Percolation in the Secrecy Graph: Bounds on the Critical Probability and Impact of Power Constraints

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***Abstract*—Secrecy graphs model the connectivity of wireless networks under secrecy constraints. Directed edges in the graph are present whenever a node can talk to another node securely in the presence of eavesdroppers. In the case of infinite networks, a critical parameter is the maximum density of eavesdroppers that can be accommodated while still guaranteeing an infinite component in the network, i.e., the *percolation threshold*. We focus on the case where the location of the nodes and the eavesdroppers are given by Poisson point processes, with and without power constraints. We present bounds for different types of percolation, including in-, out- and undirected percolation.**

I. Introduction

To assess the impact of secrecy constraints in wireless networks, we have recently introduced a random geometric graph, the so-called *secrecy graph*, that represents the network or communication graph including only links over which secure communication is possible [5]. We assume that a transmitter can choose the rate such that it can communicate to any receiver that is closer than any of the eavesdroppers. If a power constraint is imposed, the maximum edge length is upper bounded by some value $\rho < \infty$. A natural topic for investigation is the maximum eavesdropper density at which infinite components cease to exist. If they do exist, end-to-end secure communication at a nonzero rate is likely to be possible. Since the resulting graph is directed, there are different types of components, including in-, out-, and undirected components. In each case, the percolation threshold (in terms of the density of eavesdroppers) is different.

II. Model

Our model is as follows. Let $\mathcal{P}$ and $\mathcal{P}'$ be independent Poisson processes, of intensities 1 and λ respectively, in $\mathbb{R}^d$. The case $d = 2$ provides a good example. We will call the points of $\mathcal{P}$ *black points* and the points of $\mathcal{P}'$ *red points*. Now define a directed graph, the *directed secrecy graph* $\vec{G}_{\text{sec}}$, on vertex set $\mathcal{P}$, by sending a directed edge from $x \in \mathcal{P}$ to $y \in \mathcal{P}$ if there is no point of $\mathcal{P}'$ in the open ball $D(x, \|x-y\|)$ centered at x with radius $\|x-y\|$. If there is a power constraint, all edges longer than some maximum value $\rho < \infty$ are removed.

The motivation for this construction is that $x \in \mathcal{P}$ can send a message to $y \in \mathcal{P}$ without being overheard by an eavesdropper from $\mathcal{P}'$. For more details, see [5], where the model was originally defined.

Our main aim in this paper is to study the critical value(s) of λ for various types of percolation in $\vec{G}_{\text{sec}}$ in the plane (precise definitions will be given later), first without power constraint (Section III) and second with power constraint (Section IV).

III. Percolation without Power Constraint

For a model of an infinite undirected random graph, *percolation* is said to occur if an infinite component occurs with positive probability. (In fact, this probability is almost always 1 by Kolmogorov's 0-1 law—see Theorem 1.) Since $\vec{G}_{\text{sec}}$ is a directed graph, there are several things we could mean by "component", which lead to several definitions of percolation. Following [1], we distinguish five distinct events. First, write G_{sec} for the undirected graph obtained from $\vec{G}_{\text{sec}}$ by removing the orientations of the edges and replacing any resulting double edges by single edges, and G'_{sec} for the undirected graph obtained from $\vec{G}_{\text{sec}}$ by including only those edges xy for which both $\vec{xy} \in \vec{G}_{\text{sec}}$ and $\vec{yx} \in \vec{G}_{\text{sec}}$. We write $\mathbf{U}$ for the event that G_{sec} has an infinite component, $\mathbf{O}$ for the event that $\vec{G}_{\text{sec}}$ has an infinite out-component, $\mathbf{I}$ for the the event that $\vec{G}_{\text{sec}}$ has an infinite in-component, $\mathbf{S}$ for the event that $\vec{G}_{\text{sec}}$ has an infinite strongly connected subgraph, and $\mathbf{B}$ for the event that G'_{sec} has an infinite component. Here, an out (resp. in)-component is a subgraph with a spanning subtree whose edges are all directed away from (resp. towards) a root vertex, and a strongly connected subgraph is one where there are directed paths from x to y for all x and y in the subgraph. As noted in [1], we have the following implications:

$$\mathbf{B} \Rightarrow \mathbf{S} \Rightarrow (\mathbf{I} \text{ and } \mathbf{O}), \qquad (\mathbf{I} \text{ or } \mathbf{O}) \Rightarrow \mathbf{U}. \tag{1}$$

Let $\mathbf{X}$ denote any of $\mathbf{U}, \mathbf{O}, \mathbf{I}, \mathbf{S}$ or $\mathbf{B}$, and let $p_{\mathbf{X}}(\lambda, d) = \mathbb{P}(\mathbf{X})$.

