ON THE TOTAL CURVATURE OF SEMIALGEBRAIC GRAPHS

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ABSTRACT. We define the total curvature of a semialgebraic graph $\Gamma \subset \mathbb{R}^3$ as an integral $K(\Gamma) = \int_\Gamma d\mu$, where μ is a certain Borel measure completely determined by the local extrinsic geometry of Γ. We prove that it satisfies the Chern-Lashof inequality $K(\Gamma) \geq b(\Gamma)$, where $b(\Gamma) = b_0(\Gamma) + b_1(\Gamma)$, and we completely characterize those graphs for which we have equality. We also prove the following unknottedness result: if $\Gamma \subset \mathbb{R}^3$ is homeomorphic to the suspension of an n-point set, and satisfies the inequality $K(\Gamma) < 2 + b(\Gamma)$, then Γ is unknotted. Moreover, we describe a simple planar graph G such that for any $\varepsilon > 0$ there exists a knotted semialgebraic embedding Γ of G in $\mathbb{R}^3$ satisfying $K(\Gamma) < \varepsilon + b(\Gamma)$.

CONTENTS

INTRODUCTION

The total curvature of a simple closed C^2-curve Γ in $\mathbb{R}^3$ is the quantity

$$K(\Gamma) = \frac{1}{\pi} \int_\Gamma |k(s)\,|ds|,$$

where $k(s)$ denotes the curvature function of Γ and $|ds|$ denotes the arc-length along C. In 1929 W. Fenchel [9] proved that for any such curve Γ we have the inequality

$$K(\Gamma) \geq 2, \tag{F}$$

with equality if and only if C is a planar convex curve.

Two decades later, I. Fáry [8] and J. Milnor [17] gave probabilistic interpretations of the total curvature. Milnor's interpretation goes as follows.

Any unit vector $\boldsymbol{u} \in \mathbb{R}^3$ defines a linear function $h_{\boldsymbol{u}} : \mathbb{R}^3 \to \mathbb{R}$, $x \mapsto (\boldsymbol{u}, x)$, where $(-,-)$ denotes the inner product in $\mathbb{R}^3$. For a generic $\boldsymbol{u}$, the restriction of $h_{\boldsymbol{u}}$ to Γ is a Morse function. We denote by $w_\Gamma(\boldsymbol{u})$ the number of critical points of this function. Then

$$K(\Gamma) = \frac{1}{\text{area}\,(S^2)} \int_{S^2} w_\Gamma(\boldsymbol{u})\,|d\boldsymbol{u}|,$$

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where S^2 denotes the unit sphere in $\mathbb{R}^3$ and $|du|$ the Euclidean area density on S^2. Since any function on Γ has at least two critical points (a minimum and a maximum) the inequality (F) is obvious. Moreover, they show that if $K(\Gamma)$ is not too large, then Γ cannot be knotted. More precisely, if $K(\Gamma) < 4$ then Γ cannot be knotted.

Soon after, in 1957, Chern and Lashof [5] proved higher dimensional generalizations of the results of Fenchel, Fáry and Milnor. Fix a compact k-dimensional submanifold $\Gamma \subset \mathbb{R}^{n+1}$. Again, any unit vector $\boldsymbol{u} \in \mathbb{R}^{n+1}$ defines a linear function $h_{\boldsymbol{u}}$ on $\mathbb{R}^{n+1}$. For generic $\boldsymbol{u}$ restriction of $h_{\boldsymbol{u}}$ to Γ is a Morse function. We denote by $w_\Gamma(\boldsymbol{u})$ the number of its critical points. Observe that if $M_{\boldsymbol{u}}(t)$ denotes the Morse polynomial of $h_{\boldsymbol{u}}|_\Gamma$ then $w_\Gamma(t) = M_{\boldsymbol{u}}(t)|_{t=1}$. We set

$$K(\Gamma) = \frac{1}{\operatorname{area}(S^n)} \int_{S^n} w_\Gamma(\boldsymbol{u})\, |du|,$$

where S^n denotes the unit sphere in $\mathbb{R}^{n+1}$.

The Morse inequalities imply that $w_\Gamma(\boldsymbol{u}) \geq \sum_{j=0}^k b_j(\Gamma)$ for generic $\boldsymbol{u}$, where $b_j(\Gamma)$ are the Betti numbers of Γ. In particular, we obtain the Chern-Lashof inequality

$$K(\Gamma) \geq \sum_{j=0}^{k} b_j(\Gamma). \tag{CL}$$

Chern and Lashof proved that, much like in the case of curves, the quantity $K(\Gamma)$ can be expressed as an integral

$$K(\Gamma) = \int_\Gamma \rho_\Gamma(x) |dA_\Gamma(x)|,$$

where $\rho_\Gamma(x)$ can be explicitly computed from the second fundamental form of the embedding $\Gamma \hookrightarrow \mathbb{R}^{n+1}$, and $|dA_\Gamma|$ is the Euclidean area density on Γ. Additionally, they proved that $K(\Gamma) = 2$ if and only if Γ^k is a convex hypersurface of an affine $(k+1)$-dimensional plane in $\mathbb{R}^{n+1}$. The embedding $\Gamma^k \hookrightarrow \mathbb{R}^{n+1}$ is called *tight* if we have equality in (CL). The subject of tight embeddings continues to be an active area of research (see e.g. [2, 13, 15]).

In this paper we extend the Chern-Lashof approach to singular one dimensional compact semialgebraic subsets of $\Gamma \subset \mathbb{R}^3$. They can be visualized as graphs embedded in some "tame"[1] fashion in $\mathbb{R}^3$. There are several competing proposals of what should constitute the total curvature of such a graph (see e.g. [11, 20]) but they don't seem to fit the elegant mold created by Chern and Lashof. Our approach addresses precisely this issue and its uses an approach based on stratified Morse theory pioneered by T. Banchoff [1] and N. Kuiper [14] for special cases of stratified spaces, more precisely, PL spaces. Here are the main ideas and results.

Consider a compact, connected one-dimensional semialgebraic subset $\Gamma \subset \mathbb{R}^3$. We fix a Whitney stratification of Γ, i.e., we fix a finite subset $V \subset \Gamma$ such that the complement is a finite disjoint union of C^2 arcs. Then, for a generic $\boldsymbol{u} \in S^2$ the restriction of $h_{\boldsymbol{u}}$ to Γ is a stratified Morse function in the sense of Lazzeri [16] and Goreski-MacPherson [10]. We denote by $M_{\boldsymbol{u}}(t)$ its stratified Morse polynomial and we set

$$w_\Gamma(\boldsymbol{u}) = M_{\boldsymbol{u}}(t)|_{t=1}.$$

The stratified Morse inequalities imply that $w_\Gamma(\boldsymbol{u}) \geq b_0(\Gamma) + b_1(\Gamma) = 1 + b_0(\Gamma)$ and we define the total curvature of Γ to be

$$K(\Gamma) = \frac{1}{\operatorname{area}(S^2)} \int_{S^2} w_\Gamma(\boldsymbol{u})\, |du|.$$

Clearly the total curvature satisfies the Chern-Lashof inequality (CL), and we say that Γ is *tight* if we have equality.

[1] For example, tameness would prohibit "very wavy" edges.

In Theorem 2.5 we give an explicit description of $K(\Gamma)$ in terms of infinitesimal and local invariants of Γ which shows that the total curvature is independent of the choice of the Whitney stratification. The can be given a characterization similar in spirit to the approach in [17]. More precisely (see Corollary 2.7) the number $\mu(\Gamma)=\frac{1}{2}\big(K(\Gamma)+\chi(\Gamma)\big)$ is equal to the average number of local minima of the family of functions $h_{\boldsymbol{u}}|_\Gamma$, $\boldsymbol{u}\in S^2$. Following the terminology in [17] we will refer to $\mu(\Gamma)$ as the *crookedness* of Γ. The Chern-Lashof inequality can be rephrased as $\mu(\Gamma)\geq 1$.

In Corollary 2.12 we proved that if the vertices of Γ have degrees ≤ 3 then our integral curvature coincides (up to a multiplicative factor) with the integral curvature recently introduced by Gulliver and Yamada [11]. In general there does not seem to be a simple relationship between these two notions of integral curvature.

We also investigate the structure of one-dimensional tight semialgebraic sets. We observe that Γ is tight if and only if it satisfies Banchoff's *two-piece property*: the intersection of Γ with any closed half-space is either empty, or connected. Using this observation we were able to give a complete description of the tight one dimensional semialgebraic subsets of $\mathbb{R}^3$. More precisely, in Theorem 3.1 we prove that they are of two types.

• *Type S: Straight.* In this case all the edges are *straight* line segments. Moreover, there exists a convex polyhedron (canonically determined by Γ such that the following hold (see [13, Lemma 2.4])

(a) The 1-skeleton of P is contained in Γ.
(b) Any vertex v of Γ which is not a vertex of P has the property lies in the convex hull of its neighbors.

• *Type C: Curved.* In this case some of the edges of Γ have nontrivial curvature. Then Γ is contained in a plane P and there exists a closed convex semialgebraic curve $B\subset P$ with the following properties.

(a) $B\subset\Gamma$.
(b) $\Gamma\setminus B$ is a union of line segments contained in the region R bounded by B.
(c) The complement of Γ in the region bounded by B is a finite union of convex open subsets of the plane P.

In particular, this gives a positive answer to a question raised at the end of [11, Sec. 4].

We also discuss knottedness issues. In Theorem 4.1 we prove that if $\Gamma\subset\mathbb{R}^3$ is a semialgebraic subset of $\mathbb{R}^3$ homeomorphic to the suspension of an n-point set, and $\mu(\Gamma)<2$, then Γ is isotopic to a planar embedding of this suspension. The case $n=2$ was first proved by Fáry [8] and Milnor [17], while the case $n=3$ was investigated Gulliver-Yamada [11] who proved the unknottedness under the more stringent requirement $\mu(\Gamma)<\frac{3}{2}$.

The situation is dramatically different for slightly more complicated graphs. Consider a graph which is homeomorphic to the union of a round circle and two parallel chords. We show that for every $\varepsilon>0$ there exists a knotted PL-embedding $\Gamma_\varepsilon\hookrightarrow\mathbb{R}^3$ of this graph such that

$$\mu(\Gamma_\varepsilon)<1+\varepsilon.$$

In Question 4.4 describe a possible candidate of a class of planar graphs for which the knottedness results of Milnor ought to hold. For the reader's convenience, we have included a brief appendix containing some basic facts about semialgebraic sets used throughout the paper.

Notations. In this paper, we will denote by $(-,-)$ the inner product in $\mathbb{R}^3$, by $|\bullet|$ the corresponding Euclidean norm. For any finite set S we will denote its cardinality by $\#S$.

1. One dimensional stratified Morse theory

Suppose Γ is a compact connected 1-dimensional semi-algebraic subset of $\mathbb{R}^3$. It can be identified *non canonically* with a graph as follows. We fix a finite subset $V\subset\Gamma$ called the *vertex set* such that the complement $\Gamma \setminus V$ is a finite disjoint union of real analytic, bounded semialgebraic arcs without self intersections connecting different points in V. We will refer to these arcs as *open edges* and we will denote by $E = E(\Gamma)$ the set of open edges. Note that this definition excludes the existence edges with identical endpoints although we allow for multiple edges between two given points.

