappa\,;\lambda)$ has an elementary submodel of type $(\kappa'; \lambda')$. Modulo the consistency of some lagre cardinal hypotheses, the following facts prove the independence of certain instances of Chang's conjecture:

- (Vaught) If V=L, then Chang's Conjecture fails between $(\omega_2;\omega_1)$ and $(\omega_1;\omega)$ and between $(\omega_3;\omega_2)$ and $(\omega_2;\omega_1)$.
- (Silver) Con(ZFC + 'there exists a Ramsey Cardinal') $\implies$ Con(ZFC + 'Chang's Conjecture holds between $(\omega_2;\omega_1)$ and $(\omega_1;\omega)$').
- (Koepke) If Chang's Conjecture holds between $(\omega_3;\omega_2)$ and $(\omega_2;\omega_1)$, then there is an inner model with a measurable cardinal.

It is important to note that the models used in getting such independence results tend to be models of ZFC, or at least of certain fragments of ZFC (so instead of using the full strength of Chang's Conjecture—e.g., "for *every* model of type $(\kappa;\lambda)$"—we focus on particular transitive sets which reflect some portion of $\mathrm{Th}(V;\in)$, where V is the universe of sets). Even Robinson's result involves coding the axiom of extensionality into the theory in question.

All of this suggests that those of us model theorists who are interested in multi-cardinal phenomena, but are perhaps less interested in the model theory of ZFC, should restrict the classes of models at which we look. If we "toss out" particularly hard models, like those of set theory, we might find some interesting multi-cardinal phenomena in the models left over. And these phenomena might be amenable to model theoretic, as opposed to set theoretic, investigation.

In this dissertation, we investigate multi-cardinal phenomena in models whose theories are stable. There are two reasons for focusing on this class of models. First, it is known that stable theories have quite a few multi-cardinal models. Let us say that a formula $\phi(x;\bar{a})$ is a **two-cardinal formula for T** if there is a model $M \models T$ such that $\bar{a} \in M$ and $\omega \leq |\phi(M;\bar{a})| < |M|$. Then we know the following:

- (Shelah) If T is ω-stable but not uncountably catagorical, then T has a two-cardinal formula.
- (Hrushovski) If T is countable, superstable and has the finite cover property, then T has a two-cardinal formula.
- (Hrushovski) If T is stable and has the finite cover property, then T^{eq} has a two-cardinal formula.

These results show that stable theories provide a rich setting for investigating multi-cardinal phenomena (though having shown this, we shall proceed to develop techniques for investigating such phenomena which have nothing to do with the details of the above results).

Second, focusing on stable theories allows us to make use of the theory of forking, and this theory provides powerful tools for investigating the phenomena indicated by the above results. As we will see, the notions of independence investigated in the theory of forking are closely related to the kinds of independence involved in multi-cardinal phenomena—that is, the kinds of independence which allow us to manipulate the size of some parts of a model while leaving other parts of the model fixed (see section 2.1 for more detailed discussion of the interaction between these two kinds of independence). This makes the class of stable theories a natural domain in which to examine multi-cardinal phenomena.

1.2 Notation & Conventions

Throughout the dissertation, we work with a fixed theory T in a fixed language $\mathfrak{L}$. T is complete, and we assume that all models are models of T (so, all models have signature $\mathfrak{L}$). We also assume the existence of a "monster model" $\mathbb{M}$ for T: $\mathbb{M}$ is a saturated model in some cardinality larger than any of the cardinals in which we are interested (or, even just a λ-big model for some sufficiently large λ, see [H]). Since $\mathbb{M}$ is saturated, it is both universal and strongly homogenous. So, we can regard all smaller models as elementary submodels of $\mathbb{M}$, and we can regard maps between smaller models as restrictions of automorphisms of $\mathbb{M}$.

For the most part, our arguments do not require any of the additional structure afforded by expanding our theory and working in $\mathbb{M}^{''}$. But at several key points this style of argument becomes crucial—see especially our use of $\mathbf{F^{na}}$ in chapters two and three. Since this expansion does not adversely affect any of the properties we are interested in—stability, rank, cardinality of models, the isolation relations associated with prime model theory, etc.—we find it convienient to assume that all of our arguments take place in $\mathbb{M}^{''}$. As this assumption is pervasive, we abuse notation and write "$\mathbb{M}$" for $\mathbb{M}^{''}$, "T" for T^{eq}, and "$\mathfrak{L}$" for $\mathfrak{L}^{\mathbf{eq}}$. For further information on $\mathbb{M}^{''}$, see [Sh, III §6] or [Mk, Part B].

Throughout the dissertation, $M, N, \ldots$ denote models and $A, B, \ldots$ denote subsets of models. If M is a model, then we use "M" also to denote the domain of M. Finite tuples are denoted by $\bar{a}, \bar{b}, \bar{c}, \ldots$, and we use $\bar{a} \in A$ to mean that every element of $\bar{a}$ is a member of A. We use $\alpha, \beta, \gamma, \ldots$ to denote ordinals; $\kappa, \lambda, \mu, \ldots$ to denote infinite cardinals; m and n to denote natural numbers; and i, j, k and l to denote either ordinals or natural numbers depending on the context.

Our notation concening stability theory follows that of [Mk]. In particular, we use $a \downarrow_A b$ to mean that $tp(a, A \cup b)$ does not fork over A, and we use $A \rhd_C B$ to mean that A dominates B over C. $S_n(A)$ will denote the collection of types over A in variables $x_1, \ldots, x_n$, and $Aut_A(\mathbb{M})$ will denote the automorphisms of $\mathbb{M}$ which fix A pointwise. Similarly, $S^*_n(A)$ will denote the collection of strong types over A in variables $x_1, \ldots, x_n$, and $Aut^*_A(\mathbb{M})$ will denote the set of automorphisms of $\mathbb{M}$ which fix strong types over A (see [Sh, III §2] or [Mk, Part 2] for information on strong types). Finally, we follow Shelah in using $\lambda(T)$ to denote the first cardinal in which T is stable (so always $\lambda(T) \leq 2^{|T|}$).

Often, we are concerned with our ability to define various sets inside of $\mathbb{M}$. We will say that a set is definable if it is of the form $\psi(\mathbb{M})$ for some ψ having parameters in $\mathbb{M}$, and we will say that a set is ∞-definable if it is an intersection of $< |\mathbb{M}|$ definable sets. A set is definable (∞-definable) over A if all the parameters needed to define it come from A. A subset A of M is definable in M if there is a definable A' such that $A = M \cap A'$ and all the parameters needed to define A' come from M (similarly for ∞-definable). Typically, the "in M" part of this notation will be clear from context and will be omitted.

In several parts of the dissertation, we will need to work with various sequences of quasi-indiscernibles:

Definition 1.2.1 *Let $\langle I_i \mid i < \delta \rangle$ be a sequence of linearly ordered sets. We say that $\langle I_i \mid i < \delta \rangle$ is a sequence of $\underline{(n,m)\text{-indiscernibles}}$ iff for every strictly increasing $f : m \to \delta$ and every $\phi(\bar{x}_1, \ldots, \bar{x}_m)$ a formula taking n-tuples in each $\bar{x}_i$ spot, and every $\bar{a}_1, \ldots, \bar{a}_m$ and $\bar{b}_1, \ldots, \bar{b}_m$ sequences of n-tuples, if $\bar{a}_i$ and $\bar{b}_i$ come (elementwise) from $I_{f(i)}$ and are increasing in the order on $I_{f(i)}$, then $\mathbb{M} \models \phi(\bar{a}_1, \ldots, \bar{a}_m)$ iff $\mathbb{M} \models \phi(\bar{b}_1, \ldots, \bar{b}_m)$.*

If the ϕ's in this definition are allowed to have parameters from A, then we say that $\langle I_i \mid i < \delta \rangle$ is a sequence of (n,m)-indiscernibles over A. If $\langle I_i \mid i < \delta \rangle$ is a sequence of (n,m)-indiscernibles (over A) for every n and m, then we say $\langle I_i \mid i < \delta \rangle$ is a sequence of (ω,ω)-indiscernibles (over A).

Let $\langle I_i \mid i < \delta \rangle$ be a sequence of (n,m)-indiscernibles. For each $i < \delta$, let $\langle a^i_j \mid j < \omega \rangle$ be an increasing sequence of elements from I_i. Let Δ be the (infinitary) type of $\{a^i_j \mid j < \omega,\ i < \delta\}$ in free variables $\{x^i_j \mid j < \omega,\ i < \delta\}$. The **$(\mathbf{n},\mathbf{m})$-type** of $\langle I_i \mid i < \delta \rangle$ is the set of all $\phi \in \Delta$ such that there are at most m superscripts on the x^i_j represented in ϕ and such that for fixed i, there are at most n subscripts on the x^i_j represented in ϕ. Note that this definition is independent of our initial choice of the a^i_j, but it does depend on viewing $\langle I_i \mid i < \delta \rangle$ as a set of (n,m)-indiscernibles. When there is no ambiguity, we will often drop the "(n,m)" and talk simply of the type of $\langle I_i \mid i < \delta \rangle$

Vaught's notion of a model's being of type $(\kappa\,;\lambda_0, \ldots, \lambda_n)$ has two drawbacks. First, it only makes sense for finite sequences of cardinals (since the cardinals of a type are arranged in decreasing order); second, it assumes that the relevant subsets of a model M are picked out by predicates (rather than being, say, ∞-definable over M). To deal with these drawbacks, we modify Vaught's definition somewhat. Throughout the dissertation, we let $\langle \mathbb{A}_i \mid i < \delta_0 \rangle$ be a fixed collection of subsets of $\mathbb{M}$. Then:

Definition 1.2.2 *A model M is of $\underline{\text{type}}$ $(\kappa\,;\lambda_0, \ldots, \lambda_i, \ldots)_{i<\delta_0}$, if $|M| = \kappa$ and for every $i < \delta_0$, $|M \cap \mathbb{A}_i| = \lambda_i$. M is an $\underline{F\text{-model}}$ if $F : \delta_0 \to CARD$ and the type of M is $(\kappa; F(0), \ldots, F(i), \ldots)_{i<\delta_0}$ for some κ.*

So, we do not require that there be any relations between the sets $M \cap \mathbb{A}_i$; and the only restriction on the cardinals in a type $(\kappa\,;\lambda_0, \ldots, \lambda_i, \ldots)_{i<\delta_0}$ is the implicit one: for all i, $\kappa \geq \lambda_i$. Nor do we require that the $\mathbb{A}_i$ be definable (or even ∞-definable). However, if the $\mathbb{A}'_i s$ *are* supposed to be definable, or ∞-definable, then we will usually insist that they be definable using only parameters from M.

1.3 An Extension of Vaught's Theorem

In [V 65], Vaught proved that any theory in a countable language which admits some $(\beth_\omega(\kappa);\kappa)$ admits every $(\chi;\lambda)$. In this section, we generalize Vaught's theorem in three ways: we examine the situation for uncountable languages, we examine what happens when the subsets of our models are ∞-definable (rather than simply being picked out by predicates in $\mathfrak{L}$), and we examine some extensions Vaught's theorem from the two-cardinal case to the δ-cardinal case.

We begin by fixing some notation (to be used only in this section). Let κ be the cardinality of $\mathfrak{L}$, let $\langle \Gamma_i(x) \mid i < \delta \rangle$ be a fixed sequence of (perhaps partial) types over $\emptyset$, and let M be a fixed model. We say $\langle I_i \mid i < \delta \rangle$ is a set of (n, m, Γ)-indiscernibles if

- Each I_i is a linearly ordered subset of $\Gamma_i(M)$ where $\Gamma_i(M) \equiv_{df} \{m \in M : \text{for every } \phi(x) \in \Gamma, \mathbb{M} \models \phi(>)\}$.
- For every j, $\langle I_i \mid j \leq i < \delta \rangle$ is a set of (n, m)-indiscernibles over $\bigcup_{k<j} \Gamma_k(M)$.

When necessary, we replace $\mathfrak{L}$ by some Skolem expansion of $\mathfrak{L}$; at such points, we assume that M remains a model for the expanded language.

A key tool in the proof of Vaught's theorem is a result on partitions which was proved by Erdös and Rado. To state this result, we need to introduce some more notation. We will write,

$$\lambda \longrightarrow (\mu)^n_\nu \tag{1.1}$$

to mean that if X is a linearly ordered set of cardinality λ and $[X]^n$ is partitioned into ν pieces, then there exists $Y \subset X$ such that $|Y| = \mu$ and all the (increasing) n-tuples from Y are in the same piece of the given partition. This notation has the property that whenever a fact of the form 1.1 holds, it will still hold when λ is made larger and/or when μ, ν and n are made smaller.

Given this notation, the original version of the Erdös-Rado theorem can be written as

$$(\beth_{n-1}(\lambda))^+ \longrightarrow (\lambda^+)^n_\lambda \tag{1.2}$$

For our purposes in this section, it is useful to modify this version of the theorem a little. By replacing λ in the original theorem with $\beth_\alpha(\lambda)$ and then manipulating the cardinals on both sides of the arrow (making those on the left larger, and those on the right smaller) we obtain.

$$\beth_{\alpha+n}(\lambda) \longrightarrow (\beth_\alpha(\lambda))^n_{2^\lambda} \tag{1.3}$$

Note the immediate relevance to model theory. If A is a set of size $\lambda \geq |\mathfrak{L}|$, then there are at most 2^λ n-types over A. This induces a natural partition on $[X]^n$ for any $X \supset A$. So, if $|X| \geq \beth_{\alpha+n}(\lambda)$, we can find a $Y \subset X$ such that $|Y| = \beth_\alpha(\lambda)$ and Y is n-indiscernible over A. The next lemma extends this idea to find (n, m, Γ)-indiscernibles.

Lemma 1.3.1 *Let $|\Gamma_0(M)| > \beth_{\alpha+nm}(\kappa)$ and $|\Gamma_i(M)| > \beth_{\alpha+nm}(\sup_{j<i}(|\Gamma_j(M)|))$ for $0 < i < \delta$. Then we can find a sequence of (n, m, Γ)-indiscernibles $\langle I_i \mid i < \delta\rangle$ such that $|I_0| \geq \beth_\alpha(\kappa)$ and for $i > 0$, $|I_i| \geq \beth_\alpha(\sup_{j<i}(|\Gamma_j(M)|))$.*

Remark: If desired, we can make each I_i live in some specified subset of $\Gamma_i(M)$, providing that this subset has cardinality greater than $\beth_{\alpha+nm}(\sup_{j<i}(|\Gamma_j(M)|))$).

Proof of Lemma. The proof is by induction on m keeping n fixed but letting α be variable. Let $m = 1$ and let α be such that $|\Gamma_0(M)| > \beth_{\alpha+n}(\kappa)$ and $|\Gamma_i(M)| > \beth_{\alpha+n}(\sup_{j<i}(|\Gamma_j(M)|))$. Then we apply the Erdös-Rado theorem as above to find in each $\Gamma_i(M)$ a set of n-indiscernibles over $\bigcup_{j<i}(\Gamma_j(M))$ of size $\beth_\alpha(\sup_{j<i}(|\Gamma_j(M)|))$ (for $i = 0$, we get cardinality $\beth_\alpha(\kappa)$).

So let $m > 1$ and let α be such that the hypothesis of the lemma hold. By the induction hypothesis for $\alpha + n$ in place of α, we can find $(n, m-1, \Gamma)$-indiscernibles $\langle I_i \mid i < \delta\rangle$ such that $|I_0| \geq \beth_{\alpha+n}(\kappa)$ and for $i > 0$, $|I_i| \geq \beth_{\alpha+n}(\sup_{j<i}(|\Gamma_j(M)|))$. For each i, let J_i be an (ordered) n-tuple from I_i; and let $J = \bigcup_{i<\delta} J_i$. Applying the Erdös-Rado theorem again, we can find within each I_i an ${I_i}'$ such that $|{I_i}'| \geq \beth_\alpha(\sup_{j<i}(|\Gamma_j(M)|))$ and ${I_i}'$ is n-indiscernible over $J \cup \bigcup_{j<i}(\Gamma_j(M))$ (for $i = 0$, we get $|I_0'| \geq \beth_\alpha(\kappa)$ and I_0' n-indiscernible over J).

Now I claim that $\langle {I_i}' \mid i < \delta\rangle$ is the desired sequence of (n, m, Γ)-indiscernibles. For let $\bar{a}_1, \ldots, \bar{a}_m$ and $\bar{b}_1, \ldots, \bar{b}_m$ be sequences of n-tuples such that for some strictly increasing $f : m \to \delta$, $\bar{a}_i$ and $\bar{b}_i$ come from $I'_{f(i)}$ and are increasing in the order on $I'_{f(i)}$. Let $\phi(\bar{x}_1, \ldots, \bar{x}_m)$ be a formula with parameters from $\bigcup_{j<f(1)} \Gamma_j(M)$ which takes n-tuples in each $\bar{x}_i$ spot. Then we have,

$$M \models \phi(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_m) \iff M \models \phi(\bar{a}_1, \bar{J}_{f(2)} \ldots, \bar{J}_{f(m)}) \tag{1.4}$$

$$\iff M \models \phi(\bar{b}_1, \bar{J}_{f(2)} \ldots, \bar{J}_{f(m)}) \tag{1.5}$$

$$\iff M \models \phi(\bar{b}_1, \bar{b}_2 \ldots, \bar{b}_m) \tag{1.6}$$

Here, (1.4) and (1.6) follow from the fact that $\langle I_i \mid f(1) < i < \delta\rangle$ is a set of $(n, m-1)$-indiscernible over $\bigcup_{j \leq f(1)} \Gamma_j(M)$; (1.5) follows from the fact that $I'_{f(1)}$ is n-indiscernible over $J \cup \bigcup_{j<f(1)}(\Gamma_j(M))$. $\square$

Ideally, we would like to extend this this lemma to generate a large set of (ω, ω, Γ)-indiscernibles in M. If $\mathfrak{L}$ were skolemized, this would be enough to get the multi-cardinal results we are after: by stretching (and/or shrinking) these sets of indiscernibles and then closing under Skolem functions we could build models of arbitrary type.

Unfortunately, this does not always work: no matter how large the "splits" between the cardinalities of the Γ_i's get, we may be unable to ensure the existence of (ω, ω, Γ)-indiscernibles in M. We can, however, use the techniques of our lemma to finitely approximate such a set of indiscernibles. If we are careful, this will be sufficient for proving our multi-cardinal results.

Lemma 1.3.2 *Let $\mu = (\beth(\kappa + |\delta|))^+$ and let $|\Gamma_i(M)| > \beth_\mu(\sup_{j<i}(|\Gamma_j(M)|))$ for all i. Then there exists a set of (ω, ω)-indiscernibles $\langle I_i \mid i < \delta\rangle$ such that each I_i is a subset of $\Gamma_i(\mathbb{M})$ and*

(†) *Whenever f is a function in $\mathfrak{L}$, $\bar{a}$ and $\bar{a}'$ are "matching" tuples from $\bigcup_{j<i<\delta} I_i$, and $\bar{b}$ is a tuple from $\bigcup_{k\leq j} I_k$:* $\mathbb{M} \models \Gamma_j(f(\bar{b},\bar{a})) \implies \mathbb{M} \models f(\bar{b},\bar{a}) = f(\bar{b},\bar{a}')$.

Proof. We begin by constructing a sequence of approximations to the desired set of indiscernibles (or, more precisely, a sequence of sequences of aproximations to this set). We build $\langle \langle \langle I_i \mid i < \delta \rangle_j \mid j < \mu \rangle_k \mid 1 \leq k < \omega \rangle$, satisfying the following conditions:

1. for every i, j, k, $|I_{i,j,k}| \geq \beth_j(\sup_{l<i}(|\Gamma_l(M)|))$.

2. for every j, k, $\langle I_i \mid i < \delta \rangle_{j,k}$ is a set of (k, k, Γ)-indiscernibles.

3. for fixed k, all of the $\langle I_i \mid i < \delta \rangle_{j,k}$ share a common type.

4. for $k < k'$, the type of $\langle I_i \mid i < \delta \rangle_{1,k'}$ extends that of $\langle I_i \mid i < \delta \rangle_{1,k}$.

Construction: (by induction on k) Let $k = 1$ and let $j < \mu$. By the last lemma, we can find a set of $(1, 1, \Gamma)$-indiscernibles $\langle I_i \mid i < \delta \rangle_j$ such that for every i, $|I_i| \geq \beth_j(\sup_{l<i}(|\Gamma_l(M)|))$ (note that $j < \mu \Rightarrow j + 1 < \mu$).

As there are at most $\beth(\kappa + |\delta|)$ different types for $(1, 1)$-indiscernibles, we can find a sequence of j's which is cofinal in μ such that the associated $\langle I_i \mid i < \delta \rangle_j$ all share the same type. Relabeling this sequence along μ (instead of some cofinal subset of μ) we get the desired $\langle \langle I_i \mid i < \delta \rangle_j \mid j < \mu \rangle_1$.

So let $k > 1$. Again, let $j < \mu$. By the induction hypothesis, we know that each $I_{i,j+kk+1,k-1}$ has cardinality $\geq \beth_{j+kk+1}(\sup_{l<i}(|\Gamma_l(M)|))$. So, by our last lemma (and the remarks following that lemma), we can find a set of (k, k, Γ)-indiscernibles $\langle I_i \mid i < \delta \rangle_j$ such that for every i, $|I_i| \geq \beth_j(\sup_{l<i}(|\Gamma_l(M)|))$ and $I_i \subset I_{i,j+kk+1,k-1}$.

Again, there are at most $\beth(\kappa + |\delta|)$ different types for (k, k)-indiscernibles, so from here our argument proceeds as above. Note that conditions 1–3 are preserved explicitly here; and by choosing $I_i \subset I_{i,j+kk+1,k-1}$, we ensure that condition 4 is preserved as well.