Theorem 1. *For all values of λ and d, and all choices of $\mathbf{X}$, $p_{\mathbf{X}}(\lambda, d)$ is either 0 or 1.*

Proof: The Poisson process is ergodic, and so the probability of any translation invariant event, such as percolation, is automatically 0 or 1. ■

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A complete proof from first principles is given in [14], where the uniqueness of the infinite cluster also has been established.

Since, for a fixed instance of $\mathcal{P}$, adding points to $\mathcal{P}'$ can only remove edges from $\vec{G}_{\mathrm{sec}}$, the probability $p_{\mathbf{X}}(\lambda, d)$ is non-increasing in λ. Define the *critical intensity* $\lambda_{\mathbf{X},d}$ by the formula

$$\lambda_{\mathbf{X},d} = \inf\{\lambda : p_{\mathbf{X}}(\lambda, d) = 0\} = \sup\{\lambda : p_{\mathbf{X}}(\lambda, d) = 1\}$$

and write (just for this paper) $\lambda_{\mathbf{X}} = \lambda_{\mathbf{X},2}$. We reiterate that *increasing* λ *decreases* the probability of percolation, in our formulation of the model. From (1), we have

$$\lambda_{\mathbf{B}} \le \lambda_{\mathbf{S}} \le \min\{\lambda_{\mathbf{I}}, \lambda_{\mathbf{O}}\}, \qquad \max\{\lambda_{\mathbf{I}}, \lambda_{\mathbf{O}}\} \le \lambda_{\mathbf{U}}. \tag{2}$$

Our first aim is to provide bounds on $\lambda_{\mathbf{X}}$. While doing this, we survey various methods that have been used for other continuum percolation models. They are from [4], and [10], on percolation in the Gilbert disc model, and from [1] and [6], on percolation in the k-nearest neighbour model.

A. Branching processes ([4], [6], [10])

For both the Gilbert disc model and the k-nearest neighbour model (the “traditional models”), the basic method is as follows. We start with a vertex x of $\mathcal{P}$, grow the cluster containing x in “generations”, and compare the growing cluster to a branching process. For the most natural way of doing this (details below), the branching process has more points than the cluster, so, in all dimensions, if the branching process dies out, so will the cluster. We can now use classical results which tell us when certain branching processes die out. Consequently, in all dimensions, branching processes give lower bounds for thresholds in the traditional models, i.e., they show that for certain parameters, percolation *does not* occur.

In the following, we will describe the method for the Gilbert disc model, although it is almost the same as for the k-nearest neighbour model. Assume that the origin O is a point of $\mathcal{P}$. First pick the points of $\mathcal{P}$ within distance r of O – these are the first generation. The second generation are the points of $\mathcal{P}$ which are each within distance r of some first generation point, but are not in the first generation themselves (i.e., they are not within distance r of O). The third generation are the points of $\mathcal{P}$ not belonging to the first two generations, but which are each within distance r of some second generation point, and so on. The associated branching process is obtained by setting each offspring size distribution to be $\mathrm{Po}(\pi r^2)$, so that we are essentially growing the same cluster containing O, but ignoring the fact that the various discs we have scanned for points actually overlap. In [4], Gilbert argues that if $\pi r^2 \le 1$, the branching process dies out with probability 1, so that the critical area for percolation is at least 1. When $\pi r^2 > 1$, it is possible to calculate (numerically) the probability that the branching process dies out, so this gives an upper bound on the probability that O belongs to an infinite component.