A *semialgebraic graph* is a compact, connected 1-dimensional semialgebraic set together with a choice of vertex set satisfying the above properties.[2] Clearly, on the same semialgebraic set we can define multiple structures of semialgebraic sets.

If Γ is a semialgebraic graph with vertex set V, then the degree of a vertex v, denoted by $\deg(v)$ is the number of edges incident to v. Then

$$\chi(\Gamma) = \#V - \#E = \frac{1}{2}\sum_{v\in V}\big(\, 2-\deg(v)\,\big). \tag{1.1}$$

Since Γ is connected we deduce

$$b_1(\Gamma) = 1-\chi(G) = 1-\#V+\#E. \tag{1.2}$$

Note that the degree of a vertex p can be also defined as the cardinality of the intersection of Γ with a sphere of sufficiently small radius centered at p. This definition makes sense even for points $p \in \Gamma\setminus V$, and for such points we have $\deg(p) = 2$. The equality (1.1) can be rewritten as

$$\chi(\Gamma) = \sum_{p\in\Gamma}\big(\, 2-\deg(p)\,\big) \tag{1.3}$$

For every $p \in \Gamma$ we define $\boldsymbol{N}_p \subset S^2$ as follows. For $q \in \Gamma \setminus \{p\}$ denote by $\overrightarrow{\rho_p(u)}$ the unit vector $\overrightarrow{\rho_p(q)} := \frac{1}{|\vec{pq}|}\vec{pq}$. We obtain in this fashion o semialgebraic map

$$\rho_p : \Gamma \setminus v \to S^2.$$

Now set

$$\boldsymbol{\tau} \in \boldsymbol{N}_p \Longleftrightarrow \exists \text{ sequence } (q_k)_{k\geq 1} \subset \Gamma\setminus\{p\},\ \ \boldsymbol{\tau} = \lim_{k\to\infty}\overrightarrow{\rho_p(q_k)}.$$

Since Γ is semialgebraic, the set $\boldsymbol{N}_p$ is finite for any $p \in \Gamma$. We will refer to the vectors in $\boldsymbol{N}_p$ as the *interior unit tangent vectors* to Γ at v. The union of half-lines at p in the directions given by $\boldsymbol{\tau} \in \boldsymbol{N}_p$ is called the *tangent cone* to Γ at p.

Any unit vector $\boldsymbol{u} \in S^2$ defines a linear map (height function)

$$h_{\boldsymbol{u}} : \mathbb{R}^3 \to \mathbb{R},\ \ h_u(x) := (\boldsymbol{u}, x).$$

A unit vector $\boldsymbol{u}$ is called Γ-*nondegenerate* if the restriction of $h_{\boldsymbol{u}}$ to Γ is a stratified Morse function with respect to the vertex-edge stratification. More precisely, (see [10]) this means that the restriction of $h_{\boldsymbol{u}}$ to the interior of any edge has only nondegenerate critical points and moreover

$$(\boldsymbol{u},\boldsymbol{\tau}) \neq 0,\ \ \forall v \in V,\ \ \boldsymbol{\tau} \in \boldsymbol{N}_v.$$

The complement in S^2 of the set of Γ-nondegenerate vectors is called the *discriminant set* of the semialgebraic graph and it is denoted by Δ_Γ. The discriminant set clearly depends on the choice of graph structure, but one can prove (see [10, 12] and Lemma A.2) that Δ_Γ is a closed semialgebraic subset of S^2 of dimension ≤ 1. In particular, most unit vectors $\boldsymbol{u}$ are Γ-nondegenerate.

If $\boldsymbol{u}$ is a nondegenerate vector then for every $p \in \Gamma$ and every $\varepsilon > 0$ we set

$$L^{\pm}_{\varepsilon}(p,\boldsymbol{u}) := \Big\{ q \in \Gamma;\ \ |p-q| = \varepsilon,\ \ \pm\big(h_{\boldsymbol{u}}(q) - h_{\boldsymbol{u}}(p)\,\big) > 0 \Big\}, \tag{1.4}$$

[2] The graph structure is a special Whitney stratification of Γ.

$$d_p^{\pm}(\boldsymbol{u}) := \lim_{\varepsilon \searrow 0} \# L_\varepsilon^{\pm}(p, \boldsymbol{u}).$$

Observe that $d_p^+(\boldsymbol{u}) + d_p^-(\boldsymbol{u}) = \deg(p)$. In Figure 1 we have $d_v^+(\boldsymbol{u}) = 2$, $d_v^-(\boldsymbol{u}) = 3$, while $\boldsymbol{N}_v$ consists of four vectors because two of the edges are tangent at v.

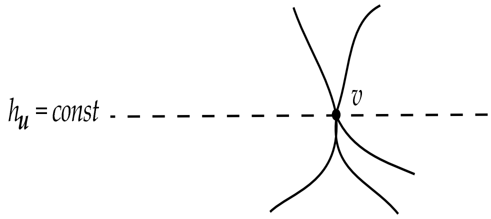

FIGURE 1. *The neighborhood of a critical point.*

Suppose that $\boldsymbol{u}$ is a Γ-nondegenerate unit vector. A *stratified critical point* of $h_{\boldsymbol{u}}$ on Γ is a point $p \in \Gamma$ which is either a vertex, or a critical point of the restriction of $h_{\boldsymbol{u}}$ to one of the edges. We denote by $\mathbf{Cr}(\boldsymbol{u})$ the set[3] of stratified critical points of $h_{\boldsymbol{u}}$. The points in $\Gamma \setminus \mathbf{Cr}(\boldsymbol{u})$ are called *regular points* of $h_{\boldsymbol{u}}$. Observe that if p is a regular point, then $d_p^-(\boldsymbol{u}) = 1$.

Note that $\mathbf{Cr}(\boldsymbol{u})$ is a finite subset of Γ containing the vertices. We set $\mathbf{Cr}^1_{\boldsymbol{u}} := \mathbf{Cr}_{\boldsymbol{u}} \setminus V$. In other words, $Cr^1_{\boldsymbol{u}}$ consists of the points in the interiors of edges where the tangent vector to the edge is perpendicular to $\boldsymbol{u}$.

The set $\mathbf{Cr}^1(\boldsymbol{u})$ decomposes into a set of local minima $\mathbf{Cr}^1_{\min}(\boldsymbol{u})$, and a set of local maxima $\mathbf{Cr}^1_{\max}(\boldsymbol{u})$. The set of vertices V further decomposes as $V = V_{\min}(\boldsymbol{u}) \cup V^*(\boldsymbol{u})$, where $V_{\min}(u)$ consists of the vertices of Γ which are local minima of $h_{\boldsymbol{u}}$, and $V^*(\boldsymbol{u})$ is its complement. The set $V_{\min}(-\boldsymbol{u})$ is the set of local maxima of $h_{\boldsymbol{u}}$, and for this reason we will denote it by $V_{\max}(\boldsymbol{u})$.

Note that for every nondegenerate unit vector $\boldsymbol{u}$, and every $c \in \mathbb{R}$ the sublevel set $\{h_{\boldsymbol{u}} \leq c\}$ is a also a semialgebraic graph, and the level set $\{h_{\boldsymbol{u}} = c\}$ consists of finitely many points. If a level set $\{h_{\boldsymbol{u}} = c\}$ contains only regular points then $\{h_{\boldsymbol{u}} \leq c - \varepsilon\}$ is homeomorphic to $\{h_{\boldsymbol{u}} \leq c + \varepsilon\}$ for all sufficiently small ε.

If the level set $\{h_{\boldsymbol{u}} = c\}$ contains the critical points $p_1, \dots, p_k$, then $\{h_{\boldsymbol{u}} \leq c + \varepsilon\}$ is homotopic to the set obtained from $\{h_{\boldsymbol{u}} \leq c - \varepsilon\}$ by separately conning off each of the sets $L_\delta^-(\boldsymbol{u}, p_i)$, $i = 1, \dots, k$. The points p_i will be the vertices of the added cones. Here we define the cone over the empty set to be a single point. For example, if the level set $\{h_{\boldsymbol{u}} = c\} \cap \Gamma$ contains k local minima, and the sublevel set $\{h_{\boldsymbol{u}} \leq c - \varepsilon\} \cap \Gamma$ is connected for all $\varepsilon > 0$ sufficiently small, then the sublevel set $\{h_{\boldsymbol{u}} \leq c + \varepsilon\} \cap \Gamma$ will have $k + 1$ connected components for all $\varepsilon > 0$ sufficiently small.

To every critical point $p \in \Gamma$ we associate its *Morse polynomial* $M_{\boldsymbol{u}}(t, p) \in \mathbb{Z}[t]$ according to the rule

$$M_{\boldsymbol{u}}(t, p) := \begin{cases} 1 & \text{if } p \text{ is a local minimum of } h_{\boldsymbol{u}} \\ (d_v^-(\boldsymbol{u}) - 1)t & \text{otherwise.} \end{cases}$$

Observe that $M_{\boldsymbol{u}}(t, p) = 0$ if p is a regular point. Let us observe that $M_{\boldsymbol{u}}(t, p)$ is the Poincaré polynomial of the topological pair $(\operatorname{cone}(L_\varepsilon^-(p, \boldsymbol{u})), L_\varepsilon^-(p, \boldsymbol{u}))$ where $\operatorname{cone}(L_\varepsilon^-(p, \boldsymbol{u}))$ denotes the cone over $L_\varepsilon^-(p, \boldsymbol{u})$ and ε is sufficiently small

The *homological weight* of the critical point $p \in \mathbf{Cr}(\boldsymbol{u})$ is then defined as the integer

$$w(p, \boldsymbol{u}) := M_{\boldsymbol{u}}(t, p)|_{t=1}.$$

[3] The same unit vector may be nondegenerate for several graph structures and the critical sets corresponding to these graph structures could be different.

The *Morse polynomial* of $\boldsymbol{u}$ is defined as

$$M_{\boldsymbol{u}}(t) = M_{\boldsymbol{u},\Gamma}(t) := \sum_{p\in \mathbf{Cr}(\boldsymbol{u})} M_{\boldsymbol{u}}(t,p) = \sum_{p\in\Gamma} M_{\boldsymbol{u}}(t,p).$$

The *homological weight* of $\boldsymbol{u}$ is then the integer

$$w(\boldsymbol{u}) = w_\Gamma(\boldsymbol{u}) := M_{\boldsymbol{u}}(1) = \sum_{p\in \mathbf{Cr}(\boldsymbol{u})} w(p,\boldsymbol{u}).$$

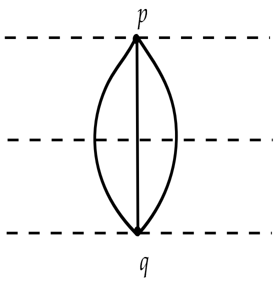

FIGURE 2. *A stratified Morse function on a θ-graph.*

Remark 1.1. The homological weight $w_\Gamma(\boldsymbol{u})$ is in general different from the number of critical points. In Figure 2 we have depicted a stratified Morse function $h_{\boldsymbol{u}}$ on a planar graph. The vector $\boldsymbol{u}$ lies in the plane of the graph, it is perpendicular to the dotted lines and points upwards. This function has precisely two stratified critical points p and q but its homological weight is $w(\boldsymbol{u}) = w(\boldsymbol{u},p) + w(\boldsymbol{u},q) = 2+1 = 3$. Its Morse polynomial is $M_{\boldsymbol{u}}(t) = 1+2t$. Note that $M_{\boldsymbol{u}}(-1) = -1 = \chi(\Gamma)$.□

We define a partial order $\preccurlyeq$ on the vector space $\mathbb{R}[t]$ by declaring $P \preccurlyeq Q$ if and only if there exists a polynomial R with *nonnegative coefficients* such that

$$Q - P = (1+t)R.$$

The following result follows immediately from the main theorems of stratified Morse theory [10].

