Seiberg-Witten invariants of rational homology 3-spheres. Part I

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Introduction

Throughout the paper we will use the notations in [26].

$H^k(M, g)$ will denote the space of k-forms on the compact oriented manifold M with respect to the metric g

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1 Three dimensional monopoles

1.1 The 3-dimensional Seiberg-Witten equations

Suppose M is an oriented 3-manifold. To formulate the Seiberg-Witten we will need to fix additional geometric data.

- A Riemann metric g.
- A *spin*^{c} structure σ.
- A real co-closed form η.
- A smooth function $\mu : M \to \mathbb{R}$.

The *spin*^{c} structure σ determines a bundle of spinors $\mathbb{S}_\sigma \to M$. We denote by $\det \sigma$ the determinant line bundle of $\mathbb{S}_\sigma$. Fix a Hermitian metric on $\det \sigma$ and denote by $\mathcal{A}_\sigma$ the space of connections on $\det \sigma$ compatible with the Hermitian metric.

We have a Clifford multiplication map

$$\boldsymbol{c} : T^*M \otimes \mathbb{C} \to \mathrm{End}(\mathbb{S}_\sigma), \;\; \alpha \mapsto \boldsymbol{c}(\alpha).$$

We normalize this multiplication so that

$$\boldsymbol{c}(dv_g) = -1.$$

This is equivalent to the identity

$$\boldsymbol{c}(\alpha) = \boldsymbol{c}(*\alpha), \;\; \forall \alpha \in \Omega^1(M).$$

If $\mathbf{i}\alpha$ is a purely imaginary one form on M then $\mathbb{C}(\mathbf{i}\alpha)$ is a traceless symmetric endomorphism of $\mathbb{S}_\sigma$.

The configuration space $\mathcal{C} = \mathcal{C}_\sigma$ consists of pairs $\mathsf{C} := (\psi, A) \in \Gamma(\mathbb{S}_\sigma) \times \mathcal{A}_\sigma$. The group $\mathcal{G} := \mathrm{Map}\,(M.S^1)$ of gauge trasformations acts on $\mathcal{C}$ according to the rule

$$\gamma \cdot (\psi, A) = \Big(\gamma \cdot \psi, A - 2\frac{d\gamma}{\gamma}\Big).$$

For every configuration $\mathsf{C} = (\psi, A)$ we denote by $\mathbf{Stab}(\mathsf{C}) \subset \mathcal{G}$ the stabilizer of C with respect to the $\mathcal{G}$-action. It is known that

$$\mathbf{Stab}(\mathsf{C}) = \{1\}, \; S^1$$

and

$$\mathbf{Stab}(\psi, A) = S^1 \Longleftrightarrow \psi \equiv 0.$$

The configurations with notrivial stabilizer are called reducibles. The others are called irreducible. We will denote the set of irreducible/reducible configurations by $\mathcal{C}^{irr}/\mathcal{C}^{red}$.

Any connection $A \in \mathcal{A}_\sigma$ canonically determines a formally selfadjoint Dirac operator

$$\mathfrak{D}_A : \Gamma(\mathbb{S}_\sigma) \to \Gamma(\mathbb{S}_\sigma).$$

The *three-dimensional monopole* is a configuration satisfying the (η, μ)-perturbed Seiberg-Witten equations

$$\begin{cases} \mathfrak{D}_A\psi + \mu f = 0 \\ \boldsymbol{c}(*F_A + \mathbf{i}\eta) = \frac{1}{2}q(\psi) \end{cases} . \tag{$SW_{\eta,\mu}$}$$

where $q(\psi) \in \mathrm{End}(\mathbb{S}_\sigma)$ is the traceless symmetric endomorphism acting according to the rule

$$\Gamma(\mathbb{S}_\sigma) \ni \phi \mapsto \langle \phi, \psi\rangle\psi - \frac{1}{2}|\psi|^2\phi.$$

Equivalently, the (μ, η)-monopoles are zeros of the Seiberg-Witten map

$$SW = SW_{g,\mu,\eta} : \mathcal{C} \to \mathcal{C},\ \ \mathsf{C} = (\psi, A) \mapsto \Big(\mathfrak{D}_A\psi + \mu\psi, \frac{1}{2}\boldsymbol{c}^{-1}\big(\, q(\psi)\,\big) - (*F_A + \mathbf{i}\eta)\Big).$$

Denote by $\mathrm{End}_0(\mathbb{S}_0)$ the space of traceless symmetric endomorphisms of $\mathbb{S}_\sigma$. For latter use let us mention a few basic properties of the quadratic map q. We define the real, pointwise inner product on the space of symmetric endomorphisms of $\mathbb{S}_\sigma$ to be

$$\langle T, S\rangle := \mathbf{Re}\ \mathrm{tr}\,(TS).$$

Then

$$|\boldsymbol{c}(\mathbf{i}\alpha)|^2 = 2|\alpha|^2,\ \ \forall \alpha \in \Omega^1(M) \tag{1.1a}$$

$$\langle q(\psi)\psi, \psi\rangle = |q(\psi)|^2 = \frac{1}{2}|\psi|^4. \tag{1.1b}$$

$$\langle T, q(\psi)\rangle = \langle T\psi, \psi\rangle,\ \ \forall T \in \mathrm{End}_0(\mathbb{S}_\sigma),\ \ \psi \in \Gamma(\mathbb{S}_\sigma). \tag{1.1c}$$

We denote by $\mathcal{Z}_\sigma = \mathcal{Z}_\sigma(g, \eta, \mu)$ the set of monopoles. $\mathcal{Z}_\sigma$ is a $\mathcal{G}$-invariant subset of $\mathcal{C}_\sigma$ and we denote by $\mathfrak{M}_\sigma = \mathfrak{M}_\sigma(g, \eta, \mu)$ the space of orbits

$$\mathfrak{M}_\sigma := \mathcal{Z}_\sigma/\mathcal{G}_\sigma.$$

The sets $\mathcal{Z}_\sigma^{irr}/\mathcal{Z}_\sigma^{red}$ and $\mathfrak{M}_\sigma^{irr}/\mathfrak{M}_\sigma^{red}$ are defined in an obvious fashion.

1.2 Admissible 3-manifolds

We will focus our attention on a special class of 3-manifolds, the admissible ones. An oriented 3 manifold is called *admissible* if either $\partial M = \emptyset$ or it is noncompact and the complement M_∞ of some compact subset is diffeomorphic to a semi-infinite cylinder

$$\mathbb{R}_+ \times \text{disjoint union of tori}$$

A *spin*^{c} structure is called admissible if $\det \sigma \mid_{\partial M}$ is trivial.