This method can be used to give an upper bound of $\lambda_{\mathbf{O}} \le 1$ for the secrecy graph model. In fact, for oriented out-percolation, we have the following result:

Proposition 2. *The probability $\theta_{\mathbf{O}}(\lambda)$ that O belongs to an infinite out-component in the secrecy graph satisfies*

$$\theta_{\mathbf{O}}(\lambda) \le \max\{0, 1 - \lambda\}.$$

Proof: See [14]. ■

In higher dimensions, the cluster is approximated better and better by the appropriate branching process, at least for the Gilbert and k-nearest neighbour models. This is because the distances from a point $p \in \mathcal{P}$ to its two nearest neighbours in $\mathcal{P}$ converge in distribution to a (common) deterministic limit, and because the overlap between the balls centered at a parent and at its child gets smaller and smaller, as $d \to \infty$. There is a slight complication in that the error (between the model and a branching process) is only asymptotically negligible over finitely many generations. Therefore, in both [6] and [10], oriented lattice percolation is brought in to establish asymptotic thresholds for percolation. The results are that in sufficiently high dimension, $k = 2$ gives percolation for the k-nearest neighbour model, and that the critical volume in the Gilbert model tends to 1 as $d \to \infty$.

For the secrecy graph, we have

Theorem 3. *If $\lambda \ge 1$, then, for all d, $\theta_{\mathbf{O},d}(\lambda) = 0$. If $\lambda < 1$, then $\theta_{\mathbf{O},d}(\lambda) \to 1 - \lambda$ as $d \to \infty$.*

The first part of the theorem follows from the above proposition. The proof of the second part, where we assume $\lambda < 1$, is lengthy and can also be found in [14].

Although the branching method seems to be tailored for oriented out-percolation, it also gives bounds via (2).

B. Lattice percolation ([4], [6], [12], [13])

Two variants of the basic method, applied to the Gilbert model, are described in Gilbert’s original paper [4]. For both variants, fix a connection radius r. First, if we consider the square lattice with bonds of length $r/2$, and make the state of a bond e open iff there is at least one point of $\mathcal{P}$ in the square whose *diagonal* is e, then bond percolation in the lattice implies percolation in the Gilbert model. Second, if we consider the hexagonal lattice where the hexagons have side length $r/\sqrt{13}$, and make the state of a hexagon open iff it contains a point of $\mathcal{P}$, then face percolation in the hexagonal lattice implies percolation in the Gilbert model. Using the fact that the critical probabilities for both bond percolation in the square lattice and face percolation in the hexagonal lattice are equal to $1/2$, one thus obtains upper bounds on the critical area πr_c^2 of about 17.4 and 10.9, respectively.

Häggström and Meester [6] used this method to show that, for fixed d, percolation occurs in the k-nearest neighbour model for sufficiently large k. Pinto and Win [12] (see [13] for more details) applied it to show that percolation occurs in all versions of the secrecy graph model when λ is sufficiently small. For the latter application, one needs to use *dependent percolation*, which means that the bounds are rather weak. Their method can be used to derive a bound which is two orders of magnitude away from the likely truth.

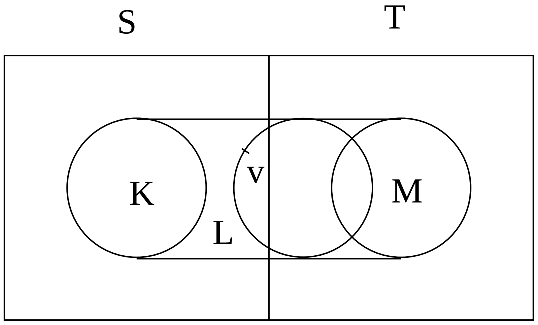

Fig. 1. The rolling ball method

So while lattice percolation has generally been used to show that percolation *does occur* in these models, it can also be used to show that percolation *does not occur* in the secrecy graph if λ is sufficiently large.

C. The rolling ball method ([1])

This is a method designed to show that percolation *does occur* for certain parameter ranges in various models. It was applied in [1] to prove upper bounds for critical values of k in the k-nearest neighbour model.

The method involves comparison with 1-independent percolation and carries through almost entirely for the secrecy graph. We will only need to modify some of the equations from [1]: however, for completeness, we include a full account of the method here. First, we state precisely what we mean by a *1-independent percolation model.*

Definition 4. *A bond percolation model on $\mathbb{Z}^2$ is said to be 1-independent if, whenever E_1 and E_2 are sets of edges at graph distance at least 1 from each other (i.e., if no edge of E_1 is incident to any edge of E_2), the state of the edges in E_1 is independent of the state of the edges in E_2.*

We will use the following theorem, proved in [2].