For every $k < \omega$, view $\langle I_i \mid i < \delta \rangle_{1,k}$ as a set of (k, k)-indiscernibles and let Δ_k be its (k, k)-type. Let $\Delta = \bigcup_{k<\omega} \Delta_k$. Clearly, Δ is a consistent set of formulas. Let $\langle I_i \mid i < \delta \rangle$ realize Δ (where each I_i is of the form $\langle c^i_j \mid 1 \leq j < \omega \rangle$).

Claim: *$\langle I_i \mid i < \delta \rangle$ is the desired set of (ω, ω)-indiscernibles.*

Proof of Claim. For each k, the fact that $\Delta \supset \Delta_k$ ensures that $\langle I_i \mid i < \delta \rangle$ is a set of (k, k)-indiscernibles; since this holds for every $k < \omega$, $\langle I_i \mid i < \delta \rangle$ is a set of (ω, ω)-indiscernibles. Further, since every element in I_i has the same type as the elements of $I_{i,1,1}$, I_i is a subset of $\Gamma_i(\mathbb{M})$.

So we only need to ensure that $\langle I_i \mid i < \delta \rangle$ satisfies (†). Let f be a function in $\mathfrak{L}$, let $\bar{a}$ and $\bar{a}'$ be "matching" n-tuples from $\bigcup_{j<i<\delta} I_i$, and let $\bar{b}$ be an m-tuple from $\bigcup_{k\leq j} I_k$. Suppose, towards a contradiction, that $\mathbb{M} \models \Gamma_j(f(\bar{b},\bar{a})) \cup \{f(\bar{b},\bar{a}) \neq f(\bar{b},\bar{a}')\}$. Let $\bar{c}, \bar{c}', \bar{d}$ be tuples from $\langle I_i \mid i < \delta \rangle_{2,2n+m}$ which "match" $\bar{a}, \bar{a}', \bar{b}$ (vis-a-vis all the relevant orders). Since $\Delta \supset \Delta_{2n+m}$, $M \models \Gamma_j(f(\bar{d},\bar{c})) \cup \{f(\bar{d},\bar{c}) \neq f(\bar{d},\bar{c}')\}$. Because $\langle I_i \mid i < \delta \rangle_{2,2n+m}$ is a set of $(2n + m, 2n + m, \Gamma)$-indiscernibles, and because there are at least $\beth_2(|\Gamma_j(M)|)$

distinct sequences in $\langle I_i \mid i < \delta \rangle_{2,2n+m}$ which "match" $\bar{c}$, we generate $\beth_2(|\Gamma_j(M)|)$ distinct elements of $\Gamma_j(M)$. Since this is a contradiction, (†) must be satisfied. □ (claim, lemma)

Theorem 1.3.3 *Let $\mu = (\beth(\kappa + |\delta|))^+$ and let $|\Gamma_i(M)| > \beth_\mu(\sup_{j<i}(|\Gamma_j(M)|))$ for all i. Then for any nondecreasing $f : \delta \to CARD$ such that $\kappa + |\delta| \leq f(0)$, there exists a model N such that for every $i < \delta$, $|\Gamma_i(N)| = f(i)$.*

Proof. W.L.O.G. we may assume that $\mathfrak{L}$ is completely skolemized (so existential formulas are witnessed by Skolem functions, not just by Skolem terms). If $\mathfrak{L}$ is not completely skolemized, we simply replace $\mathfrak{L}$ by a complete Skolem expansion which preserves M as a model; this expansion does not change the cardinality of $\mathfrak{L}$. Let $\langle I_i \mid i < \delta \rangle$ be a set of (ω, ω)-indiscernibles as from the last lemma. By compactness, we can replace $\langle I_i \mid i < \delta \rangle$ with a sequence $\langle J_i \mid i < \delta \rangle$ such that $\langle J_i \mid i < \delta \rangle$ has the same type as $\langle I_i \mid i < \delta \rangle$ and for every i, $|J_i| = f(i)$.

Let N be the model generated by closing $\bigcup_{i<\delta} J_i$ under Skolem functions. Then I claim that N is the desired model. For let $i < \delta$. As $J_i \subset N \cap \Gamma(\mathbb{M})$, we have $|\Gamma(N)| \geq f(i)$. Further, every element in $\Gamma(N)$ is of the form $F(\bar{a}, \bar{b})$ where $\bar{a} \in \bigcup_{i<j<\delta} J_j$and $\bar{b} \in \bigcup_{j \leq i} J_j$. By our choice of $\langle J_i \mid i < \delta \rangle$, this form is independent of the choice of $\bar{a}$, at least up to a matching in all the relevant orders. But, there are only $|\delta|$ different types for $\bar{a}$ vis-a-vis these orders. So, there are at most $\kappa \cdot |\delta| \cdot (\sup_{j \leq i} f(j))$ distinct elements in $\Gamma(N)$. As $\kappa \cdot |\delta| \cdot (\sup_{j \leq i} f(j)) \leq f(i)$, we are done. □

So, when we are dealing with ∞-definable sets, it does not matter how many sets are in question. As long as we can find a model where the cardinality differences between these sets is large enough, we can find a model of any type we desire. To summarize this section, then, and to make all the accumulated background assumptions explicit, we give the following:

Corollary 1.3.4 *Suppose that each $\mathbb{A}_i$ in $\langle \mathbb{A}_i \mid i < \delta_0 \rangle$ is ∞-definable over $\emptyset$. Let $\mu = (|\mathfrak{L}| + |\delta_0|)^+$. Suppose that there exists an F-model for some F such that for every $i < \delta_0$, $F(i) > \beth_\mu(\sup_{j<i}(F(j)))$. Then for any nondecreasing $F' : \delta_0 \to CARD$ such that $|\mathfrak{L}| + |\delta_0| \leq F'(0)$, there will also be an F' model.*

Chapter 2

Admitting Cardinals in Stable Theories

In [Sh69], Shelah proved that any stable theory which admits some $(\kappa;\lambda)$ where $\kappa > \lambda$ admits every $(\kappa';\lambda')$ for $\kappa' \geq \lambda' \geq |\mathfrak{L}|$ (in Shelah's result, λ and λ' measure subsets which are picked out by some unary predicate P). Later, Forrest proved an n-cardinal version of this result under the (significantly stronger) assumption of ω-stability, see [F].

In [La], Lachlan proved that for countable theories we can get the following strengthening of Shelah's result: if $P(M) \subset N \precneqq M$, then there exists M' such that $M \precneqq M'$ and $P(M') \subset N$. Harnik showed that Lachlan's theorem can be generalized to the case in which M and N are λ-compact for some $\lambda \geq \mu(T)$ and P is replaced by some type $\Gamma(x)$ of cardinality $< \lambda$; here, $\mathfrak{L}$ is allowed to be uncountable, but M is not allowed to be an arbitrary model (even when Γ is finite), see [Ha].

Each of these results exploits two features of stable theories: the presence of a nice notion of independence (non-forking) and the existence of a nice theory of prime models (which varies from proof to proof). The strategy in this chapter is to isolate the distinct roles these two features play in explaining multi-cardinal phenomena. In section 2.1, we show that one class of multi-cardinal phenomena is essentially equivalent to a certain (quite strong) variety of forking independence. This result leads to several generalizations of the results mentioned above. Most notably, it facilitates extensions from the two-cardinal case to the δ-cardinal case for arbitrary δ. Also, it permits some weakening of the hypotheses traditionally used in two-cardinal theorems (see the remarks following theorem 2.3.3).

In 2.4, we distinguish a second class of multi-cardinal phenomena. This class of phenomena has relatively little to do with forking independence and is best analyzed by looking at the details of certain "prime model" constructions. Recognizing this allows us to prove a second δ-cardinal theorem for infinite sets of formulas.

Throughout this chapter, T is assumed to be stable.

2.1 Two-Cardinal Models and Foreign Types

In this section, we examine the relationship between forking independence and the independence involved in multi-cardinal phenomena. In particular, we show that the existence of a certain kind of multi-cardinal phenomena is equivalent to the existence of a certain kind of forking independence. The following definition, due to Hrushovski, is central:

Definition 2.1.1 *Let $\Gamma(\bar{x})$ be a partial type over A and let $p \in S(B)$. We say that p is foreign to $\Gamma(x)$ if for every $A' \supset A \cup B$ and every $p' \in S(A')$ a non-forking extension of p, if $\mathbb{M} \models p'(a)$ and $\mathbb{M} \models \Gamma(\bar{c})$, then $a \downarrow_{A'} \bar{c}$.*

Remarks: When working with foreign types, the following facts are quite useful. We state them without proof, as their proofs are simple exercises in manipulating the non-forking relation.

1. Let p be foreign to Γ. If p' is a non-forking extension of p, p' is foreign to Γ. If p does not fork over A and $p \upharpoonright A$ is stationary, then $p \upharpoonright A$ is foreign to Γ.
2. Let p be stationary and foreign to Γ. Then for any α, $p^{(\alpha)}$ is foreign to Γ.
3. Let p be foreign to $\Gamma_i(x)$ for $i < n$. Let $\Gamma(x_1, \ldots, x_n) = \Gamma_{i_1}(x_1) \cup \ldots \cup \Gamma_{i_n}(x_n)$. Then p is foreign to Γ.

Our next goal is to prove that the existence of multi-cardinal phenomena can entail the existence of (certain instances of) foreignness. We prove this through a series of lemmas.

Lemma 2.1.2 *Let M be a model, let $\Gamma(x)$ be a partial type over M, and let $a \in \mathbb{M}$ be such that $a \models \Gamma$. Suppose that one of the following conditions holds:*

1. *M is λ-compact and $|\Gamma| < \lambda$.*
2. *M is λ-saturated and Γ is over some $A \subset M$ such that $|A| < \lambda$.*

Then $a \downarrow_{\Gamma(M)} M$.

Proof. Let $\theta(x, \overline{m}) \in tp(a, M)$. As $\Gamma(x) \cup \{\theta(x, \overline{m})\}$ is consistent, it must be realized in M (either because $|\Gamma(x) \cup \{\theta(x, \overline{m})\}| < \lambda$, or because $|A \cup \{\overline{m}\}| < \lambda$). Further, it must be realized by some $m \in \Gamma(M)$. So, for any $N \supset \Gamma(M)$, $\theta(x, \overline{m})$ is realized in N. This entails that $\theta(x, \overline{m})$ does not fork over $\Gamma(M)$. As θ was arbitrary, we are done. □

Lemma 2.1.3 *Let M be a λ-compact model. Let $\Gamma(x)$ be a partial type over M such that $|\Gamma| < \lambda$ and let $p \in S(M)$. Then the following conditions are equivalent*

1. *There exists $a \models p$ such that for every $b \models \Gamma$, $a \downarrow_M b$.*
2. *For every $a \models p$ and every $b \models \Gamma$, $a \downarrow_M b$.*
3. *p is foreign to Γ*

Proof. We prove $1 \Rightarrow 2$, $3 \Rightarrow 1$, and $\neg 3 \Rightarrow \neg 2$.

$1 \Rightarrow 2$: Let a be as in 1, and let $a' \models p$. Let $F \in Aut_M \mathbb{M}$ take a to a'. Since F fixes $\Gamma(\mathbb{M})$ (as a set), a' inherits the relevant property.

$3 \Rightarrow 1$: Trivial.

$\neg 3 \Rightarrow \neg 2$: Suppose p is not foreign to Γ. Then we can find $A \supset M$, p' a non-forking extension of p in $S(A)$, $a \models p'$ and $b \models \Gamma$ such that $tp(a, A \cup b)$ forks over A. Since $tp(a, A \cup \{b\})$ is not an heir of p, there is some $\theta(x, y, \overline{x}, \overline{m}) \in \mathfrak{L}(\mathfrak{M})$ and $\overline{a} \in A$ such that, $\models \theta(a, b, \overline{a}, \overline{m})$ but for no $m, \overline{m}' \in M$, $\models \theta(a, m, \overline{m}', \overline{m})$.

Let d_p be a defining scheme for p. For each $\Delta(x)$, a finite sequence of formulas from $\Gamma(x)$, we define

$$\Delta^{\dagger}(a, \overline{a}, \overline{m}, \overline{m}_1, \ldots, \overline{m}_n) \equiv \exists x[\theta(a, x, \overline{a}, \overline{m}) \wedge \psi_1(x, \overline{m}_1) \wedge \ldots \wedge \psi_n(x, \overline{m}_n)] \tag{2.1}$$

where $\Delta = \langle \psi_1, \ldots, \psi_n \rangle$. For any such Δ, we clearly have

$$\models (d_p x)\Delta^{\dagger}(x, \overline{a}, \overline{m}, \overline{m}_1, \ldots, \overline{m}_n). \tag{2.2}$$

View (2.2) as a schema which produces $|\Gamma|$ formulas over $M \cup \overline{a}$. As M is λ-compact, we can find $\overline{m}' \in M$ such that for all Δ as above,

$$\models (d_p x)\Delta^{\dagger}(x, \overline{m}', \overline{m}, \overline{m}_1, \ldots, \overline{m}_n). \tag{2.3}$$

So, by the definition of $\Delta^{\dagger}$ and some basic facts on defining schemes, we have for any $\psi_1, \ldots \psi_n$ in Γ,

$$\models \exists x[\theta(a, x, \overline{m}', \overline{m}) \wedge \psi_1(x, \overline{m}_1) \wedge \ldots \wedge \psi_n(x, \overline{m}_n)]. \tag{2.4}$$

By compactness, then, we can find some $b' \models \Gamma$ such that $\models \theta(a, b', \overline{m}', \overline{m})$. And by our original choice of θ, this entails that $tp(a, M \cup b')$ forks over M. But this contradicts 2. $\square$

Lemma 2.1.4 *Let M be $\mathbf{F}^a_\lambda$-saturated for $\lambda \geq \kappa(T)$. Let $A \subset M$ such that $|A| < \lambda$ and let $\Gamma(x)$ be a partial type over A. If $p \in S(M)$, then the following conditions are equivalent.*

1. *There exists $a \models p$ such that for every $b \models \Gamma$, $a \downarrow_M b$.*

2. *For every $a \models p$ and every $b \models \Gamma$, $a \downarrow_M b$.*

3. *p is foreign to Γ*

Proof. Similar to the previous lemma. The key thing to notice is that M being $\mathbf{F}^a_{\kappa(T)}$-saturated means that M is good. Hence, in the proof of $\neg 3 \Rightarrow \neg 2$, we can choose our defining scheme to be over some $A' \subset M$ such that $|A'| < \kappa(T)$. So, the set of formulas generated in the analog of (2.2) will be over $A \cup A' \cup \{\overline{a}\}$. As $|A \cup A'| < \lambda$, and $\mathbf{F}^a_\lambda$-saturation implies λ-saturation, we can proceed as in the previous lemma. $\square$

Remark: Recall that for $\lambda > \aleph_0$, $\mathbf{F}^a_\lambda$-saturation is the same as λ-saturation. And if T is ω-stable, then $\mathbf{F}^a_{\aleph_0}$-saturation is the same as $\aleph_0$-saturation. So in general, the amount of saturation needed in this lemma is just barely enough to bound the size of Dom(Γ). However, if Dom(Γ) is finite and T is not ω-stable, then we may need a little bit more.

Lemma 2.1.5 *Let $N \prec M$ and let $\Gamma(x)$ be a partial type over N such that $\Gamma(M) = \Gamma(N)$. Let $a \in M \setminus N$, let $p = tp(a, N)$, and suppose that one of the following conditions holds:*

1. *M and N are λ-compact and $|\Gamma| < \lambda$.*
2. *M and N are $\mathbf{F}^a_\lambda$-saturated for some $\lambda \geq \kappa(T)$ and Γ is over some A such that $|A| < \lambda$.*

Then p is foreign to Γ.

Proof. Suppose first that condition 1 holds. Let $b \models \Gamma$. If $b \in M$, then trivially $a \downarrow_N b$. If $b \notin M$, then by 2.1.2, $b \downarrow_{\Gamma(M)} M$. As $\Gamma(M) \subset N$ and $a \in M$, $b \downarrow_N a$. As b is arbitrary here, 2.1.3 implies that p is foreign to Γ.

Suppose next that condition 2 holds. Then a similar argument will go through if we substitute the reference to 2.1.3 by one to 2.1.4 (also we note that M $\mathbf{F}^a_\lambda$-saturated implies M λ-saturated when we cite 2.1.2). $\square$

Remarks: If M is λ-compact ($\mathbf{F}^a_\lambda$-saturated) and the cardinality of M is significantly greater than that of $\Gamma(M)$ then we can find $N \precneqq M$ such that N is also λ-compact ($\mathbf{F}^a_\lambda$-saturated) and $\Gamma(M) \subset N$. By the last lemma, this is enough to insure the existence of types foreign to Γ. So, we have one half of the relationship between multi-cardinal phenomena and forking-independence: at least under certain conditions, the existence of multi-cardinal phenomena, vis-a vis a partial type Γ, entails the existence of complete, stationary types foreign to Γ.

To exploit this relationship between multi-cardinal phenomena and forking independence, we need a better idea of what the difference between the cardinalities of M and $\Gamma(M)$ has to be in order to generate the N mentioned in 2.1.5. The following list gives a number of conditions which guarantee the existence of such an N (and so, also, the existence of types foreign to Γ). Items 1–3 work for arbitrary stable theories; items 4–5 require that T be totally transcendental.

1. M is λ-compact, $|\Gamma| < \lambda$ and $|M| > (|\Gamma(M)| + |T|)^{<\lambda}$.
2. M is λ-compact, $|\Gamma| < \lambda$ and $|M| > \lambda(T) + |\Gamma(M)|^{<\kappa(T)}$.
3. M is $\mathbf{F}^a_\lambda$-saturated for some $\lambda \geq \kappa(T)$, Γ is over some A such that $|A| < \lambda$, and $|M| > \lambda(T) + |\Gamma(M)|^{<\kappa(T)}$.
4. M is λ-compact, $|\Gamma| < \lambda$ and $|M| > |\Gamma(M)| + |T|$.
5. M is $\mathbf{F}^a_\lambda$-saturated, Γ is over A where $|A| < \lambda$, and $|M| > |\Gamma(M)| + |T|$.

The proofs that these conditions work are trivial, and we omit them here.

Next, we want to see that the relationship between multi-cardinal phenomena and forking independence runs both ways. Recall that two stationary types, p and q, are **parallel** iff there is a third type r which is a non-forking extension of both p and q. By the remarks following definition 2.1.1, we can see that foreignness is preserved under parallelism–i.e. if p and q are parallel and p is foreign to Γ then q is also foreign to Γ. Recall also that if p is stationary and foreign to Γ, then $p^{(\alpha)}$ is foreign to Γ for any α. Using these facts, we get the following theorem.

Theorem 2.1.6 *Let p be stationary, let Γ be a partial type over some set A, let $\lambda > |A| + |T|$, and let $\alpha > 0$ be arbitrary. Then the following are equivalent:*

1. *p is foreign to Γ*
2. *There exist $\mathbf{F}^a_{|A|^+}$-saturated models $N \precneqq M$ such that $A \cup \Gamma(M) \subset N$ and for some $\overline{m} \in M \setminus N$, $tp(\overline{m}, N)$ is parallel to p.*
3. *Like 2, except that M and N are $\mathbf{F}^a_\lambda$-saturated and $|M| > \beth_\alpha(|N|)$.*
4. *Like 2, except that M and N are $|\Gamma|^+$-compact.*

Proof. $3 \Rightarrow 2$ and $3 \Rightarrow 4$ are trivial; $2 \Rightarrow 1$ and $4 \Rightarrow 1$ follow from 2.1.5 plus the comments preceeding this theorem. So, we only need $1 \Rightarrow 3$.

Let N be $\mathbf{F}^a_\lambda$-saturated such that $A \cup Dom(p) \subset N$ and let q be the non-forking extension of p to N. Let I be a Morley Sequence for q of length $\beth_{\alpha+1}(|N|)$, and let $N[I]$ be $\mathbf{F}^a_\lambda$-prime over $N \cup I$. Since $tp(I, N)$ is foreign to Γ, $N[I] \cap \Gamma(\mathbb{M}) \subset N$. Setting $M = N[I]$ and letting $\overline{m}$ be an arbitrary element of I, we are done. $\square$

Corollary 2.1.7 *Let Γ be a partial type over A. Then the following are equivalent:*

1. *There exists a complete, stationary type p which is foreign to Γ.*
2. *For arbitrarily large λ and α there exist $\mathbf{F}^a_\lambda$-saturated models M such that $A \subset M$ and $|M| > \beth_\alpha(|\Gamma(M)|)$.*

Proof. $1 \Rightarrow 2$ is a trivial application of the theorem. $2 \Rightarrow 1$ follows from the remarks immediately after lemma 2.1.5. $\square$

So, with respect to a particular type p and partial type Γ, the above theorem completely characterizes the relationship between two-cardinal phenomena and forking independence. The corollary characterizes the relationship in more general terms (i.e. without mentioning a specific p). Note that all of the conditions listed two pages back are entailed by 2 in the corollary, and each of them entails 1.

2.2 Prime Models

The results of the last section give us a general strategy for moving from one multi-cardinal model to another. We use the first model to insure the existence of a collection of types foreign to the ∞-definable sets in question; then we use some prime model theory to build models with the desired cardinalities.

This strategy depends on the existence of a nice theory of prime models (for instance, we would like our theory to allow the construction of prime models in as many cardinalities as possible). The present section surveys some results in and around the theory of prime models. Some of these results will remain unused until we get to chapter 3, but it seems convenient to include them here, rather than including a second section on prime models later in the dissertation. With the exception of 2.2.8, all of the results in this section are essentially present in the existing literature.