Theorem 1.2 (Morse inequalities)**.** *Denote by $P_\Gamma(t)$ the Poincaré polynomial of Γ, $P_\Gamma(t) = 1 + b_1(\Gamma)t$. Then for every nondegenerate vector $\boldsymbol{u} \in S^2$ we have*

$$M_{\boldsymbol{u}}(t) \succcurlyeq P_\Gamma(t).$$

In particular,

$$M_{\boldsymbol{u}}(-1) = P_\Gamma(-1) = \chi(\Gamma) \tag{1.5}$$

and

$$w_\Gamma(\boldsymbol{u}) \geq 1 + b_1(\Gamma). \tag{1.6}$$

□

Definition 1.3. A Γ-nondegenerate vector $\boldsymbol{u}$ is called (Γ-)*perfect* if $M_{\boldsymbol{u}}(t,\Gamma) = P_\Gamma(t)$. □

The situation in Figure 2 corresponds to a perfect nondegenerate unit vector.

Lemma 1.4. *A nondegenerate vector $\boldsymbol{u}$ is perfect if and only if $w_\Gamma(\boldsymbol{u}) = P_\Gamma(1) = 1 + b_1(\Gamma)$.*

Proof. Observe that since $M_{\boldsymbol{u}}(t,\Gamma)$ and $P_\Gamma(t)$ are polynomials of degree 1, they are completely determined by their values at two different t's. Since $M_{\boldsymbol{u}}(t,\Gamma)|_{t=-1} = P_\Gamma(-1)$ we deduce that a vector $\boldsymbol{u}$ is perfect if and only if $M_{\boldsymbol{u}}(1) = P_\Gamma(1)$. □

Proposition 1.5. *Suppose $\boldsymbol{u}$ is a* Γ-nondegenerate *vector $\boldsymbol{u} \in S^2$. Then the following statements are equivalent.*

(a) The vector $\boldsymbol{u}$ is perfect.

(b) For any $c \in \mathbb{R}$ the sublevel set $\{h_{\boldsymbol{u}} \leq c\} \cap \Gamma$ is connected.

Proof. We set $m(\boldsymbol{u}) = \min_{p\in\Gamma} h_{\boldsymbol{u}}(p)$. Observe that $M_{\boldsymbol{u}}(0)$ is the number of local minima of $h_{\boldsymbol{u}}$.

(b) $\Longrightarrow$ (a) The sublevel set $\{h_{\boldsymbol{u}} \leq m(\boldsymbol{u})\}$ is connected. It coincides with the set of absolute minima of $h_{\boldsymbol{u}}$. Since $\boldsymbol{u}$ is nondegenerate this is a discrete set consisting of a single point p_0. All the sublevel sets $\{h_{\boldsymbol{u}} \leq c\} \cap \Gamma$ are connected so that $h_{\boldsymbol{u}}$ cannot have a local minimum other than the absolute minimum.

Indeed, as c increases away from $m(\boldsymbol{u})$, the first time c encounters a critical value c_0 such that the level set $\{h_{\boldsymbol{u}} = c_0\}$ contains local minima, then the number of components of $\{h_{\boldsymbol{u}} \leq c_0\} \cap \Gamma$ increases by exactly the number of the local minima.

This proves that $1 = M_{\boldsymbol{u}}(0) = P_\Gamma(0)$. On the other hand, $M_{\boldsymbol{u}}(-1) = P_\Gamma(-1) = \chi(\Gamma)$. Since both $M_{\boldsymbol{u}}(t)$ and $P_\Gamma(t)$ are polynomials of degree 1 we conclude that $M_{\boldsymbol{u}}(t) = P_\Gamma(t) = 1$.

(a) $\Longrightarrow$ (b) Since $\boldsymbol{u}$ is perfect we deduce that $M_{\boldsymbol{u}}(0) = P_\Gamma(0) = 1$ so that $h_{\boldsymbol{u}}$ has a unique local minimum. In particular, the sublevel set $\{h_{\boldsymbol{u}} \leq m(\boldsymbol{u})\} \cap \Gamma$ consists of single point and thus it is connected. To conclude run in reverse the topological argument in the proof of the implication (b) $\Longrightarrow$ (a). □

Definition 1.6. (a) A connected semialgebraic *graph* $\Gamma \subset \mathbb{R}^3$ is called *tight* if *all* the Γ-nondegenerate vectors $\boldsymbol{u}$ are Γ-perfect, i.e.,

$$w_\Gamma(\boldsymbol{u}) = 1 + b_1(\Gamma), \ \ \forall \boldsymbol{u} \in S^2 \setminus \Delta_\Gamma.$$

(b) A compact connected 1-dimensional semialgebraic *set* $\Gamma \subset \mathbb{R}^3$ is called *tight* if it is tight for some choice of graph structure.

(c) A semi algebraic set in $\mathbb{R}^3$ is called *tightable* if it admits a semialgebraic embedding as a tight semialgebraic graph. □

Proposition 1.7. *Suppose $\Gamma \subset \mathbb{R}^3$ is a semialgebraic graph. Then the following statements are equivalent.*

(a) The graph Γ is tight.

(b) The intersection of Γ with **almost any** *closed half-space is either empty or connected.*

(c) The intersection of Γ with **any** *closed half-space is either empty or connected.*

Proof. Observe first that the intersection of Γ with a closed half-space is a set of the form $\{h_{\boldsymbol{u}} \leq c\} \cap \Gamma$ for some $\boldsymbol{u} \in S^2$ and $c \in \mathbb{R}$.

The implications (c) $\Longrightarrow$ (a), (b) and (a) $\Longrightarrow$ (b) follow immediately from Proposition 1.5. It suffice to prove only the implication (b) $\Longrightarrow$ (c). We use an argument inspired by the proof of [15, Thm. 3.11].

Denote by $\mathcal{U}$ the set of vectors $\boldsymbol{u} \in S^2$ such that for any $c \in \mathbb{R}$ the sublevel set $\{h_{\boldsymbol{u}} \leq c\}$ is connected. The set $\mathcal{U}$ is dense in S^2.

Fix a vector $\boldsymbol{u} \in S^2$ and a real number c. Then there exists a sequence of nondegenerate perfect vectors $\boldsymbol{u}_n \in \mathcal{U}$ and a sequence of positive real numbers $(r_n)_{n\geq 1}$ satisfying the following properties.

- $\lim_{n\to\infty} \boldsymbol{u}_n = \boldsymbol{u}$.
- $\lim_{n\to\infty} r_n = 0$
- $\{h_{\boldsymbol{u}} \leq c\} \cap \Gamma \subset \{h_{\boldsymbol{u}_{n+1}} \leq c + r_{n+1}\} \cap \Gamma \subset \{h_{\boldsymbol{u}_n} \leq c + r_n\} \cap \Gamma, \forall n \geq 1$.

The sets $X = \{\, h_{\boldsymbol{u}} \leq c\,\} \cap \Gamma$ and $X_n = \{h_{\boldsymbol{u}_n} \leq c + r_n\} \cap \Gamma$ are closed semialgebraic subsets of Γ satisfying the conditions

$$X \subset X_{n+1} \subset X_n, \;\; \forall n \geq 1 \;\; \text{and} \;\; X = \bigcap_{n\geq 1} X_n$$

We conclude that (see [19, §6.6-§6.8])

$$\check{H}^0(X, \mathbb{Z}) = \varinjlim_n \check{H}^0(X_n, \mathbb{Z}),$$

where $\varinjlim$ denotes the inductive limit and $\check{H}^\bullet(-, \mathbb{Z})$ denotes the Čech homology with integral coefficients. For semialgebraic sets the Čech cohomology coincides with the usual singular cohomology, and all the sets X_n are connected so that $\check{H}^0(X_n, \mathbb{Z}) = \mathbb{Z}$ for all n. The above equality implies that X is connected. □

Remark 1.8. The property (c) in Proposition 1.7 is usually referred to as the *two-piece property* (TPP for brevity). The subsets an Euclidean space satisfying this property are known as 0-*tight sets*. □

Using the above proposition and Lemma [13, Lemma 2.4] we obtain the following result.

Corollary 1.9. *Suppose that Γ is a semialgebraic graph such that all its edges are straight line segments. Then Γ is tight if and only if there exists a convex polyhedron P such that the following hold.*

(a) $\Gamma \subset P$.

(b) The vertices and edges of P are contained in Γ.

(c) All the edges of Γ are straight line segments.

(c) If v is a vertex of Γ which is not a vertex of P then v lies in the convex hull of its neighbors. □

Corollary 1.10. *Any plane, closed, convex semi-algebraic curve is tight.* □

2. Total curvature

Fix a compact connected semialgebraic graph $\Gamma \subset \mathbb{R}^3$. For every integrable function $S^2 \ni \boldsymbol{u} \mapsto f(\boldsymbol{u}) \in \mathbb{R}$ we denote by $\langle f(\boldsymbol{u})\rangle \in \mathbb{R}$ its average. We would like to investigate the average $\langle w_\Gamma(\boldsymbol{u})\rangle$ and thus we would like to know that the function

$$S^2 \setminus \Delta_\Gamma \ni \boldsymbol{u} \mapsto w_\Gamma(\boldsymbol{u}) \in \mathbb{Z}$$

is integrable. This is a consequence of the following result proved in the Appendix.

Proposition 2.1. *The function $S^2 \setminus \Delta_\Gamma \ni \boldsymbol{u} \mapsto w(\boldsymbol{u}) \in \mathbb{Z}$ is semialgebraic. In particular, it is bounded and measurable.* □

Following [5] we define the *total curvature* of Γ to be average $K(\Gamma)$ of this function,

$$K(\Gamma) := \langle w_\Gamma(\boldsymbol{u})\rangle = \frac{1}{4\pi}\int_{S^2} |w(\boldsymbol{u})| d\sigma(\boldsymbol{u})|,$$

where $|d\sigma|$ denotes the Euclidean area density on the unit sphere S^2. For the definition and (1.6) we obtain the following generalization of Fenchel's inequality.