We will use the cylindrical language of [26]. A *cylindrical structure* on M is a choice of a diffeomorphism

$$M_\infty \cong \mathbb{R}_+ \times \text{disjoint union of tori.}$$

Fix a cylindrical structure. We will denote by t the outgoing longitudinal coordinate along the cylindrical neck. A metric $\hat{g}$ on M will be called *admissible* along the neck it has the form

$$\hat{g} = dt^2 + g_0 + t\text{-exponentially small perturbation}$$

where g_0 denotes a flat metric on a torus. All the choices of perturbations η, w will be assumed asymptotically cylindrical meaning that they are cylindrical modulo an exponentially decaying term.

At this point it is convenient to study in some detail the form of the Seiberg-Witten equations on a cylinder $\mathbb{R} \times \Sigma$, where Σ is a compact oriented surface. We will use the "$\widehat{\ }$"-conventions of [26]. Thus, the quantities defined over the 3-manifold $\mathbb{R} \times \Sigma$ will be indicated by a "$\widehat{\ }$". The absence of a "$\widehat{\ }$" will indicate a quantity defined over the slice T^2. For example, $\hat{d}$ denotes the exterior derivative over $\mathbb{R} \times T^2$ whike d denotes the exterior derivative over T^2. They are related by the identity

$$\hat{d} = dt \wedge \partial_t + d.$$

We fix a metric g on Σ and we form $\hat{g} = dt^2 + g$ over $\mathbb{R} \times \Sigma$. We will denote by $\hat{\sigma}$ a cylindrical *spin*^{c} structure on $\mathbb{R} \times \Sigma$. It induces a *spin*^{c} structure on Σ. We get two bundles of spinors, $\mathbb{S}_{\hat{\sigma}}$ over the cylinder and $\mathbb{S}_\sigma := \mathbb{S}_{\hat{\sigma}} \mid_\Sigma = \partial_\infty \mathbb{S}_{\hat{\sigma}}$, and two Clifford multiplications

$$\hat{\boldsymbol{c}} : \Omega^*(\mathbb{R} \times \Sigma) \otimes \mathbb{C} \to \mathrm{End}(\mathbb{S}_{\hat{\sigma}})$$

and

$$\boldsymbol{c} : \Omega^*(\Sigma) \otimes \mathbb{C} \to \mathrm{End}(\mathbb{S}_\sigma).$$

They are related by the equality

$$\boldsymbol{c}(\alpha) = \hat{\boldsymbol{c}}(dt)\hat{\boldsymbol{c}}(\alpha), \ \ \forall \alpha \in \Omega^1(\Sigma).$$

We set $J := \hat{\boldsymbol{c}}(dt)$ so that the last equality can be rewritten as

$$\boldsymbol{c}(\alpha) = J\hat{\boldsymbol{c}}(\alpha), \ \ \forall \alpha \in \Omega^1(\Sigma).$$

Fix a local oriented orthonormal frame (e_1, e_2) of the tangent bundle of Σ and denote by (e^1, e^2) the dual coframe.

We denote by $\mathbb{S}^\pm_{\hat{\sigma}}$ the $\mp\mathbf{i}$-eigensub-bundles of $\mathbb{S}^\pm_{\hat{\sigma}}$ defined by J. Correspondingly, any spinor $\hat{\psi} \in \Gamma(\mathbb{S}_{\hat{\sigma}})$ splits as

$$\hat{\psi} = \hat{\psi}^+ \oplus \hat{\psi}^-, \ \ \hat{\psi}^\pm \in \Gamma(\mathbb{S}^\pm_{\hat{\sigma}})$$

The operator J has the block decomposition

$$J = \begin{bmatrix} -\mathbf{i} & 0 \\ 0 & \mathbf{i} \end{bmatrix}$$

Now note that

$$\{\boldsymbol{c}(e^i), J\} := \boldsymbol{c}(e^i)J + J\boldsymbol{c}(e^i) = (J\hat{\boldsymbol{c}}(e^i))J + J^2\hat{\boldsymbol{c}}(e^i) = 0$$

and

$$\boldsymbol{c}(e^1 \wedge e^2) = \boldsymbol{c}(dv_g) = \boldsymbol{c}(e^1)\boldsymbol{c}(e^2) = J\hat{\boldsymbol{c}}(e^1)J\hat{\boldsymbol{c}}(e^2) = \hat{\boldsymbol{c}}(e^1)\hat{\boldsymbol{c}}(e^2) = J.$$

Thus

$$\boldsymbol{c}(e^i)\Gamma(\mathbb{S}_\sigma^\pm) \subset \Gamma(\mathbb{S}_\sigma^\mp)$$

and

$$\boldsymbol{c}(e^1 \wedge e^2)\phi_\pm = \mp\mathbf{i}\phi_\pm, \ \ \forall \phi_\pm \in \Gamma(\mathbb{S}_\sigma^\pm).$$

Thus $\mathbb{S}_\sigma^+ \oplus \mathbb{S}_\sigma^-$ is the $\mathbb{Z}_2$-graded bundle of spinors naturally associated to the *spin*^{c}structure σ. If we set $L := \mathbb{S}_\sigma^+$ then

$$\mathbb{S}_\sigma^- \cong L \otimes K_\Sigma^* \ \text{ and } \ \det\sigma = L^2 \otimes K_\Sigma^*$$

where K_Σ denotes the canonical line bundle of Σ.

Fix a reference connection A_0 on $\det\sigma \to \Sigma$. Suppose $\hat{\mathsf{C}} = (\hat{\psi}, \hat{A})$ is a monopole on $\mathbb{R} \times \Sigma$. We can write

$$\hat{A} = \mathbf{i}f(t)dt + \mathbf{i}a(t) + A_0$$

where $f(t)$ is a path of real valued smooth functions on Σ and $a(t)$ is a path of real valued smooth one forms on Σ. Also, it will be convenient to think of $\hat{\psi}$ as a path $\psi(t)$ of spinors on $\mathbb{S}_\sigma$. Set $A(t) := A_0 + \mathbf{i}a(t)$. Then

$$\hat{\mathfrak{D}}_{\hat{A}} = \hat{\boldsymbol{c}}(dt)(\partial_t + \mathbf{i}f) + \hat{\boldsymbol{c}}(e^1)\nabla_{e_1}^{\hat{A}} + \hat{\boldsymbol{c}}(e^2)\nabla_{e_2}^{\hat{A}}$$

$$= J\Big(\partial_t + \mathbf{i}f - \boldsymbol{c}(e^1)\nabla_{e_1}^{A(t)} - \boldsymbol{c}(e^2)\nabla_{e_2}^{A(t)}\Big) = J\Big(\partial_t - \mathfrak{D}_{A(t)}\Big).$$