Throughout the dissertation, our notation concerning prime model construction will follow that in [Sh]; also, we will assume a basic familiarity with the results there (see chapter IV in particular). The nicest results concerning prime models involve two particular notions of isolation: $\mathbf{F}^a_\lambda$ and $\mathbf{F}^t_\lambda$. In dealing with these notions, the following notation is helpful:

Definition 2.2.1 *A notion of isolation is* <u>*standard*</u> *if it is of the form* $\mathbf{F}^x_\lambda$ *where* $x = t$*, and* $\lambda \geq \mu(T)$*; or where* $x = t$*,* $\lambda = \aleph_0$ *and* T *has Skolem functions; or where* $x = a$ *and* $\lambda \geq \kappa(T)$*. In any case, we assume that* λ *is regular.*

Note here that $\mathbf{F}^t_\lambda$-saturation is the same as λ-compactness and that $\mathbf{F}^a_\lambda$-saturation is almost the same as λ-saturation (see the remarks following 2.1.4).

Standard isolation notions have several properties which make them useful for our purposes. The following three are of particular importance:

1. For any A, there is an $\mathbf{F}$-prime, $\mathbf{F}$-constructible model over A.

2. For any A, the $\mathbf{F}$-prime model over A is unique up to isomorphism over A.

3. If M is $\mathbf{F}$-saturated and $M[A]$ is $\mathbf{F}$-prime over $M \cup A$, then $A \triangleright_M M[A]$.

The proofs that properties 1-3 hold are standard and can be found in chapters IV and V of [Sh] or in chapters IX and X of [Ba]. Property 3 is actually a consequence of a stronger property: for standard $\mathbf{F}$, if M is $\mathbf{F}$-saturated and $A \downarrow_M B$, then $tp(M[A], M \cup A) \vdash tp(M[A], M \cup A \cup B)$. This stronger property, along with 2 above, allows us to prove the following:

Proposition 2.2.2 *Let* $\mathbf{F}$ *be standard, let* M_0 *be* $\mathbf{F}$*-saturated, and let* $\langle E_i \mid i < \delta \rangle$ *be independent over* M_0*. For* $x = 1, 2$ *and* $i \leq \delta$*, let* M^x_i *satisfy the following conditions:* $M^x_0 = M_0$*,* M^x_{i+1} *is* $\mathbf{F}$*-prime over* $M^x_i \cup E_i$*, and for limit* i*,* M^x_i *is* $\mathbf{F}$*-prime over* $\bigcup_{j<i} M^x_j$*. Let* E *denote* $\bigcup_{i<\delta} E_i$*. Then,*

1. M^x_δ *is* $\mathbf{F}$*-prime over* $M_0 \cup E$*.*

2. There exists $F \in Aut_{(M_0 \cup E)}\mathbb{M}$ *such that for every* $i \leq \delta$*,* $F(M^1_i) = M^2_i$

Proof. 1. (Baldwin) For $i < \delta$, let $\vec{A}_i$ be a construction of M_i over $\bigcup_{j<i} M_i \cup \bigcup_{j<i} E_j$. The property mentioned above ensures that $\vec{A}_i$ is actually a construction over $\bigcup_{j<i} M_i \cup E$. So, pasting all these constructions together—in the obvious order—we get a construction of M^x_δ over $M_0 \cup E$. This gives 1.

2. We build a sequence $\langle F_i \mid i \leq \delta \rangle$ with the following properties:

   a. For every i, $F_i \in Aut_{(M_0 \cup E)}\mathbb{M}$

   b. For $k \leq i$, $F_i(M^1_k) = M^2_k$

   c. For $i < j$, $F_i \restriction (M^1_i \cup E) = F_j \restriction (M^1_i \cup E)$

When we are done, F_δ will be the desired map.

We start by letting F_0 be the identity on $\mathbb{M}$. Given F_i, we notice that $F_i(M^1_{i+1})$ is isomorphic to M^2_{i+1} over $M^2_i \cup E_i$. Further, $\bigcup_{j>i} E_j \downarrow_{M_0} M^2_i \cup E_i$ (by 1 in this proposition and property 3 above). So, $F_i(M^1_{i+1})$ is isomorphic to M^2_{i+1} over $M^2_i \cup E$. Letting $G \in Aut\ \mathbb{M}$ witness this, we set $F_{i+1} = G \circ F_i$.

For i limit, we first define a map G as follows. For $m \in (\bigcup_{j<i} M^1_j \cup E)$, we let $G(m) = m'$ iff $F_j(m) = m'$ for arbitrarily large $j < i$. G is clearly an elementary map from $\mathbb{M}$ to $\mathbb{M}$. Let G' be an automorphism of $\mathbb{M}$ extending G.

From here we proceed as in the first case: as $G'(M^1_i)$ is isomorphic to M^2_i over $\bigcup_{j<i} M^2_j$, and as $\bigcup_{j\geq i} E_j \downarrow_{M_0} \bigcup_{j<i} M^2_j$, we get that $G'(M^1_i)$ is isomorphic to M^2_i over $\bigcup_{j<i} M^2_j \cup E$. Let G'' be an automorphism of $\mathbb{M}$ witnessing this fact and set $F_i = G'' \circ G'$. $\square$

Many useful properties of $\mathbf{F}^a_\lambda$ can be "relativised" to work within a particular model M (where M is not necessarily $\mathbf{F}^a_\lambda$-saturated). The following definition fixes some notation for working in such a context.

Definition 2.2.3 *Let A, B and C be subsets of D. We say that A is $\mathbf{F}^a_\lambda$-saturated in D ($A \prec^a_\lambda D$) if whenever q is a partial type almost over some $A' \subseteq A$ where $|A'| < \lambda$, if q is realized in D, then q is realized in A. We say A is $\mathbf{F}^a_\lambda$-maximal in D if no $d \in D \setminus A$ is $\mathbf{F}^a_\lambda$-isolated over A. We say that A dominates B over C in D ($A \rhd^D_C B$) if for any $\bar{d} \in D$,*

$$\bar{d} \downarrow_C A \implies \bar{d} \downarrow_C B$$

If $D = \mathbb{M}$, then these notions reduce to the ordinary ones. A is $\mathbf{F}^a_\lambda$-maximal in $\mathbb{M}$ iff A is $\mathbf{F}^a_\lambda$-saturated in $\mathbb{M}$ iff A is $\mathbf{F}^a_\lambda$-saturated; $A \rhd^{\mathbb{M}}_C B$ iff $A \rhd_C B$.

For our purposes, these notions are useful primarily when D is a λ-compact model for some $\lambda \geq \kappa_r(T)$. Many facts concerning the $\mathbf{F}^a_{\kappa(T)}$-saturated case carry over to this case. The following are of particular importance.

Fact 2.2.4 *Let $\lambda \geq \kappa(T)$. Suppose M is λ-compact, and suppose A is $\mathbf{F}^a_\lambda$-maximal in M. Then A is also a λ-compact model.*

Proof. Let $\Gamma(x)$ be a partial type over A such that $|\Gamma| < \lambda$. We set $\Gamma = \Gamma_0$ and try to find an increasing sequence of partial types $\langle \Gamma_i(x) \mid i < \kappa(T) \rangle$ such that:

1. Each Γ_i is consistent and over A.

2. For every i, there is a unique $\varphi \in \Gamma_{i+1} \setminus \Gamma_i$.

3. For every i, the φ mentioned in 2 forks over dom(Γ_i).

Suppose we succeed here. Let $b \models \bigcup_{i<\kappa(T)} \Gamma_i(x)$ and let $p_i = tp(b, dom(\Gamma_i))$. Then $\langle p_i \mid i < \kappa(T)\rangle$ will be a forking chain of length $\kappa(T)$.

Since this is a contradiction, there must be some $i < \kappa(T)$ such that the requisite φ_i cannot be found. This means that for any $b \models \Gamma_i(x)$, $b \downarrow_{dom(\Gamma_i)} A$. So, $tp(b, A) \in \mathbf{F}^a_\lambda(dom(\Gamma_i))$. Since M is λ-compact, we can find $b \in M$ such that $b \models \Gamma_i(x)$; so $tp(b, A)$ is $\mathbf{F}^a_\lambda$-isolated. And as A is $\mathbf{F}^a_\lambda$ maximal in M, $b \in A$ as desired. $\Box$

Fact 2.2.5 *Let $\lambda \geq \kappa_r(T)$, let A be $\mathbf{F}^a_\lambda$-maximal in D, let b and A' be subsets of D such that $b \notin A$. If $tp(b, A \cup A')$ is $\mathbf{F}^a_\lambda$-isolated, then $b \not\downarrow_A A'$.*

Proof. Suppose $b \downarrow_A A'$. Then $tp(b, A)$ is $\mathbf{F}^a_\lambda$-isolated. This contradicts the maximality of A in D. $\Box$

Fact 2.2.6 *Let $\lambda \geq \kappa_r(T)$, let A be $\mathbf{F}^a_\lambda$-maximal in D, and suppose $A' \downarrow_A D$. If $b \notin A$ and $tp(b, A \cup A')$ is $\mathbf{F}^a_\lambda$-isolated, then $b \notin D$.*

Proof. Suppose $b \in D$. As $tp(b, A \cup A')$ is $\mathbf{F}^a_\lambda$-isolated and $b \downarrow_A A'$, $tp(b, A)$ is $\mathbf{F}^a_\lambda$-isolated. Again, this contradicts the maximality of A in D. $\Box$

Fact 2.2.7 *Let $\lambda \geq \kappa_r(T)$, let N be $\mathbf{F}^a_\lambda$-saturated in D, let $A_1 \subset \mathbb{M}$, $A_2 \subset D$ such that $A_1 \downarrow_N A_2$. If $tp(b, N \cup A_1)$ is $\mathbf{F}^a_\lambda$-isolated, then*

$$stp(b, N \cup A_1) \vdash tp(b, N \cup A_1 \cup A_2).$$

Proof. W.L.O.G. we may assume A_2 is finite and $|A_1| < \lambda$. Let $B_1 \subseteq N$ such that $|B_1| < \lambda$ and $stp(b, B_1 \cup A_1) \vdash stp(b, N \cup A_1)$; let $B_2 \subseteq N$ such that $|B_2| < \lambda$ and $A_1 \cup A_2 \downarrow_{B_2} N$; let $B = B_1 \cup B_2$.

Suppose, towards a contradiction, that $stp(b, N \cup A_1) \nvdash tp(b, N \cup A_1 \cup A_2)$. Then $stp(b, B \cup A_1) \nvdash tp(b, N \cup A_1 \cup A_2)$. Let $b_1, b_2 \models stp(b, B \cup A_1)$ and let $\theta(x, \bar{n}, \bar{a}_1, \bar{a}_2)$ be such that $\mathbb{M} \models [\theta(b_1, \bar{n}, \bar{a}_1, \bar{a}_2) \wedge \neg\theta(b_2, \bar{n}, \bar{a}_1, \bar{a}_2)]$. Set $B' = B \cup \bar{n}$.

As N is $\mathbf{F}^a_\lambda$-saturated in D, there is $A \subset N$ such that $stp(A, B') = stp(A_2, B')$ (recall here that A_2 is finite). As $A \downarrow_{B'} A_1$ and $A_2 \downarrow_{B'} A_1$, this entails that $stp(A, B' \cup A_1) = stp(A_2, B' \cup A_1)$. Let $G \in Aut^*_{B' \cup A_1}\mathbb{M}$ such that $G(A_2) = (A)$. Then, $stp(b, B \cup A_1) = stp(G(b_1), B \cup A_1) = stp(G(b_2), B \cup A_1)$. Furthermore, $\mathbb{M} \models [\theta(G(b_1), \bar{n}, \bar{a}_1, G(\bar{a}_2)) \wedge \neg\theta(G(b_2), \bar{n}, \bar{a}_1, G(\bar{a}_2))]$. This contradicts the fact that $stp(b, B_1 \cup A_1) \vdash stp(b, N \cup A_1)$. $\Box$

Remarks: (1) This proof of 2.2.7 is almost a verbatim copy of the proof of V 3.2 in [Sh]. The only new point here is the realization that we do not need N $\mathbf{F}^a_\lambda$-saturated in $\mathbb{M}$. When $A_2 \subset D$ we can get by with N $\mathbf{F}^a_\lambda$-saturated in D.

(2) Note that the proof of 2.2.7 only requires that $A_2 \subset D$. A_1 can live outside of D. If we modify the definition of relative $\mathbf{F}^a_\lambda$-saturation to let N be $\mathbf{F}^a_\lambda$-saturated in D even when $N \not\subset D$ (i.e. for every q almost

over some $A \subseteq N$ where $|A| < \lambda$, if q is realized in D, q is realized in N), then we can have N outside D as well.

(3) 2.2.7 is a generalization of the property mentioned before 2.2.2 and it has similar consequences. In particular, if N and D are as in 2.2.7 and $N[A] \subset D$ is $\mathbf{F}^a_\lambda$-constructible over $N \cup A$, then $A \rhd^D_N N[A]$. Also, the first part of 2.2.2 goes through in this context: if M_0 is $\mathbf{F}^a_\lambda$-saturated in $M_0 \cup \bigcup_{i<\delta} E_i$ and each M^x_i is $\mathbf{F}^a_\lambda$-constructible over $\bigcup_{j<i}(M^x_j \cup E_j)$, then M^x_δ is $\mathbf{F}^a_\lambda$-constructible over $M_0 \cup \bigcup_{i<\delta} E_i$.

Along with these standard notions of isolation, there are two non-standard notions which we will occasionally use. The first is Shelah's $\mathbf{F}^l_{|T|}$ when $|T|$ is regular. Recall that,

$$(p, B) \in \mathbf{F}^l_\lambda \quad \text{iff} \quad \text{for every } \psi \text{ there is } p_\psi \subseteq p \restriction B,\ |p_\psi| < \lambda, \text{ such that}$$
$$p_\psi \vdash p \restriction \psi \text{ and } |B| \leq \lambda + |T|;\ cf(\lambda) > |T| \Rightarrow |B| < \lambda.$$

$\mathbf{F}^l_{|T|}$ does not admit prime models (much less unique prime models) over arbitrary sets. It does, however, satisfy the following useful properties.

1. Over any A, there exists a $|T|$-compact, $\mathbf{F}^l_{|T|}$-constructible, $\mathbf{F}^l_{|T|}$-atomic M.

2. If N is $\mathbf{F}^t_{|T|^+}$-saturated and $A \subset N$, then we may have $M \prec N$ (M as in 1).

3. If M is $|T|$-compact and N is $\mathbf{F}^l_{|T|}$-atomic over $M \cup A$, then $A \rhd_M N$.

Here, property 1 is standard (see [Sh] IV), and property 2 follows trivially from the definition of an $\mathbf{F}^l_{|T|}$-construction. For $|T| = \aleph_0$, property 3 is well-known (see [Ba], X). For $|T| > \aleph_0$, we give the following:

Proposition 2.2.8 *Let $\lambda > \aleph_0$ be regular, let M be λ-compact, and let N be $\mathbf{F}^l_\lambda$-atomic over $M \cup A$. Then, $A \rhd_M N$.*

Proof. Suppose not. Let $b \in \mathbb{M}$ such that $b \downarrow_M A$ and $b \not\downarrow_M N$. As $tp(b, N)$ is not a coheir of $tp(b, M)$, there is a formula $\theta(x, \overline{n}, \overline{m})$ such that $\mathbb{M} \models \theta(b, \overline{n}, \overline{m})$ but for no $m \in M$ does $\mathbb{M} \models \theta(m, \overline{n}, \overline{m})$.

Let $p = tp(\overline{n}, M \cup A)$. For every $\psi \in \mathfrak{L}$ choose a specific $p_\psi \subseteq p$ such that $|p_\psi| < \lambda$ and $p_\psi \vdash p \restriction \psi$. Define a function F from complete sublanguages of $\mathfrak{L}$ to complete sublanguages of $\mathfrak{L}$ as follows: $\phi \in F(\mathfrak{L}')$ iff all of the symbols in ϕ also occur in some formula of p_ψ where $\psi \in \mathfrak{L}'$.

Build an increasing sequence of languages $\langle \mathfrak{L}_i \mid i < \omega \rangle$ as follows: $\mathfrak{L}_0 \subset \mathfrak{L}$ is minimal such that $\theta \in \mathfrak{L}_0$, and for $i > 0$, $\mathfrak{L}_i$ is minimal such that $F(\mathfrak{L}_{i-1}) \cup \mathfrak{L}_{i-1} \subset \mathfrak{L}_i$. Let $\mathfrak{L}' = \bigcup_{i<\omega} \mathfrak{L}_i$ and note that $|\mathfrak{L}'| < \lambda$.

Reduct $\mathbb{M}$ to the language $\mathfrak{L}'$. M remains a λ-compact model for this language, and we still have $b \downarrow_M A$. Also we have that $\bigcup_{\psi \in \mathfrak{L}'} p_\psi \vdash p \restriction \mathfrak{L}'$. Since $|\bigcup_{\psi \in \mathfrak{L}'} p_\psi| < \lambda$, $tp(\overline{n}, M \cup A)$ is $\mathbf{F}^t_\lambda$ isolated over $M \cup A$. And as $\mathbf{F}^t_\lambda$ is standard here, we get $A \rhd_M A \cup \{\overline{n}\}$. Thus $b \downarrow_M A \cup \{\overline{n}\}$. This makes $tp(b, M \cup A \cup \{\overline{n}\})$ a coheir of $tp(b, M)$; so for some $m \in M$, $\mathbb{M} \models \theta(m, \overline{n}, \overline{m})$. And this is a contradiction. □

Our final "prime model" notion was introduced by Shelah and Buechler in [BuSh]. The notion only works in the superstable case, and the assumption that we are working in $\mathbb{M}^{\text{eq}}$ is important when employing this notion. In particular, "algebraic closure" always refers to algebraic closure in $\mathbb{M}^{eq}$. We begin with some preliminaries.

We say $N \subset_{na} M$ if for all $a \in N$ and $\theta(x, a)$ such that $\theta(N, a) \neq \theta(M, a)$, there is a $b \in \theta(N, a) \setminus acl(a)$. The following facts concerning $\subset_{na}$ are easily verified:

1. Let $A \subset M$. There exists N such that $A \subset N \subset_{na} M$ and $|N| = |A| + |T|$.
2. $M_0 \subset_{na} M_1 \subset_{na} M_2 \implies M_0 \subset_{na} M_2$.
3. Let $\langle N_i \mid i < \delta \rangle$ be an increasing chain such that $N_i \subset_{na} M$ for every i. Then $\bigcup_{i<\delta} N_i \subset_{na} M$

For our purposes, the main result concerning $\subset_{na}$ is the following:

Theorem 2.2.9 *Let T be superstable, let $N \subset_{na} M$, and let $N \subset A \subset M$. Then there exists $N' \subset_{na} M$ such that $A \subset N'$, $|N'| = |A| + |T|$ and $A \rhd_N N'$.*

The proof of this theorem can be found in [BuSh]. It involves techniques which go well beyond the scope of this dissertation, so we do not include it here. Note that there is nothing in the theorem which suggests that N' is prime over A for any notion of isolation resembling one of Shelah's **F**'s. Nevertheless, we will often abuse notation and say that N' is $\mathbf{F}^n$-prime over A; also, we will use $N[A]$ to denote N' even though N' is not unique over $N \cup A$.

To employ any of these "prime model" notions effectively, we need some idea of the cardinalities in which prime models can live. Let **F** be a notion of isolation, and let κ be a cardinal. We let $\mathbf{F}(\kappa)$ be the least κ' such that for every A with $|A| \leq \kappa$, every **F**-constructible model over A has cardinality $\leq \kappa'$. For $\mathbf{F}^n$ this definition does not make sense, so we simply stipulate that $\mathbf{F}^n(\kappa) = \kappa + |T|$.

Note that for any of the notions of isolation we have considered, $\mathbf{F}(\mathbf{F}(\kappa)) = \mathbf{F}(\kappa)$. In general, prime models for these notions will live in cardinals for which $\mathbf{F}(\kappa) = \kappa$. We will call such cardinals **F**-good. The following list gives some conditions which ensure that a cardinal is **F**-good for one of our notions of isolation:

1. $(\mathbf{F}=\mathbf{F}^t_\lambda)$ $\mathbf{F}(\kappa) = \kappa$ if either $\kappa = (\kappa + |T|)^{<\lambda}$ or $\kappa \geq \lambda + \lambda(T)$ and $\kappa = \kappa^{<\kappa(T)}$.
2. $(\mathbf{F}=\mathbf{F}^a_\lambda)$ $\mathbf{F}(\kappa) = \kappa$ if $\kappa \geq \lambda + \lambda(T)$ and $\kappa = \kappa^{<\kappa(T)}$.
3. $(\mathbf{F}=\mathbf{F}^l_{|T|})$ $\mathbf{F}(\kappa) = \kappa$ if either $\kappa = \kappa^{<|T|}$ or $\kappa = \lambda(T) + \kappa^{<\kappa(T)}$.
4. $(\mathbf{F}=\mathbf{F}^n)$ $\mathbf{F}(\kappa) = \kappa$ if $\kappa \geq |T|$.

The proofs that these conditions work are trivial, and we omit them here.

2.3 Multi-Cardinal Theorems I

In this section, we assume that each $\mathbb{A}_\beth$ in $\langle \mathbb{A}_\beth \mid \beth < \delta_\nvdash \rangle$ is ∞-definable over B_i by some $\Gamma_i(x)$. To avoid repetition, we use **HYP** to denote the following hypothesis:

HYP For each $i < \delta_0$, there exists a pair $\overline{M}_i \subsetneq \overline{N}_i$ such that $\bigcup_{j \le i} B_j \subset \overline{M}_i$, $\bigcup_{j<i} \Gamma_j(\overline{N}_i) \subset \overline{M}_i$, $\Gamma_i(\overline{N}_i) \neq \Gamma_i(\overline{M}_i)$ and either of the following holds:

1. $\overline{M}_i$ and $\overline{N}_i$ are λ-compact and for $j < i$, $|\Gamma_j| < \lambda$.
2. $\overline{M}_i$ and $\overline{N}_i$ are $\mathbf{F}^a_\lambda$-saturated for $\lambda \ge \kappa(T)$, and for $j < i$, $|B_j| < \lambda$.

