Lagrangian spheres and Dehn twists

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In this paper we answer a question raised in [8] as to whether, at least in a simple situation, symplectic Dehn twists provide the only method of constructing symplectically knotted Lagrangian spheres in a symplectic 4-manifold.

We first recall the arrangement from [8]. Let W be the Stein manifold formed by adding to the unit cotangent bundle of the two-sphere, T^1S^2, a single 2-handle along the Legendrian curve in a single fiber of the boundary. As a Stein manifold it carries a symplectic structure which has a conformally expanding vector field whose flow exists for all time. The symplectic structure is the Kähler form associated to a plurisubharmonic exhaustion function and all such forms are equivalent up to symplectomorphism (see [4]). Alternatively W can be realized as the plumbing of two copies of T^1S^2. Namely we take two copies of T^1S^2 and identify the cotangent fibers projecting to a disk D in S^2 with a product $D \times E$ in each copy. We then identify the two copies of $D \times E$, preserving the product structure but reversing the factors. The resulting symplectic manifold W has two Lagrangian spheres L_1 and L_2 coming from the zero-sections in the T^1S^2 (or, in the previous description, the original zero-section and the stable manifold of the index 2 critical point in the added handle).

The following was proven in [8].

Theorem 1 *Let L be a Lagrangian sphere in W which is homologous to one of* *the zero-sections L_1. Then there exists a symplectomorphism ϕ of W such that $\phi(L) = L_1$.*

Now, associated to each of the Lagrangian spheres L_1 and L_2 is a compactly supported symplectomorphism called a (symplectic) Dehn twist τ_{L_i}. Up to Hamiltonian diffeomorphism it can be arranged to have support in an arbitrarily small neighborhood of L_i and acts as the antipodal map on L_i. Again up to Hamiltonian diffeomorphism one checks that $\tau_{L_1}(L_2)$ is equal to the Lagrangian connected sum of L_1 and L_2 but $\tau^2_{L_1}$ is smoothly isotopic to the identity, and so $\tau^2_{L_1}(L_2)$ is smoothly isotopic to L_2. Nevertheless P. Seidel showed in [14] that $\tau^2_{L_1}(L_2)$ is not Hamiltonian isotopic to L_2, it is a symplectically knotted Lagrangian isotopic to L_2, as is $\tau^{2r}_{L_1}(L_2)$ for all $r \neq 0$. A natural question is whether these are the only examples of such Lagrangian knots, and we will show that this in indeed the case.

Theorem 2 *Let L be a Lagrangian sphere in W. Then there exists a composition of Dehn twists τ such that $\tau(L)$ is Hamiltonian isotopic to L_1 or L_2.*

Now, in a Stein manifold a Lagrangian isotopy can be composed with a conformally contracting vector field (the negative gradient of the plurisubharmonic exhaustion) so as to lie in an arbitrarily small neighborhood of the union of the stable manifolds of the critical points. Also, a theorem of Weinstein says that a Lagrangian sphere (or two Lagrangian spheres intersecting transversally in a single point) have tubular neighborhoods unique up to symplectomorphism. Thus Theorem 1 in [8] about Lagrangian spheres in T^*S^2 implies the following.

Theorem 3 *Let L_1 be a Lagrangian sphere in a symplectic 4-manifold M. Then any other Lagrangian sphere $L \subset M$ which is sufficiently C^0 close to L_1 is Hamiltonian isotopic to L_1.*

The conclusions of this paper give the following.

Theorem 4 *Let L_1 and L_2 be two Lagrangian spheres in a symplectic 4-manifold M, intersecting transversally in a single point. Then for any other Lagrangian sphere $L \subset M$ which is sufficiently C^0 close to $L_1 \cup L_2$ there exists a composition τ of the Dehn twists τ_{L_1} and τ_{L_2} about L_1 and L_2 such that $\tau(L)$ is Hamiltonian isotopic to L_1 or L_2.*

Proof of Theorem 2

By Weinstein's Theorem a Lagrangian 2-sphere has self-intersection -2, thus Lagrangian spheres in W are homologous to either L_1, L_2 or $L_1 \sharp L_2$. Up to Hamiltonian isotopy $L_1 \sharp L_2 = \tau_{L_2}(L_1)$ and so it suffices to prove the result assuming that L is homologous to L_1.

