

## Imposing and Testing Equality Constraints in Models

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Last revised February 15, 2015

*Overview.* We have previously discussed how to impose and test various restrictions on models. In this section we will extend this discussion by explaining how to test whether two or more coefficients within a model are equal; we'll also show how to test more complicated sorts of equality constraints.

*Test for equality of parameters within a model.* Suppose you wish to test

$$H_0: \beta_1 = \beta_2$$

$$H_A: \beta_1 \neq \beta_2$$

It may be helpful to note that this is the same as testing

$$H_0: \beta_1 - \beta_2 = 0$$

$$H_A: \beta_1 - \beta_2 \neq 0$$

That is, you want to test whether two variables have equal effects. For example, in a model of family decision-making, you might hypothesize that wives have the same amount of influence as their husbands. Or, you might want to test whether time spent in one type of activity has the same effect as time spent in another activity. There are at least 2 ways of doing this.

*Option 1. Wald Test.* Wald tests are computed using the estimated coefficients and the variances/covariances of the estimates from the unconstrained model. A nice feature of Wald tests is that they only require the estimation of one model. This is the approach used by Stata's `test` command, where it is quite easy and simple to use.

Here is the rationale for this approach: Recall that testing  $\beta_{Educ} = \beta_{Jobexp}$  is equivalent to testing  $\beta_{Educ} - \beta_{Jobexp} = 0$ . Also recall that

$$V(X \pm Y) = V(X) + V(Y) \pm 2 \text{COV}(X, Y) = \sigma^2_{X \pm Y}$$

This implies that

$$V(b_{Educ} - b_{Jobexp}) = V(b_{Educ}) + V(b_{Jobexp}) - 2 \text{COV}(b_{Educ}, b_{Jobexp})$$

Hence, an appropriate test statistic for this problem is

$$F_{1, N-K-1} = \left( \frac{(b_{Educ} - b_{Jobexp}) - (\beta_{Educ} - \beta_{Jobexp})_0}{\sqrt{s^2_{b_{Educ}} + s^2_{b_{Jobexp}} - 2s_{b_{Educ}, b_{Jobexp}}}} \right)^2 = \left( \frac{(b_{Educ} - b_{Jobexp})}{\sqrt{s^2_{b_{Educ}} + s^2_{b_{Jobexp}} - 2s_{b_{Educ}, b_{Jobexp}}}} \right)^2$$

We have seen this idea many times before: Observed Value – Value Predicted by the Null (the predicted difference in this case being zero) divided by the estimated standard error of the estimator. Since an F test is being reported, all of this is squared.

**Stata Example.** Here is a modified version of the income/education/job experience example we have been using. I have reworked the data so that it is now a sample of 100 blacks and four hundred whites. We want to test whether a year of job experience (JOBEXP) has the same effect on income as a year of education (EDUC).

```
. use https://www3.nd.edu/~rwilliam/statafiles/blwh.dta, clear
. reg income educ jobexp
```

Source	SS	df	MS	Number of obs =	500
Model	32798.4018	2	16399.2009	F( 2, 497) =	1103.96
Residual	7382.84742	497	14.8548238	Prob > F =	0.0000
				R-squared =	0.8163
				Adj R-squared =	0.8155
Total	40181.2493	499	80.5235456	Root MSE =	3.8542

income	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]
educ	1.94512	.0436998	44.51	0.000	1.859261 2.03098
jobexp	.7082212	.0343672	20.61	0.000	.6406983 .775744
_cons	-7.382935	.8027781	-9.20	0.000	-8.960192 -5.805678

```
. test educ = jobexp
```

```
( 1) educ - jobexp = 0
```

```
F( 1, 497) = 564.98
Prob > F = 0.0000
```

The F statistic is highly significant, which means we reject the hypothesis that the two effects are equal.

To make sure that Stata did things correctly, you can use the `vce` command after running `regress` to get the variances and covariances (`estat vce` also works)

```
. vce
```

	educ	jobexp	_cons
educ	.00191		
jobexp	.000191	.001181	
_cons	-.027719	-.018488	.644453

Hence,

$$F_{1,N-K-1} = \left( \frac{(b_{Educ} - b_{Jobexp})}{\sqrt{s^2 b_{Educ} + s^2 b_{Jobexp} - 2s_{b_{Educ}, b_{Jobexp}}}} \right)^2 = \left( \frac{(1.94512 - .7082212)}{\sqrt{.00191 + .001181 - 2 * .000191}} \right)^2$$

$$= \left( \frac{1.236973312}{.052038447} \right)^2 = 23.77037317^2 = 565.03$$

Incidentally, in Stata, if you want to see what the constrained parameter estimates look like, add the `coef` parameter to the test command, e.g.

```
. test educ = jobexp, coef
```

```
( 1) educ - jobexp = 0
```

```
F( 1, 497) = 564.98  
Prob > F = 0.0000
```