To unify notation when proving this section's main theorem, we adopt the following framework. We work with classes of models, K, which are preserved under automorphisms of $\mathbb{M}$. Associated with each K is a nondecreasing function $F_K : CARD \to CARD$ such that for every κ, $F_K(F_K(\kappa)) = F_K(\kappa)$; cardinals such that $F_K(\kappa) = \kappa$ will be called K-good. Also, K and F_K must satisfy the following conditions:

1. For any A, there is a K-model $M \supset A$ such that $|M| \le F_K(|A|)$.
2. If M is a K-model and $M \subset A$, then there is a K-model $M[A] \supset M \cup A$ such that $|M[A]| \le F_K(|M \cup A|)$ and $A \rhd_M M[A]$
3. If $\langle M_i \mid i < \alpha \rangle$ is an increasing sequence of K-models, then there is a K-model $N \supset \bigcup_{i<\alpha} M_i$ such that $|N| = F_K(|\bigcup_{i<\alpha} M_i|)$ and for $j < \alpha$, $\bigcup_{i<\alpha} M_i \rhd_{M_j} N$.

We begin by noting several classes of models which satisfy these conditions.

Lemma 2.3.1 *The conditions outlined above are satisfied when:*

1. *K is the class of $\mathbf{F}$-saturated models for some standard $\mathbf{F}$ and $F_K(\kappa) = \mathbf{F}(\kappa)$.*
2. *K is the class of $|T|$-compact models and $F_K(\kappa) = \mathbf{F}^l_{|T|}(\kappa)$.*
3. *K is the class of models $M \prec_{na} \mathbb{M}$, and $F_K(\kappa) = \kappa + |T|$.*

Proof. 1 and 2 are essentially trivial. In 1, we just take prime models in all cases. In 2, we take $|T|$-compact models which are $\mathbf{F}^l_{|T|}$-constructible over the relevant sets. For 3, condition 1 is also trivial; condition 2 follows from 2.2.9; and condition 3 is trivial if we simply let $N = \bigcup_{i<\alpha} M_i$. $\square$

Theorem 2.3.2 *Suppose* **HYP** *holds. For any non-decreasing $G : \delta_0 \to CARD$ such that $|G(0)| \ge |\bigcup_{i<\delta_0} B_i| + |\delta_0| + \kappa(T)$ and each $G(i)$ is K-good, there exists an K-model M which is also a G-model. If $\kappa(T) = \kappa^+$ and $|\bigcup_{i<\delta_0} B_i| + |\delta_0| \le \kappa$, then we only need $|G(0)| \ge \kappa$.*

Proof. By **HYP**, we can find for each $i < \delta_0$ some $a_i \in \Gamma_i(\overline{N}_i) \setminus \Gamma_i(\overline{M}_i)$ ($\overline{N}_i$ and $\overline{M}_i$ as in **HYP**). By 2.1.5, $tp(a_i, \overline{M}_i)$ is foreign to Γ_j for every $j < i$. Let p_i be a complete stationary type parallel to $tp(a_i, \overline{M}_i)$ and over some C_i such that $|C_i| < \kappa(T)$. So, p_i is also foreign to Γ_j for $j < i$.

We construct by induction an increasing sequence of K-models $\langle M_i \mid i \le \delta_0 \rangle$ such that for each i, $|M_{i+1}| = |M_{i+1} \cap \mathbb{A}_{\sqsupset}| = \mathbb{G}(\sqsupset)$, and for every $j < i$ and $b \in \mathbb{A}_j$, $b \downarrow_{M_{j+1}} M_i$. We begin by letting M_0 be an arbitrary K-model of cardinality $\le G(0)$ which contains $\bigcup_{i<\delta_0} (B_i \cup C_i)$.

Suppose we have M_i. Let q_i be a non-forking extension of p_i to M_i, and let I_i be a Morley Sequence for q of length $G(i)$. Let $M_{i+1} = M_i[I_i]$ (as in condition B. on class K). Clearly the cardinality constraints on M_{i+1} and $\mathbb{A}_\beth$ are satisfied. For the independence constraints, suppose $j < i+1$ and let $b \in \mathbb{A}_j$. If $j = i$, then it is trivial that $b \downarrow_{M_{j+1}} M_i$; so we can assume that $j < i$. As q_i is foreign to Γ_j, $tp(I_i, M_i)$ is also foreign to Γ_j. As $I_i \rhd_{M_i} M_i[I_i]$, $b \downarrow_{M_i} M_i[I_i]$. So by the induction hypothesis, $b \downarrow_{M_{j+1}} M_i[I_i]$.

Suppose i is limit. By condition 3 on K models, we can find a K-model $N \supset \bigcup_{j<i} M_j$ such that $|N| = F_K(|\bigcup_{j<i} M_j|)$ and for $k < i$, $\bigcup_{j<i} M_j \rhd_{M_{k+1}} N$. Because G is non-decreasing and $G(i)$ is K-good, we have that $|N| \leq G(i)$. Let $k < i$ and let $b \in \mathbb{A}_\neg$. By the induction hypothesis and the finite character of forking, $b \downarrow_{M_{k+1}} \bigcup_{j<i} M_j$. So, $b \downarrow_{M_{k+1}} N$. Letting $M_i = N$ we finish the construction.

Now let $M = M_{\delta_0}$ and let $i < \delta_0$. As $b \downarrow_{M_{i+1}} M$ for every b in $\mathbb{A}_\beth$, $M \cap \mathbb{A}_\beth \subseteq \mathbb{M}_{\beth+\nvDash}$. So $|M \cap \mathbb{A}_\beth| = |\mathbb{M}_{\beth+\nvDash} \cap \mathbb{A}_\beth| = \mathbb{G}(\beth)$. As desired, then, M is a G-model. $\square$

Corollary 2.3.3 *Suppose* **HYP** *holds. Let $i_0 < \delta_0$ and let M be a K-model such that for every $i \geq i_0$ there exists $a_i \in \Gamma_i(\overline{N}_i) \setminus \Gamma_i(\overline{M}_i)$ and $p_i \in S(M)$ such that p_i is parallel to $tp(a_i, \overline{M}_i)$. Let $G : \delta_0 \to CARD$ such that:*

1. *for $i < i_0$, $G(i) = |M \cap \mathbb{A}_\beth|$*
2. *for $i \geq j \geq i_0$, $G(i)$ is K-good and $G(i) \geq G(j) \geq |M|$.*

Then there exists a K-model M' such that $M \prec M'$, M' is a G-model and for $i < i_0$, $M' \cap \mathbb{A}_\beth = \mathbb{M} \cap \mathbb{A}_\beth$.

Proof. Just like the proof of 2.3.2. Letting M be a base for the construction, we build $\langle M_i \mid i_0 < i \leq \delta_0 \rangle$ as in the theorem (making use of the obvious p_i's). Clearly the resulting model is a G-model. And just as the original construction did not add new elements of $\mathbb{A}_\beth$ at any stage past $i+1$, this construction does not add new elements of $\mathbb{A}_\beth$ for $i < i_0$. $\square$

Corollary 2.3.4 *Suppose* **HYP** *holds. For any non-decreasing $G : \delta_0 \to CARD$ such that $|\mathfrak{L}| + |\bigcup_{i<\delta_0} B_i| + |\delta_0| \leq G(0)$, there is a G-model.*

Proof. Set $\mu = (|\mathfrak{L}| + |\bigcup_{i<\delta_0} B_i| + |\delta_0|)^+$. Let K be the class of $\mathbf{F}^{\mathbf{a}}_{\kappa_{\mathbf{r}}(\mathbf{T})}$-saturated models. Choose $F : \delta_0 \to CARD$ such that for every $i < \delta_0$, $F(i) > \beth_\mu(sup_{j<i}(F(j)))$ and $F(i)$ is K-good. By the theorem, there exists an F-model M.

Expand our language by adding constants for elements of $\bigcup_{i<\delta_0} B_i$. Note that M is still a model for the expanded language $\mathfrak{L}'$, that $|\mathfrak{L}'| \leq |\mathfrak{L}| + |\bigcup_{\mathfrak{i}<\delta_\mathfrak{o}} \mathfrak{B}_\mathfrak{i}|$, and that each $\mathbb{A}_\beth$ is ∞-definable over $\emptyset$ in $\mathfrak{L}'$. The desired G-model follows from 1.3.4. $\square$

Remarks: (1) For $\delta_0 = 2$, these corollaries give some standard theorems. If $\mathbb{A}_\nvdash$ is definable and $\mathbb{A}_\nvDash = \mathbb{M}$, then 2.3.4 gives Shelah's theorem from [Sh69] (if $\delta_0 = n$ and T is ω-stable, then the same corollary gives Forrest's generalization). If we assume also that T is countable, that K is the class of all models, and let

$M = \overline{N}_1$, then 2.3.3 gives Lachlan's theorem. To get Harnik's extension of Lachlan's result, let K be the class of $\mathbf{F}^{\mathbf{t}}_{\lambda}$-saturated models for $\lambda \geq \mu(T)$, let $\mathbb{A}_{\nvdash}$ be ∞-definable by some $\Gamma(x)$ where $|\Gamma| < \lambda$, and let $M = \overline{N}_1$. 2.3.3 gives the result. For a host of similar results using somewhat different machinery, see [Sh] V.6

(2) The results here do more than extend these theorems from the 2-cardinal case to the δ-cardinal case. First, they allow this extension to go through even when there is no "initial model" to begin with. That is, **HYP** does not require a single highly saturated model which takes care of all our foreignness at once.

Second, the approach taken here allows us to separate the details of our prime model constructions from the multi-cardinal problems which motivate them. We can use one kind of saturation/compactness to get our foreign types, and then employ a theory of prime models which relates to an entirely different kind of saturation/compactness. This is especially important when we start with a model pair which has a level of saturation/compactness which is enough for generating foreignness, but insufficient for serving as a base for further prime model constructions—e.g. when we start with a pairs of mere models for an uncountable, stable, non-superstable theory and each $\mathbb{A}_{\daleth}$ is defined by a predicate.

(3) For the purpose of obtaining multi-cardinal transfer results, **HYP** is a relatively weak hypothesis. In particular, the differences in the sizes of N_i and $\bigcup_{j<i}(N_i \cap \mathbb{A}_{\daleth})$ needed to ensure the existence M_i in **HYP** are often quite small. If T is superstable, for instance, then any cardinal splits above $\lambda(T)$ allow us to build the model pairs required in **HYP** (for any degree of saturation). If T is totally transcendental, then any splits above $|T|$ allow us to build these pairs. Similarly, if each $A_{\,i}$ is definable by a single formula, then arbitrary cardinal splits do the job. In no case, do we need cardinal splits which make $|N_i| > (\lambda(T) + |\bigcup_{j<i}(N_i \cap \mathbb{A}_{\daleth})|^{<\kappa(\mathbb{T})})^+$.

2.4 Multi-Cardinal Theorems II

In this section, we examine a class of multi-cardinal phenomena which has, at least on the surface, relatively little to do with forking independence. We analyze this class by looking at the details of certain prime model constructions which use standard $\mathbf{F}$'s. Roughly, we will try to mimic the techniques of section 1.3: prime model theory will replace that section's use of Skolem functions, and we will use stability theory instead of the Erdös-Rado theorem to generate sets of indiscernibles. As in section 2.3 we assume that each $\mathbb{A}_{\daleth}$ in $\langle \mathbb{A}_{\daleth} \mid \daleth < \delta_{\nvdash} \rangle$ is ∞-definable over some set A.

We begin with some preliminaries on indiscernible sets in stable theories. Recall that a Morley Sequence over A is a set I such that I is independent over A and every element of I satisfies the same strong type over A. Recall also that any Morley Sequence over A is indiscernible over A. The following summarizes some basic facts on manipulating Morley Sequences:

Proposition 2.4.1 (Facts on Morley Sequences) *MS = Morley Sequence*

1. *Suppose $I \downarrow_{A_I} A$. Then I is a MS over A if and only if I is a MS over A_I.*

2. *Let I be a MS over A. Then there exists $A_I \subset A$ such that $|A_I| < \kappa(T)$ and $I \downarrow_{A_I} A$.*

3. *Let I be a MS over A and let B be arbitrary. There exists $I_0 \subset I$ such that $|I_0| \leq |B| + \kappa(T)$ and $I \setminus I_0 \downarrow_A B$. If $|B| < cf(\kappa(T))$, we can have $|I_0| < \kappa(T)$.*

Proof. 1. Easy and standard.

2. Let $i \in I$ be arbitrary and let A_I be such that $|A_I| < \kappa(T)$ and $i \downarrow_{A_I} A$. Since I is indiscernible over A, $i' \downarrow_{A_I} A$ for every $i' \in I$. Suppose, towards a contradiction, that $I \not\downarrow_{A_I} A$. Let $\bar{\imath} = (i_1, \ldots, i_n)$ be a sequence of minimal length such that $\bar{\imath} \not\downarrow_{A_I} A$. By minimality of $\ln(\bar{\imath})$, $\{i_1, \ldots, i_{n-1}\} \downarrow_{A_I} A$. And since I is independent over A, $i_n \downarrow_{A_I} A \cup \{i_1, \ldots, i_{n-1}\}$. So, $i_n \downarrow_{A_I \cup \{i_1, \ldots, i_{n-1}\}} A \cup \{i_1, \ldots, i_{n-1}\}$. Thus, $\bar{\imath} \downarrow_{A_I} A$ for a contradiction.

3. Clearly, we can find an I_0 of the desired size such that $B \downarrow_{A \cup I_0} I$. As I is independent over A, $I \setminus I_0 \downarrow_A I_0$. Thus, $I \setminus I_0 \downarrow_A B$ as desired. □

Proposition 2.4.1 gives us some tools for manipulating Morley Sequences once we find them. We now show that if a set C is large enough, then we can find equally large Morley Sequences inside of C.

Lemma 2.4.2 (Baldwin) *Let $A \subset C$ such that $|A| < \lambda = |C|$. If $\kappa(T) < cf(\lambda)$, then there exist $D, E \subset C$ such that $|D| < \lambda$, $|E| = \lambda$ and E is independent over $A \cup D$.*

Proof. Choose by induction a sequence of subsets of C, $\langle C_i | \ i < \kappa(T) \rangle$ such that for each i, C_i is maximally independent (in C) over $A \cup \bigcup_{j<i} C_j$. Suppose that for each i, $|C_i| < \lambda$. Since $\kappa(T) < cf(\lambda)$, we can find $c \in C \setminus (\bigcup_{i<\kappa(T)} C_i)$. Letting $p_i = tp(c, A \cup \bigcup_{j<i} C_j)$, we get a forking sequence $\langle p_i | \ i < \kappa(T) \rangle$. As this is a contradiction, we can find a least i such that $|C_i| = \lambda$; let $D = \bigcup_{j<i} C_j$ and $E = C_i$ to finish. □

Lemma 2.4.3 *Let $A \subset C$ such that $|A| < \lambda = |C|$. Let λ be regular, and suppose that T is stable in arbitrarily large $\lambda' < \lambda$. Then there exist $I, A_I \subset C$ such that:*

1. *$|I| = \lambda$ and $|A_I| < \kappa(T)$*

2. *$I \downarrow_{A_I} A$*

3. *I is a Morley Sequence over A_I (hence, over $A \cup A_I$ as well)*

Proof. By the previous lemma, there exist $D, E \subset C$ such that $|D| < \lambda$, $|E| = \lambda$ and E is independent over $A \cup D$. As $|A \cup D| < \lambda$ and as T is stable in arbitrarily large $\lambda' < \lambda$, $|S_s(A \cup D)| < \lambda$. So we can find $I \subset E$ such that $|I| = \lambda$ and all the i's in I realize the same strong type over $A \cup D$. Thus, I is a Morley Sequence over $A \cup D$.

By 2.4.1 (2), there exists $A_I \subset A \cup D$ such that $|A_I| < \kappa(T)$ and $I \downarrow_{A_I} A \cup D$. By 2.4.1 (1), I is still a Morley Sequence over $A \cup A_I$. □

As in section 2.3, we avoid repetition by using **F-HYP** (where $\mathbf{F} = \mathbf{F}^x_\lambda$ is some standard notion of isolation) to denote the following hypothesis:

F-HYP There exists an $\mathbf{F}$-saturated model $M \supset A$ such that for every $i < \delta_0$, $|M \cap \mathbb{A}_i| > \lambda(T) + [\mathbf{F}(|A| + |\delta_0| + \kappa(T) + |\bigcup_{j<i} M \cap \mathbb{A}_j|)]^{<\kappa(T)}$. If $\kappa(T) = \kappa^+$ and $|A| + |\delta_0| \leq \kappa$, then we only require that $|M \cap \mathbb{A}_0| > \lambda(T) + [\mathbf{F}(\kappa)]^{<\kappa(T)}$.

Theorem 2.4.4 *Let $\mathbf{F} = \mathbf{F}^x_\lambda$ be standard, and suppose* **F-HYP** *holds. Then for any nondecreasing $G : \delta_0 \to CARD$ such that $G(0) \geq \mathbf{F}(|A| + |\delta_0| + \kappa(T))$ and each $G(i)$ is $\mathbf{F}$-good, there exists an $\mathbf{F}$-saturated G-model, M'.*

Remark: If $\kappa(T) = \kappa^+$ and $|A| + |\delta_0| \leq \kappa$, then we only need $G(0) \geq \mathbf{F}(\kappa)$.

Proof. Using lemma 2.4.3, we choose a sequence $\langle (I_i, B_i) \mid i < \delta_0 \rangle$ such that:

1. $I_i \subset M \cap \mathbb{A}_i$ is a Morley Sequence over B_i where $B_i \subset M$ and $|B_i| < \kappa(T)$.
2. $|I_0| = [\lambda(T) + (\mathbf{F}(|A| + |\delta_0| + \kappa(T)))^{<\kappa(T)}]^+$, $|I_i| = [|\bigcup_{j<i} (M \cap \mathbb{A}_i)|^{<\kappa(T)}]^+$.
3. $I_i \downarrow_{B_i} A \cup \bigcup_{j<i} (M \cap \mathbb{A}_i)$.

Next, we construct a sequence $\langle M_i \mid i \leq \delta_0 \rangle$ by induction: $M_i \prec M$ is $\mathbf{F}$-prime over $A \cup \bigcup_{j<\delta_0} B_i \cup \bigcup_{j<i} [(M \cap \mathbb{A}_j) \cup M_j]$. Note that for every i, $|M_i| < |I_i|$. By 2.4.1 (3), we can find $I'_i \subset I_i$ such that $|I'_i| = |M_i|$ and $I_i \setminus I'_i \downarrow_{B_i} M_i$. Choose $J_i \subset (I_i \setminus I'_i)$ such that $|J_i| = \lambda$.

Finally, we constuct a sequence $\langle N_i \mid i \leq \delta_0 \rangle$ by induction: $N_i \prec M_i$ is $\mathbf{F}$-prime over $A \cup \bigcup_{j<\delta_0} B_i \cup \bigcup_{j<i} [N_j \cup J_j]$. Note that for $i < \delta_0$, $J_i \downarrow_{N_0} N_i$ (as $J_i \downarrow_{B_i} M_i$, $N_i \subset M_i$ and $B_i \subset N_0$). Let $N = N_{\delta_0}$.

Claim 1: *For any $i < \delta_0$, $N \cap \mathbb{A}_\sqsupset \subset \mathbb{N}_{\sqsupset+\nvdash}$.*

Proof of Claim 1: Note first that $\langle J_i \mid i < \delta_0 \rangle$ is independent over N_0. Thus, by 2.2.2 (1), we may view N as $N_{i+1}[\bigcup_{j>i} J_j]$. By construction, $\bigcup_{j>i} J_j \downarrow_{N_o} M_{i+1}$. So since $M \cap \mathbb{A}_\sqsupset \subset \mathbb{M}_{\sqsupset+\nvdash}$ and $N_0 \subset N_{i+1} \subset M_{i+1}$, $\bigcup_{j>i} J_j \downarrow_{N_{i+1}} \cup (M \cap \mathbb{A}_\sqsupset)$. Therefore, as $\bigcup_{j>i} J_j \rhd_{N_{i+1}} N$, we get $N \cap \mathbb{A}_\sqsupset \subset \mathbb{N}_{\sqsupset+\nvdash}$ as desired. □ (claim 1)

For each $i < \delta_0$, let $a_i \in J_i$ and let $p_i = tp(a_i, N_0)$. Let $\langle E_i \mid i < \delta_0 \rangle$ be an independent sequence over N_0 such that each E_i is a Morley Sequence for p_i of length $G(i)$. Let M' be $\mathbf{F}$-prime over $N_0 \cup \bigcup_{i<\delta_0} E_i$.

Claim 2: *M' is a G-model.*

Proof of Claim 2: Suppose not. Let $i_0 < \delta_0$ be least such that $|M' \cap \mathbb{A}_{\sqsupset_\nvdash}| > \mathbb{G}(\sqsupset_\nvdash)$ (remember: $E_i \subset M' \cap \mathbb{A}_\sqsupset$ ensures that $|M' \cap \mathbb{A}_\sqsupset| \geq \mathbb{G}(\sqsupset)$ for every i). Let $E_\leq = \bigcup_{i \leq i_0} E_i$ and $E_> = \bigcup_{i > i_0} E_i$. By 2.2.2 (1), we can view M' as $N_0[E_\leq][E_>]$. Note that $|N_0[E_\leq]| \leq \mathbf{F}(|N_0 \cup \bigcup_{i \leq i_0} E_i|) = G(i_0)$. So, there exists $b \in (M' \cap \mathbb{A}_{\sqsupset_\nvdash}) \setminus \mathbb{N}_\nvdash[\mathbb{E}_\leq]$.