Corollary 2.2.
$$K(\Gamma)\geq 1+b_1(\Gamma),$$
with equality if and only if Γ is tight. □

We want give a description of $K(\Gamma)$ in terms of local geometric invariants of Γ. Observe that for any nondegenerate vector $\boldsymbol{u}\in S^2$ we have
$$w(\boldsymbol{u})=\#\,\mathbf{Cr}^1(\boldsymbol{u})+\sum_{v\in V}w(\boldsymbol{u},v)=\sum_{e\in E}\#\big(\,\mathbf{Cr}^1(\boldsymbol{u})\cap e\,\big)+\sum_{p\in V}w(\boldsymbol{u},p).$$
For every (open) edge of Γ we denote by $K(e)$ the total curvature
$$K(e):=\frac{1}{\pi}\int_e|k_e(s)||ds|,$$
where $|ds|$ denotes the arclength along e and k_e denotes the curvature along e. Then (see [5] or [15, §2]) we have
$$\big\langle\,\#(\,\mathbf{Cr}^1(\boldsymbol{u})\cap e\,)\,\big\rangle=K(e),$$
so that,
$$\big\langle\,\#\,\mathbf{Cr}^1(\boldsymbol{u})\,\big\rangle=\sum_{e\in E}K(e). \tag{2.1}$$
For every vertex $p\in V$, and any nondegenerate unit vector $\boldsymbol{u}$ we set
$$\lambda(\boldsymbol{u},p)=\begin{cases}2 & p\in V_{\min}(\boldsymbol{u})\\ d_p^-(\boldsymbol{u}) & p\in V\setminus V_{\min}(\boldsymbol{u}).\end{cases}$$
Then
$$w(\boldsymbol{u},p)=\lambda(\boldsymbol{u},p)-1\ \text{ and }\ \sum_{p\in V}w(\boldsymbol{u},p)=\sum_{p\in V}\lambda(\boldsymbol{u},p)-\#V,$$
so that,
$$\sum_{p\in V}\langle w(\boldsymbol{u},p)\rangle=\sum_{p\in V}\langle\lambda(\boldsymbol{u},p)\rangle-\#V. \tag{2.2}$$

Definition 2.3. For every point $p\in\Gamma$ we define
$$\Sigma_\Gamma^+(p):=\big\{\boldsymbol{u}\in S^2;\ \ (\boldsymbol{u},\boldsymbol{\tau})>0,\ \ \forall\boldsymbol{\tau}\in\boldsymbol{N}_p\,\big\}.$$
The closure of $\Sigma_\Gamma^+(p)$ is a geodesic polygon on S^2, and we denote by $\sigma_\Gamma^+(p)$ its area. □

Remark 2.4. Let us observe that if $p\in\Gamma$ is not a vertex, then $\sigma_\Gamma^+(p)=0$. □

For every nondegenerate unit vector $\boldsymbol{u}$, and any vertex p we have
$$\lambda(\boldsymbol{u},p)+\lambda(-\boldsymbol{u},p)=\deg(p)+\begin{cases}2 & \boldsymbol{u}\in\Sigma_\Gamma^+(p)\cup-\Sigma_\Gamma^+(p)\\ 0 & \text{otherwise},\end{cases}$$
and we conclude that
$$2\langle\lambda(\boldsymbol{u},p)\rangle=\langle\lambda(\boldsymbol{u},p)\rangle+\langle\lambda(-\boldsymbol{u},p)\rangle=\deg(p)+\frac{1}{\pi}\sigma_\Gamma^+(p).$$
Hence,
$$\sum_{p\in V}\langle\lambda(\boldsymbol{u},p)\rangle=\frac{1}{2}\sum_{p\in V}\deg p+\frac{1}{2\pi}\sum_{p\in V}\sigma_\Gamma^+(p)=\#E+\frac{1}{\pi}\sum_{p\in V}\sigma_\Gamma^+(p).$$

Using the equality (2.2) we deduce

$$\sum_{p\in V}\langle w(\boldsymbol{u},p)\rangle = \#E-\#V+\frac{1}{2\pi}\sum_{p\in V}\sigma_\Gamma^+(p) = \frac{1}{\pi}\sum_{p\in V}\sigma_\Gamma^+(p)-2+\big(1+b_1(\Gamma)\big).$$

We have thus proved the following result.

Theorem 2.5. *The total curvature of the compact connected semialgebraic graph Γ is given by*

$$K(\Gamma)=\frac{1}{2\pi}\sum_{p\in V}\sigma_\Gamma^+(p)+\sum_{e\in E}K(e)-\chi(\Gamma)=\frac{1}{2\pi}\sum_{p\in V}\sigma_\Gamma^+(p)+\sum_{e\in E}K(e)-2+b_1(\Gamma)+1.$$

In particular, the graph Γ is tight if and only if

$$\sum_{p\in V}\sigma_\Gamma^+(p)+2\sum_{e\in E}\int_e|k_e(s)|\,|ds|=2\pi. \qquad \square$$

Corollary 2.6. *The total curvature of Γ is independent of the choice of vertex set V.* $\square$

Proof. Indeed, Remark 2.4 implies that

$$K(\Gamma)=\frac{1}{2\pi}\sum_{p\in\Gamma}\sigma_\Gamma^+(p)+\sum_{e\in E}K(e)-\chi(\Gamma).$$

Neither one of the three summands depends on the choice of vertex set. $\square$

Corollary 2.7. *Suppose Γ is a compact connected semialgebraic graph in $\mathbb{R}^3$. For every Γ-nondegenerate unit vector $\boldsymbol{u}\in S^2$ we denote by $\mu(\boldsymbol{u})$ the number of local minima of the function $h_{\boldsymbol{u}}$ on Γ, and by $\mu(\Gamma)$ the average, $\mu(\Gamma):=\langle\mu(\boldsymbol{u})\rangle$.Then*

$$\mu(\Gamma)=\frac{1}{2}\big(K(\Gamma)+\chi(\Gamma)\big)=\frac{1}{4\pi}\sum_{p\in V}\sigma_\Gamma^+(p)+\frac{1}{2\pi}\sum_{e\in E}\int_e|k(s)|\,|ds|. \tag{2.3}$$

Proof. Observe that

$$2\mu(\boldsymbol{u})=2M_{\boldsymbol{u}}(0)=M_{\boldsymbol{u}}(1)+M_{\boldsymbol{u}}(-1)=w(\boldsymbol{u})+\chi(\Gamma).$$

The equality (2.3) is obtained by averaging the above identity. $\square$

Corollary 2.8. *The total curvature of a plane, closed, convex semi-algebraic curve is equal to* 2. $\square$

Corollary 2.9. *Suppose Γ is a planar, convex semialgebraic arc with endpoints $p_0\neq p_1$. Let $\theta_0,\theta_1\in[0,\pi]$ denote the angles at p_0 and respectively p_1 between the arc Γ and the line determined by π_0. Then*

$$\int_\Gamma|k(s)\,|ds|=\theta_0+\theta_1.$$

Proof. Denote by $\hat{\Gamma}$ the closed curve obtained from Γ by connecting p_0 to p_1 by a straight line segment. Then $\hat{\Gamma}$ is a plane convex curve and using Theorem 2.5 we deduce

$$2=K(\hat{\Gamma})=\frac{1}{\pi}\int_\Gamma|k(s)\,|ds|+\frac{1}{2\pi}\big((2\pi-2\theta_0)+(2\pi-2\theta_1)\big)$$

which implies the claimed equality. $\square$

Corollary 2.10. *Suppose the graph Γ is contained in a plane and it is the union of a closed, convex semialgebraic curve B and a finite union of line segments contained in the region R bounded by B such that $R\setminus\Gamma$ is a finite union of convex sets. Then Γ is tight.*

Proof. The curve B is a disjoint union of (open) edges and vertices of B. Denote by V_B (respectively E_B) the collection of vertices (respectively open) edges contained in B and by V_B' (respectively E_B') the collection of vertices (respectively open edges) not contained in B. If $v\in V_B'$ then the cone generated by $\boldsymbol{N}_v$ is the plane of the graph Γ so that $\sigma_\Gamma^+(v)=0$. Similarly, if $e\in E_B'$, then $K(e)=0$ since e is a straight line segment. Hence

$$K(\Gamma)=\frac{1}{2\pi}\sum_{v\in V_B}\sigma_\Gamma^+(B)+\sum_{e\in E_B}K(e)-2\chi(\Gamma)=K(B)-\chi(\Gamma)=2-\chi(\Gamma)=1+b_1(\Gamma).$$

This proves that Γ is tight. □

Remark 2.11. Observe that when Γ is a polygonal simple close curve then the formula (2.3) specializes to Milnor's formula [17, Thm. 3.1].

(b) Recently, Gulliver and Yamada have proposed in [11] a different notion of total curvature. For every nondegenerate vector $\boldsymbol{u}$ and every vertex p they define a defect at p to be the integer

$$\delta(\boldsymbol{u},p)=\big(d_p^-(\boldsymbol{u},p)-d_p^+(\boldsymbol{u})\big)^+,\quad x^+:=\max(x,0),\quad .$$

The Gulliver-Yamada total curvature is then the real number $T(\Gamma)$ defined by

$$\frac{1}{\pi}T(\Gamma)=\sum_{p\in V}\langle\,\delta_+(\boldsymbol{u},p)\,\rangle+\sum_{e\in E}K(e).$$

Observe that

$$\delta(\boldsymbol{u},p)+\delta(-\boldsymbol{u},p)=|d_p^-(\boldsymbol{u},p)-d_p^+(\boldsymbol{u})|$$

so that

$$\frac{1}{\pi}T(\Gamma)=\frac{1}{2}\sum_{p\in V}\big\langle\,|d_p^-(\boldsymbol{u},p)-d_p^+(\boldsymbol{u})|\,\big\rangle+\sum_{e\in E}K(e)$$

Let us show that when all vertices have degrees ≤ 3, then

$$\frac{1}{\pi}T(\Gamma)=K(\Gamma). \tag{2.4}$$

Indeed, for every vertex $p\in V$, and every unit vector $\boldsymbol{u}$, such that both $\boldsymbol{u}$ and $-\boldsymbol{u}$ are regular, we have

$$\delta(\boldsymbol{u},p)+\delta(-\boldsymbol{u},p)=w(p,\boldsymbol{u})+w(p,-\boldsymbol{u})=\begin{cases}3 & \boldsymbol{u}\in\Sigma_p^+(\Gamma)\cup-\Sigma_p^+(\Gamma)\\ 1 & \text{otherwise.}\end{cases}$$

The equality (2.4) follows by averaging the above identity.

In general $K(\Gamma)\neq\frac{1}{\pi}T(\Gamma)$. To see this consider the graph Γ_0 obtained by joining the north pole to the south pole of the unit sphere by n meridians of longitudes $\frac{2k\pi}{n}$, $k=1,\dots,n$. If n is an odd integer then

$$|d_p^-(\boldsymbol{u},p)-d_p^+(\boldsymbol{u})|=1$$

for any vertex p and almost all $\boldsymbol{u}$'s. In this case we have $\sigma_p^+(\Gamma_0)=0$ for any vertex p and we deduce

$$K(\Gamma_0)=\sum_{e\in E}K(e)+n-2.$$

On the other hand

$$\frac{1}{\pi}T(\Gamma_0) = 1 + \sum_{e\in E} K(e).$$

(c) Taniyama [20] has proposed another notion of total curvature $T_1(\Gamma)$ given by

$$\frac{1}{\pi}T_1(\Gamma) = \sum_{e\in E} K(e) + \sum_{p\in V}\theta(p),$$

where for every vertex p the quantity $\theta(p)$ is the sum

$$\sum_{\boldsymbol{\tau}_0\neq\boldsymbol{\tau}_1\in \boldsymbol{N}_p}\bigl(\pi - \measuredangle(\boldsymbol{\tau}_0,\boldsymbol{\tau}_1)\bigr).$$

We can check by direct computation that $K(\Gamma_0) \neq \frac{1}{\pi}T(\Gamma_0) \neq \frac{1}{\pi}T(\Gamma_1)$. It seems that Taniyama's definition is closer in the spirit to the approach of I. Fáry [8] where he showed that the total curvature of simple closed curve in $\mathbb{R}^3$ is equal to the average of the total curvatures of its projections on all the two dimensional planes. □

From (2.4) and Corollary 2.2 we obtain the following generalization of the first half of [11, Thm.2].