We will extend the approach in [8] which established Theorem 1 and so recall that here. We will think of the non-compact end of W as a copy of $[0, \infty) \times M$ where M carries a contact structure with contact form α and the symplectic structure on the end is given by $\omega = d(e^t \alpha)$. The manifold M is a Lens space $L(3, 2)$ and α a canonical contact structure. Let J be a tame almost-complex structure on W such that on $[0, \infty) \times M$ the almost-complex structure J is translation invariant, preserves the contact planes on M and satisfies $J(\frac{\partial}{\partial t}) = X$, where X is the Reeb vector field on M associated to α. Then W admits a finite energy foliation with respect to J. For definitions and background on finite energy curves we refer to the series of papers [10], [11], [12], [13], [3]. The following is Theorem 16 from [8].

Theorem 5 *For any such J, the almost-complex manifold (W, J) can be foliated by finite energy planes. Exactly three planes in the foliation, E_0, E_1, E_2, are asymptotic to a particular Reeb orbit x_0. The other finite energy planes are all asymptotic to $3x_0$. After choosing orientations for L_1 and L_2 we may assume that $E_i \bullet L_j = -\delta_{ij}$ and $E_0 \bullet L_j = 1$ for $i, j = 1, 2$.*

In fact the finite energy foliation is described quite explicitly in [8], in particular in terms of the intersection of the finite energy planes with a level $\{R\} \times M$, for R large. The rigid planes E_i intersect $\{R\} \times M$ transversally in a certain tubular neighborhood U of the Reeb orbit x_0. The boundary of U is foliated by circles in an S^1-family of finite energy planes and this family divides W into two pieces. We assume that the piece foliated by planes disjoint from U is disjoint from all of the Lagrangian spheres, and when we vary J it will always be fixed in this region.

Given the foliation of W the proof in [8] proceeded to construct a plurisubharmonic exhaustion function for W. The plurisubharmonic function has three critical points, one of index 0 and two of index 2. With respect to the Kähler structure associated to the plurisubharmonic function the two stable manifolds form Lagrangian spheres intersecting in a single point. We will identify our symplectic structure ω_0 on W with the Kähler structure coming from a J_0-plurisubharmonic function, where J_0 is a fixed almost-complex structure. Then the stable manifolds of the index 2 critical points correspond to L_1 and L_2.

Now, the rigid planes E_i and the finite energy foliation are determined by a given almost-complex structure J. However there are ambiguities in the construction of the plurisubharmonic function. Up to contractible choices these are essentially equivalent to choosing families of surfaces in U diffeomorphic to a sphere with three disks removed. One boundary of such a surface should coincide with the intersection of a finite energy plane with the boundary of U and the other three boundaries with the intersections of the E_i with U. It can be seen that U can be singularly foliated by such surfaces, the foliation being smooth away from the E_i. The surfaces can be chosen in an essentially canonical way given the position of the $E_i \cap U$, and after they are chosen there exists a corresponding plurisubharmonic function with the required properties.

Suppose that J is another almost-complex structure tamed by ω_0. Let ω_1 be the symplectic form corresponding to a J-plurisubharmonic function. Then there are two natural symplectomorphisms from (W, ω_1) to (W, ω_0). Since both ω_0 and ω_1 tame the same almost-complex structure, convex linear combinations of the two forms are also symplectic and so by Moser's theorem we can generate a symplectomorphism between them. On the other hand, given a plurisubharmonic exhaustion its gradient flow is conformally symplectic with respect to the corresponding symplectic form. Therefore we get another symplectomorphism by first identifying neighborhoods of the stable manifolds using Weinstein's Theorem and extending this to a global symplectomorphism using the gradient flows (see [4] for these ideas). Composing this symplectomorphism with the inverse of the Moser diffeomoephism gives a symplectomorphism of (W, ω_0) determined by a tame almost-complex structure J. If $J = J_0$ and the surfaces in U are chosen in the same way then we may assume that this map is the identity.

Now let L be a Lagrangian sphere in W homologous to L_1. It was shown in [8] that there exists an almost-complex structure J satisfying the requirements above such that the corresponding unstable manifold of one of the index 2 critical points is disjoint from L. Furthermore, under the Moser map $(W, \omega_0) \to (W, \omega_1)$ the Lagrangian L can be arranged to stay disjoint from this unstable manifold. Therefore by Theorem 1 in [8], composing with a Hamiltonian diffeomorphism we may assume that the Moser map takes L to one of the stable manifolds. Thus the symplectomorphism of (W, ω_0) described above maps L onto L_1. This outlines the proof of Theorem 1 above taken from [8].