Since M' is $\mathbf{F}$-atomic over $N_0[E_\leq] \cup E_>$, we can find $B \subset N_0[E_\leq]$ and $E \subset E_>$ such that $tp(b, N_0[E_\leq] \cup E_>) \in \mathbf{F}(B \cup E)$ (so, $|B \cup E| < \lambda$). Let $C \subset N_0[E_\leq]$ be completely closed in some construction of $N_0[E_\leq]$ over $N_0 \cup E_\leq$. W.L.O.G. $|C \cap E_i| = \lambda$ for each $i \leq i_0$ and $N_0 \subset C$. Let $N_0[C] \prec N_0[E_\leq]$ be $\mathbf{F}$-prime over C.

Let $\langle E'_i \mid i_0 < i < \delta_0 \rangle$ be a sequence such that for each i, $E'_i \subset E_i$, $|E'_i| = \lambda$, and $E \subset \bigcup_{i>i_0} E'_i$. Let $E' = \bigcup_{i>i_0} E'_i$. Note that $tp(b, N_0[C] \cup E') \in \mathbf{F}(B \cup E)$. Let $N_0[C][E'] \prec M'$ be $\mathbf{F}$-prime over $N_0[C] \cup E'$ and contain b.

Note that there is an $F \in Aut_{N_0}\mathbb{M}$ such that for $i \leq i_0$, $F(C \cap E_i) = J_i$ and for $i > i_0$, $F(E_i') = J_i$. So by 2.2.2 (2), we can find an $F' \in Aut_{N_0}\mathbb{M}$ such that $F'(N') = N$ and, more importantly, $F'(N_0[C]) = N_{i_0+1}$. As F' fixes A pointwise, and as $\mathbb{A}_{\beth_{\nvDash}}$ is ∞-definable over A, F' fixes $\mathbb{A}_{\beth_{\nvDash}}$ as a set. So $F'(b) \in \mathbb{A}_{\beth_{\nvDash}} \setminus \mathbb{N}_{\beth_{\nvDash}+\nVdash}$. But this contradicts $N \cap \mathbb{A}_{\beth_{\nvDash}} \subset N_{i_0+1}$. $\Box$ (claim 2, theorem)

Corollary 2.4.5 *Let* $\mathbf{F} = \mathbf{F}^x_\lambda$ *be standard, and let* M *witness the fact that* **F-HYP** *holds. Let* $G : \delta_0 \to CARD$ *such that*

1. *for* $i < i_0$, $G(i) = |M \cap \mathbb{A}_\beth|$
2. *for* $i \geq j \geq i_0$, $G(i)$ *is* $\mathbf{F}$*-good and* $G(i) \geq G(j) \geq |M|$.

Then there exists a $\mathbf{F}$*-saturated* M' *such that* $M \prec M'$, M' *is a* G*-model and for* $i < i_0$, $M' \cap \mathbb{A}_\beth = \mathbb{M} \cap \mathbb{A}_\beth$.

Remark: We do not need the full strength of **F-HYP** for this result. As long as M respects the cardinality constraints in **F-HYP** for $i \geq i_0$, the proof will work.

Proof. Almost the same as the proof of 2.4.4. The modifications are as follows. Instead of having $N = N_0[\bigcup_{i<\delta_0} J_i]$, let $N = M_i[\bigcup_{i_0 \leq i < \delta_0} J_i]$. When we "stretch" the J_i to get E_i, we only stretch for $i \geq i_0$. Then, letting $M' = M_i[\bigcup_{i_0 \leq i < \delta_0} E_i]$, the proof goes through as before. $\Box$

Corollary 2.4.6 *Let* $\mathbf{F} = \mathbf{F}^x_\lambda$ *be standard, and let* M *witness the fact that* **F-HYP** *holds. Suppose that for every* $i < \delta_0$, $|M \cap \mathbb{A}_\beth|$ *is regular and* T *is stable in arbitrarily large* $\nu < |M \cap \mathbb{A}_\beth|$. *Let* $G : \delta_0 \to CARD$ *be nondecreasing such that:*

1. $G(0) \geq \mathbf{F}(|A| + |\delta_0| + \kappa(T))$.
2. *for* $i < \delta_0$, $G(i) \leq |M \cap \mathbb{A}_\beth|$ *and* $G(i)$ *is* $\mathbf{F}$*-good.*

Then there exists an $\mathbf{F}$*-saturated* G*-model,* M' *such that* $M' \prec M$.

Proof. Again we follow closely the proof of 2.4.4. The modifications are as follows. First, when choosing our initial Morley Sequences $\langle I_i \mid i < \delta_0 \rangle$, we insist that $|I_i| = |M \cap \mathbb{A}_\beth|$ (this is possible by 2.4.3 and our conditions on M). When choosing our second collection of Morley Sequences $\langle J_i \mid i < \delta_0 \rangle$, we insist that $|J_i| = G(i)$.

Given $\langle J_i \mid i < \delta_0 \rangle$, we construct $\langle N_i \mid i \leq \delta_0 \rangle$ as in the theorem. Note that for each $i < \delta_0$, $|N_{i+1}| = |N_{i+1} \cap \mathbb{A}_\beth| = \mathbb{G}(\beth)$. Note also that the argument for Claim 1 in our original proof still goes through in this context. So, $N \cap \mathbb{A}_\beth \subset \mathbb{N}_{\beth+\nVdash}$ for every $i < \delta_0$. Letting $M' = N \prec M$ we obtain a G-model as desired. $\Box$

Remark: The requirement that $|M \cap \mathbb{A}_\beth|$ is regular and T is stable in arbitrarily large $\nu < |M \cap \mathbb{A}_\beth|$ is irrelevant for $i = 0$. If this requrement is eliminated, we simply include $G(0)$ elements of $M \cap \mathbb{A}_{\nvDash}$ in N_0 and let J_0 have cardinality $\emptyset$.

Corollary 2.4.7 *Let* $\mathbf{F} = \mathbf{F}^t_{|T|^+}$ *and suppose that* **F-HYP** *holds. Then for any nondecreasing* $G : \delta_0 \to CARD$ *such that* $G(0) \geq \mathbf{F}^l_{|T|}(|A| + |\delta_0| + \kappa(T))$ *and each* $G(i)$ *is* $\mathbf{F}^l_{|T|}$*-good, there exists an* $|T|$*-compact* G*-model,* M'.

Remark: If $\kappa(T) = \kappa^+$ and $|A| + |\delta_0| \leq \kappa$, then we only need $G(0) \geq \mathbf{F}^{\mathbf{l}}_{|\mathbf{T}|}(\kappa)$.

Proof. Let $G' : \delta_0 \to CARD$ be increasing such that for each $i < \delta_0$, $G(i) < G'(i)$ and $G'(i)$ is $F^t_{|T|^+}$-good. By theorem 2.4.4, we can find M' an $F^t_{|T|^+}$-saturated G'-model. By the proof of that theorem, we can assume that there is a sequence $\langle (M_i, E_i) \mid i \leq \delta_0 \rangle$ such that:

1. for $i > 0$, M_i is prime over $\bigcup_{j<i} (M_j \cup E_j)$; $M' = M_{\delta_0}$.
2. for every i, $E_i \subset \mathbb{A}_\beth$ is a Morley Sequence over M_0; $|E_i| = G'(i)$;
   and $E_i \downarrow_{M_0} M_i$.
3. for every i, $M' \cap \mathbb{A}_i \subset M_{i+1}$.

For $i < \delta_0$, let $E'_i \subset E_i$ such that $|E'_i| = G(i)$ and let $B_i \subset M_0$ such that $|B_i| < \kappa(T)$ and $E'_i \downarrow_{B_i} M_0$. Construct a sequence $\langle N_i \mid i \leq \delta_0 \rangle$ by induction: $N_i \prec M_i$ is $\mathbf{F}^{\mathbf{l}}_{|\mathbf{T}|}$-constructible and $|T|$-compact over $A \cup \bigcup_{j<\delta_0} B_j \cup \bigcup_{j<i} (N_i \cup E'_i)$. Let $N = N_{\delta_0}$.

Claim: *For all* $i < j \leq \delta_0$*,* $M_i \downarrow_{N_i} N_j$.

Proof of Claim. For fixed i, let j be minimal such that this fails. Suppose first that j is limit. Then by the finite character of forking, $M_i \downarrow_{N_i} \bigcup_{k<j} N_k$. But, since N_j is $\mathbf{F}^{\mathbf{l}}_{|\mathbf{T}|}$-constructible over $\bigcup_{k<j} N_j$, we have $\bigcup_{k<j} N_j \triangleright_{N_i} N_j$. So, $M_i \downarrow_{N_i} N_j$ for a contradiction.

So let $j = k + 1$. As $E_k \downarrow_{M_0} M_k$ and $E_k \downarrow_{B_k} M_0$, we get $E_k \downarrow_{N_k} M_k$ (remember $N_k \supset B_k$). As $E_k \triangleright_{N_k} N_{k+1}$, we have $N_{k+1} \downarrow_{N_k} M_k$. So as $M_i \subset M_k$, we get $M_i \downarrow_{N_k} N_{k+1}$. By the induction hypothesis, $M_i \downarrow_{N_i} N_k$. Therefore, $M_i \downarrow_{N_i} N_{k+1}$ for a contradiction. □ (claim)

Note that as $E'_i \subset N$, we have $|N \cap \mathbb{A}_\beth| \geq \mathbb{G}(\beth)$ for all i. Further for any i, $|N_{i+1}| = G(i)$. By the claim, $M_{i+1} \downarrow_{N_{i+1}} N$. So since $N \cap \mathbb{A}_\beth \subset \mathbb{M}_{\beth+\not\vdash}$, $N \cap \mathbb{A}_\beth \subset \mathbb{N}_{\beth+\not\vdash}$. Therefore, $|N \cap \mathbb{A}_\beth| = \mathbb{G}(\beth)$. Letting $M' = N$ we are done. □

Corollary 2.4.8 *Let* $\mathbf{F} = \mathbf{F}^x_\lambda$ *be standard, and suppose* **F-HYP** *holds. Then for any nondecreasing* $G : \delta_0 \to CARD$ *such that* $|\mathfrak{L}| + |\bigcup_{i<\delta_0} B_i| + |\delta_0| \leq G(0)$*, there is a* G*-model.*

Proof. The same idea as the proof of 2.3.4. Use 2.4.4 in place of 2.3.2. □

Remarks: (1) When T is ω-stable and $\mathbf{F} = \mathbf{F}^{\mathbf{t}}_\omega$, special cases of the above corollaries lead to standard results. For $\delta_0 = n$, 2.4.8 gives Forrest's generalization of Shelah's two cardinal theorem. For $\delta_0 = 2$, 2.4.6 gives a theorem originally proved by Lascar, see [Ls].

(2) Theorem 2.4.4 is weaker than theorem 2.3.2 in three ways. First, **F-HYP** requires that we start with a single model which witnesses all of the relevant cardinality splits simultaneously. **HYP** allows us to

use different models to witness different cardinality splits. Indeed, **HYP** does not require actual cardinality splits: a pair of models $P(M) \subset N \supsetneq M$ can witness **HYP** even though $|P(M)| = |N| = |M|$.

Second, **F-HYP** requires that the model we start out with have a relatively high degree of saturation: it must be saturated for some standard **F**. This is true even when the sets $\langle \mathbb{A}_\beth \mid \beth < \delta_\nvDash \rangle$ are definable in some simple manner—i.e. by single formulas. In such cases, **HYP** would allow us to start with models having a low degree of saturation (in the single formula case, we would just need models).

Finally, 2.4.4 allows us to start with some **F**-saturated multi-cardinal model and construct another model with the same (or lesser) degree of saturation. We cannot use 2.4.4 to increase the saturation of our models: we cannot, for instance, start with an $\mathbf{F}^{\mathbf{a}}_{\lambda}$-saturated model and use 2.4.4 to construct an $\mathbf{F}^{\mathbf{a}}_{\lambda^+}$-saturated model. In contrast, 2.3.2 allows us to start with models having some low degree of saturation and then construct models with arbitrarily high degrees of saturation.

(3) Theorem 2.4.4 has, however, one major advantage over theorem 2.3.2. The main hypothesis of 2.4.4, **F-HYP**, involves no restrictions on the way the sets in $\langle \mathbb{A}_\beth \mid \beth < \delta_\nvDash \rangle$ are defined (other than that they are ∞-definable over the M mentioned in **F-HYP**). Therefore, if each $\mathbb{A}_\beth$ requires a large amount of information to define, a model M can witness **F-HYP** without witnessing **HYP**.

To illustrate, suppose that T is countable and that we are only interested in comparing the cardinalities of two ∞-definable sets, $\mathbb{A}_\nvDash$ and $\mathbb{A}_\nVdash$. In this case, any ω_1-saturated model M such that $M \supset A$ and $|M \cap \mathbb{A}_\nVdash| > |\mathbb{M} \cap \mathbb{A}_\nvDash|^\omega$ will witness $\mathbf{F}^{\mathbf{t}}_{\omega_1}$**-HYP**. In contrast, if $|A| > \omega$ and A is minimal such that $\mathbb{A}_\nvDash$ is definable over A, then we would need a model which is at least $|A|^+$-saturated to witness **HYP**.

Thus, whenever $|T|$ is small in comparison to the sizes of the Γ_i's used to define the $\mathbb{A}_\beth$'s, we can use 2.4.4 to get results which 2.3.2 cannot get.

Chapter 3

Chang's Conjecture in Stable Theories

The first result on Chang's conjecture in the context of stable theories was obtained by Lascar in [Ls]. Lascar showed that if T is ω-stable and M is a model of type $(\kappa; \lambda)$ where $\kappa > \lambda$ and κ is regular, then for any κ' and λ' such that $\kappa' \leq \kappa$, $\lambda' \leq \lambda$ and $\omega \leq \lambda' \leq \kappa'$, there exists a model N of type $(\kappa'; \lambda')$ such that $N \prec M$. In [Sh], Shelah extended this result to the superstable case and eliminated the requirement that κ be regular (replacing this requirement with some somewhat messier conditions on the relationships between the various cardinals in question).

We have already examined one result in the general neighborhood of Chang's conjecture: in chapter 2, corollary 2.4.6 provides a strong generalization of Lascar's theorem. If we examine the proof of this corollary carefully, we will see that it depends on our ability to build arbitrarily large Morley Sequences within each set of the form $M \cap \mathbb{A}_{\beth}$, $i < \delta_0$. As a result, the corollary places some fairly strong conditions on the cardinalities of these sets: for every $i < \delta_0$, $|M \cap \mathbb{A}_{\beth}|$ must be regular and T must be stable in arbitrarily large $\mu < |M \cap \mathbb{A}_{\beth}|$. The second of these conditions fails, even for T superstable, if we are working with sets of cardinality $\leq \lambda(T)$. It also fails if T is not superstable and we are working with a set of size λ where $\nu < \lambda \leq \nu^{<\kappa(T)}$ for some ν.

To get around these problems, we develop some techniques for eliminating the use of Morley Sequences from proofs like that of 2.4.6. In section 3.1 we focus on the task of building "large" sets which are independent from some specified smaller sets. That is, suppose $|A| > |B|$; then we would like to find sets A', B' such that,

- $A' \subseteq A$ and $|A'|$ is as large as possible (ideally, $|A'| = |A|$).
- $|B'|$ is as small as possible (ideally, $|B'| < \kappa(T)$).
- $A' \downarrow_{B'} B$.

In 3.1 and 3.2, we classify fairly precisely the extent to which this project can be carried out. In 3.1, we give some general conditions under which sets of the form A' and B' can be found (see 3.1.4). In 3.2, we illustrate why these conditions are likely to be the best available: for cases where the conditions fail, we provide stratagies for building models of ZFC in which counterexamples can be found (but, see the remarks following our GCH examples for a gap in this classification).

In section 3.3, we apply the results of 3.1 and prove several 2-cardinal versions of Chang's Conjecture. We also show how these theorems can be used to get results on admitting cardinals. In 3.4, we extend the results of 3.3 from the 2-cardinal case to δ-cardinal case and prove two separate δ-cardinal theorems. Finally, we examine a further result on admitting cardinals which follows from the two theorems just mentioned (see 3.4.4).

Throughout this chapter, T is assumed to be stable.

3.1 Independence and Cardinality I

For convenience in stating several of the results in this section we adopt the following piece of notation. We say that $\underline{\dagger(\mu,\nu,\lambda)}$ holds whenever,

- $\kappa(T) \leq cf(\mu) \leq \mu \leq \nu$.
- λ is regular, and $\lambda \leq \kappa_r(T)$.
- there is no μ' such that $\mu \leq \mu' \leq \nu$ and $\lambda \leq cf(\mu') < \kappa(T)$.

Note here that if $\lambda = \kappa_r(T)$, then $\dagger(\mu,\nu,\lambda)$ holds trivially (providing, of course, that $\kappa(T) \leq cf(\mu) \leq \mu \leq \nu$). Note also that if $\dagger(\mu,\nu,\lambda)$ holds, $\mu \leq \mu' \leq \nu' \leq \nu$ and $cf(\mu') \geq \kappa(T)$, then $\dagger(\mu',\nu',\lambda)$ holds as well. See the remarks following theorem 3.1.4 for discussion of the motivation for this definition.

Lemma 3.1.1 *Let A, B, C and B_C be subsets of $\mathbb{M}$ such that $cf(|B|) \geq \kappa(T)$ and $C \downarrow_{B_C} B$. Then we can partition A into $cf(|B|)$ pieces (some of which may be empty), $\langle A_i \mid i < cf(|B|)\rangle$, such that for each A_i there is a $B_i \subseteq B$ where $|B_i| < |B|$ and $A_i \cup C \downarrow_{B_i \cup B_C} B$.*

Remark: The proof of this lemma is essentially due to Shelah; it is contained within his proof of IX 1.4 in [Sh].

Proof of Lemma. Let $\nu = cf(|B|)$ and let $\langle B_i \mid i \leq \nu)\rangle$ be an increasing, continuous sequence of subsets of B such that $|B_i| < |B|$ for $i < \nu$, and $B_\nu = B$. We define an increasing sequence of subsets of A, $\langle A_i \mid i < \nu)\rangle$, by induction: A_i is a maximal subset of A such that $\bigcup_{j<i} A_j \subseteq A_i$ and $A_i \cup C \downarrow_{B_i \cup B_C} B$.

Now I claim that $\bigcup_{i<\nu} A_i = A$. For suppose $a \in A \setminus (\bigcup_{i<\nu} A_i)$. Then for each i, $tp(A_i \cup C \cup \{a\}, B \cup B_C)$ forks over $B_i \cup B_C$

$$\implies tp(a, A_i \cup C \cup B \cup B_C) \text{ forks over } A_i \cup C \cup B_i \cup B_C.$$

$$\implies \exists j > i \text{ such that } tp(a, A_i \cup C \cup B_j \cup B_C) \text{ forks over } A_i \cup C \cup B_i \cup B_C.$$

$$\implies \exists j > i \text{ such that } tp(a, A_j \cup C \cup B_j \cup B_C) \text{ forks over } A_i \cup C \cup B_i \cup B_C.$$

So, letting $p_i = tp(a, A_i \cup C \cup B_i \cup B_C)$, we have that a cofinal subsequence of $\langle P_i \mid i < \nu \rangle$ is a forking sequence. And as $\kappa(T) \leq \nu$, this gives a contradiction.

Replace each A_i with $A_i \setminus \bigcup_{j<i} A_j$. Then $\langle A_i \mid i < \nu \rangle$ is a partition of A; and for each $i < \nu$, $A_i \cup C \downarrow_{B_i \cup B_C} B$. As $|B_i| < |B|$, we are done. $\square$

Lemma 3.1.2 (The Partition Lemma) *Let B, C and B_C be subsets of $\mathbb{M}$ such that $C \downarrow_{B_C} B$. Suppose that $\dagger(\mu, |B|, \lambda)$ holds. Then for arbitrary A, we can partition A into $|B|^{<\lambda}$ pieces, $\langle A_i \mid i < |B|^{<\lambda} \rangle$, such that for each A_i there is a $B_i \subseteq B$ where $|B_i| < \mu$ and $A_i \cup C \downarrow_{B_i \cup B_C} B$.*

Proof. We take cases on the regularity of μ.

Case 1. (μ regular) We fix C, B_C, μ and λ and proceed by induction on $|B| \geq \mu$. If $|B| = \mu$, then we can apply lemma 3.1.1 to get a partition of A into $cf(|B|)$ pieces (where A is arbitrary). Adding empty pieces to the partition as needed, we get a partition of size $|B|^{<\lambda}$ as desired.

So let $\mu < |B|$ and assume that the lemma holds for every B' such that $\mu \leq |B'| < |B|$. If $\dagger(\mu, \nu, \lambda)$ fails for some $\mu \leq \nu \leq |B|$ then we have a contradiction (as in that case, $\dagger(\mu, |B|, \lambda)$ fails as well). So, $\dagger(\mu, \nu, \lambda)$ must hold for every $\mu \leq \nu \leq |B|$. We take subcases on $cf(|B|)$.

Subcase a. ($cf(|B|) \geq \kappa(T)$): By lemma 3.1.1 we can partition A into $cf(|B|)$ pieces $\langle A_i \mid i < cf(|B|) \rangle$, such that for each A_i there is a $B_i \subseteq B$ where $|B_i| < |B|$ and $A_i \cup C \downarrow_{B_i \cup B_C} B$. W.L.O.G., we may assume $|B_i| \geq \mu$ for each $i < cf(|B|)$.

For every $i < cf(|B|)$, we can apply the induction hypothesis and partition A_i as $\langle A_{i,j} \mid j < |B_i|^{<\lambda} \rangle$ such that each $A_{i,j}$ has an associated $B_{i,j} \subset B_i$ where $|B_{i,j}| < \mu$ and $A_{i,j} \cup C \downarrow_{B_{i,j} \cup B_C} B_i$. By forking transitivity, $A_{i,j} \cup C \downarrow_{B_{i,j} \cup B_C} B$. Note that the number of such $A_{i,j}$ is at most $|B| \cdot |B|^{<\lambda} = |B|^{<\lambda}$. So, letting $P = \{A_{i,j} \mid i < cf(|B|) \text{ and } j < |B|^{<\lambda}\}$ be our partition, we are done.