Corollary 2.12. *Suppose Γ is a connected, semialgebraic graph such that each of its vertices has degree ≤ 3. If $T(\Gamma)$ denotes the Gulliver-Yamada total curvature of Γ then*

$$T(\Gamma) = \pi K(\Gamma) \geq \pi\bigl(1 + b_1(\Gamma)\bigr).$$ □

Remark 2.13. Suppose S is a topological space which admits an embedding as a semi-algebraic graph in $\mathbb{R}^3$. We define

$$\mu_0(S) := \inf\bigl\{\,\mu(\varphi(S)\,);\ \ \varphi: S\to\mathbb{R}^3 \ \text{ is an embedding as a semialgebraic graph}\,\bigr\}. \tag{2.5}$$

Note that $\mu_0(S)\geq 1$. Clearly $\mu_0(S)$ is a topological invariant of S. If S is tightable (see Definition 1.6(c) then $\mu_0(S) = 1$, but the converse is not true.

To see this, consider the topological space Σ_n obtained as the suspension of a n-point set. Denote by $P_\pm$ the two suspension points. Theorem 3.1 implies that Σ_n is not tightable if $n > 3$. However, for every $\varepsilon > 0$ we can find semialgebraic embeddings $\varphi: \Sigma_n\hookrightarrow\mathbb{R}^2$ such that $\mu\bigl(\varphi(\Sigma_n)\bigr) < 1+\varepsilon$.

To see this, we embed Σ_n in the plane as union of a circle with $(n-2)$ arcs connecting the diametrically opposed points $P_\pm$ (see Figure 3). Assume that the arcs are C^2, they do not intersect in the interior and they are C^2-close to the diameter $[P_-P_+]$. Then $\mu(\Sigma_n)\approx 1$.

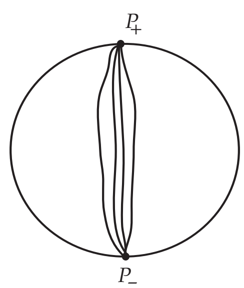

FIGURE 3. *A suspension of a n-point set with small crookedness.*

Let us observe that we can find semialgebraic sets S with arbitrarily large $\mu_0(S)$. Suppose S is a wedge of $n > 1$ circles $C_1, \dots, C_n$. We denote by w the common point of these n-circles.

For any semialgebraic embedding $\varphi : S \to \mathbb{R}^3$ as semi-algebraic graph we obtain n pairs of unit vectors at w, $\boldsymbol{n}_i^\pm \in \boldsymbol{N}_w$, $i = 1, \dots, n$. The vectors $\boldsymbol{n}_i^\pm$ are tangent to the component C_i^φ of the wedge, $C_i^\varphi = \varphi(C)$; see Figure 4.

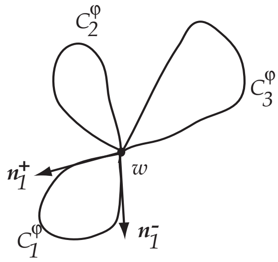

FIGURE 4. *A wedge of circles.*

Let $\alpha_i \in [0, \pi]$ denote the angle between $\boldsymbol{n}_i^\pm$. If $S^\varphi := \varphi(S)$ then

$$\mu(S^\varphi) = \frac{1}{4\pi}\sigma_w^+(S^\varphi) + \frac{1}{2\pi}\sum_{i=1}^n \int_{C_i^\varphi} |k(s)|ds|$$

$$= \frac{1}{4\pi}\sigma_w^+(S^\varphi) + \sum_{i=1}^n \big(\mu(C_i^\varphi) - \frac{1}{2\pi}(\pi - \alpha_i)\big) \geq \sum_{i=1}^n \mu(C_i^\varphi) - \frac{n}{2} \geq \frac{n}{2}$$

In fact

$$\mu_0(S) = \frac{n}{2}.$$

Indeed,for every $\varepsilon > 0$ we can embed S as a planar wedge of circles, such that the components C_i^φ are convex, the solid angle $\sigma_w^+(S^\varphi)$ is trivial and the angles α_i are small, $\alpha_i < \frac{2\pi\varepsilon}{n}$. Then

$$\mu(S^\varphi) = \sum_{i=1}^n \mu(C_i^\varphi) - \frac{n}{2} + \frac{1}{2\pi}\sum_{i=1}^n \alpha_i = \frac{n}{2} + \frac{1}{2\pi}\sum_{i=1}^n \alpha_i < \frac{n}{2} + \varepsilon. \qquad \square$$

3. TIGHTNESS

We want to give a complete explicit classification of tight semi-algebraic graphs in $\mathbb{R}^3$.

Theorem 3.1. *A semialgebraic graph Γ is tight if an only if its of one the following types: type C described in Corollary 2.10, and type S described in Corollary 1.9.*

Proof. Suppose Γ is a connected semialgebraic graph. As usual we denote by V the set of vertices and by E the set of edges. We denote by $\boldsymbol{R}_\Gamma \subset S^2$ the set of Γ-regular unit vectors.

Assume Γ is tight. According to Proposition 1.7 this means that the intersection of Γ with any closed half-space is connected.

For every point p on an (open) edge e we denote by L_p the affine line tangent to the edge e at p and by $k(p)$ the curvature of e at p. If $k(p) \neq 0$ we denote $\boldsymbol{n}(p)$ the normal vector to e at p. We set

$$\mathcal{C}(\Gamma) := \{p \in \Gamma \setminus V;\ \ k(p) \neq 0\}.\big(V \cup F(\Gamma)\big),\ \ \mathcal{C}(e) := \mathcal{C}(\Gamma) \cap e,\ \ \forall e \in E.$$

We will refer to $\mathcal{C}(\Gamma)$ as the *curved region* of Γ since it consists of the points along edges where the curvature is nonzero. The set $\mathcal{C}(e)$ is semialgebraic, open in e and consists of finitely many arcs.

If $\mathcal{C}(\Gamma) = \emptyset$ then all the edges of Γ are straight line segments and the theorem reduces to Corollary 1.9. Thus in the sequel we will assume that $\mathcal{C}(\Gamma) \neq \emptyset$.

Lemma 3.2. *Suppose p is a point on an open edge of the tight graph Γ such that $k(p) \neq 0$. Then the graph Γ is contained in the affine osculator plane, i.e., the affine plane determined by the affine tangent line L_p and the normal vector $\boldsymbol{n}(p)$, and it is situated entirely in one of the closed half-planes determined by L_p.*

Proof. Consider a unit vector $\boldsymbol{u}$ such that $(\boldsymbol{u}, \boldsymbol{n}(p)) > 0$. Set $c = h_{\boldsymbol{u}}(p)$ The intersection between the half-space $\{h_{\boldsymbol{u}} \leq c\}$ and Γ is closed, connected and semialgebraic. The curve selection theorem implies that and p is an isolated point of this intersection. Hence the intersection consists of a single point so that Γ is contained in the half-space

$$H^+_{\boldsymbol{u},p} = \{q \in \mathbb{R}^3;\ \ (\boldsymbol{u}, q-p) \geq 0\}.$$

Thus

$$\Gamma \subset \bigcap_{(\boldsymbol{u},\boldsymbol{n}(p))>0} H^+_{\boldsymbol{u},p}.$$

This intersection is a half-plane in the osculator plane determined by the affine tangent line L_p. □

Using a similar argument as in the proof of Lemma 3.2 we deduce the following result.

Lemma 3.3. *The edges of Γ are convex arcs.* □

For every vertex v of Γ we denote by $\mathcal{C}_v$ the convex cone spanned by the vectors in $\boldsymbol{N}_v$ and by $\mathcal{C}_v^*$ is dual cone

$$\mathcal{C}_v^* = \big\{\boldsymbol{p} \in \mathbb{R}^3;\ \ (\boldsymbol{p}, \boldsymbol{\tau}) \geq 0;\ \ \forall \boldsymbol{\tau} \in \boldsymbol{N}_v\,\big\}.$$

The intersection of $\mathcal{C}_v^*$ with the unit sphere is the closure of the open region $\Sigma^+_\Gamma(v)$ introduced in Definition 2.3.

Lemma 3.4. *For every vertex v of Γ we have $\Gamma \subset \mathcal{C}_v$.*

Proof. Note first that $\Sigma^+_\Gamma(v) = \emptyset$ if and only if $\sigma^+_\Gamma(v) = 0$. If $\sigma^+_\Gamma(v) = 0$ then $\mathcal{C}_v = \mathbb{R}^3$ and the statement is obvious. Assume $\sigma^+_\Gamma(v) > 0$.

Then $\Sigma^+_\Gamma(v)$ is an open subset of S^2 and $\Sigma^+_\Gamma(v) \setminus \Delta_\Gamma$ is dense in $\Sigma^+_\Gamma(v)$. For any $\boldsymbol{u} \in \Sigma^+_\Gamma(v) \setminus \Delta_\Gamma$ the function $h_{\boldsymbol{u}}$ has a local minimum at v, and since Γ is tight, it has to be the absolute minimum. In other words Γ is contained in the half-space $H^+_v(\boldsymbol{u})$ so that

$$\Gamma \subset \bigcap_{\boldsymbol{u}\in\Sigma^+_\Gamma(v)\setminus\Delta_\Gamma} H^+_v(\boldsymbol{u}) = \bigcap_{\boldsymbol{u}\in\Sigma^+_\Gamma(v)} H^+_v(\boldsymbol{u})$$

(cl=closure)

$$= \bigcap_{\boldsymbol{u}\in \mathrm{cl}\,(\Sigma^+_\Gamma(v))} H^+_v(\boldsymbol{u}) = (\mathcal{C}_v^*)^* = \mathcal{C}_v,$$

where at the last step we have used the fact that a closed convex cone coincides with its bidual. □

We set

$$V_{ext} = V_{ext}(\Gamma) := \big\{v \in V;\ \ \sigma^+_\Gamma(v) > 0\,\big\},$$

and we will refer to the vertices in V_{ext} as *extremal vertices*.

Lemma 3.5. *Suppose $\mathcal{C}(\Gamma) \neq \emptyset$. Then there exists a semialgebraic closed convex curve B such that the following hold.*

(a) $B \subset \Gamma$.
(b) Γ is contained in the convex hull R of B.
(c) $\Gamma \setminus B$ is a union of straight line segments.
(d) The components of $R \setminus \Gamma$ are convex open sets.

Proof. Since $\mathcal{C}(\Gamma) \neq \emptyset$ Lemma 3.2 implies that Γ must be a planar. For every $v \in V$ we denote by $\boldsymbol{A}_v$ the intersection of $\mathcal{C}_v$ the unit circle in the plane containing Γ of the curve. Set $\theta_v := \text{length}\,(\boldsymbol{A}_v)$. Denote by V'_{ext} the set of vertices such that $\theta_v \leq \pi$. Clearly $V_{ext} \subset V'_{ext}$ but in general the inclusion is strict. For example, the vertices p and v in Figure 5 belong to V'_{ext}, but not to V_{ext}. The vertex b belongs to V_{ext}.