Theorem 5. *If every edge in a 1-independent bond percolation model on $\mathbb{Z}^2$ is open with probability at least 0.8639, then, almost surely, there is an infinite open component. Moreover, if B is a bounded region of the plane, there is, almost surely, a cycle of open edges surrounding B.*

We will use the first part of the theorem for our lower bounds, and the second part for our upper bounds.

For simplicity, let us first consider the case of $\mathbf{B}$-percolation. Later, we will indicate the modifications necessary for the other types.

Consider the rectangular region consisting of two adjacent squares S, T shown in Figure 1. Both S and T have side length $2r + 2s$, where r and s are to be chosen later. Also, T may be to the right, left, above or below S, in which case Figure 1 should be rotated accordingly. We define the *basic good event* $E_{\mathbf{B},S,T}$ to be the event that every black point u in the central disc K of S is joined to at least one black point in the central disc M of T by a path in G'_{sec}, regardless of the state of the Poisson processes outside $S \cup T$, and moreover that K contains at least one black point.

Now consider the following percolation model on $\mathbb{Z}^2$. Each vertex $(i, j) \in \mathbb{Z}^2$ corresponds to a square $[Ri, R(i+1)] \times [Rj, R(j+1)]$ in $\mathbb{R}^2$, where $R = 2r + 2s$, and an edge is open between adjacent vertices (corresponding to squares S and T) if *both* the corresponding basic good events $E_{\mathbf{B},S,T}$ and $E_{\mathbf{B},T,S}$ hold. Note that this is a 1-independent model on $\mathbb{Z}^2$, and that percolation in this model implies percolation in the original one. Since, by Theorem 5, the critical probability for any 1-independent model is at most 0.8639, if we can show that, for some r, s, λ,

$$\mathbb{P}(E_{\mathbf{B},S,T}) \geq 0.93195$$

it will follow that

$$\mathbb{P}(E_{\mathbf{B},S,T} \cap E_{\mathbf{B},T,S}) \geq 0.8639$$

by symmetry, and hence we will have shown that $\lambda_{\mathbf{B}} \geq \lambda$.

To bound the probability that a basic good event fails, we proceed as follows. Let K, L and M be as in Figure 1. (L is the region between the two discs K and M.) Define $E'_{\mathbf{B},S,T}$ to be the event that for every black point $v \in K \cup L$, there is a black point u such that i) $uv \in E(G'_{\text{sec}})$ ii) $\|u - v\| \leq s$ and iii) $u \in D_v$, where D_v is the disc of radius r inside $K \cup L \cup M$ with v on its K-side boundary (the middle disc in Figure 1). If we let F_S be the event that there is at least one black point in K, then we have (see [1] for background)

$$E'_{\mathbf{B},S,T} \cap F_S \subset E_{\mathbf{B},S,T}$$

and so

$$E^C_{\mathbf{B},S,T} \subset (E'_{\mathbf{B},S,T})^C \cup F^C_S$$

so that, since $\mathbb{P}((E'_{\mathbf{B},S,T})^C)$ is bounded by the expected number of points v such that i), ii) or iii) fail,

$$\mathbb{P}(E^C_{\mathbf{B},S,T}) \leq e^{-\pi r^2} + 2r(2r + 2s)p_{\mathbf{B},r,s}$$

where $p_{\mathbf{B},r,s}$ is the probability that i), ii) or iii) fail for some fixed v.