If we set

$$\varepsilon := \frac{1}{\sqrt{2}}(e^1 + \mathbf{i}e^2), \ \ \partial_A = \frac{1}{\sqrt{2}}\varepsilon \wedge (\nabla_{e_1}^A - \mathbf{i}\nabla_{e_2}^A),$$

$$\bar{\varepsilon} = \frac{1}{\sqrt{2}}(e^1 - \mathbf{i}e^2), \ \ \bar{\partial}_A = \frac{1}{\sqrt{2}}\bar{\varepsilon} \wedge (\nabla_{e_1}^A + \mathbf{i}\nabla_{e_2}^A)$$

then we have (see [25])

$$\hat{\boldsymbol{c}}(\bar{\varepsilon}) = \sqrt{2}\begin{bmatrix} 0 & 0 \\ \bar{\varepsilon} & 0 \end{bmatrix}, \ \ \hat{\boldsymbol{c}}(\varepsilon) = \sqrt{2}\begin{bmatrix} 0 & -\varepsilon \\ 0 & 0 \end{bmatrix}$$

Thus

$$\boldsymbol{c}(\bar{\varepsilon}) = J\hat{\boldsymbol{c}}(\bar{\varepsilon}) = \sqrt{2}\begin{bmatrix} 0 & 0 \\ \mathbf{i}\bar{\varepsilon} & 0 \end{bmatrix}, \ \ \boldsymbol{c}(\varepsilon) = J\hat{\boldsymbol{c}}(\varepsilon) = \sqrt{2}\begin{bmatrix} 0 & \mathbf{i}\varepsilon \\ 0 & 0 \end{bmatrix}$$

so that

$$\mathfrak{D}_A = \sqrt{2}\begin{bmatrix} 0 & -\mathbf{i}\bar{\partial}_A^* \\ \mathbf{i}\bar{\partial}_A & 0 \end{bmatrix}$$

The first of the Seiberg-Witten equations $(SW_{\eta,\mu})$ can be rewritten as

$$\begin{cases} (\partial_t+\mu+\mathbf{i}f)\psi_+ + \mathbf{i}\sqrt{2}\bar{\partial}^*_{A(t)}\psi_- = 0 \\ \\ (\partial_t+\mu+\mathbf{i}f)\psi_- - \mathbf{i}\sqrt{2}\bar{\partial}_{A(t)}\psi_+ = 0 \end{cases} \tag{1.2}$$

The cylindrical structure also affects the second equation in $(SW_{\eta,\mu})$. With respect to the decomposition $\mathbb{S}_{\hat{\sigma}} = \mathbb{S}^+_{\hat{\sigma}} \oplus \mathbb{S}^-_{\hat{\sigma}}$ the endomorphism $q(\hat{\psi}) = q(\hat{\psi}_+ \oplus \hat{\psi}_-)$ has the form (see [25, Sec. 2.1])

$$q(\psi) = \begin{bmatrix} \frac{1}{2}(|\psi_+|^2 - |\psi_-|^2) & \psi_+ \otimes \bar{\psi}_- \\ \\ \bar{\psi}_+ \otimes \psi_- & \frac{1}{2}(|\psi_-|^2 - |\psi_+|^2) \end{bmatrix}. \tag{1.3}$$

Every complex 1-form on Σ decomposes uniquely as

$$\alpha = \alpha^{1,0} + \alpha^{0,1}, \;\; \alpha^{1,0} \in \Omega^{1,0}(\Sigma), \;\; \alpha^{0,1} \in \Omega^{0,1}(\Sigma).$$

If α is real then $\alpha^{0,1} = \overline{\alpha^{1,0}}$. Using these observations in (1.3) we deduce

$$\hat{\boldsymbol{c}}^{-1}(q(\psi)) = \frac{\mathbf{i}}{2}\Big(|\psi_+|^2 - |\psi_-|^2\Big)dt + \frac{1}{\sqrt{2}}\Big(\bar{\psi}_+ \otimes \psi_- - \psi_+ \otimes \bar{\psi}_-\Big). \tag{1.4}$$

Observe now that

$$\hat{*}e^1 = -dt\wedge e^2 = -dt \wedge *e^1, \;\; \hat{*}e^2 = dt \wedge e^1 = -dt \wedge *e^2$$

$$\hat{*}(dt\wedge e^1) = e^2 = *e^1, \;\; \hat{*}(dt \wedge e^2) = -e^1 = *e^2$$

and

$$\hat{*}e^1 \wedge e^2 = dt \wedge *(e^1 \wedge e^2).$$

Using the equality

$$F_{\hat{A}} = F_{A_0} + \mathbf{i}da + dt \wedge \Big(\mathbf{i}\dot{a} - \mathbf{i}df\Big)$$

we now deduce

$$\hat{*}F_{\hat{A}} = dt \wedge \Big(*F_{A_0} + \mathbf{i} * da\Big) + \mathbf{i} * (\dot{a} - df).$$

We can also decompose the perturbation term η as

$$\eta = \eta_0 dt + \eta_1, \;\; \eta_0 := \partial_t \lrcorner\, \eta.$$

The equality

$$\hat{*}F_{\hat{A}} + \mathbf{i}\eta = \frac{1}{2}\hat{\boldsymbol{c}}^{-1}\big(q(\psi)\big)$$

can now be rewritten as

$$\begin{cases} *da = \frac{1}{4}\Big(|\psi_+|^2 - |\psi_-|^2\Big) + \mathbf{i} * F_{A_0} - \eta_0 \\ \\ *\dot{a} - *df = -\frac{\mathbf{i}}{2\sqrt{2}}\Big(\bar{\psi}_+ \otimes \psi_- - \psi_+ \otimes \bar{\psi}_-\Big) - \eta_1 \end{cases} \tag{1.5}$$

Now observe that

$$*\varepsilon = -\mathbf{i}\varepsilon, \text{ and } *\bar{\varepsilon} = \mathbf{i}\bar{\varepsilon}$$

where $*$ denotes the extension by complex linearity of the real Hodge $*$-operator to complex valued forms. Observe that $\bar{\psi}_+ \otimes \psi_- \in \Omega^{0,1}$, $\psi_+ \otimes \bar{\psi}_- \in \Omega^{1,0}$ and $*^2 = -1$ on 1-forms. Applying $*$ to the second equation in (1.5) we can rewrite these equations as follows:

$$\begin{cases} *da = \frac{1}{4}\Big(|\psi_+|^2 - |\psi_-|^2\Big) + \mathbf{i} * F_{A_0} - \eta_0 \\ \\ \dot{a} - df = -\frac{1}{2\sqrt{2}}\Big(\bar{\psi}_+ \otimes \psi_- + \psi_+ \otimes \bar{\psi}_-\Big) + *\eta_1 \end{cases}. \tag{1.6}$$

The equations (1.2) and (1.6) can be further simplified by assuming the configuration $\hat{\mathsf{C}}$ is in temporal gauge, i.e. $f \equiv 0$. In this case we have

$$\begin{cases} (\partial_t + \mu)\psi_+ + \mathbf{i}\sqrt{2}\bar{\partial}^*_{A(t)}\psi_- = 0 \\ \\ (\partial_t + \mu)\psi_- - \mathbf{i}\sqrt{2}\bar{\partial}_{A(t)}\psi_+ = 0 \\ \\ \dot{a} = -\frac{1}{2\sqrt{2}}\Big(\bar{\psi}_+ \otimes \psi_- + \psi_+ \otimes \bar{\psi}_-\Big) + *\eta_1 \\ \\ *F_{A(t)} = \frac{\mathbf{i}}{4}\Big(|\psi_+|^2 - |\psi_-|^2\Big) + \mathbf{i}\eta_0 \end{cases} \tag{1.7}$$