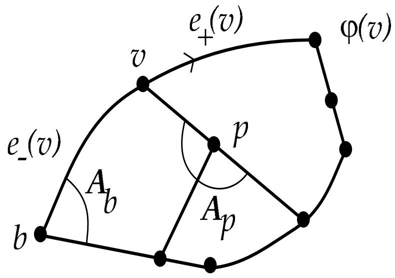

FIGURE 5. *A tight graph and its set V'_{ext}.*

Note that if $\theta_v > \pi$ then $\theta_v = 2\pi$. Moreover

$$\sigma^+_\Gamma(v) = 2(\pi - \theta_v).$$

For $v \in V'_{ext}$ the angle $\boldsymbol{A}_v$ is spanned by two vectors $\boldsymbol{\tau}_\pm \in \boldsymbol{N}_v$, where $\boldsymbol{\tau}_\pm$ are ordered such that the counterclockwise angle from $\boldsymbol{\tau}_-$ to $\boldsymbol{\tau}_+$ is $\leq \pi$. The two vectors $\boldsymbol{\tau}_\pm$ correspond to two edges $e_\pm(v)$ incident to v; see Figure 5.

If we start at v and travel along the edge $e_+(v)$ we encounter another vertex $\varphi(v)$ of Γ. From Lemma 3.2 and 3.4 we deduce that during our travel the graph Γ is situated to the right of the affine tangent lines to $e_+(v)$ oriented by the direction of motion; see Figure 5. This implies that $\varphi(v) \in V'_{ext}$ and $e_+(v) = e_-(\varphi(v))$.

We have thus obtained a map $\varphi : V'_{ext} \to V'_{ext}$ such that v is connected to $\varphi(v)$ by the edge $e_+(v)$ and Γ is situated to the right of the edge $e_+(v)$ oriented by the motion from v to $\varphi(v)$.

Suppose $S \subset V'_{ext}$ is a minimal φ-invariant subset of V_{ext}. Then we can label the vertices in S as $s_1, s_2, \ldots, s_k$, so that

$$k = |S|, \;\; v_2 = \varphi(v_1), \ldots, \;\; v_k = \varphi(v_{k-1}), \;\; v_1 = \varphi(v_k).$$

The succession of edges $e_+(v_1), \ldots, e_+(v_k)$ determines a closed, clockwise oriented curve. It is convex because it is situated on one side of the affine tangent lines to the smooth points of this curve and it is contained in each of the angles $\boldsymbol{A}_{v_i}$. Denote by B this closed convex curve.

The graph Γ is contained in the region R bounded by B. If e is an (open) edge of Γ not intersecting B, then e must be a line segment. Indeed, if $p \in e$ is a point where $k(p) \neq 0$, then the line $\mathcal{N}_p$ through p determined by $\boldsymbol{n}(p)$ intersects B in two different points q_1, q_2 which, according to Lemma 3.2, are situated in the same half-plane determined by the affine tangent line L_p. In particular, the point p is not contained on the closed segment $[p_1, p_2]$. This is a contradiction because the intersection of the line $\mathcal{N}_p$ with R is precisely the segment $[p_1, p_2]$ which implies that Γ has a point p not contained in R. Arguing in a similar fashion we deduce that $V_{ext} \subset B$.

For every vertex $v \in V$ the set $\boldsymbol{N}_v$ is contained in the unit circle in the plane of Γ. Thus, we can cyclically counterclockwise order the vectors in $\boldsymbol{N}_v$. If $v \in V \setminus V'_{ext}$ then the (counterclockwise) angle between two consecutive vectors in $\boldsymbol{N}_v$ (with respect to this cyclic ordering) is $< \pi$ because in this case the planar cone spanned by $\boldsymbol{N}_v$ must coincide with the plane of Γ.

Suppose D is a component of $R \setminus \Gamma$. Its boundary is a disjoint union of finitely many vertices and open edges of Γ. We fix the clockwise orientation of ∂D. At each vertex $v \in V \cap \partial D$ we have two edges an incoming and an outgoing edge contained in ∂D. They determine two vectors $\boldsymbol{\tau}_0, \boldsymbol{\tau}_1 \in \boldsymbol{N}_v$. Since D is a connected component of the complement of Γ in R, these two vectors must be successive vectors in $\boldsymbol{N}_v$ with respect to the counterclockwise cyclic order on $\boldsymbol{N}_v$.

If $v \in B \cup V'_{ext}$ then the angle between these vectors is $\leq \pi$ because the curve B is convex. If v is in the interior of R and $v \in V \setminus V'_{ext}$ then as we have seen above, the counterclockwise angle between these vector must be smaller than π. This proves that D is a convex open region. □

This completes the proof of Theorem 3.1. □

Remark 3.6. Because tight sets satisfy the two-piece-property (see Remark 1.8) we can use the characterization of planar 0-tight planar sets in [15, Thm. 1.3] to give an alternate proof of Lemma 3.5. □

Remark 3.7. We chose to work in the category of semialgebraic sets because they may be familiar to a larger audience. In fact all the results in this paper are valid in any o-minimal category. We refer to [7] for more information about o-minimal sets and maps. □

4. Knottedness

As we mentioned in the introduction, Milnor proved that the knotting of a circle $C \subset \mathbb{R}^3$ requires a substantial amount of curvature. More precisely, if $\mu(C) < 2$ then C cannot be knotted. Gulliver and Yamada [11] extended this result to singular situations. Let us define a θ-graph to be a graph homeomorphic to the union of a round circle and a diameter. Such a graph is 3-regular, and for these graphs the Gulliver-Yamada total curvature coincides with the notion of total curvature introduced in this paper. Theorem 2 in [11] can be rephrased as follows. If $\Gamma \subset \mathbb{R}^3$ is a semialgebraic graph homeomorphic to a θ-graph, and $\mu(\Gamma) < \frac{3}{2}$, then the embedding of Γ is isotopic to a planar embedding.

Denote by Σ_n the suspension of an n-point set. In Figure 6 we have depicted a planar embedding of Σ_5. We will refer to the two distinguished points of the suspension as the *poles*. Observe that Σ_2 is a circle, and Σ_3 is a θ-graph. As explained in Remark 2.13, this set admits semialgebraic embeddings Γ with $\mu(\Gamma) \approx 1$. Our next result generalizes the unknottedness results of Milnor [17] and Gulliver-Yamada [11].

Theorem 4.1. *Suppose $\Gamma \subset \mathbb{R}^3$ is a semialgebraic subset homeomorphic to the suspension of n-points. If $\mu(\Gamma) < 2$ then Γ is unknotted, i.e., it is isotopic with a planar embedding of the suspension.*

Proof. Fix a vertex set V on Γ. The poles belong to the vertex set. The complement of the poles in Γ is a semialgebraic set with n connected components. We will refer to these as the *meridians* of the suspension. They are (open) semialgebraic arcs without self intersections.

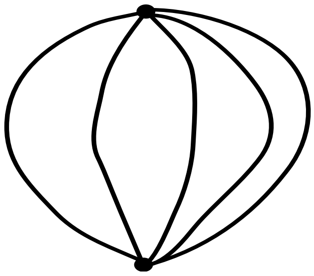

FIGURE 6. *The suspension of* 5 *points.*

Denote by $\mathcal{U} \subset S^2$ the set consisting of Γ-nondegenerate unit vectors $\boldsymbol{u}$ such that $h_{\boldsymbol{u}}$ has at least two local minima on Γ. The set $\mathcal{U}$ is semi-algebraic. Since $\mu(\Gamma) < 2$ we deduce that

$$\mathrm{Area}\,(\mathcal{U}) < \frac{1}{2}\mathrm{Area}\,(S^2).$$

If $\mathcal{M} := S^2 \setminus \mathcal{U}$ then $\mathcal{M}$ is semialgebraic and $\mathrm{Area}\,(\mathcal{M}) > \frac{1}{2}\mathrm{Area}\,(S^2)$. Hence

$$\mathrm{Area}\,\big(\,\mathcal{M} \cap (-\mathcal{M})\,\big) > 0.$$

Since $\mathcal{M} \cap (-\mathcal{M})$ is semialgebraic, and has nonzero area, we deduce that it has nonempty interior. We denote this interior by $\mathcal{V}$. For any $\boldsymbol{u} \in \mathcal{V}$ the function $h_{\boldsymbol{u}}$ has a unique local minimum and a unique local maximum on Γ. From the results in [18] we deduce that we can choose $\boldsymbol{u} \in \mathcal{V}$ satisfying the additional condition that the restriction of $h_{\boldsymbol{u}}$ on $\mathbf{Cr}(\boldsymbol{u})$ is injective.

The poles of Γ are critical points of $h_{\boldsymbol{u}}$ so that $h_{\boldsymbol{u}}$ has distinct values on these poles. We label the poles by $P_\pm$ so that $h_{\boldsymbol{u}}(P_+) > h_{\boldsymbol{u}}(P_-)$. We set $m_\pm := h_{\boldsymbol{u}}(P_\pm)$. We need to distinguish two cases.

A. *The poles are the only local extrema of* $h_{\boldsymbol{u}}$ *on* Γ. In this case, the restriction of $h_{\boldsymbol{u}}$ on each meridian induces a semialgebraic homeomorphic between that meridian and the open interval (m_-, m_+). The graph is then a braid with the top and bottom capped-off; see Figure 7.

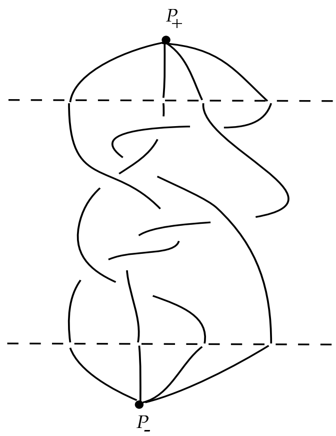

FIGURE 7. *A braided embedding of* Σ_4.

Such a braided embedding is isotopic to a planar embedding because a braid with the top capped off can be untwisted by an isotopy which keeps the bottom of the braid fixed. This can be see easily for the basic braids $\sigma_i^{\pm}$; see Figure 8. By Artin's classical result (see e.g. [3, Thm. 1.8]) any braid is a composition of such elementary braids. Thus, we can inductively untwist any capped braid.

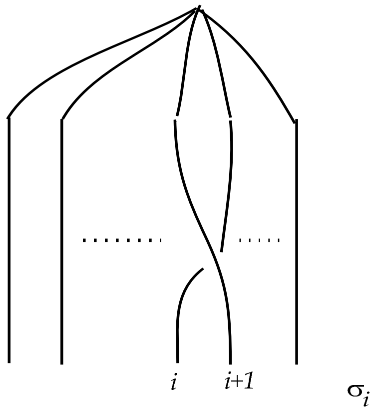

FIGURE 8. *The elementary braid σ_i with the top capped off.*

B. *One of the poles $P_\pm$ is not an absolute extremum of $h_{\boldsymbol{u}}$.* We will reduce this case to the previous one. More precisely, we will prove that we can isotop Γ an embedding of Σ_n such that $h_{\boldsymbol{u}}$ is a stratified Morse function with a unique relative maximum at P_+ and a unique relative minimum located at p_-.