To bound $p_{\mathbf{B},r,s}$, we consider the probability that the vertex u closest to v inside D_v fails one of i), ii) or iii) (or does not exist). Suppose some $u \in D_v$ does exist, and write $t = \|u - v\|, A = B(v,t), B = B(v,t) \cap D_v$ and $C = B(u,t)$. Let $p_{\mathbf{B}}(u)$ be the probability that u is the closest point to v inside D_v, but that $uv \notin G'_{\text{sec}}$. Then

$$p_{\mathbf{B}}(u) = (1 - e^{-\lambda|A \cup C|})e^{-|B|} \tag{3}$$

and also

$$p_{\mathbf{B},r,s} \leq e^{-|D_v \cap B(v,s)|} + \int_{u \in D_v \cap B(v,s)} p_{\mathbf{B}}(u)\, du$$

so that

$$\mathbb{P}(E^C_{\mathbf{B},S,T}) \leq e^{-\pi r^2} + 2r(2r + 2s)\Bigg(e^{-|D_v \cap B(v,s)|} + \int_{u \in D_v \cap B(v,s)} (1 - e^{-\lambda|A \cup C|})e^{-|B|}\, du\Bigg) \tag{4}$$

$\mathbf{X}$	λ	r	s	p
$\mathbf{U}$	0.002	1.659	3.15	0.0669
$\mathbf{O}$	0.0008	1.658	3.15	0.0677
$\mathbf{B}$	0.0005	1.657	3.15	0.0680

TABLE I
UPPER BOUNDS ON $p = \min_{r,s} \mathbb{P}(E^C_{\mathbf{X},S,T})$ (VALUES OF p ROUNDED UP.)

and the right hand side can be minimized over all r and s, with λ fixed. The result is shown in Table 1, in row $\mathbf{B}$.

The calculation for the cases $\mathbf{U}$ and $\mathbf{O}$ is exactly analogous, using the graphs G_{sec} and $\vec{G}_{\text{sec}}$ respectively. The analogues of (3) are

$$p_{\mathbf{U}}(u) = (1 - e^{-\lambda|A|} - e^{-\lambda|C|} + e^{-\lambda|A\cup C|})e^{-|B|} \quad (5)$$

and

$$p_{\mathbf{O}}(u) = (1 - e^{-\lambda|A|})e^{-|B|} \quad (6)$$

respectively, and the natural analogue of (4) applies. The results of the optimization are shown in Table 1.

As proved in [1], the bound for $\lambda_{\mathbf{O}}$ in fact applies to $\lambda_{\mathbf{S}}$ and $\lambda_{\mathbf{I}}$ as well. In conclusion, we have proved the following theorem.

Theorem 6. $\lambda_{\mathbf{U}} \geq 0.002, \lambda_{\mathbf{O}} \geq 0.0008, \lambda_{\mathbf{I}} \geq 0.0008, \lambda_{\mathbf{S}} \geq 0.0008$ *and* $\lambda_{\mathbf{B}} \geq \ 0.0005$.

D. High confidence results ([1])

This method gives both upper and lower bounds for percolation thresholds in the k-nearest neighbour model. It involves computing a certain high dimensional integral using Monte Carlo methods, and so is not fully rigorous. The approach carries over essentially completely for the secrecy graph.

The lower bound method (corresponding to the upper bound method for the k-nearest neighbour model) may be summarized as follows. Given a trial value of λ, which we wish to show is a lower bound on one of the percolation thresholds $\lambda_{\mathbf{U}}, \lambda_{\mathbf{O}}$ or $\lambda_{\mathbf{B}}$, we choose trial values of r and s. Then we generate a random instance of $\mathcal{P} \cup \mathcal{P}'$ inside $S \cup T$ and test for the following conditions: i) for more than half of the black points $v \in K$, there are paths (in $G_{\text{sec}}, \vec{G}_{\text{sec}}$ or G'_{sec} for the cases $\mathbf{X} = \mathbf{U}, \mathbf{O}, \mathbf{B}$) to more than half the black points in M, regardless of the state of $\mathcal{P} \cup \mathcal{P}'$ outside $S \cup T$; ii) for more than half of the black points $v \in M$, there are paths to more than half the black points in K, regardless of the state of $\mathcal{P} \cup \mathcal{P}'$ outside $S \cup T$. As before, it is clear that this is a 1-independent model on the bonds joining adjacent squares, and that percolation in this model implies percolation in the original one. Consequently, if these conditions hold with probability at least 0.8639, then percolation occurs. The condition that the path should be independent of the process outside $S \cup T$ is simply obtained by ignoring any edges of $uv \in E(\vec{G}_{\text{sec}}(S\cup T))$ where $\|u - v\| > \text{dist}(u, \partial(S\cup T))$, since only edges uv with $\|u - v\| \leq \text{dist}(u, \partial(S\cup T))$ are guaranteed to exist in $\vec{G}_{\text{sec}}$. ∂A denotes the boundary of $A \subset \mathbb{R}^2$.