Constrained coefficients

	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]	
educ	1.160275	.0286252	40.53	0.000	1.104171	1.21638
jobexp	1.160275	.0286252	40.53	0.000	1.104171	1.21638
_cons	-3.166146	.7829306	-4.04	0.000	-4.700662	-1.63163

Especially if you have tried to test a fairly complicated hypothesis, it is good to look at the constrained coefficients to make sure you specified things correctly, e.g. if for some reason you hypothesized that  $\beta_{\text{Educ}} = 3 * \beta_{\text{Jobexp}} - 1$ ,

```
. test educ = 3*jobexp - 1, coef
```

```
( 1) educ - 3 jobexp = -1
```

```
F( 1, 497) = 59.09  
Prob > F = 0.0000
```

Constrained coefficients

	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]	
educ	1.848938	.0418704	44.16	0.000	1.766873	1.931002
jobexp	.9496459	.0139568	68.04	0.000	.9222911	.9770007
_cons	-9.381235	.7595261	-12.35	0.000	-10.86988	-7.892591

test also makes it easy to test simpler hypotheses, e.g.

```
. test educ = 2
```

```
( 1) educ = 2
```

```
F( 1, 497) = 1.58  
Prob > F = 0.2098
```

Conversely, Stata's `testnl` command lets you test complicated nonlinear sorts of relationships among coefficients, e.g. if for some reason you hypothesize  $\text{Sqrt}(\beta_{\text{Educ}}) = \beta_{\text{Jobexp}}$ ,

```
. testnl sqrt(_b[educ]) = _b[jobexp]
```

```
(1) sqrt(_b[educ]) = _b[jobexp]
```

```
F(1, 497) = 365.49  
Prob > F = 0.0000
```

*Option 2: Incremental F Test.* The incremental F test is another approach. Appendix A reviews incremental F tests in general, and Appendix B shows the math involved for testing equality constraints; in this section we will simply outline the logic. Some key advantages of this approach are that (a) you can use it with most statistical software, and (b) even though this isn't always the easiest approach, it is important to understand it because the strategy used here is similar to the strategy that is optimal for other statistical techniques like logistic regression. You can proceed as follows:

- Regress Y on X1, X2, and any other IVs in the model. Store the results. We refer to this as the *unconstrained* model, because the effects of X1 and X2 are *not* constrained to be equal. That is, you are estimating the model

$$y = \alpha + \beta_1 X_1 + \beta_2 X_2 + \sum_{k=3}^K \beta_k X_k + \varepsilon$$

- Compute a new variable that is equal to the sum of the two variables you hypothesize to have equal effects, e.g. `gen sum12 = x1 + x2`
- Run a second regression in which you regress Y on SUM12 and any other IVs in the model. (Do NOT include X1 and X2 though.) Store the results. We refer to this as the *constrained* model, because, by adding X1 and X2 together, you are forcing their estimated effects to be equal (i.e. only one beta is being estimated for both variables). That is, you are estimating the model

$$y = \alpha + \beta_1 (X_1 + X_2) + \sum_{k=3}^K \beta_k X_k + \varepsilon$$

In practice, we estimate this via

$$y = \alpha + \beta_1 \text{sum12} + \sum_{k=3}^K \beta_k X_k + \varepsilon$$

- Do an incremental F test. As Appendices make clear, you are basically testing whether the  $R^2$  from the unconstrained model (where coefficients are not equal) significantly differs from the  $R^2$  from the constrained model (where coefficients have been constrained to be equal). If they do significantly differ then you reject the hypothesis that the coefficients are equal.
- This procedure can be easily modified to test similar hypotheses. For example, if you hypothesize that X1 and X2 have equal but opposite effects, compute a variable like `DIFF12 = X1 - X2`. If you think that the effect of X1 is twice that of X2, compute something like `WSUM12 = 2X1 + X2`. If you hypothesize that 3 variables have equal effects, compute `SUM123 = X1 + X2 + X3`. (Note that J=2 in this case.)