Subcase b. ($cf(|B|) < \kappa(T)$): We begin by choosing an increasing, continuous sequence $\langle B_i \mid i \leq cf(|B|) \rangle$ of subsets of B such that $|B_0| = \mu$, $|B_i| < |B|$ for $i < cf(|B|)$, and $B_{cf|B|} = B$. By induction, we construct a sequence of partitions, $\langle P_i \mid i \leq cf(|B|) \rangle$, satisfying the following conditions:

1. Each P_i is a partition of A into $|B_i|^{<\lambda}$ pieces.
2. For each $A_{i,j}$ in P_i there exists $B_{i,j} \subset B_i$ such that $|B_{i,j}| < \mu$ and $A_{i,j} \cup C \downarrow_{B_{i,j} \cup B_C} B_i$.
3. If $j > i$, then P_j refines P_i

When the construction is finished, $P_{cf|B|}$ will be the desired partition.

Construction: For $i = 0$, we apply the original induction hypothesis and partition A over B_0 so as to satisfy conditions 1 and 2.

For $i = j+1$, we let $P_j = \langle A_{j,k} \mid k < |B_j|^{<\lambda}\rangle$. As $|B_i| < |B\,|$, we can again apply our original induction hypothesis and partition each $A_{j,k}$ into $|B_i|^{<\lambda}$ pieces over B_i such that each $A_{j,k,l}$ has an associated $B_{j,k,l} \subseteq B_i$ where $|B_{j,k,l}| < \mu$ and $A_{j,k,l} \cup C \downarrow_{B_{j,k,l} \cup B_C} B_i$. We then set,

$$P_i = \{A_{j,k,l} : \ k < |B_j|^{<\lambda} \text{ and } l < |B_i|^{<\lambda}\}.$$

Clearly, P_i satisfies conditions 2 and 3. Since $(|B_j|^{<\lambda}) \cdot (|B_i|^{<\lambda}) = |B_i|^{<\lambda}$, condition 1 is satisfied as well.

For i limit, we define P_i through its associated equivalence relation. Using the obvious notation, we set:

$$a \sim_i c \Longleftrightarrow \text{ for every } j < i,\ a \sim_j c.$$

Clearly, P_i satisfies condition 3. Condition 1 follows from the computation:

$$|P_i| \leq \prod_{j<i} |P_j| \leq \prod_{j<i} |B_j|^{<\lambda} \leq \prod_{j<i} |B_i|^{<\lambda} = (|B_i|^{<\lambda})^{|i|} = |B_i|^{<\lambda}.$$

For the final step of this computation, we need to know that $i < \lambda$. But as $cf(|B\,|) < \kappa(T)$ and $\dagger(\mu, |B\,|, \lambda)$ holds, we must have $cf(|B\,|) < \lambda$. So, since $i \leq cf(|B|)$, $i < \lambda$.

For condition 2, note that for each $A_{i,k} \in P_i$ and each $j < i$ we can find some $A_{j,k} \in P_j$ and an associated $B_{j,k} \subset B_j$ such that $A_{i,k} \subset A_{j,k}$, $|B_{j,k}| < \mu$ and $A_{j,k} \cup C \downarrow_{B_{j,k} \cup B_C} B_j$. By continuity, $A_{i,k} \cup C \downarrow_{\bigcup_{j<i} B_{j,k} \cup B_C} B_i$. Finally, since $i \leq cf(|B|) < \kappa(T) \leq \mu$, we get $|\bigcup_{j<i} B_{j,k}| < \mu$.

Case 2. (μ singular) If $|B\,| = \mu$, then we can simply apply lemma 3.1.1. So suppose $|B\,| > \mu$ and let A be arbitrary. By Case 1, we can partition A into $|B\,|^{<\lambda}$ pieces, $\langle A_i \mid i < |B\,|^{<\lambda}\rangle$, such that each A_i has an associated $B_i \subset B$ where $|B_i| < \mu^+$ and $A_i \cup C \downarrow_{B_i \cup B_C} B$. W.L.O.G., we may assume that each B_i has cardinality μ.

By lemma 3.1.1, we can partition each A_i as $\langle A_{i,j} \mid j < cf(\mu)\rangle$ such that each $A_{i,j}$ has an associated $B_{i,j} \subset B_i$ where $|B_{i,j}| < \mu$ and $A_{i,j} \cup C \downarrow_{B_{i,j} \cup B_C} B_i$. By forking transitivity, then, $A_{i,j} \cup C \downarrow_{B_{i,j} \cup B_C} B$. Further, the total number of such $A_{i,j}$ is at most $cf(\mu) \cdot |B\,|^{<\lambda} = |B\,|^{<\lambda}$. □

Remark: The proof of this lemma is complicated by the need to prepare for some constructions employed in proving case three of 3.1.4. The main ideas in the lemma's proof come through more clearly when this proof is done for the special case mentioned in the next theorem.

Theorem 3.1.3 *Let A and B be arbitrary subsets of $\mathbb{M}$. Then we can partition A into $|B\,|^{<\kappa_r(T)}$ pieces $\langle A_i \mid i < |B\,|^{<\kappa_r(T)}\rangle$, such that for each A_i there is a $B_i \subseteq B$ where $|B_i| < \kappa_r(T)$ and $A_i \downarrow_{B_i} B$.*

Proof. We apply lemma 3.1.2 letting $C = B_C = \emptyset$ and letting $\mu = \lambda = \kappa_r(T)$. Note that because $\lambda = \kappa_r(T)$, $\dagger(\mu, |B\,|, \lambda)$ is trivially satisfied. □

Theorem 3.1.4 *Let A and B be subsets of $\mathbb{M}$ such that $|A| > |B|$. Let $\kappa \leq |A|$, let $\mu < |B|$ and suppose that one of the following conditions holds:*

1. $\kappa < |A|$, *and there exists* λ *such that* $\dagger(\mu^+, |B|, \lambda)$ *holds and* $|B|^{<\lambda} < |A|$.

2. $\kappa = |A|$, *and there exists* λ *such that* $\dagger(\mu^+, |B|, \lambda)$ *holds and* $|B|^{<\lambda} < cf(|A|)$.

3. $\kappa = |A|$, *and there exists* λ *such that* $\dagger(\mu^+, |B|, \lambda)$ *holds and* $|B|^{<\lambda} < |A|$ *and* $cf(|A|) < \mu^+$.

Then there exists $A' \subset A$, $B' \subset B$ *such that* $|A'| = \kappa$, $|B'| = \mu$ *and* $A' \downarrow_{B'} B$.

Proof. Suppose first that condition 1 holds. By the partition lemma, 3.1.2, we can partition A into $|B|^{<\lambda}$ pieces, $\langle A_i \mid i < |B|^{<\lambda}\rangle$, such that for each A_i there is a $B_i \subseteq B$ where $|B_i| < \mu^+$ and $A_i \downarrow_{B_i} B$. W.L.O.G., each B_i has cardinality μ; and as $|B|^{<\lambda} < |A|$, one of the A_i sets has size at least κ. Making this set smaller as necessary, we are done.

Suppose next that condition 2 holds. Again, we employ 3.1.2 to partition A into $|B|^{<\lambda}$ pieces, $\langle A_i \mid i < |B|^{<\lambda}\rangle$, such that for each A_i there is a $B_i \subseteq B$ where $|B_i| < \mu^+$ and $A_i \downarrow_{B_i} B$. Again we may assume that each B_i has cardinality μ. As $|B|^{<\lambda} < cf(|A|)$, one of the A_i sets must have size $|A|$.

Finally, suppose that condition 3 holds. Let $\langle \kappa_k \mid k < cf(\kappa)\rangle$ be increasing such that $\kappa = \sum_{k<cf(\kappa)} \kappa_k$. We construct by induction a sequence $\langle (A_k, B_k) \mid k < cf(\kappa)\rangle$ such that for each k: $A_k \subset A$ and $|A_k| \geq \kappa_k$; $B_k \subset B$ and $B_k = \mu$; $\bigcup_{j\leq k} A_k \downarrow_{\bigcup_{j\leq k} B_k} B$. Assume we have constructed this sequence for $j < k$. By forking continuity, we know that $\bigcup_{j<k} A_k \downarrow_{\bigcup_{j<k} B_j} B$. By 3.1.2, we can partition A into $|B|^{<\lambda}$ pieces, $\langle A'_i \mid i < |B|^{<\lambda}\rangle$, such that for each A'_i there is a $B'_i \subseteq B$ where $|B'_i| < \mu^+$ and $A'_i \cup \bigcup_{j<k} A_j \downarrow_{B_i \cup \bigcup_{j<k} B_j} B$. W.L.O.G., each B'_i has cardinality μ; and as $|B|^{<\lambda} < |A|$, one of the A'_i sets has cardinality at least κ_k. Letting this A'_i be A_k and letting the associated B'_i be B_k, we finish the construction.

Let $A' = \bigcup_{k<cf(\kappa)} A_k$ and $B' = \bigcup_{k<cf(\kappa)} B_k$. By forking continuity, $A' \downarrow_{B'} B$. Further, $|A'| = \sum_{k<cf(\kappa)} \kappa_k = \kappa$. Finally, $\mu \leq |B'| \leq cf(\kappa) \cdot \mu = \mu$. $\square$

Remark: Some comments are in order concerning the role $\dagger$ plays in this theorem. Suppose that A and B are as in the theorem and that $|A|$ is regular (so we can ignore cofinality issues). If T is superstable, then we can find the desired A' and B' without invoking lemma 3.1.2 and without mentioning $\dagger$. We simply apply lemma 3.1.1 repeatedly to construct a sequence $\langle (A_i, B_i) \mid i < \omega\rangle$ such that:

- $(A_0, B_0) = (A, B)$; $i < j < \omega \Longrightarrow A_j \subset A_i, B_j \subset B_i$.
- for each $i < \omega$, $|A_i| = |A|$; if $|B_i| \geq \omega$, then $|B_{i+1}| < |B_i|$.
- $A_{i+1} \downarrow_{B_{i+1}} B_i$.

As the cardinals are well-founded, we eventually reach an n such that $|B_n| < \omega$. By forking monotonicity, $A_n \downarrow_{B_n} B$. Shrinking A_n and expanding B_n as necessary, we obtain the desired A' and B'.

If T is not superstable, then this argument will fail whenever we attempt to run the construction through some singular cardinal with cofinality $< \kappa(T)$ (as lemma 3.1.1 cannot be applied in this situation). So, we have to turn to the sort of argument found in lemma 3.1.2, case 1, subcase b. This latter argument introduces a factor of $|B_i|^{cf(|B_i|)}$ into the computation; hence, as the cofinality in question becomes larger, the construction becomes more difficult. $\dagger(\mu^+, |B|, \lambda)$ picks out the largest problematic cofinality for a cardinal between the one we begin with, $|B|$, and the one we wish to end up with, μ. It serves, therefore, to measure the difficulty of the required construction.

Under certain conditions, the hypotheses involving $\dagger$ in the above theorem are trivially satisfied; in such cases, the theorem itself can be stated in a somewhat smoother fashion. The following three corollaries provide the most significant "smooth" versions of the theorem.

Corollary 3.1.5 *Let A, B, κ and μ be as in the theorem, and suppose that T is superstable. Suppose that one of the following holds:*

1. $\kappa < |A|$
2. $\kappa = |A|$ *and* $|B| < cf(|A|)$
3. $\kappa = |A|$ *and* $cf(|A|) < \mu^+$.

Then there exists $A' \subset A$, $B' \subset B$ such that $|A'| = \kappa$, $|B'| = \mu$ and $A' \downarrow_{B'} B$.

Proof. Since $\kappa_r(T) = \omega$, $\dagger(\mu^+, |B|, \kappa_r(T))$ holds automatically. So, letting $\lambda = \omega$, conditions 1-3 in the corollary are equivalent to conditions 1-3 in theorem 3.1.4 (note that for $\lambda = \omega$, $|B|^{<\lambda} = |B|$). Thus, we can simply apply the theorem. □

Remark: Shelah has an alternate proof of this result which makes extensive use of large independent sets [Sh] V, 6.16-6.17. Shelah's proof does not generalize to the non-superstable case.

Corollary 3.1.6 *Let A, B, κ and μ be as in the theorem, let $\mu^+ \geq \kappa_r(T)$, and suppose there are no singular cardinals between μ and $|B|$. Suppose that one of the following holds:*

1. $\kappa < |A|$
2. $\kappa = |A|$ *and* $|B| < cf(|A|)$
3. $\kappa = |A|$ *and* $cf(|A|) < \mu^+$.

Then there exists $A' \subset A$, $B' \subset B$ such that $|A'| = \kappa$, $|B'| = \mu$ and $A' \downarrow_{B'} B$.

Proof. Since there are no singular cardinals between μ and $|B|$, $\dagger(\mu^+, |B|, \omega)$ holds automatically. Again, we note that $|B|^{<\omega} = |B|$ and apply the theorem. □

Corollary 3.1.7 *Let A, B, κ and μ be as in the theorem, let $\mu^+ \geq \kappa_r(T)$, and suppose that $|A| > |B|^{<\kappa_r(T)}$. Suppose that one of the following holds:*

1. $\kappa < |A|$

2. $\kappa = |A|$ *and* $|B|^{<\kappa_r(T)} < cf(|A|)$

3. $\kappa = |A|$ *and* $cf(|A|) < \mu^+$.

Then there exists $A' \subset A$, $B' \subset B$ *such that* $|A'| = \kappa$, $|B'| = \mu$ *and* $A' \downarrow_{B'} B$.

Proof. Apply the theorem, noting that $\dagger(\mu^+, |B|, \kappa_r(T))$ holds automatically. $\square$

3.2 Independence and Cardinality II

In this section, we give reasons for thinking that theorem 3.1.4 is the best we can expect to prove within the confines of ZFC. To avoid unnecessary notation when stating our results, we restrict ourselves to the countable language case. For some comments concerning the uncountable case, see the remarks at the end of this section.

For convenience, we adopt the following notation: we say that $\underline{\ddagger(\kappa, \kappa', \lambda, \lambda')}$ holds if for every set A of cardinality κ and every set B of cardinality λ, we can find $A' \subset A$ and $B' \subset B$ with cardinalities κ' and λ' respectively such that $A' \downarrow_{B'} B$. Note that questions concerning $\ddagger(\kappa, \kappa', \lambda, \lambda')$ are only interesting when $\kappa' \leq \kappa$, $\lambda' < \lambda$, and $\lambda < \kappa$; hence, we will always take these conditions for granted.

We begin by showing that 3.1.4 gives optimal results under the assumption that T is superstable. Note that by corollary 3.1.5 the only cases in which $\ddagger(\kappa, \kappa', \lambda, \lambda')$ can fail (under our superstability assumption) are cases in which κ is singular, $\kappa' = \kappa$, and $\lambda' < cf(\kappa) \leq \lambda$. Fix some particular κ, κ', λ, and λ' satisfying these conditions.

Let $\mathfrak{L} = \{\mathfrak{P}, \mathfrak{Q}, \mathfrak{g}\}$ where P and Q are unary predicates and G is a binary relation. Let a model for $\mathfrak{L}$ be given as follows:

- $P(M) = \lambda$; $Q(M) = \lambda \times \kappa$; $M = P(M) \cup Q(M)$.
- G is a function from $Q(M)$ to $P(M)$ such that $G((\alpha, \beta)) = \alpha$.

Here is the intuitive idea. $P(M)$ and $Q(M)$ are infinite, disjoint sets; $P(M)$ has no intrinsic structure, and $Q(M)$ uses G to associate an infinite set of (otherwise undifferentiated) elements to each element of $P(M)$. Let $T = Th(M)$; it is easy to check that T is superstable (indeed ω-stable) and quantifier eliminable.

Let $\langle \kappa_i \mid i < cf(\kappa) \rangle$ be increasing and cofinal in κ. Let N be a submodel of M such that $P(N) = P(M)$ and

$$(\alpha, \beta) \in Q(N) \Longleftrightarrow \text{either } \ \alpha < cf(\kappa) \ \& \ \beta < \kappa_\alpha \quad \text{or} \quad \beta < \omega.$$

So, $|Q(N)| = \kappa$ and $|P(N)| = \lambda$. However, suppose $A' \subset Q(N)$ such that $|A'| = \kappa$. Then a trivial combinatorial argument shows that $G(A')$ must have cardinality at least $cf(\kappa)$. Further, it is easy to see that given any $B' \subset P(N)$ such that $G(A') \not\subset B'$, $tp(A', P(N))$ forks over B' (as witnessed, for instance, by the formula "x=G(y)"). So, $Q(N)$ and $P(N)$ witness the failure of $\ddagger(\kappa, \kappa', \lambda, \lambda')$ as desired.

If we eliminate the assumption that T is superstable, then there are several additional cases in which $\ddagger(\kappa, \kappa', \lambda, \lambda')$ can fail (i.e. several additional cases which are not ruled out by 3.1.4). Initially, it is convenient, to consider these cases under the assumption that GCH holds. Under this assumption, there are only three types of cases in which $\ddagger(\kappa, \kappa', \lambda, \lambda')$ can fail:

1. $\kappa' = \kappa$; and $\lambda' < cf(\kappa) \leq \lambda$.

2. $cf(\lambda) = \omega$; $\kappa = \lambda^+$.

3. $\kappa' = \kappa$; $cf(\lambda) = \omega$; and $cf(\kappa) = \lambda^+$.

The first of these cases can be "explained" using the same model we used for the superstable case. So we turn to the second, and fix some particular κ, κ', λ, and λ' satisfying those conditions.

Suppose that $\kappa' \geq \lambda$. Let $\mathfrak{L} = \{\mathfrak{P}, \mathfrak{Q}, \langle \mathfrak{F}_\mathfrak{i} \mid \mathfrak{i} < \omega \rangle\}$ where P and Q are unary predicates and G and the F_i's are binary relations. Let a model N for $\mathfrak{L}$ be given as follows:

- $P(N) = \lambda$; $Q(N) = {}^\omega\lambda$; $N = P(N) \cup Q(N)$.

- $F_i : Q(N) \to P(N)$ by $F_i(\eta) = \eta(i)$.

Here is the intuitive idea. $P(N)$ and $Q(N)$ are disjoint sets. $P(M)$ has no intrinsic structure; elements of $Q(M)$ "code up" sets of size $\leq \omega$ in $P(M)$ via the sequence $\langle F_i \mid i < \omega \rangle$. Let $T = Th(M)$. It is easy to check that T is stable and quantifier eliminable and that $\kappa(T) = \omega_1$. Similarly, it is clear that $|Q(N)| = \lambda^+$ and $|P(N)| = \lambda$.

Let A' be an arbitrary subset of $Q(N)$ such that $|A'| = \kappa'$. Then, since no set of size $< \lambda$ can have κ' distinct subsets, $|\bigcup_{i<\omega} F_i(A')| = \lambda$. Further, it is easy to see that given any $B' \subset P(N)$ such that $\bigcup_{i<\omega} F_i(A') \not\subset B'$, $tp(A', P(N))$ forks over B' (as witnessed by some formula of the form "$x = F_i(y)$"). So, $Q(N)$ and $P(N)$ witness the failure of $\ddagger(\kappa, \kappa', \lambda, \lambda')$ as desired.

Remark: Shelah also discusses (or at least mentions) this model in connection with two-cardinal phenomena. See [Sh], V.6 and IX.1.

We turn now to case three, and fix some particular κ, κ', λ, and λ' satisfying the relevant conditions. We we build a countermodel by combining the two models we have already examined. Let $\mathfrak{L} = \{\mathfrak{P}, \mathfrak{Q}, \mathfrak{R}, \mathfrak{G}, \langle \mathfrak{F}_\mathfrak{i} \mid \mathfrak{i} < \omega \rangle\}$ where P, Q and R are unary predicates and G and the F_i's are binary relations. Let a model M for $\mathfrak{L}$ be given as follows:

- $P(M) = \lambda$; $Q(M) = {}^\omega\lambda$; $R(M) = {}^\omega\lambda \times \kappa$; $M = P(M) \cup Q(M) \cup R(M)$.

- $G : R(M) \to Q(M)$ by $G(\eta, \beta) = \eta$; $F_i : Q(M) \to P(M)$ by $F_i(\eta) = \eta(i)$.

Here is the intuitive idea. $P(M)$, $Q(M)$ and $R(M)$ are disjoint sets. $P(M)$ has no intrinsic structure; elements of $Q(M)$ "code up" sets of size $\leq \omega$ in $P(M)$ via the sequence $\langle F_i \mid i < \omega \rangle$; $R(M)$ associates size

κ set of (otherwise undifferentiated) elements to each element in $Q(M)$. Let $T = Th(M)$. As usual, T is stable and quantifier eliminable; $\kappa(T) = \omega_1$.

Let $\langle \kappa_i \mid i < \lambda^+ \rangle$ be increasing and cofinal in κ. Let $\langle \eta_i \mid i < \lambda^+ \rangle$ be an enumeration of $Q(M)$. Let N be a submodel of M such that $P(N) = P(M)$, $Q(N) = Q(M)$ and

$$(\eta_i, \beta) \in R(N) \Longleftrightarrow \beta < \kappa_i$$

So, $|R(N)| = \kappa$, $|Q(N)| = \lambda^+$ and $|P(N)| = \lambda$. However, suppose A' is an arbitrary subset of $R(N)$ such that $|A'| = \kappa$. Combining the arguments from the last two cases, we can see that $|\bigcup_{i<\omega} (F_i \circ G)(A')| = \lambda$. Further given any $B' \subset P(N)$ such that $\bigcup_{i<\omega} (F_i \circ G)(A') \not\subset B'$, we have that $tp(A', P(N))$ forks over B' (as witnessed by some formula of the form "$x = (F_i \circ G)(y)$"). So, $R(N)$ and $P(N)$ witness the failure of $\ddagger(\kappa, \kappa', \lambda, \lambda')$ as desired.