Now suppose that we choose a family of almost-complex structures J_t, $0 \leq t \leq 1$ with $J_1 = J$. Choosing corresponding plurisubharmonic exhaustion functions we get a family ϕ_t of symplectomorphisms of (W, ω_0) with $\phi_1(L) = L_1$. To get a smooth family of symplectomorphisms our foliations of U must be chosen to vary smoothly with t. If this could be done in such a way that the foliation corresponding to J_0 is the standard one then ϕ_0 would be the identity and one would in fact construct a Hamiltonian isotopy from L to L_1 (any smooth isotopy of Lagrangian spheres can be realized by a global Hamiltonian flow).

We are prevented from choosing such families of foliations if the relative positions of the $E_i \cap U$ rotate for $0 \leq t \leq 1$. This is the only obstruction.

Suppose now that we carry out this procedure starting with $\tau(L)$ rather than L, where τ is a composition of Dehn twists. Actually we can assume that τ is a composition of even powers of Dehn twists, so it is isotopic to the identity. In this case we can choose $J_1 = \tau(J)$ and connect this to $\tau(J_0)$ through the family $\tau(J_t)$ since τ will be a compactly supported symplectomorphism. We observe that the intersection of $\tau(J_t)$ finite energy planes with U are exactly the same as the intersections of the J_t finite energy planes. Therefore to understand the new family of intersections $E_i \cap U$ it suffices to understand the intersections $E_i \cap U$ for a family of tame almost-complex structures connecting $\tau(J_0)$ and J_0.

First consider T^*S^2 with its standard symplectic form. This again can be thought of as a Stein manifold with open end symplectomorphic to $[0, \infty) \times N$ where $N = \mathbb{R}P^3$ with its standard contact form. The Reeb flow here can be identified with the geodesic flow on S^2. We fix a tame almost-complex structure J invariant under the natural action of $\mathrm{Isom}(\mathrm{S}^2)$. Then as described in [5], and used in [6] and [7], T^*S^2 admits a finite energy foliation with all planes asymptotic to multiples of a Reeb orbit y_0 corresponding to, say, the equator on S^2. The foliation now contains two rigid planes E_i asymptotic to the single orbit y_0 and all other finite energy planes in the foliation are asymptotic to $2y_0$. The rigid planes will project to opposite hemispheres on the S^2. Now rotation about the axis perpendicular to the equator preserves y_0 and J and so also the rigid finite energy planes. It follows that each intersects the zero-section at either the north or south pole and intersects the tubes of radius r, denoted T^rS^2, in circles projecting to parallels on S^2. The square τ^2 of the symplectic Dehn twist about the zero-section can be thought of as the Hamiltonian flow of $H = \frac{1}{2}|p|^2$ if the cotangent vector has length $|p| \leq 2\pi$ and the identity if $|p| \geq 2\pi$. This map τ^2 is isotopic to the identity through symplectomorphisms τ_t^2 where τ_t^2 is equal to the Hamiltonian flow of $H(tp)$ for $|p| \leq \frac{2\pi}{t}$ and the identity for $|p| \geq \frac{2\pi}{t}$. We observe that $\tau_t^2(J)$ for $0 < t \leq 1$ give a family of tame almost-complex structures converging to J as $t \to 0$. In fact, for R sufficiently large, $\tau_t^2(J)|_{T^{\geq R}S^2}$ is approximately equal to J for all t since τ_t^2 acts as the geodesic flow on a fixed level (which we can assume to preserve the relevent CR structure) and is approximately translation invariant for R large. Therefore after a small adjustment we will think of τ_t^2 as a compactly supported variation of J. In a level $T^R S^2$ let us choose coordinates (x, y) in a cross-section A transverse to our Reeb orbit at $(0, 0)$ such that our rigid J-holomorphic planes intersect in points $(\pm\epsilon, 0)$. Then we observe that for $0 < t \leq 1$ the positions of $\tau_t^2(E_i) \cap A$ perform one complete rotation. Since the space of almost-complex structures is contractible, any family connecting $\tau^2(J)$ and J will have the same effect on the intersections.

Returning to our original situation, a family of almost-complex structures J_t on W connecting $\tau_{L_1}^2(J_0)$ and J_0 can be chosen to be fixed away from a neighborhood of L_1, in particular near the rigid curve E_2. So in following this path the position of E_2 remains unchanged and we claim that E_0 and E_1 rotate their position once.