**EXAMPLE.** First, we estimate the unconstrained model. `INCOME` is regressed on `EDUC` and `JOBEXP`, yielding the following:

```
. use "https://www3.nd.edu/~rwilliam/statafiles/blwh.dta", clear
. reg income educ jobexp
```

Source	SS	df	MS	Number of obs =	500
Model	32798.4018	2	16399.2009	F( 2, 497) =	1103.96
Residual	7382.84742	497	14.8548238	Prob > F =	0.0000
				R-squared =	0.8163
				Adj R-squared =	0.8155
Total	40181.2493	499	80.5235456	Root MSE =	3.8542

income	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]	
educ	1.94512	.0436998	44.51	0.000	1.859261	2.03098
jobexp	.7082212	.0343672	20.61	0.000	.6406983	.775744
_cons	-7.382935	.8027781	-9.20	0.000	-8.960192	-5.805678

```
. est store unconstrained
```

Now, we estimate the constrained model. First, we compute  $JOBED = EDUC + JOBEXP$ . Then, we regress  $INCOME$  on  $JOBED$ .

```
. gen jobed = jobexp + educ
. reg income jobed
```

Source	SS	df	MS	Number of obs =	500
Model	24405.6951	1	24405.6951	F( 1, 498) =	770.43
Residual	15775.5542	498	31.6778196	Prob > F =	0.0000
				R-squared =	0.6074
				Adj R-squared =	0.6066
Total	40181.2493	499	80.5235456	Root MSE =	5.6283

income	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]	
jobed	1.160275	.0418016	27.76	0.000	1.078146	1.242405
_cons	-3.166146	1.143318	-2.77	0.006	-5.412468	-.9198242

```
. est store constrained
```

We can now use Maarten Buis's `ftest` command, which can be downloaded from SSC. `ftest` lets you do F tests of nested models. It is particularly useful when models are nested (as in this case) but cannot be estimated via the `nestreg` command.

```
. ftest unconstrained constrained
Assumption: constrained nested in unconstrained
```

```
F( 1, 497) = 564.98
prob > F = 0.0000
```

**Conclusion.** If you are doing OLS regression and you are using Stata, Option 1 (Wald tests) is probably the easiest way to go. HOWEVER, if you are using a maximum likelihood technique like logistic regression, a modified version of Option 2 (using a chi-square statistic instead of F) tends to be optimal.

Also, while we have primarily talked about testing the equality of 2 coefficients, e.g.  $\beta_1 = \beta_2$ , we have also seen that much more complicated sorts of tests are possible. You can also do simpler tests, like  $\beta_1 = 3$ .

Of course, any test you do should have a rationale behind it. You don't do tests just because they are possible, you do them because there are substantive reasons that motivate them.

## Appendix A: Incremental F Tests about a subset of coefficients.

We sometimes wish to test hypotheses concerning a subset of the variables in a model. For example, suppose a model includes 3 demographic variables (X1, X2, and X3) and 2 personality measures (X4 and X5). We may want to determine whether the personality measures actually add anything to the model, i.e. we want to test

$$H_0: \beta_4 = \beta_5 = 0$$

$$H_A: \beta_4 \text{ and/or } \beta_5 \neq 0$$

One way to proceed is as follows.

1. Estimate the model with all 5 IVs included. This is known as the *unconstrained model*. Retrieve the values for SSE and/or  $R^2$  (hereafter referred to as  $SSE_u$  and  $R^2_u$ .) [NOTE: If using the  $R^2$  values, copy them to several decimal places so your calculations will be accurate.]
2. Estimate the model using only the 3 demographic variables. We refer to this as the *constrained model*, because the coefficients for the excluded variables are, in effect, constrained to be 0. Retrieve the values for SSE and/or  $R^2$  (hereafter referred to as  $SSE_c$  and  $R^2_c$ ).
3. Compute the following:

$$F_{J, N-K-1} = \frac{(R_u^2 - R_c^2) * (N - K - 1)}{(1 - R_u^2) * J} = \frac{(SSE_c - SSE_u) / J}{SSE_u / (N - K - 1)} = \frac{(SSE_c - SSE_u) * (N - K - 1)}{SSE_u * J}$$

where J = the number of constraints imposed (in this case, 2) and K = the number of variables in the *unconstrained* model (in this case, 5). Put another way, J = the error d.f. for the constrained model minus the error d.f. for the unconstrained model.

If J = 1, this procedure will lead you to the same conclusions a two-tailed T test would (the above F will equal the  $T^2$  from the unconstrained model. )

If J = K, i.e. all the IVs are excluded in the constrained model, the incremental F and the Global F become one and the same; that is, the global F is a special case of the incremental F, where in the constrained model all variables are constrained to have zero effect. You can see this by noting that, if there are no variables in the model,  $R^2 = 0$ .