Form the configuration space $\mathcal{C}_\sigma$ on Σ consisting of triples

$$\mathsf{C} = (\psi_+, \psi_-, A) \in \Gamma(\mathbb{S}^+_\sigma) \times \Gamma(\mathbb{S}^-_\sigma) \times \mathcal{A}_\sigma.$$

Define

$$\Upsilon = \Upsilon_\sigma : \mathcal{C}_\sigma \to \Omega^0(\Sigma, \mathbf{i}\mathbb{R}), \ (\psi_+, \psi_-, A) \mapsto *F_A - \frac{\mathbf{i}}{4}\Big(|\psi_+|^2 - |\psi_-|^2\Big).$$

Note that the last equation in (1.7) can be rewritten as

$$\Upsilon(\mathsf{C}(t)) = \mathbf{i}\eta_0. \tag{1.8}$$

Lemma 1.1. *Suppose the perturbation terms are compactly supported. If $\hat{\mathsf{C}} = (\hat{\psi}, \hat{A})$ is a smooth configuration such that $\hat{A}$ is temporal, $\hat{A} = A_0 + \mathbf{i}a(t)$, and satisfying the first three equations in (1.7) (with $f = 0$) then for large $|t|$ we have*

$$\frac{d}{dt}\Upsilon(\mathsf{C}(t)) = 0$$

where $\mathsf{C}(t) = \hat{\mathsf{C}}\,|_{t\times\Sigma} = (\psi(t), A_0 + a(t)) \in \mathcal{C}_\sigma$. Moreover, the first three equations (1.7) with $f = 0$ describe the ascending gradient flow of the functional

$$\mathfrak{E} = \mathfrak{E}_\sigma : \mathcal{C}_\sigma \to \mathbb{R}, \ \mathfrak{E}(\psi, A) = \frac{1}{2}\int_\Sigma \langle \mathfrak{D}_A\psi, \psi\rangle dv(g)$$

Proof Observe first that for large $|t|$ we have $\mu = 0$, $\eta = 0$. We have

$$\frac{d}{dt}\Upsilon(\mathsf{C}(t)) = *\mathbf{i}d\dot{a} - \frac{\mathbf{i}}{4}\frac{d}{dt}\Big(|\psi_+|^2 - |\psi_-|^2\Big)$$

Now take the the exterior derivative of the third equation in (1.7). We obtain

$$d\dot{a} = -\frac{1}{2\sqrt{2}}d\Big(\bar{\psi}_+ \otimes \psi_- + \psi_+ \otimes \bar{\psi}_-\Big)$$

Now observe that $\bar{\psi}_+ \otimes \psi_- \in \Omega^{0,1}(\Sigma)$ and $\psi_+ \otimes \bar{\psi}_- \in \Omega^{1,0}(\Sigma)$. We deduce

$$d\Big(\bar{\psi}_+ \otimes \psi_- + \psi_+ \otimes \bar{\psi}_-\Big) = \partial(\bar{\psi}_+ \otimes \psi_-) + \overline{\partial(\bar{\psi}_+ \otimes \psi_-)}$$

We will deal only with the first term. Using the Hodge identity $\partial_A = \mathbf{i}dv_g \wedge \bar{\partial}_A^*$ we deduce

$$\partial(\bar{\psi}_+ \otimes \psi_-) = (\partial_A\bar{\psi}_+) \wedge \psi_- + \bar{\psi}_+(\partial_A\psi_-) = \Big((\partial_A\bar{\psi}_+) \wedge \psi_- + \mathbf{i}dv_g \wedge \bar{\psi}_+(\bar{\partial}_A^*\psi_-)\Big)$$

(use the *conjugate linear* Hodge operator $*_c$ such that $*_c\psi_+ = dv_g \wedge \bar{\psi}_+$ and the equality $\bar{\psi}_+ = \Lambda(*_c\psi_+)$ where Λ denotes the contraction by the Kähler form)

$$= \Big\{\big(\partial_A\Lambda(*_c\psi_+)\big) \wedge \psi_- + \mathbf{i}(*_c\psi_+)(\bar{\partial}_A^*\psi_-)\Big\}$$

(use the Hodge identities $\partial_A\Lambda = \Lambda\partial_A - \mathbf{i}\bar{\partial}_A^*$, $\bar{\partial}_A^* = - *_c \bar{\partial}_A *_c$ and $*_c^2 = 1$ on even forms.)

$$= \Big(\mathbf{i}\big(*_c(\bar{\partial}_A\psi_+)\big) \wedge \psi_- + \mathbf{i}(*_c\psi_+)(\bar{\partial}_A^*\psi_-)\Big)$$

(use the first two equations in (1.7), $\bar{\partial}_A^*\psi_- = \frac{\mathbf{i}}{\sqrt{2}}\dot{\psi}_+$, $\bar{\partial}_A\psi_+ = -\frac{\mathbf{i}}{\sqrt{2}}$)

$$= \Big(\mathbf{i} *_c (-\frac{\mathbf{i}}{\sqrt{2}}\dot{\bar{\psi}}_-) \wedge \psi_- - \frac{1}{\sqrt{2}}(*_c\psi_+)\dot{\psi}_+ = -\frac{1}{\sqrt{2}}\Big((*_c\psi_+)(\dot{\psi}_+) + \big(*_c\dot{\psi}_-\big) \wedge \psi_-\Big).$$

We conclude

$$\begin{aligned} &d\Big(\bar{\psi}_+ \otimes \psi_- + \psi_+ \otimes \bar{\psi}_-\Big) \\ &= -\frac{1}{\sqrt{2}}\Big((*_c\psi_+)(\dot{\psi}_+) + \big(*_c\dot{\psi}_-\big) \wedge \psi_-\Big) - \frac{1}{\sqrt{2}}\overline{\Big((*_c\psi_+)(\dot{\psi}_+) + \big(*_c\dot{\psi}_-\big) \wedge \psi_-\Big)} \\ &\quad -\frac{1}{\sqrt{2}}\Big((*_c\psi_+)(\dot{\psi}_+) - \psi_- \wedge \big(*_c\dot{\psi}_-\big)\Big) - \frac{1}{\sqrt{2}}\overline{\Big((*_c\psi_+)(\dot{\psi}_+) - \psi_- \wedge \big(*_c\dot{\psi}_-\big)\Big)} \\ &= -\frac{1}{\sqrt{2}}\frac{d}{dt}(|\psi_+|^2 - |\psi_-|^2)dv_g. \end{aligned}$$

Hence

$$*\mathbf{i}d\dot{a} = \frac{\mathbf{i}}{4}\frac{d}{dt}\Big(|\psi_+|^2 - |\psi_-|^2\Big)$$

thus proving the first part of the lemma. To prove the second part we only need to compute the L^2-gradient of $\mathfrak{E}$.