Suppose that the unique relative maximum is achieved at a point q located on one of the meridians. Denote this meridian by M. The hyperplane H determined by $\boldsymbol{u}$ and containing P_+ intersects the meridian M at a unique point r; see Figure 9. (If there were several points of intersection with M, the function $h_{\boldsymbol{u}}$ would have several local maxima on this meridian.)

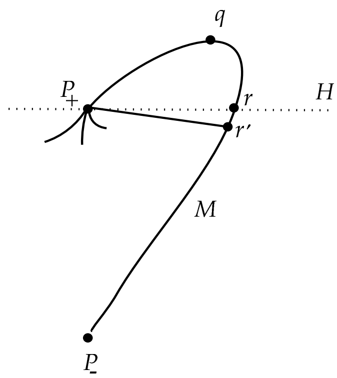

FIGURE 9. *Deforming Γ to a braided embedding.*

The pole P_+ is a stratified critical point and, since the function $h_{\boldsymbol{u}}$ does not have local maxima on meridians different from M we deduce that we can find an open, convex polyhedral cone $\mathcal{C}$ with vertex at P_+ satisfying the following conditions.

- The cone $\mathcal{C}$ is situated below the hyperplane H, i.e., it is contained in the open half-space $\{h_{\boldsymbol{u}} < m_+\}$.
- The cone $\mathcal{C}$ contains all the meridians other than M.

We deduce that if a point r' on the meridian M is sufficiently close to r, then the open line segment (P_+r') does not intersect any of the meridians situated below H. Now choose $r' \in M$ sufficiently close to r and situated slightly below H; see Figure 9. Moreover, we can assume that M is smooth at r' and thus r' is not a critical point of $h_{\boldsymbol{u}}$. Clearly we can deform the arc P_+qr' to the segment $[P_+r']$ while keeping the endpoint fixed such that during the deformation we do not intersect the meridians different from M. We have thus isotoped Γ to an embedding with the property that $h_{\boldsymbol{u}}$ is a stratified Morse function with no local maxima along the meridians.

Arguing similarly, we can isotop Γ such that $h_{\boldsymbol{u}}$ has no local minima along the meridians. Thus we have isotoped ourself to case **A**. This concludes the proof of Theorem 4.1. □

FIGURE 10. *A surgery from the singular figure eight to a trefoil knot.*

Remark 4.2. (a) The special case $b = 2$ is due to Fáry [8] and Milnor [17]. The case $n = 3$ was investigated by Gulliver and Yamada [11] where they established the unknottedness of a theta graph whose crookedness satisfies $\mu < \frac{3}{2}$.
(b) Using the crookedness lowering trick in [17, Lemma 1.2] one can prove that if Γ is a semialgebraic embedding of Σ_n with $\mu(\Gamma) \leq 2$, then either Γ planar, or it is isotopic to an embedding Γ' with $\mu(\Gamma') < \mu(\Gamma)$. This shows that if $\mu(\Gamma) \leq 2$ then Γ is unknotted.

The upper bound 2 on crookedness in Theorem 4.1 cannot be improved. For example, if Γ is the figure eight curve in the left-hand side of Figure 10 then $\mu(\Gamma) = 2$. We can find surgeries of this figure eight curve producing the trefoil knot in the right-hand side while barely changing the total curvature so that $\mu(\mathrm{trefoil}) \approx 2$. Essentially, this surgery replaces a straight line segment with an arc of circular helix with the same endpoints, and axis parallel to the segment. If the slope of this arc of helix is very large then the total curvature is very small. More precisely the total curvature of the arc

$$[0, 2\pi c] \ni s \mapsto \big(\, a\cos\frac{s}{c}, a\sin\frac{s}{c}, \frac{bs}{c}\,\big) \in \mathbb{R}^3, \;\; c = \sqrt{a^2+b^2},$$

is $\frac{2a}{c} = \frac{2}{\sqrt{1+m^2}}$ where $m = \frac{b}{a}$ is its slope.

□

Remark 4.3. The above result is not valid for graphs that are not suspensions of finite sets. In fact the situation is dramatically different. For such graphs, it is possible that they are almost tight $\mu(\Gamma)$ is very close to 1 and still be knotted. We describe below such an instance.

Consider the planar semialgebraic set Γ_a depicted in Figure 11(a). In this case $\mu(\Gamma) = 1$. We shrink the horizontal edges to obtain a set Γ_b as in Figure 11(b) still satisfying $\mu(\Gamma_b) = 1$. We denote by r the common length of the two horizontal edges.

Now deform the vertical edge AB to obtain the broken line AVB as in Figure 11(c). The triangle AVB is isosceles. The new semialgebraic set Γ_c satisfies

$$\mu(\Gamma_c) = 1 + \frac{\theta}{\pi},$$

where θ denotes the angle between AB and VA. Continue deforming the broken line AVB until V crosses the vertical chord CD, and push V a bit more to the left to obtain the semialgebraic set Γ_d depicted in Figure 11(d).

If we continue to denote by θ the (new) angle between VA and AB then

$$\mu(\Gamma_d) = 1 + \frac{\theta}{\pi}.$$

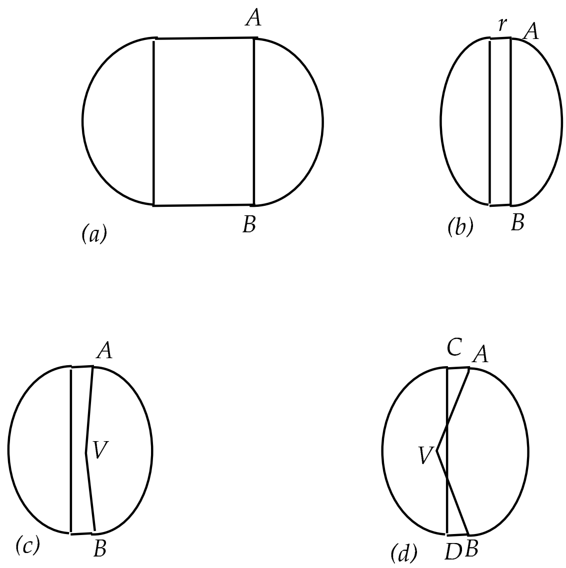

FIGURE 11. *Deforming a simple tight set.*

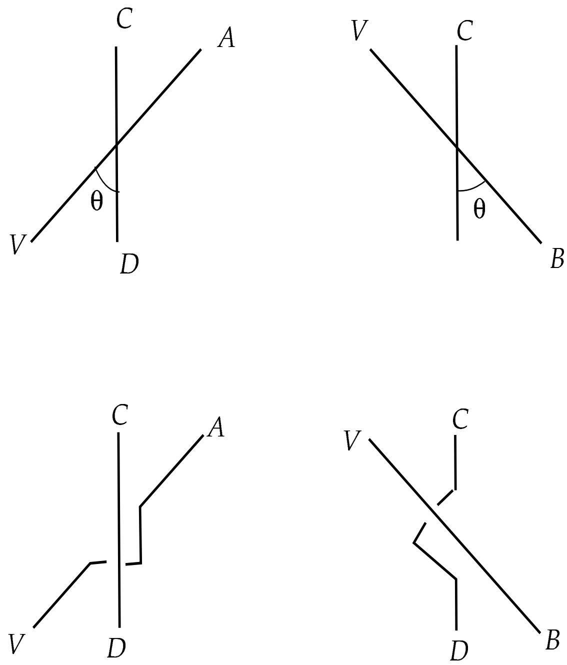

FIGURE 12. *Resolving a crossing as an under/overcrossing.*

On the left-had side of Figure 12 we describe how to replace the intersection between the lines VA and CD with by creating a small (under) "dimple" in the arc AB. The new arc VA will go below CD. We can arrange that resulting change in $\mu(\Gamma)$ due to this dimple is $< \frac{\varepsilon}{4}$. In the right-hand side of Figure 13 we replace similarly the intersection of VB and CD with a crossing of VB over the new arc CD. Again we can arrange that the change in $\mu(\Gamma)$ due to this surgery is $< \frac{\varepsilon}{4}$.

After all these transformations we obtain a subset in $\mathbb{R}^3$ homeomorphic to the original set Γ_a and isotopic to the set depicted in Figure 13. We observe two disjoint circles forming a Hopf link.

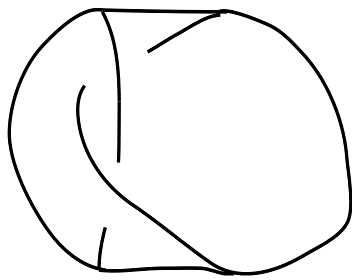

FIGURE 13. *The formation of a Hopf link*

By choosing the length r in Figure 11(b) sufficiently small, we can arrange that the angle θ between VA and CD in Figure 11(d) is $< \frac{\pi\varepsilon}{2}$. After performing all these operations we obtain a semialgebraic Γ set isotopic with the set in Figure 13 and satisfying

$$\mu(\Gamma) < 1 + \frac{\theta}{\pi} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < 1 + \varepsilon.$$

This set cannot be unknotted to the set Γ_a because of the presence of the nontrivial Hopf link. $\square$

Question 4.4. The last remark leads to a natural question. Suppose Γ is a planar graph with the following property.

(**P**) For any planar semialgebraic embedding $\Gamma' \hookrightarrow \mathbb{R}^2$, and any two bounded connected components R_1, R_2 of the complement $\mathbb{R}^2 \setminus \Gamma'$, the closures $\overline{R}_1$ and $\overline{R}_2$ have a point in common.

For example, the suspension of a n-point set Σ_n (Figure 6) or the wedge of circles (Figure 4) has this property. We believe that the following is true.

*Suppose Γ is a planar graph satisfying property (**P**). Then the following hold.*
(a) For any $\varepsilon > 0$ there exists a planar embedding $\varphi : \Gamma \hookrightarrow \mathbb{R}^2$ such that $\mu\big(\varphi(\Gamma)\big) < \mu_0(\Gamma) + \varepsilon$, where $\mu_0(\Gamma)$ is the topological invariant defined in (2.5).
(b) Any semialgebraic embedding $\Gamma \hookrightarrow \mathbb{R}^3$ is isotopic to a planar embedding if $\mu(\Gamma) < \mu_0(\Gamma) + 1$.
$\square$

APPENDIX A. BASICS OF REAL SEMI-ALGEBRAIC GEOMETRY

We want to present a few basic facts of real algebraic geometry used throughout the paper. For proofs and more details we refer to [4, 7, 6].

A subset $A \subset \mathbb{R}^n$ is called *semialgebraic* if it is a finite union of sets described by finitely many polynomial inequalities. We denote by $\mathcal{S}^n$ the collection of semi-algebraic subsets of $\mathbb{R}^n$ and we set $\mathcal{S} = \cup_{n \geq 1} \mathcal{S}^n$.

Using the canonical embedding $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ we can regard $\mathcal{S}^n$ as a subcollection of $\mathcal{S}^{n+1}$. If $A \in \mathcal{S}^m$ and $B \in \mathcal{S}^n$, then a (possibly discontinuous) map $f : A \to B$ is semialgebraic if its graph $\Gamma_f \subset A \times B$ is a semialgebraic set. Sometimes we will refer to the semialgebraic sets/functions as *definable* sets/ functions.