The probability that conditions i) and ii) are satisfied can be expressed as a complicated multiple integral, whose value we would like to be greater than 0.8639, for some r and s.

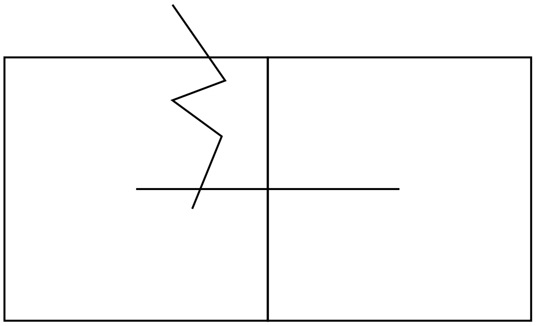

Fig. 2. Forbidden path for upper bound method

This is the integral we estimate using Monte Carlo methods. Using a computer program we generated many instances, and counted the proportion of times these conditions held. From these we calculated the confidence level, i.e., the probability p that these results (or better) could be obtained, if the true value of the integral was less than 0.8639. In all cases p was less than 10^{-25}: the detailed results appear in Table 2. It turns out that the method for the $\mathbf{X} = \mathbf{O}$ case actually applies to the cases $\mathbf{X} = \mathbf{S}$ and $\mathbf{X} = \mathbf{I}$ as well, and the results obtained are as follows.

Theorem 7. *With high confidence,* $\lambda_{\mathbf{B}} \geq 0.09, \lambda_{\mathbf{O}} \geq 0.11, \lambda_{\mathbf{I}} \geq 0.11, \lambda_{\mathbf{S}} \geq 0.11$ *and* $\lambda_{\mathbf{U}} \geq 0.20$.

The upper bound method (corresponding to the lower bound method for the k-nearest neighbour model) is as follows. For suitable r and s, we generate instances of $\mathcal{P}$ and $\mathcal{P}'$ in $S \cup T$, and check whether, regardless of the state of the processes outside $S \cup T$, there is no path (in $G_{\text{sec}}, \vec{G}_{\text{sec}}$ or G'_{sec} for the cases $\mathbf{X} = \mathbf{U}, \mathbf{O}, \mathbf{B}$) from outside $S \cup T$ that crosses the line segment joining the center of S to the center of T (see Figure 2). We define a 1-independent percolation model on $\mathbb{Z}^2$ by declaring an edge open if this condition holds for the corresponding rectangle $S \cup T$. If an edge is open with probability at least 0.8639, then, from Theorem 5, there are open cycles surrounding any bounded region of the plane. Consequently, if there was an infinite $\mathbf{X}$-component starting in some such bounded region, it would have to cross an open cycle, and in particular cross the central line segment in one of the rectangles $S \cup T$ corresponding to an open edge in this cycle. This contradicts the condition for that edge to be open, and so percolation cannot occur if the edges are open with probability at least 0.8639.

The results of these simulations are also shown in Table 2, and so we have the following result.

Theorem 8. *With high confidence,* $\lambda_{\mathbf{B}} \leq 0.13, \lambda_{\mathbf{O}} \leq 0.17, \lambda_{\mathbf{I}} \leq 0.17, \lambda_{\mathbf{S}} \geq 0.17$ *and* $\lambda_{\mathbf{U}} \leq 0.27$.

IV. PERCOLATION WITH POWER CONSTRAINTS

To model power constraints, we remove from all the original types of secrecy graphs all edges with length larger than the maximum transmission radius ρ. As it turns out, many of the

$\mathbf{X}$	bound	value	r	s	successes	trials	confidence
$\mathbf{U}$	lower	0.20	90	10	1480	1500	10^{-66}
$\mathbf{O}$	lower	0.11	60	0	963	1000	10^{-25}
$\mathbf{B}$	lower	0.09	80	0	2159	2250	10^{-51}
$\mathbf{U}$	upper	0.27	110	0	4296	4600	10^{-51}
$\mathbf{O}$	upper	0.17	110	0	3689	4000	10^{-25}
$\mathbf{B}$	upper	0.13	125	0	6226	6750	10^{-45}

TABLE II
RESULTS OF MONTE-CARLO SIMULATIONS. (ALL CONFIDENCES ROUNDED UP.)

proposed techniques can also be used to derive bounds on the case with power limit.