Observe that

$$\frac{d}{dt}|_{t=0}\ \mathfrak{E}(\psi + t\dot{\psi}, A + \mathbf{i}t\dot{a}) = \int_\Sigma \langle \mathfrak{D}_A\psi, \dot{\psi}\rangle dv(g) + \frac{1}{4}\int_\Sigma \langle \boldsymbol{c}(\mathbf{i}\dot{a})\psi, \psi\rangle dv(g).$$

To proceed further observe that if we locally decompose $\dot{a} = u\varepsilon + \bar{u}\bar{\varepsilon}$, u complex valued function, we have

$$\begin{aligned}\langle \boldsymbol{c}(\mathbf{i}\dot{a})\psi, \psi\rangle &= \langle \boldsymbol{c}(\mathbf{i}u\varepsilon)\psi, \psi\rangle + \langle \boldsymbol{c}(\mathbf{i}\bar{u}\bar{\varepsilon})\psi, \psi\rangle \\ &= -\sqrt{2}\,\mathbf{Re}\,\langle u, \psi_+ \otimes \bar{\psi}_-\rangle - \sqrt{2}\,\mathbf{Re}\,\langle \bar{u}, \bar{\psi}_+ \otimes \psi_-\rangle \\ &= -\sqrt{2}\,\mathbf{Re}\,\big\langle \dot{a}, \psi_+ \otimes \bar{\psi}_- + \bar{\psi}_+ \otimes \psi_-\big\rangle.\end{aligned}$$

Thus

$$\nabla\mathfrak{E}(\psi, A) = \Big(\mathfrak{D}_A\psi, -\frac{1}{2\sqrt{2}}(\psi_+ \otimes \bar{\psi}_- + \bar{\psi}_+ \otimes \psi_-)\Big).\ \blacksquare$$

2 Seiberg-Witten invariants of closed 3-manifolds

2.1 Generalities

Suppose M is a compact oriented 3-manifold. Fix a Riemann metric g, a *spin*^{c} structure σ, a real co-closed 1-from η and a real function μ. We would like to introduce some natural structures on the configuration space $\mathcal{C}_\sigma$ and the set of monopoles $\mathfrak{M}_\sigma$. We will denote by $L^{k,p}$ the Sobolev spaces of distributions k-times differentiable with derivatives in L^p.

We re-define the configuration space $\mathcal{C}_\sigma$ to include some information about the regularity of the configuration. Thus $\mathcal{C}_\sigma$ will stand for the space of $L^{2,2}$-configurations (ψ, A). In this statement we have tacitly assumed we have chosen a fixed *smooth* reference connection A_0 on $\det\sigma$. We want to be more specific about this choice.

By Chern-Weil theory, for every connection A on $\det\sigma$ the differential form $\frac{\mathbf{i}}{2\pi}F_A$ represents the integral homology class $c_\sigma := c_1(\det\sigma)$. For every differential form α on M we will denote by $[\alpha] = [\alpha]_g$ its harmonic part in the Hodge decomposition. We now choose the reference connection to be the unique smooth connection A_0 such that

$$F_{A_0} = [F_{A_0}] = -2\pi\mathbf{i}c_\sigma \in H^2(M, \mathbb{R}).$$

Note that

$$[F_A] = [F_{A_0}],\ \ \forall A \in \mathcal{A}_\sigma.$$

The gauge group is defined as

$$\mathcal{G} := L^{3,2}(M, S^1).$$

Fix a base point $*$ on M and denote by $\mathcal{G}(*) \subset \mathcal{G}$ the group of gauge transformations based at $*$, i.e. gauge transformations γ such that $\gamma(*) = 1$. The perturbation parameters η, μ are chosen to have $L^{k,2}$ regularity, with k sufficiently large so that they have as many classical derivatives as we need.

Let us first note that the monopoles have a variational interpretation. More precisely the Seiberg-Witten map $SW_{g,\mu,\eta}$ is the L^2-gradient of the energy functional (see [26])

$$\mathcal{E} : \mathcal{C}_\sigma \to \mathbb{R},\ \ \mathcal{E}(\mathsf{C}) = \mathcal{E}(\psi, A)$$

$$= \frac{1}{2}\int_M (A-A_0)\wedge(F_A+F_{A_0}) + \frac{1}{2}\int_M \Big(\langle \mathfrak{D}_A\psi,\psi\rangle + \mu|\psi|^2\Big)dv_g - \int_M \langle A-A_0, \mathbf{i}\eta\rangle dv_g$$

This functional is not $\mathcal{G}$-invariant but satisfies

$$\mathcal{E}(\mathsf{C}) - \mathcal{E}(\gamma\cdot\mathsf{C}) = 8\pi^2\int_M \deg\gamma\wedge c_1(\det(\sigma)) - 4\pi\int_M \deg\gamma\wedge[*\eta]$$

where $\deg\gamma := \frac{1}{2\pi}\gamma^*(d\theta)\in\Omega^1(M)$.

For every $\mathsf{C}\in\mathcal{C}_\sigma$ we denote by

$$\mathfrak{L}_\mathsf{C} : T_1\mathcal{G}_\sigma \to T_\mathsf{C}\mathcal{C}_\sigma$$

the infinitesimal action at C

$$\mathfrak{L}_\mathsf{C}(\mathbf{i}f) := \frac{d}{dt}|_{t=0}\, e^{\mathbf{i}tf}\cdot\mathsf{C} = (\mathbf{i}f\psi, A - 2\mathbf{i}df).$$

Its formal (L^2) adjoint is

$$T_\mathsf{C}\mathcal{C}_\sigma \ni \dot{\mathsf{C}} \mapsto \mathfrak{L}^*_\mathsf{C}\dot{\mathsf{C}} = \mathfrak{L}^*_\mathsf{C}(\dot\psi, \mathbf{i}\dot a) = -2\mathbf{i}d^*\dot a - \mathbf{i}\,\mathbf{Im}\,\langle\psi,\dot\psi\rangle.$$

We can identify $\ker\mathfrak{L}_\mathsf{C}$ with the Lie algebra of the stabilizer $\mathbf{Stab}\,(\mathsf{C})$ with respect to the $\mathcal{G}_\sigma$ action.

Since $\mathcal{C}_\sigma$ is an affine space we can identify the tangent space $T_\mathsf{C}\mathcal{C}_\sigma$ with $\mathcal{C}_\sigma$ via the map

$$\dot{\mathsf{C}} \mapsto \mathsf{C}+\dot{\mathsf{C}}.$$

Define the *slice* $\mathcal{S}_\mathsf{C}\subset T_\mathsf{C}\mathcal{C}_\sigma\cong\mathcal{C}_\sigma$ at C by

$$\mathcal{S}_\mathsf{C} := \ker\mathfrak{L}^*_\mathsf{C}\cap L^{2,2}.$$

More generally, we set $\mathcal{S}^r_\mathsf{C} := \ker\mathfrak{L}^*_\mathsf{C}\cap L^{r,2}$. The slice at C is equipped with a natural $\mathbf{Stab}\,(\mathsf{C})$-action and we have the following result (see [26]).