We list below some of the basic properties of the semialgebraic sets.

• The collection $\mathcal{S}$ is closed under boolean operations: union, intersection complement, cartesian product.

• (Tarski-Seidenberg) If $T:\mathbb{R}^m\to\mathbb{R}^n$ is an affine map, then $T(\mathcal{S}^m)\subset\mathcal{S}^n$.
• The image and preimage of a definable set via a definable map is a definable set.
• (*Piecewise smoothness of one variable semialgebraic functions.*) If $f:(0,1)\to\mathbb{R}$ is a semialgebraic function, then there exists

$$0=a_0<a_1<a_2<\cdots<a_n=1$$

such that the restriction of f to each subinterval (a_{i-1},a_i) is real analytic and monotone. Moreover f admits right and left limits at any $t\in[0,1]$.
• (*Closed graph theorem.*) Suppose X is a semialgebraic set and $f:X\to\mathbb{R}^n$ is a semialgebraic bounded function. Then f is continuous if and only if its graph is closed in $X\times\mathbb{R}^n$.
• (*Curve selection.*) If A is a definable set, and $x\in \boldsymbol{cl}(A)\setminus A$, then there exists an $\mathcal{S}$ definable continuous map

$$\gamma:(0,1)\to A$$

such that $x=\lim_{t\to 0}\gamma(t)$.
• Any definable set has finitely many connected components, and each of them is definable.
• Suppose A is a definable set, p is a positive integer, and $f:A\to\mathbb{R}$ is a definable function. Then A can be partitioned into finitely many definable sets $S_1,\dots,S_k$, such that each S_i is a C^p-manifold, and each of the restrictions $f|_{S_i}$ is a C^p-function.
• (*Triangulability.*) For every compact definable set A, and any finite collection of definable subsets $\{S_1,\dots,S_k\}$, there exists a compact simplicial complex K, and a definable homeomorphism

$$\Phi:K\to A$$

such that all the sets $\Phi^{-1}(S_i)$ are unions of relative interiors of faces of K.
• (*Definable selection.*) Suppose A,Λ are definable. Then a *definable* family of subsets of A parameterized by Λ is a definable subset

$$S\subset A\times\Lambda.$$

We set

$$S_\lambda:=\bigl\{a\in A;\;\;(a,\lambda)\in S\bigr\},$$

and we denote by Λ_S the projection of S on Λ. Then there exists a definable function $s:\Lambda_S\to A$ such that

$$s(\lambda)\in S_\lambda,\;\;\forall\lambda\in\Lambda_S.$$

• (*Dimension.*) The dimension of a definable set $A\subset\mathbb{R}^n$ is the supremum over all the nonnegative integers d such that there exists a C^1 submanifold of $\mathbb{R}^n$ of dimension d contained in A. Then $\dim A<\infty$, $\dim(X\times Y)=(\dim X)(\dim Y)$, $\forall X,Y\in\mathcal{S}$, and

$$\dim(\boldsymbol{cl}(A)\setminus A)<\dim A.$$

Moreover, if $(S_\lambda)_{\lambda\in\Lambda}$ is a definable family of definable sets then the function

$$\Lambda\ni\lambda\mapsto\dim S_\lambda$$

is definable.
• If $f:A\to B$ is a semialgebraic bijection (non necessarily continuous) then $\dim A=\dim B$.
• If $A_1,\dots,A_n$ are definable sets then

$$\dim(A_1\cup\cdots\cup A_n)=\max\bigl\{\dim A_k;\;\;1\le k\le n\bigr\}.$$

• (*Definable triviality of semialgebraic maps.*) We say that a semialgebraic map $\Phi:X\to S$ is *definably trivial* if there exists a definable set F, and a definable homeomorphism $\tau:X\to F\times S$

such that the diagram below is commutative

$$\begin{tikzcd} X \arrow[rr, "\tau"] \arrow[dr, "\Phi"'] & & S \times F \arrow[dl, "\pi_S"] \\ & S & \end{tikzcd} .$$

If $\Psi : X \to Y$ is a continuous definable map, and p is a positive integer, then there exists a partition of Y into definable C^p-manifolds $Y_1, \dots, Y_k$ such that each the restrictions

$$\Psi : \Psi^{-1}(Y_k) \to Y_k$$

is definably trivial.

From the definable triviality of semialgebraic maps we deduce the following consequence.

Corollary A.1. *If $F : A \to B$ is a continuous semialgebraic map, then the collection of fibers $(F^{-1}(b))_{b \in B}$ contains only finitely many homeomorphism types.* □

Lemma A.2. *The discriminant set Δ_Γ is a closed semialgebraic subset of S^2 of dimension ≤ 1.*

Proof. We denote by $F(\Gamma)$ the set of points on the open arcs where the curvature vanishes. The set $F(\Gamma)$ is of dimension at most 1, and thus is a finite disjoint union of (open arcs) arcs and points. Moreover the collection

$$\mathcal{L}_\Gamma := \{ L_p;\ \ p \in F(\Gamma) \}$$

consists of finitely many affine lines. We see that

$$\mathcal{C}(\Gamma) := \Gamma \setminus \big(V \cup F(\Gamma) \big).$$

The set Δ_Γ of degenerate vectors is a disjoint union

$$\Delta_\Gamma = \Delta_\Gamma^V \sqcup \Delta_\Gamma^F \sqcup \Delta_\Gamma^*$$

- Δ_Γ^V consists of unit vectors $\boldsymbol{u}$ perpendicular to some vector $\boldsymbol{\tau}$ in some $\boldsymbol{N}_v$, $v \in V$. It is a finite union of great circles on S^2
- Δ_Γ^F consists of unit vectors $\boldsymbol{u}$ perpendicular to one of the affine lines in $\mathcal{L}_\Gamma$. It is also a finite union of great circles.
- Δ_Γ^* consists of unit vectors $\boldsymbol{u}$ such that there exist a point $p \in \mathcal{C}(\Gamma)$ with the property $\boldsymbol{u} \perp \boldsymbol{n}(p), L_p$. It is 1-dimensional and thus it is a finite union of semialgebraic arcs on S^2.

Observe that for $p \in \mathcal{C}(\Gamma)$ the subspace spanned by $\boldsymbol{n}(p)$ and the tangent space $T_p e$ is the osculator plane of E at p. We denote this plane by $\mathcal{O}_p$. Then

$$\Delta_\Gamma^* = \{ \boldsymbol{u} \in S^2;\ \ \exists p \in \mathcal{C}(\Gamma):\ \ \boldsymbol{u} \perp \mathcal{O}_p \}.$$

We need to prove that Δ_Γ^* is has dimension ≤ 1. Consider the set

$$\boldsymbol{N}^*(\Gamma) = \{ (p, u) \in \mathcal{C}(\Gamma) \times S^2;\ \ \boldsymbol{u} \perp \mathcal{O}_p \}.$$

The set $\boldsymbol{N}^*(\Gamma)$ is semialgebraic, and the natural map $\boldsymbol{N}^*(\Gamma) \to \mathcal{C}(\Gamma)$ is two-to-one. In particular $\boldsymbol{N}^*(\Gamma)$ is one dimensional. The projection of $\boldsymbol{N}^*(\Gamma)$ on S^2 is the set Δ_Γ^* so that $\dim \Delta_\Gamma^* \leq 1$. □

Proof of Proposition 2.1. Suppose Γ is a compact, connected semialgebraic subset of $\mathbb{R}^3$ of dimension 1. We define

$$\mathcal{X}_\Gamma := \{ (\boldsymbol{u}, p, q, \varepsilon) \in S^2 \times \Gamma \times \Gamma \times \mathbb{R};\ \ h_{\boldsymbol{u}}(q) < h_{\boldsymbol{u}}(p),\ \ |p - q| = \varepsilon,\ \}.$$

The natural projection $\pi : \mathcal{X}_\Gamma \to S^2 \times \Gamma\mathbb{R}$, $(\boldsymbol{u}, p, q, \varepsilon) \mapsto (\boldsymbol{u}, p, \varepsilon)$ is a continuous semialgebraic map and its fiber over $(\boldsymbol{u}, p, \varepsilon)$ is the link $L^+_\varepsilon(p, \boldsymbol{u})$ defined in (1.4). From Corollary A.1 we deduce again that the are only finitely many topological types amongst these spaces.

Fix a graph structure on Γ and consider the definable subset $\mathbf{Cr} \subset (S^2 \setminus \Delta_\Gamma) \times \Gamma$

$$\mathbf{Cr} := \big\{ (\boldsymbol{u}, p) \in (S^2 \setminus \Delta_\Gamma) \times \Gamma;\;\; p \in \mathbf{Cr}(\boldsymbol{u}) \,\big\}.$$

The natural projection $\mathbf{Cr} \to S^2 \setminus \Delta_\Gamma$ is semialgebraic and the fiber over $\boldsymbol{u} \in S^2 \setminus \Delta_\Gamma$ is the critical set $\mathbf{Cr}(\boldsymbol{u})$. Corollary A.1 now implies that there are finitely many topological types in the family $\mathbf{Cr}(\boldsymbol{u})$, $u \in \mathbb{S}^2 \setminus \Delta_\Gamma$. Since the sets $\mathbf{Cr}(\boldsymbol{u})$ are finite sets we deduce

$$\sup\big\{ |\,\mathbf{Cr}(\boldsymbol{u})|;\;\; u \in S^2 \setminus \Delta_\Gamma \,\big\} > \infty.$$

Now observe that

$$M_{\boldsymbol{u}}(t) = \lim_{\varepsilon \searrow 0} \sum_{p \in \mathbf{Cr}(\boldsymbol{u})} P_{\varepsilon, p, \boldsymbol{u}}(t),$$

where $P_{\varepsilon, \boldsymbol{u}}(t)$ is the Poincaré polynomial of the topological pair $\big(\,\mathrm{cone}\,(\,L^-_\varepsilon(t,p)\,), L^-_\varepsilon(t,p)\,\big)$. The map

$$\mathbf{Cr} \times \Gamma \times \mathbb{R} \ni (\boldsymbol{u}, p, \varepsilon) \mapsto P_{\varepsilon, p, \boldsymbol{u}}(t) \in \mathbb{Z}[t]$$

is definable. In particular, its range is finite, and its level sets are semialgebraic sets. This implies that $w(\boldsymbol{u})$ is semialgebraic. □

Remark A.3. Denote by $\mathcal{S}_\Gamma \subset S^2$ the set of Γ-nondegenerate unit vectors $\boldsymbol{u}$ such that the restriction of $h_{\boldsymbol{u}}$ to $\mathbf{Cr}(\boldsymbol{u})$ is injective. The set $\mathcal{S}_\Gamma$ is open and semi-algebraic, and the functions corresponding to $\boldsymbol{u} \in \mathcal{S}_\Gamma$ are the so called stable stratified Morse functions. The results of R. Pignoni [18] imply that the function

$$\mathcal{S}_\Gamma \ni \boldsymbol{u} \mapsto M_{\boldsymbol{u}}(t) \in \mathbb{Z}$$

is constant on the connected components of $\mathcal{S}_\Gamma$. □

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