Craps

The rules: Toss two fair dice and add up the faces on top.

① If 7 or 11 happens on this first toss you win.

② If 2, 3, 12 happens on this first toss you lose.

③ If $k \in \{4, 5, 6, 8, 9, 10\}$ happens on this first toss, then toss repeatedly until the first occurrence of k or 7.

ⓐ if k occurs first you win;

ⓑ if 7 occurs first you lose.

Before we start analyzing the game, let's recall some useful formulae that we use repeatedly in this course.

A_1) $$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

with absolute convergence for $|x| < 1$.

Differentiating both sides we get ②

A_2) $\dfrac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots$

Multiplying by x, and differentiating both sides we get

B_2) $\dfrac{1+x}{(1-x)^3} = 1 + 4x + 9x^2 + 16x^3 + \cdots$

Now back to the game!

The sample space for tossing the die once is $S :=$

$$\begin{Bmatrix} (1,1), & (1,2), & \ldots, & (1,6) \\ (2,1) & \cdots & & (2,6) \\ \vdots & & & \vdots \\ (6,1) & \cdots & & (6,6) \end{Bmatrix}$$

with σ field the power set $\mathcal{S}$ of S and all points equally likely, e.g., $\mathbb{P}(\{(2,3)\}) = \frac{1}{36}$.

The sample space for the game (this choice is ~~[illegible]~~ convenient - all give the same answers) is not $\Omega := S^\infty$, i.e., the set of sequences of tosses. Why not?

! We only look at the sums of face values of the tosses. So instead we use

$S = \{2, 3, \ldots, 12\}$ with probabilities

$p(2) = \frac{1}{36}$ $\quad$ $p(12) = \frac{1}{36}$

$p(3) = \frac{2}{36}$ $\quad$ $p(11) = \frac{2}{36}$

$p(4) = \frac{3}{36}$ $\quad$ $p(10) = \frac{3}{36}$

$p(5) = \frac{4}{36}$ $\quad$ $p(9) = \frac{4}{36}$

$p(6) = \frac{5}{36}$ $\quad$ $p(8) = \frac{5}{36}$

$p(7) = \frac{6}{36}$

with σ-field $\mathcal{S}$, the power set of S consisting of all subsets of S.

Now $\Omega := S^\infty$, the space of all sequences of the sums of the face values of the dice. For the σ-field on Ω, we take the σ-field generated by the "cylinder sets"

$K := Y_1 \times \cdots \times Y_n \times S \times S \times \cdots$

or $Y_1 \times \cdots \times Y_n \times S^\infty$ for short

where $Y_i \subseteq S$ is arbitrary. We define $\mathbb{P}(K) = P(Y_1) \cdots P(Y_n)$

Let us partition Ω into convenient sets to work with

$E_j :=$ the subset of Ω with elements of the form $\{j\} \times S^\infty$ | for $j = 2, \ldots, 12$

Now let us break them up further

W for winning
L for losing

$j =$	W	L	prob
2		E_2	$\frac{1}{36}$
3		E_3	$\frac{2}{36}$
7	E_7		$\frac{6}{36}$
11	E_{11}		$\frac{2}{36}$
12		E_{12}	$\frac{1}{36}$

Let $\#_k$ be the subset of S consisting of all elements <u>not</u> equal to k or 7.

	W	L
E_4	$4\ 4\ *\ *\ *\ *\ \dots$ $4\ \#_4\ 4\ *\ *\ *\ \dots$ $4\ \#_4\ \#_4\ 4\ *\ *\ \dots$ $4\ \#_4\ \#_4\ \#_4\ 4\ *\ \dots$ W_4	$4\ 7\ *\ *\ *\ \dots$ $4\ \#_4\ 7\ *\ *\ \dots$ etc L_4
E_5	$5\ 5\ *\ *\ *$ $5\ \#_5\ 5\ *\ *$ etc W_5	$5\ 7\ *\ \dots$ $5\ \#_5\ 7\ *\ \dots$ etc L_5
E_k k not 2, 3, 7, 11, 12	$k\ k\ *\ *\ *$ $k\ \#_k\ k\ *\ *$ etc W_k	$k\ 7\ *\ *$ $k\ \#_k\ 7\ *\ *$ etc L_k

Name of set

name of set

name of set

Let's compute the probabilities of these sets.