Proposition 2.1. *There exists a small* $\mathbf{Stab}\,(\mathsf{C})$*-invariant neighborhood* U_C *of* $\mathsf{C}\in\mathcal{S}_\mathsf{C}$ *such that every orbit of* $\mathcal{G}_\sigma$ *which intersects* U_C *does so trasversally, along a single* $\mathbf{Stab}\,(\mathsf{C})$*-orbit. In particular, every* $\mathcal{G}_\sigma(*)$*-orbit intersects* U_C *transversely in at most one point.*

Set $\mathcal{B}_\sigma := \mathcal{C}_\sigma/\mathcal{G}$ and $\mathcal{B}_\sigma(*) := \mathcal{C}_\sigma/\mathcal{G}(*)$. From the above proposition we conclude that $\mathcal{B}_\sigma(*)$ is a Hilbert manifold while $\mathcal{B}_\sigma$ is smooth away from the reducible orbits. The set $\mathfrak{M}_\sigma$ is then a metric subspace of $\mathcal{B}_\sigma$ with respect to the induced $L^{2,2}$-metric. Moreover, $\mathfrak{M}_\sigma$ is compact with respect to this metric topology (see [26]). The space $\mathfrak{M}_\sigma$ also has a rich local structure.

The (η,μ)-monopoles are zeros of the smooth map

$$SW = SW_{\eta,\mu} : \mathcal{C}_\sigma \to \mathcal{C}^1_\sigma \cong T_\mathsf{C}\mathcal{C}_\sigma,\ \ (\psi, A)\mapsto (\not{\mathfrak{D}}_A\psi + \mu\psi, q(\psi) - \mathbf{c}(*F_A + \mathbf{i}\eta)$$

obtained as the formal gradient of $\mathcal{E}$. Since

$$\frac{d}{dt}|_{t=0}\,\mathcal{E}_(e^{t\mathbf{i}f}\cdot\mathsf{C}) = 0$$

we deduce

$$D_{\mathsf{C}}\mathcal{E}(\mathfrak{L}_{\mathsf{C}}\mathbf{i}f)=0 \Longleftrightarrow \Big\langle\, SW(\mathsf{C})\,,\, \mathfrak{L}_{\mathsf{C}}(\mathbf{i}f)\,\Big\rangle_{L^2}=0,\ \ \forall \mathbf{i}f\in T_{\mathbf{1}}\mathcal{G}_\sigma$$

so that

$$SW_{\eta,\mu}(\mathsf{C})\in \mathcal{S}^1_{\mathsf{C}},\ \ \forall \mathsf{C}\in \mathcal{C}_\sigma.$$

For $\dot{\mathsf{C}}\in T_{\mathsf{C}}\mathcal{C}_\sigma$ and $\mathbf{i}f\in T_1\mathcal{G}$ define

$$\mathcal{T}_{\mathsf{C}}\left[\begin{array}{c}\dot{\mathsf{C}}\\ \mathbf{i}f\end{array}\right]=\left[\begin{array}{cc}\underline{SW}_{\eta,\mu} & -\frac{1}{2}\mathfrak{L}_{\mathsf{C}}\\ & \\ -\frac{1}{2}\mathfrak{L}^*_{\mathsf{C}} & 0\end{array}\right]\left[\begin{array}{c}\dot{\mathsf{C}}\\ \\ \mathbf{i}f\end{array}\right]$$

$$:=\left[\begin{array}{c}\frac{d}{dt}|_{t=0}\ SW(\mathsf{C}+t\dot{\mathsf{C}})-\frac{1}{2}\mathfrak{L}_{\mathsf{C}}(\mathbf{i}f)\\ \\ -\frac{1}{2}\mathfrak{L}^*_{\mathsf{C}}\dot{\mathsf{C}}\end{array}\right]\in \overline{T_{\mathsf{C}}\mathcal{C}_\sigma}^{L^2}\oplus L^2(N,\mathbf{i}\mathbb{R}).$$

More explicitly, if $\mathsf{C}:=(\psi,A)$ and $\dot{\mathsf{C}}=(\dot{\psi},\mathbf{i}\dot{a})$ then

$$\mathcal{T}_{\mathsf{C}}\left[\begin{array}{c}\dot{\psi}\\ \mathbf{i}\dot{a}\\ \mathbf{i}f\end{array}\right]=\left[\begin{array}{ccc}\mathfrak{D}_A+\mu & 0 & 0\\ 0 & -*d & d\\ 0 & d^* & 0\end{array}\right]\cdot\left[\begin{array}{c}\dot{\psi}\\ \mathbf{i}\dot{a}\\ \mathbf{i}f\end{array}\right]+\left[\begin{array}{c}\frac{1}{2}\mathbf{c}(\mathbf{i}\dot{a})\psi-\frac{\mathbf{i}}{2}f\psi\\ \frac{1}{2}\dot{q}(\psi,\dot{\psi})\\ \frac{\mathbf{i}}{2}\,\mathbf{Im}\,\langle\psi,\dot{\psi}\rangle\end{array}\right] \tag{2.1}$$

Denote by $\mathcal{T}^0_{\mathsf{C}}$ the first operator on the right hand side of (2.1) and set $\mathcal{P}_{\mathsf{C}}:=\mathcal{T}_{\mathsf{C}}-\mathcal{T}^0_{\mathsf{C}}$. Notice that $\mathcal{P}_{\mathsf{C}}$ is a zeroth order operator while $\mathcal{T}_{\mathsf{C}}$ is a first order, formally selfadjoint elliptic operator.

Fix a monopole C_0. The problem of understanding the structure of $\mathfrak{M}_\sigma$ near C_0 boils down to understanding the local structure of the equation

$$SW(\mathsf{C}_0+\dot{\mathsf{C}})=0. \tag{2.2}$$

where $\mathfrak{L}^*_{\mathsf{C}_0}\dot{\mathsf{C}}=0$ and $\|\dot{\mathsf{C}}\|_{2,2}$ is very small. Set

$$H^0_{\mathsf{C}_0}:=\ker\mathfrak{L}_{\mathsf{C}_0},\ \ H^1_{\mathsf{C}_0}:=\Big\{\dot{\mathsf{C}}\in\mathcal{C}_{\mathsf{C}};\ \ \underline{SW}(\dot{\mathsf{C}})=0,\ \ \mathfrak{L}^*_{\mathsf{C}_0}\dot{\mathsf{C}}=0\Big\}$$

and denote by $\Pi_1:\mathcal{S}_{\mathsf{C}_0}\to H^1_{\mathsf{C}_0}$ the L^2-orthogonal projection. Observe that

$$\ker\mathcal{T}_{\mathsf{C}_0}=H^1_{\mathsf{C}_0}\oplus H^0_{\mathsf{C}_0}.$$

For every $r>0$ we set

$$B_{\mathsf{C}}(r):=\{\dot{\mathsf{C}}\in H^1_{\mathsf{C}};\ \ \|\dot{\mathsf{C}}\|_{L^2}<r\}.$$