Remarks: (1) So, under the assumption that T is superstable, the results given by 3.1.4 are optimal. Under the assumption that T is stable and GCH holds, 3.1.4 is very close to optimal. The only missing case occurs when $\kappa = \lambda^\omega = \lambda^+$ and $\kappa' = {\lambda'}^\omega = {\lambda'}^+$. Presently, I don't know what happens in this case.

(2) Under GCH, we do get a classification of a certain "Ramsey style" property. We say that $\ddagger'(\kappa, \lambda, \lambda')$ holds just in case $\ddagger(\kappa, \kappa, \lambda, \lambda')$ holds. This property is completely classified by 3.1.4 and the above examples.

Suppose that GCH does not hold. Then our classification of the even the $\ddagger'$ relation is no longer complete. Suppose, for instance, that the following pieces of cardinal arithmetic hold:

- for every $\nu < \mu$, $\nu^\omega < \mu$.
- $\mu < \lambda < \mu^\omega$.
- $\kappa = \mu^\omega$.
- $cf(\lambda) = \omega$

Since $cf(\lambda) = \omega$ and $\lambda^\omega = \kappa$, we cannot apply 3.1.4 to get any facts of the form $\ddagger'(\kappa, \lambda, \nu)$ for $\nu < \lambda$. However, the model we constructed (under GCH) to witness the failure of $\ddagger'(\kappa, \lambda, \nu)$ will not work if $\mu \leq \nu < \lambda$ (though it will witness the failure of $\ddagger'(\kappa, \lambda, \nu)$ for $\nu < \mu$). In that model, we can let B be an arbitrary subset of $P(N)$ such that $\mu \leq |B| < \lambda$; then the set $A = Q(N) \cap \bigcup_{i<\omega} F_i^{-1}(B)$ will have cardinality κ and $A \downarrow_B P(N)$. To get around this problem, we give the following:

Theorem 3.2.1 *Let T be the theory considered above. Then there is a model of ZFC in which*

1. $2^\omega = \aleph_{\omega+1}$
2. *There exists a model $N \models T$ such that $P(N)$ and $Q(N)$ witness the failure of $\ddagger'(\aleph_{\omega+1}, \aleph_\omega, \nu)$ for every $\nu < \aleph_\omega$.*

Proof. In L, let N be the model which witnesses the failure of $\ddagger'(\aleph_{\omega+1}, \aleph_\omega, \nu)$ for every $\nu < \aleph_\omega$. Let $\mathbb{P} = \mathbb{F}\ltimes(\aleph_{\omega+\nVdash} \times \omega, \nvDash, \omega)$ be the usual partial order for adding $\aleph_{\omega+1}$ Cohen reals. Let G be L-generic over $\mathbb{P}$; and let $M = L[G]$. Since $\mathbb{P}$ has the countable chain condition, L and M have the same cardinals and cofinalities; but in M, $2^\omega = \aleph_{\omega+1}$.

Suppose $A \in M$ and $B \in M$ are subsets of $Q(N)$ and $P(N)$ respectively such that $|A| = \aleph_{\omega+1}$, $|B| = \nu$ for some $\nu < \aleph_\omega$, and $A \downarrow_B P(N)$. In particular, then, $B \supset \bigcup_{i<\omega} F_i(A)$. Since $\mathbb{P}$ had the c.c.c. in L, we can find $B' \in L$ such that $B \subset B' \subset P(N)$ and $|B'| = |B| = \nu$.

Working in L, we let $A' = \{a \in Q(N) : \bigcup_{i<\omega} F_i(a) \subset B'\}$. We know that $|A'| = \aleph_{\omega+1}$ (since in M, $A \subset A'$). But as $\bigcup_{i<\omega} F_i(A') \subset B'$, we get $A' \downarrow_{B'} P(N)$. And this is a contradiction. $\square$

Remarks: (1) The proof of this fact is meant to be illustrative of a general style of argument. To show that we cannot expect to improve our results on $\ddagger'(\kappa, \lambda, \lambda')$, we first show that these results are optimal under certain "nice" conditions on cardinal arithmetic. We then use a forcing argument to show that they can be optimal without such "nice" arithmetic. The key trick is to use partial orders with the κ'-c.c. for some $\kappa' < \lambda$. Then covering arguments of the type illustrated above can be employed.

(2) The example given here may seem to rely heavily on two facts concerning the cardinal arithmetic of our final model. First, because the singular cardinals hypothesis holds in our initial model, $|Q(N)| = |P(N)|^+$ in our final model; second everything of interest in our final model takes place below $2^{|T|}$ (as $|T| = \omega$ and 2^ω gets blown up in the passage from L to M).

However, both of these restrictions can be avoided if we use some more complicated forcing constructions and assume the existence of large cardinals. So, for instance, Gitik and Magidor have an example of a type of forcing construction which allows them to prove the following: if V satisfies GCH, κ is a strong cardinal, and $\lambda > \kappa$ is an arbitrary cardinal, then there is a generic extension $V[G]$ such that

- no bounded subsets are added to κ
- κ changes its cofinality to ω
- $\kappa^\omega \geq \lambda$

Further, the partial order necessary for obtaining this generic extension has the κ^{++}–c.c., see [GM].

Using this forcing, we can get an example in which $Q(N) > |P(N)|^+$. First we force with the Gitik-Magidor partial order to get a universe $V[G]$ in which $\kappa^\omega > \kappa^+$ while $2^\omega = \omega_1$. We then follow this up with a c.c.c. forcing to get a universe $V[G][H]$ in which $(2^\omega)^{V[G][H]} > (\kappa^\omega)^{V[G]}$. Starting with the obvious model in $V[G]$ and applying a argument similar to that of 3.2.1 gives the result.

To get a model in which everything takes place above 2^ω, we apply the Gitik-Magidor forcing to get a universe $V[G]$ in which $\kappa^\omega > \aleph_{\kappa+\omega}$. In V we build a model for the theory used in 3.2.1 which has type $(\aleph_{\kappa+\omega+1}; \aleph_{\kappa+\omega})$. Then a covering argument similar to that used in 3.2.1 gives the result.

(3) Similar arguments work when we turn to the case of uncountable languages. If we modify the models discussed above by adding a sequence of extra functions $\langle F_i \mid \omega \leq i < \mu \rangle$—so we will use these functions to

code up sets of size μ—then we obtain stable, quantifier eliminable theories with $\kappa(T) = \mu^+$. With trivial modifications, all of the above arguments go through for these new theories.

3.3 2-Cardinal Theorems

In this section, we assume that $\delta_0 = 2$. To simplify notation, we work with a fixed model M having the property that $|M \cap \mathbb{A}_{\nVdash}| > |\mathbb{M} \cap \mathbb{A}_{\nvdash}|$, and we use κ and λ to denote $|M \cap \mathbb{A}_{\nVdash}|$ and $|M \cap \mathbb{A}_{\nvdash}|$ respectively.

To avoid proving our main theorem several times, we adopt the following general framework. We work with classes, K, of submodels of M. Associated to each K is a nondecreasing function $F_K : \{\kappa \in CARD : \kappa \leq |M|\} \to \{\kappa \in CARD : \kappa \leq |M|\}$ such that for every μ, $F_K(F_K(\mu)) = F_K(\mu)$; cardinals such that $F_K(\mu) = \mu$ will be called K-good. Also associated with K is a cardinal $\lambda(K)$ such that $\lambda(K) \leq F_K(0)$ and K is closed under unions of increasing sequences having cofinality $\geq \lambda(K)$. Finally, K and F_K must satisfy the following conditions:

1. If $A \subset M$, there exists a K-model N such that $A \subset N$ and $|N| \leq F_K(|A|)$.
2. If N is a K-model and $N \cup A \subset M$, then there is a K-model $N[A] \supset N \cup A$ such that $|N[A]| \leq F_K(|N \cup A|)$ and $A \rhd_N N[A]$.

Lemma 3.3.1 *The conditions outlined above are satisfied when:*

1. *M is an $\mathbf{F}$-saturated model for some standard $\mathbf{F}$; K is the class of $\mathbf{F}$-saturated submodels of M; $F_K(\mu) = min\{\mathbf{F}(\mu), |M|\}$; and $\lambda(K) = \lambda(\mathbf{F})$*
2. *M is an $\mathbf{F}^{\mathbf{t}}_{|\mathbf{T}|^+}$-saturated model; K is the class of $|T|$-compact submodels of M; $F_K(\mu) = min\{\mathbf{F}^l_{|T|}(\mu), |M|\}$; and $\lambda(K) = |T|$*
3. *T is superstable; M is an arbitrary model; K is the class of models $N \subset_{na} M$; $F_K(\mu) = \mu + |T|$; and $\lambda(K) = \omega$.*

Proof. For 1, we take prime models to satisfy the two conditions. For 2, condition 1 is trivial; condition 2 follows by letting $N[A]$ be $\mathbf{F}^l_{|T|}$-constructible and $|T|$-compact over $N \cup A$ (see section 2.2). For 3, condition 1 follows from the fact that $N' \subset_{na} N \subset_{na} M \Longrightarrow N' \subset_{na} M$; condition 2 follows from 2.2.9. Note that in all three cases, the condition on unions of chains follows trivially from the definition of K. $\square$

Lemma 3.3.2 *Let $A \downarrow_{B_A} B$, and let B' be arbitrary. Then there exists $A' \subset A$ such that $|A'| \leq \max(|B'|, \kappa(T))$ and $A \downarrow_{B_A \cup A'} B \cup B'$. If $|B'| < cf(\kappa(T))$, then we can have $|A'| < \kappa(T)$.*

Proof. Clearly we can find an A' with the desired cardinality such that $B' \downarrow_{B \cup A'} A$. As $A \downarrow_{B_A \cup A'} B \cup A'$, we get $A \downarrow_{B_A \cup A'} B \cup B'$ by transitivity. $\square$

Lemma 3.3.3 *Let $A, B, B_A \subset M$ and suppose that $A \downarrow_{B_A} B$. Then there exists a K-model N such that $B_A \subset N$, $|N| = F_K(|B_A|)$ and $A \downarrow_N B$.*

Proof. We construct by induction a sequence $\langle (B_i, M_i) \mid i < \lambda(K) \rangle$ such that:

1. for each i, $|B_i| \leq |M_i| = F_K(|B_A|)$.
2. for each i, $B_i \subset M_i \subset B_{i+1}$; $\langle M_i \mid i < \lambda(K) \rangle$ is increasing
3. for each i, M_i is a K-model, and $A \downarrow_{B_i} B$

We begin, by letting $B_0 = B_A$ and letting M_0 be a K-model containing B_0 such that $|M_0| = F_K(B_A)$. Suppose, then,that we have (B_j, M_j) for every $j < i$. Let $M' = \bigcup_{j<i} M_i$. As $|M'| = F_K(B_A)$, we can use the last lemma to find an $A' \subset A$ such that $|A'| = F_K(B)$ and $A \downarrow_{B_A \cup A'} M' \cup B \cup A'$. Let $B_i = M' \cup A'$ and let M_i be a K-model containing B_i such that $|M_i| = |B_i| = F_K(|B_A|)$.

Clearly, conditions 2 and 3 are preserved by this construction. Further, we never need to increase $|M_i|$ at successor stages (as $F_K(|B_A|)$ is F_K-good); and as all of our limits are taken at ordinals β such that $\beta < \lambda(K) \leq F_K(0) \leq F_K(|B_A|)$, $|M_i|$ is not increased at limits either. Hence, condition 1 is satisfied.

Let $N = \bigcup_{i<\lambda(K)} M_i = \bigcup_{I<\lambda(K)} B_i$. Clearly $B_A \subset N$ and $|N| = F_K(|B_A|)$. As K is closed under unions of length $\lambda(K)$, N is a K-model. And by forking continuity, $A \downarrow_N B$. □

Theorem 3.3.4 *Let κ' and λ' be K-good such that $\kappa' \leq \kappa$, $\lambda' < \lambda$, and $\lambda' < \kappa'$. Suppose that one of the following conditions holds:*

1. *$\kappa' < \kappa$, and there exists ν such that $\dagger(\lambda'^{+}, \lambda, \nu)$ holds and $|B|^{<\nu} < \kappa$.*
2. *$\kappa' = \kappa$, and there exists ν such that $\dagger(\lambda'^{+}, \lambda, \nu)$ holds and $|B|^{<\nu} < cf(\kappa)$.*
3. *$\kappa' = \kappa$, and there exists ν such that $\dagger(\lambda'^{+}, \lambda, \nu)$ holds and both $|B|^{<\nu} < \kappa$ and $cf(\kappa) < \lambda'^{+}$.*

Then there exists a K-model M' such that $|M' \cap \mathbb{A}_{\nVdash}| = \kappa'$ and $|M' \cap \mathbb{A}_{\nvdash}| = \lambda'$.

Proof. By 3.1.4, we can find $A \subset M \cap \mathbb{A}_{\nVdash}$ and $B \subset M \cap \mathbb{A}_{\nvdash}$ such that $|A| = \kappa'$, $|B| = \lambda'$ and $A \downarrow_B M \cap \mathbb{A}_{\nvdash}$. By lemma 3.3.3, we can find a K-model such that $|A| = F_K(|B|) = \lambda'$ and $A \downarrow_N M \cap \mathbb{A}_{\nvdash}$. By condition 2 on K-models, there exists a K-model $N[A]$ such that $|N[A]| = F_K(|N \cup A|)$ and $A \rhd_N N[A]$.

As $F_K(|N \cup A|) = F_K(\kappa') = \kappa'$, we have $|N| = \kappa'$; so as $A \subset N \cap \mathbb{A}_{\nVdash}$ and $|A| = \kappa'$, we have $|N \cap \mathbb{A}_{\nVdash}| = \kappa'$. Further, since $A \downarrow_N M \cap \mathbb{A}_{\nvdash}$ and $A \rhd_N N[A]$, we have $N[A] \downarrow_N M \cap \mathbb{A}_{\nvdash}$. In particular, $N[A] \cap \mathbb{A}_{\nvdash} \subset \mathbb{N}$. So, $|N[A] \cap \mathbb{A}_{\nvdash}| = |\mathbb{N} \cap \mathbb{A}_{\nvdash}| = \lambda'$. Letting $M' = N[A]$, we are done. □

Remarks: (1) If T is superstable, this result allows us to work inside a completely arbitrary model M and get whatever cardinalities we like (modulo some cofinality issues). We just take K to be the set of models $N \subset_{na} M$. Similarly, if T is countable and M is ω_1-saturated, then we can use $F^{l}_{|T|}$ to get whatever cardinalities we like (again, modulo some cofinality issues).

If we make the assumption that $\mathbb{A}_{\nVdash} = \mathbb{M}$ and that $\mathbb{A}_{\nvdash}$ is definable by a predicate in $\mathfrak{L}$, then the superstable version of this theorem gives V 6.17 of [Sh]. Shelah's cardinality conditions are simply those generated by 3.1.5. Note that Shelah's argument requires that $\mathbb{A}_{\nvdash}$ be definable (or, at least, ∞-definable).

(2) If M has some reasonably high degree of saturation/compactness, then we can preserve this saturation/compactness in the M' given by the theorem; we just take K to be the class of F^t_μ-saturated models where μ is the degree of saturation/compactness we wish to preserve. We "pay" for this saturation/compactness by giving up some freedom vis-a-vis the cardinalities of $M' \cap \mathbb{A}_{\nvdash}$ and $M' \cap \mathbb{A}_{\nVdash}$: under this construction, these cardinalities must be F^x_μ-good.

(3) Note that this result does not require that $\mathbb{A}_{\nvdash}$ and $\mathbb{A}_{\nVdash}$ be definable (or even ∞-definable). The argument works for completely arbitrary subsets of M.

In section 2.2, we saw that many arguments involving $\mathbf{F}^{\mathbf{a}}_\mu$-constructions can be "relativized" to work inside models which are not necessarily $\mathbf{F}^{\mathbf{a}}_\mu$-saturated. These arguments do not quite fit into the framework established above (as the interplay between relatively $\mathbf{F}^{\mathbf{a}}_\mu$-maximal models and relatively $\mathbf{F}^{\mathbf{a}}_\mu$-saturated models is not captured in the K-notation). Nevertheless, a result quite similar to 3.3.4 can be proved using these relativization techniques.

Theorem 3.3.5 *Let $\mu \geq \kappa_r(T)$ be regular and suppose that M is μ-compact. Let κ' and λ' be F^a_μ-good (or just $\kappa' = |M|$ and λ' F^a_μ-good) such that $\kappa' \leq \kappa$, $\lambda' < \lambda$, and $\lambda' < \kappa'$. Suppose that one of the following conditions holds,*

1. *$\kappa' < \kappa$, and there exists ν such that $\dagger(\lambda'^+, \lambda, \nu)$ holds and $\lambda^{<\nu} < \kappa$.*
2. *$\kappa' = \kappa$, and there exists ν such that $\dagger(\lambda'^+, \lambda, \nu)$ holds and $\lambda^{<\nu} < cf(\kappa)$.*
3. *$\kappa' = \kappa$, and there exists ν such that $\dagger(\lambda'^+, \lambda, \nu)$ holds and both $\lambda^{<\nu} < \kappa$ and $cf(\kappa) < \lambda'^+$.*

Then there exists a μ-compact model M' such that $|M' \cap \mathbb{A}_{\nVdash}| = \kappa'$, $|M' \cap \mathbb{A}_{\nvdash}| = \lambda'$, and $M' \prec M$.

Proof. By 3.1.4 we can find $A \subset M \cap \mathbb{A}_{\nVdash}$ and $B \subset M \cap \mathbb{A}_{\nvdash}$ such that $|A| = \kappa'$, $|B| = \lambda'$ and $A \downarrow_B M \cap \mathbb{A}_{\nvdash}$. Note that the class of relatively $\mathbf{F}^{\mathbf{a}}_\mu$-saturated submodels of M satisfies condition 1 from the definition of K and is closed under increasing unions of length μ; this is enough to mimic the proof of 3.3.3 to find and N such that $B \subset N \prec^a_\mu M$, $|N| = \lambda'$ and $A' \downarrow_N M \cap \mathbb{A}_{\nvdash}$.

Let $X \subset M$ be $\mathbf{F}^{\mathbf{a}}_\mu$-constructible over $N \cup A'$ and relatively $\mathbf{F}^{\mathbf{a}}_\mu$-maximal in M. By 2.2.4, X is a μ-compact model; and by 2.2.7, $X \downarrow_N M \cap \mathbb{A}_{\nvdash}$. In particular, $|X \cap \mathbb{A}_{\nvdash}| = |\mathbb{N} \cap \mathbb{A}_{\nvdash}| = |\mathbb{B}'| = \lambda'$. Since $|A'|$ is either $\mathbf{F}^{\mathbf{a}}_\mu$-good or the same as $|M|$, $|X| = |A'| = \kappa'$. So, $|X \cap \mathbb{A}_{\nVdash}| = |\mathbb{A}'| = \kappa'$. Letting $M' = X$ we are done. $\square$

This theorem provides us with an additional result on admitting cardinals which could not be obtained using the techniques of either 2.3 or 2.4. It says that if the cardinality of a $(\kappa\,;\lambda)$-model is large enough, then the actual difference in cardinality between κ and λ is largely irrelevant for admitting cardinals arguments.

Corollary 3.3.6 *Suppose that $\mathbb{A}_{\nvdash}$ and $\mathbb{A}_{\nVdash}$ are ∞-definable over some set A. Suppose also that M is $\kappa_r(T)$-compact, $\kappa > \beth_{(|A|+|\mathfrak{L}|)^+}$ and $\kappa > \lambda^{<\kappa_r(T)}$. Then for any $\kappa' > \lambda' \geq |A|$ there exists $M' \supset A$ such that $|M' \cap \mathbb{A}_{\nVdash}| = \kappa'$ and $|M' \cap \mathbb{A}_{\nvdash}| = \lambda'$*

Proof. Expand $\mathfrak{L}$ by adding constants for elements in A. Note that this does not change the fact that T is stable, nor does it change the value of $\kappa(T)$.

If λ is small—say, if $\lambda < 2^{|T|}$—then we are done by 1.3.4. So suppose that λ is large. Let $\mu = (\max\{\beth_{|\mathfrak{L}|^+}, \lambda^{<\kappa_r(T)}\})^+$. Note that, $\dagger((2^{|T|})^+, \lambda, \kappa_r(T))$ holds automatically and that $\kappa \geq \mu = cf(\mu) > \lambda^{<\kappa_r(T)}$. Note also that both μ and $2^{|T|}$ are $\mathbf{F}^{\mathbf{a}}_{\kappa_{\mathbf{r}}(\mathbf{T})}$-good. So, by 3.3.5 we can find an $M' \prec M$ such that $|M' \cap \mathbb{A}_{\not\Vdash}| = \mu$ and $|M' \cap \mathbb{A}_{\nvdash}| = \nvDash^{|\mathbb{T}|}$. Now apply 1.3.4. $\square$

Remarks: (1) This corollary is primarily useful when $\kappa_r(T) \leq |T|$. When $\kappa_r(T) > |T|$, then $\kappa_r(T)$-compact is the same as $\mathbf{F}^{\mathbf{a}}_{\kappa_{\mathbf{r}}(\mathbf{T})}$-saturated; in that case, we can ignore the conditions on the size of κ and simply apply 2.4.8.

(2) 3.3.6 is particularly nice when T is superstable. Here, the requirement that M be $\kappa_r(T)$-compact becomes trivial; similarly, the requirement that $\kappa > \lambda^{<\kappa_r(T)}$ reduces to just $\kappa > \lambda$. Thus, as long as the set $M \cap \mathbb{A}_{\not\Vdash}$ is large enough, any splits between the sizes of $M \cap \mathbb{A}_{\not\Vdash}$ and $M \cap \mathbb{A}_{\nvdash}$ will be enough to get the corollary's conclusion. This is true no matter how large the types needed to define $\mathbb{A}_{\nvdash}$ and $\mathbb{A}_{\not\Vdash}$ happen to be.