*When you can use incremental F.* In order to use the incremental F test, it must be the case that

- The sample is the same for each model estimated. This assumption might be violated if, say, missing data in variables used in the unconstrained model caused the unconstrained sample to be smaller than the constrained sample. You should be careful how missing data is getting handled in your statistical routines
- One model must be “nested” within the other; that is, one model must be a constrained, or special case, of the other. For example, if one model contains IVs X1-X5, and another model contains X1-X3, the latter is a special case of the former, where the constraints imposed are

$\beta_4 = \beta_5 = 0$ . If, however, the second model included X1-X3 and X6, it would not be nested within the first model and an incremental F test would not be appropriate.

- Other types of constraints can also be tested with an incremental F test. For example, we might want to test the hypothesis that  $\beta_1 = \beta_2$ , i.e. two variables have equal effects. We'll discuss such possibilities later.

### Other comments

- Constrained and unconstrained are relative terms. An unconstrained model in one analysis can be the constrained model in another. In reality, every model is "constrained" in the sense that more variables could always be added to it.
- Wald tests, which are easily done in Stata, are an alternative to incremental F tests.
- We are also often interested in doing such tests when estimating sequences of nested models. So, for example, Model 1 may include X1, X2 and X3; Model 2 may add X4 and X5; Model 3 adds X6 and X7; and so on. With each model we may want to test whether the variables added in that model have effects that significantly differ from 0.

## Appendix B: Incremental F Tests for equality constraints

Suppose we wish to test hypotheses like

$$H_0: \beta_4 = \beta_5$$

$$H_A: \beta_4 \neq \beta_5$$

To test for equality constraints using an incremental F test, the procedure works as follows:

- Compute a new variable that is equal to the sum of the two variables you hypothesize to have equal effects, e.g. `gen sum12 = x1 + x2`
- Regress Y on X1, X2, and any other IVs in the model. Retrieve the Unconstrained Error Sum of Squares ( $SSE_u$ ) or else the unconstrained  $R^2$  (use several decimal places). We refer to this as the *unconstrained* model, because the effects of X1 and X2 are not constrained to be equal. That is, you are estimating the model

$$y = \alpha + \beta_1 X_1 + \beta_2 X_2 + \sum_{k=3}^K \beta_k X_k + \varepsilon$$

- Run a second regression in which you regress Y on SUM12 and any other IVs in the model. (Do NOT include X1 and X2 though.) Retrieve the constrained error sum of squares ( $SSE_c$ ) or else the unconstrained  $R^2$  (use several decimal places). We refer to this as the *constrained* model, because, by adding X1 and X2 together, you are forcing their estimated effects to be equal (i.e. only one beta is being estimated for both variables). That is, you are estimating the model

$$y = \alpha + \beta_1 (X_1 + X_2) + \sum_{k=3}^K \beta_k X_k + \varepsilon$$

In practice, we estimate this via

$$y = \alpha + \beta_1 \text{Sum12} + \sum_{k=3}^K \beta_k X_k + \varepsilon$$

- Do an incremental F test. In this case,  $J = 1$ , so we get

$$F_{1, N-K-1} = \frac{(SSE_c - SSE_u) / 1}{SSE_u / (N - K - 1)} = \frac{(SSE_c - SSE_u) * (N - K - 1)}{SSE_u * 1} = \frac{(R_u^2 - R_c^2) * (N - K - 1)}{(1 - R_u^2) * 1}$$



EXAMPLE. Using the same example as before, estimate the unconstrained model first.

```
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. reg income educ jobexp
```

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				Root MSE =	3.8542	

  

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_cons	-7.382935	.8027781	-9.20	0.000	-8.960192	-5.805678

Hence,  $SSE_u = 7382.847$ ,  $R^2_u = .8163$ ,  $N = 500$ ,  $K = 2$ .

Now, we estimate the constrained model. First, we compute  $JOBED = EDUC + JOBEXP$ . Then, we regress  $INCOME$  on  $JOBED$ .

```
. gen jobed = jobexp + educ
. reg income jobed
```

Source	SS	df	MS			
Model	24405.6951	1	24405.6951	Number of obs =	500	
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income	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]	
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_cons	-3.166146	1.143318	-2.77	0.006	-5.412468	-.9198242

Thus, we get  $SSE_c = 15775.55$ ,  $R^2_c = .6074$ ,  $J = 1$ .

The incremental F test is then

$$F_{1, N-K-1} = \frac{(SSE_c - SSE_u) * (N - K - 1)}{SSE_u * 1} = \frac{(R^2_u - R^2_c) * (N - K - 1)}{(1 - R^2_u) * 1}$$

$$= \frac{(15775.55 - 7382.85) * 497}{7382.85} = \frac{(.81626 - .60739) * 497}{1 - .81626} = 565$$

This value is highly significant. Ergo, we reject the null hypothesis that education and job experience have equal effects.