A. Branching processes

As in the case without power constraint, we compare the growing cluster from a black point with a branching process to obtain a bound on $\lambda_{\mathbf{O}}$. Due to the overlap between the regions scanned for neighbors in different generations, the cluster in the secrecy graph grows more slowly than in the branching process. Consequently, since the branching process dies out w.p. 1 if the mean of its offspring size distribution is smaller than 1, the cluster in the secrecy graph dies out also if the mean out-degree is smaller than 1. This argument can be made rigorous using the same techniques as in the proof of Prop. 2 (see [14] for the details).

Proposition 9. *Let $N \triangleq \pi\rho^2$. For $N > 1$,*

$$\lambda_{\mathbf{O}} \leq 1 + \frac{\mathcal{W}(-Ne^{-N})}{N}\,,$$

where $\mathcal{W}$ is the principal branch of the Lambert W function. For $N \leq 1$, the critical intensity is undefined.

Proof: The mean out-degree with power constraint is [5] $K(\lambda) = \frac{1}{\lambda}(1 - e^{-\lambda N})$, and the upper bound for $N > 1$ is the solution of $K(\lambda) = 1$. If $N < 1$, the mean out-degree is smaller than 1 even if $\lambda = 0$, so the critical density is not defined. ■

B. The rolling ball method

In the rolling ball method, only links with length smaller than s are considered. Since the optimum s in the derivation of the bounds in Theorem 6 is 3.15 (see Table I), these bounds are, in fact, also valid for the power-constrained secrecy graph with $\rho = 3.15$.

Corollary 10. *With maximum edge length $\rho = 3.15$, $\lambda_{\mathbf{U}} \geq 0.002, \lambda_{\mathbf{O}} \geq 0.0008, \lambda_{\mathbf{I}} \geq 0.0008, \lambda_{\mathbf{S}} \geq 0.0008$ and $\lambda_{\mathbf{B}} \geq 0.0005$.*

To derive bounds for smaller radii ρ, the optimization over s and r in (4) can be constrained to $s \leq \rho$.

C. High-confidence results

In the high-confidence method, power constraints are straightforward to incorporate as well. For both lower and upper bounds, we permit only edges shorter than ρ. The results are shown in Fig. IV-C. For comparison, a simulated curve for the case $\mathbf{O}$ is included.

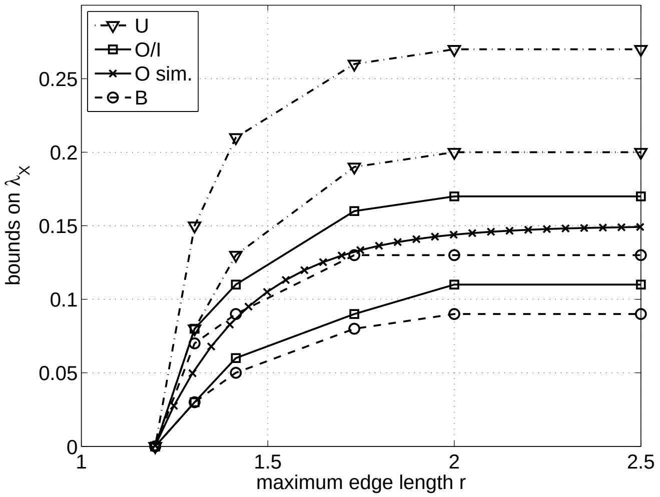

Fig. 3. Critical probabilities with power control. The smooth curve marked with x's is a simulated curve for the case $\mathbf{O}$.

V. CONCLUDING REMARKS

We have presented several methods to calculate bounds on five percolation thresholds in the Poisson secrecy graph. Due to the dependence in the model, the rigorous bounds are still rather loose; however, the high-confidence bounds derived are much tighter: the gap between the bounds is at most 55%. With power constraints, the same methods remain applicable.

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