for $k = 4, 5, 6$

noting that $\mathbb{P}(k) = \frac{k-1}{36}$

$\mathbb{P}(\neq k) = 1 - \frac{k-1}{36} - \frac{6}{36}$

$$P(W_k) = \left(\frac{k-1}{36}\right)^2 \left(1 + \left(1 - \frac{k-1}{36} - \frac{6}{36}\right) + \left(1 - \frac{k-1}{36} - \frac{6}{36}\right)^2 + \cdots\right)$$

using A.1)

$$= \left(\frac{k-1}{36}\right)^2 \left(\frac{1}{1 - \left(1 - \frac{k-1}{36} - \frac{6}{36}\right)}\right)$$

$$= \frac{(k-1)^2}{36(k+5)}$$

Similarly for $k = 8, 9, 10$ using

$\mathbb{P}(k) = \frac{13-k}{36}$ and $\mathbb{P}(\neq k) = 1 - \frac{13-k}{36} - \frac{6}{36}$

we get

$$\mathbb{P}(W_k) = \frac{(13-k)^2}{36(19-k)}$$

Note $4 \longleftrightarrow 10$

$5 \longleftrightarrow 9$

$6 \longleftrightarrow 8$

are really the same numbers

Similarly

$$\mathbb{P}(L_k) = \frac{(k-1)\cdot 6}{36\cdot(k+5)} = \frac{k-1}{6(k+5)}$$

– for $k = 4, 5, 6$

and

$$\frac{(13-k)}{6(19-k)} \quad \text{for } k = 8, 9, 10.$$

So $\mathbb{P}(\text{winning}) =$

$$\sum_{k=2}^{12} P(W_k) =$$

$$\underset{k=2}{0} + \underset{3}{0} + \underset{4}{\frac{3^2}{36\cdot 9}} + \underset{5}{\frac{4^2}{36\cdot 10}} + \underset{6}{\frac{5^2}{36\cdot 11}} + \underset{7}{\frac{6}{36}}$$

$$+ \underset{10}{\frac{3^2}{36\cdot 9}} + \underset{9}{\frac{4^2}{36\cdot 10}} + \underset{8}{\frac{5^2}{36\cdot 11}} + \underset{11}{\frac{2}{36}}$$

$$= \frac{2}{36}\left(\frac{9}{9} + \frac{16}{10} + \frac{25}{11}\right) + \frac{8}{36} = \frac{244}{495} = 0.49292929\ldots$$

As a check let's compute $P(L)$ and make sure we get $P(W)+P(L)=1$.

$$P(\text{losing}) =$$

$$\frac{1}{36} + \frac{2}{36} + \frac{3}{6\cdot 9} + \frac{4}{6\cdot 10} + \frac{5}{6\cdot 11}$$

$$\frac{3}{6\cdot 9} + \frac{4}{6\cdot 10} + \frac{5}{6\cdot 11}$$

$$= \frac{251}{495}$$

Let T be the turn the game ends on, e.g.:

$$T(7**\ast\ldots) = 1$$

$$T(6456***) = 4$$

T is the "time" the game ends. This is an example of a type of random variable central to considerations later in the semester. It is what's called a "stopping time." What is $E(T)$? A casino cares about this because besides guaranteed long term earnings (aka "your losses"), they care about the time it takes for the earnings to come in (aka "how quickly you lose").

$$E(T) = \sum_{k=1}^{\infty} k\, \mathbb{P}(T=k)$$

We can use our decomposition of Ω into the E_j:

$$\mathbb{P}(T=k) = \sum_{j=2}^{12} \mathbb{P}(\{T=k\} \cap E_j)$$

For $E_2 = \{2 * * * \ldots\}$, E_3, E_7, E_{11}, E_{12} $T=1$ and for those sets exactly this is true. So we have

$$E(T) = 1\,(P(E_2) + P(E_3) + P(E_7) + P(E_{11}) + P(E_{12}$$

$$+ \sum_{\substack{j=4,5,6,\\8,9,10}} \sum_{k=2}^{\infty} k\, P(\{T=k\} \cap E_j) =$$

$$\frac{12}{36} + \sum_{\substack{j=4,5,6,\\8,9,10}} \sum_{k=2}^{\infty} k \, \mathbb{P}(\{T=k\} \cap E_j).$$