The equation (2.2) is equivalent to the pair of equations

$$(1-\Pi_1)\Big(SW(\mathsf{C}_0+\dot{\mathsf{C}})\Big)=0,\ \ \dot{\mathsf{C}}\in\mathcal{S}_{\mathsf{C}_0},\ \ \|\dot{\mathsf{C}}\|_{2,2}\leq\varepsilon. \tag{$\dagger_\varepsilon$}$$

$$\Pi_1\Big(SW(\mathsf{C}_0+\dot{\mathsf{C}})\Big)=0,\ \ \dot{\mathsf{C}}\in\mathcal{S}_{\mathsf{C}_0},\ \ \|\dot{\mathsf{C}}\|_{2,2}\leq\varepsilon. \tag{$\dagger\dagger_\varepsilon$}$$

The local structure of $(\dagger_\varepsilon)$ can be easily analyzed using the implicit function theorem. Our next result states that the solution set of $(\dagger_\varepsilon)$ can be represented as the graph of a $\mathbf{Stab}(\mathsf{C}_0)$-equivariant map

$$\Phi_1:H^1_{\mathsf{C}_0}\to\ker\Pi_1$$

tangent to $H^1_{\mathsf{C}_0}$ at 0. We have the following result.

Proposition 2.2. *Suppose C_0 is a smooth 3-monopole. There exist $r_0 = r_0(\mathsf{C}_0) > 0$, $\varepsilon = \varepsilon(\mathsf{C}_0)$, $\nu = \nu(\mathsf{C}_0) > 0$ and a smooth $\mathbf{Stab}(\mathsf{C})$-equivariant map*

$$\Phi_1 : B_{\mathsf{C}_0}(r_0) \to \ker(1 - \Pi_1)\mathcal{S}_{\mathsf{C}_0}$$

satisfying the following requirements.

(i) $\Phi_1(0) = 0$.

(ii) Any solution $\dot{\mathsf{C}}'$ of $(\dagger_\varepsilon)$ decomposes as

$$\dot{\mathsf{C}}' = \dot{\mathsf{C}} \oplus \Phi_1(\dot{\mathsf{C}})$$

where $\dot{\mathsf{C}} = \Pi_1\dot{\mathsf{C}}' \in B_{\mathsf{C}_0}(r_0)$. In particular

$$(\mathbf{1} - \Pi_1)\Big(SW\big(\mathsf{C} + \dot{\mathsf{C}} + \Phi_1(\dot{\mathsf{C}}) \big) + \mathfrak{L}_{\mathsf{C}}\Phi_0(\dot{\mathsf{C}})\Big) = 0,$$

$\forall \dot{\mathsf{C}} \in B_{\mathsf{C}}(r)$.

(iii) $\|\Phi_1(\dot{\mathsf{C}})\|_{2,2} \leq \nu\|\dot{\mathsf{C}}\|^2$, $\|D_{\dot{\mathsf{C}}}\Phi_1(v)\|_{2,2} \leq C\|v\| \cdot \|\dot{\mathsf{C}}\|$, $\forall\, v, \dot{\mathsf{C}} \in H^1_{\mathsf{C}_0}$. *($H^1_{\mathsf{C}_0}$ is a finite dimensional spaces and thus all norms on it are equivalent.)*

Set

$$Q_{\mathsf{C}_0} : B_{\mathsf{C}_0}(r_0) \to H^1_{\mathsf{C}_0},\;\; \dot{\mathsf{C}} \mapsto \Pi_1 SW(\mathsf{C}_0 + \dot{\mathsf{C}} + \Phi_1(\dot{\mathsf{C}})).$$

Q_{C_0} is called the *Kuranishi map* at C_0. It is a $\mathbf{Stab}(\mathsf{C}_0)$-equivariant map and the above discussion shows that $Q^{-1}_{\mathsf{C}_0}/\mathbf{Stab}(\mathsf{C}_0)$ is homeomorphic to a neighborhood of C_0 in $\mathfrak{M}_\sigma$.

The reducible monopoles will play an important role in the sequel and that is why we want to describe in details some of their more salient features.

Suppose $\mathsf{C} = (0, A)$ is a reducible monopoles. We deduce

$$*F_A + \mathbf{i}\eta = 0 \Longleftrightarrow F_A + \mathbf{i} * \eta = 0.$$

We write $A = A_0 + \mathbf{i}a$, $a \in \Omega^1(M)$ and the last equality becomes

$$F_{A_0} + \mathbf{i}da + \mathbf{i} * \eta = 0.$$

The two form $*\eta$ is closed so that it can be represented as

$$*\eta = [*\eta] + d\beta = *[\eta] + d\beta,\;\; \beta \in \Omega^1(M).$$

We deduce

$$\Big(F_{A_0} + \mathbf{i}[*\eta]\Big) + \mathbf{i}(da + d\beta) = 0.$$

The first term in the left-hand-side of the above equality is a harmonic two form while the other is exact. Using the Hodge decomposition we deduce

$$[*\eta] = \mathbf{i}F_{A_0} = 2\pi c_\sigma$$

and

$$da + d\beta = 0.$$

Proposition 2.3. *Reducible monopoles exist if and only if*

$$[*\eta] = 2\pi c_\sigma. \tag{2.3}$$

Moreover, if non-empty, the space $\mathfrak{M}_\sigma^{red}$ *of equivalence classes of reducible monopoles is isomorphic to the* b_1*-dimensional torus*

$$H^1(M, \mathbb{R})/H^1(M, 4\pi\mathbb{Z}).$$

Proof Set $\eta := [\eta] + d\beta$The first part follows immediately from the observations preceding the proposition.

Suppose now $\mathfrak{M}_\sigma^{red} \neq \emptyset$. We can identify this space with the space of equivalence classes of solutions $a \in \Omega^(M)$ of the equation

$$da = -d\beta$$

modulo the equivalence relation

$$a_1 \sim a_2 \Longleftrightarrow \mathbf{i}a_1 = \mathbf{i}a_2 - \frac{2d\gamma}{\gamma}, \;\; \gamma \in \mathcal{G}.$$

By choosing one particular solution a_0 of this equation we can represent all the others as a_0 + closed one form. As γ describes $\mathcal{G}$ the family $-\frac{2d\gamma}{\gamma}$ describes all imaginary, closed 1-forms with cohomology class in $H^1(M, 4\pi\mathbb{Z})$. $\blacksquare$

Denote by $\mathcal{N}$ the set of co-closed 1-forms on M of regularity $L^{k,2}$, $k \gg 1$. Set

$$\mathcal{W}_\sigma = \mathcal{W}_\sigma(g) := \Big\{ \eta \in \mathcal{N} : \;\; [*\eta] = 2\pi c_\sigma \Big\}.$$

$\mathcal{W}_\sigma$ is a codimension b_1 affine subspace of $\mathcal{N}$.