3.4 δ-Cardinal Problems

In this section, we extend the results from 3.3 to the case in which δ_0 is infinite. As in 3.3, we work with a fixed model M, and we assume that M is an F-model for some increasing $F : \delta_0 \to CARD$.

We also carry over the K, F_K, and $\lambda(K)$ framework from section 3.3; but here we place the following additional conditions on K and F_K:

3. If N is a K-model and $A \subset N$, then there exists a K-model N' such that $A \subset N' \subset N$ and $|N'| \leq F_K(|A|)$.

4. If $\langle N_i \mid i < \alpha \rangle$ is an increasing sequence of K-models and $A \supset \bigcup_{i<\alpha} N_i$, then there is a K-model $N \supset A$ such that $|N| = F_K(|A|)$ and for $i < \alpha$, $A \rhd_{N_i} N$.

These additional conditions are satisfied by each of the classes of models mentioned in lemma 3.3.1. The only condition which is at all difficult to check is condition 4 for the class of models $N \subset_{na} M$. For this condition, we let $\overline{N} = \bigcup_{i<\alpha} N_i$, and we apply 2.2.9 to find N such that $A \subset N \subset_{na} M$ and $A \rhd_{\overline{N}} N$. Then I claim that N satisfies the demands of condition 4. For suppose that $i < \alpha$ and that $b \downarrow_{N_i} A$. Then $b \downarrow_{\overline{N}} A$. As $A \rhd_{\overline{N}} N$, we get $b \downarrow_{\overline{N}} N$. So by transitivity, $b \downarrow_{N_i} N$ (remember, $\overline{N} \subset A$).

We begin this section with a coordination lemma which helps to ensure that all of the constructions in the section are well founded. It should be thought of as an infinitary version of 3.3.2.

Lemma 3.4.1 *Let $\langle (A_i, B_i, C_i) \mid i < \alpha \rangle$ be a sequence of sets such that for each $i < \alpha$, $A_i \downarrow_{C_i} B_i$. Then,*

1. *There exists D such that $D \subseteq (\bigcup_{i<\alpha} C_i \cup \bigcup_{i<\alpha} A_i)$, $|D| \leq |\bigcup_{i<\alpha} C_i| + |\alpha| + \kappa(T)$, and for each $i < \alpha$, $A_i \downarrow_D B_i$.*

2. *If all of the A_i, B_i, and C_i are contained in M, then there exists a K-model N such that $\bigcup_{i<\alpha} C_i \subseteq N$, $|N| \leq F_K[|\bigcup_{i<\alpha} C_i + |\alpha| + \kappa(T)]$, and for each $i < \alpha$, $A_i \downarrow_N B_i$.*

3. *In 2, if all the A_i and C_i are contained in some K-model M', then we may have $N \prec M'$.*

Proof. (1) We construct by induction a sequence $\langle D_n \mid n < \omega \rangle$. $D_0 = \bigcup_{i<\alpha} C_i$. Given D_n, we employ lemma 3.3.2 and choose for each $i < \alpha$ an $A_{i,n} \subseteq A_i$ such that $|A_{i,n}| \leq |D_n| + \kappa(T)$ and $A_i \downarrow_{D_n \cup A_{i,n}} B_i$. We then let $D_{n+1} = D_n \cup \bigcup_{i<\alpha} A_{i,n}$. Note that for each n, $|D_n| \leq |\bigcup_{i<\alpha} C_i| + |\alpha| + \kappa(T)$.

Finally, we set $D = \bigcup_{n<\omega} D_n$. Clearly, $|D| \leq |\bigcup_{i<\alpha} C_i| + |\alpha| + \kappa(T)$; and by forking continuity, $A_i \downarrow_D B_i$.

(2) Follows trivially from (3)

(3) Essentially the same proof as (1). Working in M', we build a sequence of models $\langle N_k \mid k < \lambda(K) \rangle$. N_0 is just a K-model containing $\bigcup_{i<\alpha} C_i$ such that $|N_0| = F_K(|\bigcup_{i<\alpha} C_i|)$ and $N_0 \prec M'$. Given N_k, we choose for each $i < \alpha$ an $A_{i,k} \subseteq A_i$ such that $|A_{i,k}| \leq |N_k| + \kappa(T)$ and $A_i \downarrow_{N_k \cup A_{i,k}} B_i$. By condition 3 on K-models, we can find a K-model N_{k+1} such that $N_k \cup \bigcup_{i<\alpha} A_{i,k} \subset N_{k+1} \prec M'$ and $|N_{k+1}| \leq F_K(|N_k \cup \bigcup_{i<\alpha} A_{i,k}|)$. For limit i, we let $N_i = \bigcup_{k<i} N_k$.

Let $N = \bigcup_{k<\lambda(K)} N_k$. Clearly, $|N| \leq F_K(|\bigcup_{i<\kappa} C_i| + |\alpha| + \kappa(T))$. And as N is the union of an $\lambda(K)$-length chain of K-models, N is also a K-model. Finally, forking continuity insures that for each $i < \alpha$, $A_i \downarrow_N B_i$.
□

Theorem 3.4.2 *Let $G : \delta_0 \to CARD$ be non-decreasing and such that for every $i < \delta_0$, $G(i) \leq F(i)$ and $G(i) = \min\{F_K(G(i) + |\delta_0| + \kappa_r(T)), F(i)\}$. Suppose that for every $i < \delta_0$, $F(i) > (\sup_{j<i} F(j))^{<\kappa_r(T)}$ and one of the following conditions holds,*

1. *$G(i) < F(i)$*

2. *$cf(F(i)) > (\sup_{j<i} F(j))^{<\kappa_r(T)}$.*

3. *$cf(F(i)) \leq G(0)$.*

Then there exists a K-model M' which is also a G-model. If $\kappa(T) = \kappa^+$ and $\delta_0 \leq \kappa$, then we only need $G(i) = \min\{F_K(G(i) + \kappa), F(i)\}$.

Proof. For convenience, we use A_i to denote $M \cap \mathbb{A}_\daleth$. By corollary 3.1.7 we can find a sequence $\langle (E_i, B_i) \mid i < \delta_0 \rangle$ such that:

- $E_i \subset A_i$ and $|E_i| = G(i)$.
- $B_i \subset \bigcup_{j<i} A_j$ and $|B_i| < \kappa_r(T)$.
- $E_i \downarrow_{B_i} \bigcup_{j<i} A_j$.

We let E denote $\bigcup_{i<\delta_0} E_i$. By lemma 3.4.1, we can find a K-model M_0 such that $|M_0| = F_K(|\bigcup_{i<\delta_0} B_i|) \leq G(0)$ and for every $i < \delta_0$, $E_i \downarrow_{M_0} \bigcup_{j<i} A_j$.

Next, we build an increasing chain of K-models $\langle M_i \mid i \leq \delta_0 \rangle$ by induction. We try to preserve the following:

1. for $i < \delta_0$, $E_i \subset M_{i+1}$ and $|M_{i+1}| = G(i)$.

2. for $j \geq i$, $E_j \downarrow_{M_i} \bigcup\limits_{k<j} A_k$.

3. for $j < i$, $A_j \downarrow_{M_{j+1}} M_i \cup E$.

We already have M_0. So suppose we have M_j for $j < i$. By condition 4 on the class of K-models, we can find a K-model $\overline{M}_i$ such that $\bigcup\limits_{j<i} M_j \cup E \subset \overline{M}_i$ and for every $j < i$, $\bigcup\limits_{k<i} M_k \cup E \rhd_{M_j} \overline{M}_i$. By lemma 3.4.1, we can find a K-model $M_i \prec \overline{M}_i$ such that $\bigcup\limits_{j<i}(M_j \cup E_j) \subset M_i$, $|M_i| = F_K(|\bigcup\limits_{j<i}(M_j \cup E_j)|)$ and for every $j \geq i$, $E_j \downarrow_{M_i} \bigcup\limits_{k<j} A_k$.

Note that if $i = j+1$, then $E_j \subset M_i$ and $|M_i| = F_K(|E_j|) = F_K(G(j)) = G(j)$. So, condition 1 is satisfied. Further condition 2 was obtained from the application of 3.4.1. So we only need to check condition 3.

Let $j < i$. If $i = j+1$, then the result follows from condition 2. If $i > j+1$, then the induction hypothesis and forking continuity ensure that $A_j \downarrow_{M_{j+1}} \bigcup\limits_{j<i} M_j \cup E$. As $\bigcup\limits_{j<i} M_j \cup E \rhd_{M_{j+1}} \overline{M}_i$, $A_j \downarrow_{M_{j+1}} \overline{M}_i$. And as $M_i \subset \overline{M}_i$, condition 3 is satisfied.

Claim: *M_{δ_0} is a G model.*

Proof of Claim. For every $i < \delta_0$, $E_i \subset M_{i+1} \subset M_{\delta_0}$; so, $|M_{\delta_0} \cap \mathbb{A}_\daleth| \geq |\mathbb{E}_\daleth| = \mathbb{G}(\daleth)$. Further, as $A_i \downarrow_{M_{i+1}} M_{\delta_0}$, we have $M_{\delta_0} \cap \mathbb{A}_\daleth \subset \mathbb{M}_{\daleth+\nvDash}$. So since $|M_{i+1}| = G(i)$, $|M_{\delta_0} \cap \mathbb{A}_\daleth| = \mathbb{G}(\daleth)$. $\square$ (Claim, Theorem)

Remark: In our hypothesis on $G(i)$, the factor $|\delta_0| + \kappa_r(T)$ (or simply κ, if $\kappa(T) = \kappa^+$ and $\delta_0 \leq \kappa$) serves only to ensure that all of our B_i sets can fit into M'.

Notation: The proof of the next theorem makes extensive use of properties related to constructability and requires some slightly non-standard notation. Let C be constructible over A via the construction $\langle (c_i, B_i) \mid i < \alpha \rangle$ and let $C' \subset C$. We will say that $C' \subset C$ is completely closed if whenever $c_i \in C'$, $B_i \subset C'$. The complete closure of C' ($ccl(C')$) is the least completely closed C'' such that $C' \subseteq C'' \subseteq C$. Note that if $C' \subset C$ is completely closed in the construction of C over A, then C' is constructible both over A and over $C' \cap A$. Note also that for any $C' \subset C \cup A$, $|ccl(C')| \leq |C'| + \lambda(\mathbf{F})$.

Theorem 3.4.3 *Let $\mu \geq \kappa_r(T)$ be regular and suppose that M is μ-compact. Let $G : \delta_0 \to CARD$ be non-decreasing and such that for every $i < \delta_0$, $G(i) \leq F(i)$ and $G(i) = \min\{F^a_\mu(G(i) + |\delta_0| + \kappa_r(T)), F(i)\}$. Suppose that for every $i < \delta_0$, $F(i) > (\sup\limits_{j<i} F(j))^{<\kappa_r(T)}$ and one of the following conditions holds,*

1. *$G(i) < F(i)$*

2. $cf(F(i)) > (\sup_{j<i} F(j))^{<\kappa_r(T)}$.

3. $cf(F(i)) \leq G(0)$.

Then there exists a μ-compact model $M' \prec M$ which is also a G-model. If $\kappa(T) = \kappa^+$ and $\delta_0 \leq \kappa$, then we only need $G(i) = \min\{F_K(G(i) + \kappa), F(i)\}$.

Proof. Letting A_i denote $M \cap \mathbb{A}_\beth$, we choose a sequence $\langle (E_i, B_i) \mid i < \delta_0 \rangle$ as in the proof of the previous theorem. Let $E = \bigcup_{i<\delta_0} E_i$. Note that the collection of relatively $\mathbf{F}^{\mathbf{a}}_\mu$-saturated submodels of M satisfies conditions 1 and 3 from the definition of K and is closed under increasing unions of length μ; this is enough to mimic the proof of 3.4.1, 3. Hence, we can find a relatively $\mathbf{F}^{\mathbf{a}}_\mu$-saturated submodel of M, M_0, such that $|M_0| = \mathbf{F}^{\mathbf{a}}_\mu(|\bigcup_{\mathbf{i}<\delta_0} \mathbf{B_i}|) \leq \mathbf{G(0)}$ and for every $i < \delta_0$, $E_i \downarrow_{M_0} \bigcup_{j<i} A_j$.

Next, we build two increasing chains of submodels of M: $\langle M_i \mid i \leq \delta_0 \rangle$ and $\langle \overline{M}_{i+1} \mid i < \delta_0 \rangle$. We try to preserve the following:

1. $E_{i+1} \subset M_{i+1}$ and $|M_{i+1}| = G(i)$.
2. $\bigcup_{j>i+1} E_j \downarrow_{M_i} \overline{M}_{i+1} \cup \bigcup_{j\leq i+1} A_j$.
3. for $j > i$, $E_j \downarrow_{M_i} \bigcup_{k<j} A_k$.
4. M_i is $\mathbf{F}^{\mathbf{a}}_\mu$-constructible over $M_0 \cup \bigcup_{j\leq i} \overline{M}_j \cup E$.
5. M_i is relatively $\mathbf{F}^{\mathbf{a}}_\mu$-saturated in $M_i \cup E$.
6. $\overline{M}_{i+1}$ is $\mathbf{F}^{\mathbf{a}}_\mu$-constructible over $M_i \cup E$.
7. $\overline{M}_{i+1}$ is $\mathbf{F}^{\mathbf{a}}_\mu$-maximal in $\overline{M}_i \cup \bigcup_{j\leq i+1} A_j$.

We already have M_0. So suppose that we have M_i. Let N be $\mathbf{F}^{\mathbf{a}}_\mu$-constructible over $M_i \cup \bigcup_{j\leq i+1} A_j$ and $\mathbf{F}^{\mathbf{a}}_\mu$-maximal in M. Let $\overline{M}_{i+1} = M_i[E_{i+1}] \prec N$ be $\mathbf{F}^{\mathbf{a}}_\mu$-constructible and $\mathbf{F}^{\mathbf{a}}_\mu$-maximal in N. Since E is independent over M_i, this can be regarded as a construction over $M_i \cup E$ (see the remarks following 2.2.7), so 6 is satisfied. 7 is satisfied as $M_i \cup A_{i+1} \subset N$. By induction, 5 holds for M_i, so by 2.2.7 $\bigcup_{j>i+1} E_j \downarrow_{M_i} N$. So, 2 holds as well. Towards 1, note that $|\overline{M}_{i+1}| \leq \mathbf{F}^{\mathbf{a}}_\mu(|\mathbf{M_i}| + |\mathbf{E_i}|)$.

Let N' be $\mathbf{F}^{\mathbf{a}}_\mu$-constructible and over $\overline{M}_i \cup E$ and $\mathbf{F}^{\mathbf{a}}_\mu$-maximal in M.

Claim 1. *There exists $M_{i+1} \prec N'$ such that*

A. M_{i+1} *is relatively* $\mathbf{F}^{\mathbf{a}}_\mu$*-saturated in* N'.

B. M_{i+1} *is completely closed in the construction of* N' *over* $\overline{M}_i \cup E$.

C. $\overline{M}_i \cup E_i \subset M_{i+1}$ *and* $|M_{i+1}| = G(i)$.

D. for $j > i$, $E_j \downarrow_{M_i} \bigcup_{k<j} A_k$.

Proof of Claim 1. Working in N', we construct a sequence $\langle (X_j, Y_j, Z_j) \mid j < \mu \rangle$ by induction. For $j = 0$ or j limit, we let $X_j = Y_j = Z_j = \bigcup_{k<j} (X_k \cup Y_k \cup Z_k) \cup \overline{M}_i \cup E_i$. Given some (X_j, Y_j, Z_j), we find $(X_{j+1}, Y_{j+1}, Z_{j+1})$ as follows. X_{j+1} is the complete closure of Z_j in the construction of N' over $\overline{M}_i \cup E$. $Y_{j+1} \supset X_{j+1}$ is relatively $\mathbf{F}^{\mathbf{a}}_{\mu}$-saturated in N' and such that $|Y_{j+1}| = \mathbf{F}^{\mathbf{a}}_{\mu}(|\mathbf{X_{j+1}}|)$. $Z_{j+1} \supset Y_{j+1}$ is obtained by 3.4.1, 1; we require that $|Y_{j+1}| = |X_{j+1}|$ and for $k > j+1$, $E_k \downarrow_{Z_{j+1}} \bigcup_{l<k} A_l$.

Let $M_{i+1} = \bigcup_{j<\mu} Z_j$. Since M_{i+1} is an increasing union of the X_j, M_{i+1} is completely closed in the construction of N' over $\overline{M}_i \cup E$; so, condition B is satisfied. Since M_{i+1} is an increasing union of the Y_j, M_{i+1} is relatively $\mathbf{F}^{\mathbf{a}}_{\mu}$-saturated in N'; so condition A is satisfied. Since M_{i+1} is an increasing union of the Z_j, forking continuity ensures that M_{i+1} satisfies condition D. Finally, replacing Y_j with Z_j or Z_j with X_{j+1} cannot increase the size of our sets; as we start with a set of size $G(i)$ where $G(i)$ is $\mathbf{F}^{\mathbf{a}}_{\mu}$-good, replacing X_j with Y_j does not increase the size of our sets; and as all unions are at ordinals $j < \mu < |M_0| \leq G(i)$, we do not increase sizes at limits. So, condition C is satisfied as well. $\square$ (Claim 1)

Now we need to check that M_{i+1} satisfies conditions 1, 3, 4 and 5. in our main construction. Clearly conditions 1 and 3 follow from C and D in claim 1. Condition 4 follows from condition B in the claim (noting that $\overline{M}_i \cup E \subset N'$). Finally, condition 5 follows from A in the claim (noting that $M_{i+1} \cup E \subset N'$).

For i limit, let N' be $\mathbf{F}^{\mathbf{a}}_{\mu}$-constructible over $\bigcup_{j<i} M_i \cup E$ and $\mathbf{F}^{\mathbf{a}}_{\mu}$-maximal in M. Following the construction in the proof of claim 1, we find $M_i \prec N'$ such that $\bigcup_{j<i} M_j \subset M_i$, $|M_i| = \mathbf{F}^{\mathbf{a}}_{\mu}(|\bigcup_{\mathbf{j<i}} \mathbf{M_i}|)$, M_i is $\mathbf{F}^{\mathbf{a}}_{\mu}$-saturated in N', M_i is completely closed in the construction of N' over $\bigcup_{j<i} M_i \cup E$, and for every $j > i$, $E_j \downarrow_{M_i} \bigcup_{k<j} A_k$. As above, these conditions ensure that M_i satisfies conditions 1,3,4, and 5.

Claim 2. *M_{δ_0} is a G model.*

Proof of Claim 2. Note that for each $i < \delta_0$, we have that $E_i \subset \overline{M}_{i+1} \subset M_{\delta_0}$. So, $|M_{\delta_0} \cap \mathbb{A}_{\beth}| \geq |\mathbb{E}_{\beth}| = \mathbb{G}(\beth)$. By condition 4, each M_i is $\mathbf{F}^{\mathbf{a}}_{\mu}$-constructible over $M_0 \cup \bigcup_{j\leq i} \overline{M}_j \cup E$, and by condition 5, each $\overline{M}_{i+1}$ is $\mathbf{F}^{\mathbf{a}}_{\mu}$-constructible over $M_i \cup E$; so, pasting these constructions together, we find that M_{δ_0} is $\mathbf{F}^{\mathbf{a}}_{\mu}$-constructible over $\overline{M}_{i+1} \cup E$. By condition 2, $\bigcup_{j>i+1} E_j \downarrow_{M_i} \overline{M}_{i+1} \cup \bigcup_{j\leq i+1} A_j$; so by monotonicity, $\bigcup_{j>i+1} E_j \downarrow_{\overline{M}_{i+1}} \bigcup_{j\leq i+1} A_j$. Finally, by condition 7, $\overline{M}_{i+1}$ will be $\mathbf{F}^{\mathbf{a}}_{\mu}$-maximal in $\overline{M}_i \cup \bigcup_{j\leq i+1} A_j$. So by 2.2.6, $M_{\delta_0} \cap \bigcup_{j\leq i+1} A_j \subset \overline{M}_{i+1}$. As $|\overline{M}_{i+1}| = G(i)$, we are done. $\square$ (claim 2, theorem)

Corollary 3.4.4 *Suppose that each $\mathbb{A}_{\beth}$ is ∞-definable over some $A \subset M$. Let $\mu = [(|A| + |\mathfrak{L}| + |\delta_{\mathfrak{o}}|)^{++}] \cdot \delta_{\mathfrak{o}}$ and suppose that M is $\kappa_r(T)$-compact, $F(0) > \beth_{\mu}$ and for each $i < \delta_0$, $F(i) > (\sup_{j<i} F(j))^{<\kappa_r(T)}$. Then for any nondecreasing function $G : \delta_0 \to CARD$ such that $G(0) \geq |\mathfrak{L}| + |\mathfrak{A}|$, there exists a G model.*

Proof. Like the proof of 3.3.6. We begin by expanding $\mathfrak{L}$ by adding constants for elements in A, and we note that this does not change the fact that T is stable, nor does it change the value of $\kappa(T)$. Let $\mu' = (|A| + |\mathfrak{L}| + |\delta_{\mathfrak{o}}|)^+$. Choose some $G' : \delta_0 \to CARD$ such that for every $i < \delta_0$, $G'(i) < F(i)$, $G'(i) > \beth_{\mu'}(\sup_{j<i}(G'(j)))$ and $G(i)$ is $\mathbf{F}^{\mathbf{a}}_{\kappa_{\mathbf{r}}(\mathbf{T})}$-good By theorem 3.4.3, we can find $M' \prec M$ such that M' is a $\kappa_r(T)$-compact, G'-model. The result follows from 1.3.4. $\square$

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