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for E_j with $j = 4, 5, 6, 8, 9, 10$

$$T(j \underbrace{\#_j \cdots \#_j}_{(k-2)\text{ entries}} j * * \cdots) = k$$

$$T(j \underbrace{\#_j \cdots \#_j}_{(k-2)\text{ entries}} 7 * * \cdots) = k$$

So for $j = 4, 5, 6$

$$P(\{T=k\} \cap E_j) = \left(\frac{j-1}{36}\right)\left(1 - \frac{j-1}{36} - \frac{6}{36}\right)^{k-2}\left(\frac{j-1}{36} + \frac{6}{36}\right)$$

$$= \frac{(j-1)}{36}\frac{(j+5)}{36} \cdot \left(1 - \frac{j+5}{36}\right)^{k-2}$$

and for $j = 8, 9, 10$

$$P(\{T=k\} \cap E_j) = \left(\frac{13-j}{36}\right)\left(1 - \frac{19-j}{36}\right)^{k-2}\left(\frac{19-j}{36}\right)$$

So summing

$$E(T) = \frac{12}{36} +$$

$$2 \sum_{j=4}^{6} \sum_{k=2}^{\infty} k \left(\frac{j-1}{36}\right)\left(\frac{j+5}{36}\right)\left(1-\frac{j+5}{36}\right)^{k-2}$$

$$= \frac{1}{3} + 2 \sum_{j=4}^{6} \frac{(j-1)(j+5)}{(36)^2} \sum_{k=2}^{\infty} k \left(1-\frac{j+5}{36}\right)^{k-2}$$

Letting $\tilde{k} = k-1$ we have

$$= \frac{1}{3} + 2 \sum_{j=4}^{6} \frac{(j-1)(j+5)}{(36)^2} \left(\sum_{\tilde{k}=1}^{\infty} \tilde{k} \left(1-\frac{j+5}{36}\right)^{\tilde{k}-1} + \sum_{\tilde{k}=1}^{\infty} \left(1-\frac{j+5}{36}\right)^{\tilde{k}} \right)$$

using $A_1 + A_2$

$$= \frac{1}{3} + 2 \sum_{j=4}^{6} \frac{(j-1)(j+5)}{(36)^2} \left(\left(\frac{36}{j+5}\right)^2 + \left(\frac{36}{j+5}\right) \right)$$

$$= \frac{1}{3} + 2 \sum_{4}^{6} \frac{j-1}{36} \left(\frac{36}{j+5} + 1 \right) = \frac{557}{165} \approx$$

3.375757...

or about $3\frac{1}{3}$ throws.

or about $2¢/\text{toss}$ for each \$10 bet.

What about the variability? Let's compute the standard deviation

$$\sqrt{E(T^2) - E(T)^2}$$

$$E(T^2) = \frac{1}{3} + 2\sum_{j=4}^{6}\left(\frac{j-1}{36}\right)\left(\frac{j+5}{36}\right)\sum_{k=2}^{\infty} k^2\left(1-\frac{j+5}{36}\right)^{k-2}$$

Letting $\tilde{k} = k-1$ we have

$$= \frac{1}{3} + 2\sum_{j=4}^{6}\left(\frac{j-1}{36}\right)\left(\frac{j+5}{36}\right)\left[\sum_{\tilde{k}=1}^{\infty}\tilde{k}^2\left(1-\frac{j+5}{36}\right)^{\tilde{k}-1} + 2\sum_{\tilde{k}=1}^{\infty}\tilde{k}\left(\ \right)^{\tilde{k}-1} + \sum_{\tilde{k}=1}^{\infty}\left(\ \right)^{\tilde{k}-1}\right]$$

use B_2

$$= \frac{1}{3} + 2\sum_{j=4}^{6}\left(\frac{j-1}{36}\right)\left(\frac{j+5}{36}\right)\left[\frac{\left(2-\frac{j+5}{36}\right)}{\left(\frac{j+5}{36}\right)^3} + 2\left(\frac{36}{j+5}\right)^2 + \frac{36}{j+5}\right)$$

$$= \frac{61769}{3025}$$

So Variance =

$$E(T^2) - E(T)^2$$

$$= \frac{245672}{27225}$$

Standard deviation

$$\approx 3.00395\ldots$$

We will see later that this let's us estimate the "swing" in T on average.