To justify the claim, we again follow the methods of [8]. We stretch the neck a length $N \to \infty$ along the boundary of a tubular neighborhood V of L_1, symplectomorphic to a tubular neighborhood $T^{\leq r}S^2$ of the zero-section in T^*S^2. We suppose the the $J_t = J_0$ outside of V for all t. In the limit as $N \to \infty$ we have complex structures $J_{t,\infty}$ on T^*S^2 and our J_t-holomorphic finite energy foliations of W converge to $J_{t,\infty}$-holomorphic finite energy foliations of T^*S^2. These foliations may be taken to be exactly those described in the model case, in particular the limits of the rigid planes rotate positions once for $0 \leq t \leq 1$. We recall also that the limits of the rigid planes E_0 and E_1 in the completion of $W \backslash V$ converge to the same finite energy cylinder. We look at the intersections of our E_i with a 1-parameter family of surfaces intersecting this cylinder transversally. The surfaces can be chosen to be tangent to U at one end and tangent to a tube $T^r S^2$ at the other. Then for N sufficiently large our finite energy planes E_i will intersect these surfaces transversally and so their relative rotation will be the same in each. But by uniform convergence the rotation of the E_i in a $T^r S^2$ will be the same as that of the limits with respect to the $J_{t,\infty}$, in other words they rotate once. Our claim follows.

In conclusion, for a suitable choice of τ, a family of almost-complex structures connecting $\tau(J_0)$ and J_0 can produce any relative movement of the $E_i \cap U$. So given a J_1, we can find a τ such that a family of almost-complex structures J_t connecting $\tau(J_1)$ and J_0 produces no relative movement of the $E_i \cap U$. This allows us to find a smooth family of foliations of U which is standard at $t = 0$ but corresponds to a plurisubharmonic function having an unstable manifold disjoint from L at $t = 1$. Then $\tau(L)$ will be Hamiltonian isotopic to L_1 as required.

References

[1] Y. Eliashberg, Topological Characterization of Stein manifolds of dimension > 2, *Int. Journal of Math.*, 1 (1990), 29-46.

[2] Y. Eliashberg, Symplectic Geometry of plurisubharmonic functions, with notes by Miguel Abreu, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 488, *Guage theory and symplectic geometry (Montreal, PQ 1995), 49-67, Kluwer Acad. Publ., Dordrecht, 1997.*

[3] Y. Eliashberg, A. Givental and H. Hofer, Introduction to symplectic field theory, GAFA 2000 (Tel Aviv, 1999). Geom. Funct. Anal. 2000, Special Volume, Part II, 560–673.

[4] Y. Eliashberg and M. Gromov, *Convex Symplectic Manifolds*, Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989), 135-162, Proc. Sympos. Pure Math., 52, Part 2.

[5] R. Hind, Holomorphic Filling of $\mathbb{R}P^3$, *Comm. in Contemp. Math.*, 2(2000), 349-363.

[6] R. Hind, Stein fillings of Lens spaces, *Comm. in Contemp. Math.*, 6(2003), 1-17.

[7] R. Hind, Lagrangian spheres in $S^2 \times S^2$, preprint math.SG/03011092, to appear in Geom. and Funct. Anal.

[8] R. Hind, Lagrangian unknottedness in Stein surfaces, preprint.

[9] H. Hofer, K. Wysocki and E. Zehnder, A characterisation of the tight three-sphere, *Duke Math. J.*, 81(1995), no.1, 159-226.

[10] H. Hofer, K. Wysocki and E. Zehnder, Properties of pseudoholomorphic curves in symplectisations I: Asymptotics, *Ann. Inst. H. Poincaré Anal. Non Lineaire*, 13(1996), no.3, 337-379.

[11] H. Hofer, K. Wysocki and E. Zehnder, Properties of pseudoholomorphic curves in symplectisations II: Embedding controls and algebraic invariants, *Geom. Funct. Anal.*, 5(1995), no.2, 337-379.

[12] H. Hofer, K. Wysocki and E. Zehnder, Properties of pseudoholomorphic curves in symplectisations III: Fredholm theory, *Topics in nonlinear analysis*, 381-475, Prog. Nonlinear Differential Equations Appl., 35, Birkhäuser, Basel, 1999.

[13] H. Hofer, K. Wysocki and E. Zehnder, Finite energy foliations of tight three-spheres and Hamiltonian dynamics, *Ann. of Math.*, 157(2003), 125-255.

[14] P. Seidel, Lagrangian two-spheres can be symplectically knotted, *J. Diff. Geom.*, 52(1999), 145-171.