Definition 2.4. *A reducible* (η, μ)*-monopole* $(0, A)$ *is called* **regular** *if*

$$\ker(\mathfrak{D}_A + \mu) = 0.$$

As explained in [26], the Kuranishi map determined by a regular reducible monopole is $\equiv 0$. To proceed further, we need to discuss separately three cases: $b_1(M) > 1$, $b_1(M) = 1$ and $b_1(M) = 0$.

2.2 The case $b_1 > 1$

Suppose M is connected, $b_0(M) = 1$ and $b_1(M) > 1$. Since $\operatorname{codim} \mathcal{W}_\sigma > 2$ the complement

$$\mathcal{N}^0 := \mathcal{N} \setminus \mathcal{W}_\sigma$$

is an open and connected set. According to Proposition 2.3, if $\eta \in \mathcal{N}^0$ then there are no reducible (η, μ) monopoles. For this reason we will always choose $\eta \in \mathcal{N}^0$. In this section, we will also choose $\mu \equiv 0$ and we will talk only of η-monopoles.

A Sard-Smale argument leads to the following genericity result.

Proposition 2.5. *Fix a Riemann metric g on M. Then there exists a generic subset $\mathcal{N}_g^0$ of $\mathcal{N}$ such that for every $\eta \in \mathcal{N}_g^0$ and every η-monopole C we have*

$$H^0_{\mathsf{C}} = H^1_{\mathsf{C}} = 0.$$

In particular, for these η's the moduli space $\mathfrak{M}_\sigma(g, \eta)$ consists of finitely many isolated points.

Fix a generic η as in the above proposition. Then, for every $\mathsf{C} = (\psi, A) \in \mathfrak{M}_\sigma$ the self-adjoint Fredholm operator $\mathcal{T}_{\mathsf{C}}$ is invertible and thus $\det \operatorname{ind} \mathcal{T}_{\mathsf{C}}$ admits a natural orientation $\mathbf{or}_0$.

On the other hand $\det \operatorname{ind} \mathcal{T}^0\mathsf{C}$ has a natural orientation defined as follows. Observe that

$$\mathcal{T}^0_{\mathsf{C}} = \mathfrak{D}_A \oplus -\mathbf{SIGN}$$

where $\mathbf{SIGN} : \Omega^1(M) \oplus \Omega^0(M)$ is the odd-signature operator

$$\mathbf{SIGN} = \left[\begin{array}{cc} *d & -d \\ -d^* & 0 \end{array} \right]$$

Thus

$$\ker \mathcal{T}^0_{\mathsf{C}} \cong \operatorname{coker} \mathcal{T}^0_{\mathsf{C}} \cong \ker \mathfrak{D}_A \oplus \ker \mathbf{SIGN} = \ker \mathfrak{D}_A \oplus H^1(M) \oplus H^0(M).$$

and thus $\det \operatorname{ind} \mathcal{T}^0_{\mathsf{C}}$ admits a natural orientation determined by fixing an orientation on $H^*(M, \mathbb{R})$. We choose the natural one determined by the complex structure on $H^*(M)$ induced by the Hodge $*$-operator. Using the affine path

$$\mathcal{T}^t_{\mathsf{C}} := \mathcal{T}^0_{\mathsf{C}} + t\mathcal{P}_{\mathsf{C}}$$

connecting $\mathcal{T}^0_{\mathsf{C}}$ to $\mathcal{T}_{\mathsf{C}}$ we can transport the orientation on $\det \operatorname{ind} \mathcal{T}^0_{\mathsf{C}}$ to an orientation $\mathbf{or}_1$ on $\det \operatorname{ind} \mathcal{T}_{\mathsf{C}}$. The two orientations $\mathbf{or}_0$ and $\mathbf{or}_1$ differ by a sign which we denote by $\epsilon(\mathsf{C}) = \pm 1$. Using the orientation transport formula [26, Sec. 1.5.1] we deduce that this sign can be alternatively by

$$\epsilon(\mathsf{C}) = (-1)^{SF_{\mathsf{C}} + d_0}$$

where SF_{C} denotes the spectral flow along the path $\mathcal{T}^t_{\mathsf{C}}$, $0 \leq t \leq 1$, and d_0 denotes the dimension over $\mathbb{R}$ of the kernel of $\mathcal{T}^0_{\mathsf{C}}$. Clearly $d_0 \equiv 1 + b_1 \mod 2$ so that we deduce

$$\epsilon(\mathsf{C}) = (-1)^{SF_{\mathsf{C}} + b_1 + 1}. \tag{2.4}$$

Define

$$\mathbf{sw}_M(\sigma, g, \eta) = \sum_{\mathsf{C} \in \mathfrak{M}_\sigma(g, \eta)} \epsilon(\mathsf{C}).$$

As in [26, Sec. 2.3] one can show that the above count is independent of the choices (g, η) and thus defines a smooth invariant

$$\mathbf{sw}_M : Spin^c(M) \to \mathbb{Z}$$

called the *Seiberg-Witten* invariant of M.

The above function has finite support. Moreover, this function is symmetric with respect to the natural involution $\sigma \mapsto \bar{\sigma}$ on $Spin^c(M)$,

$$\mathbf{sw}_M(\sigma) = \mathbf{sw}_M(\bar{\sigma}).$$

2.3 The case $b_1 = 1$

Suppose now that M is connected but $b_1(M) = 1$. In this case $\mathcal{W}_\sigma(g)$ is a codimension 1 affine subspace of $\mathcal{N}$, i.e. a hyperplane called the *wall.* The complement of the wall is thus disconnected. In this case it is convenient to give a more computational description of the wall.

Fix an orientation of the one dimensional real vector space $H^1(M;\mathbb{R})$. Thus there exists an unique g-harmonic 1-form ω such that $\|\omega\|_{L^2(g)} = 1$ and which induces the same orientation on $H^1(M,g)$. The wall $\mathcal{W}_\sigma(g)$ can be described by the linear equation

$$\langle \eta - 2\pi * c_\sigma, \omega\rangle_{L^2} = 0.$$

We set

$$\mathcal{N}^\pm(g) = \Big\{\eta \in \mathcal{N};\ \ \pm\langle \eta - 2\pi * c_\sigma, \omega\rangle_{L^2} > 0\Big\}.$$

$\mathcal{N}^\pm(g)$ is called the positive/negative chamber. Observe that

$$\mathcal{N}(g) \setminus \mathcal{W}_\sigma(g) = \mathcal{N}^+(g) \cup \mathcal{N}^-(g).$$

Set

$$\mathcal{N} = \bigcup_g \{g\} \times \mathcal{N}(g),\ \ \mathcal{N}^\pm = \bigcup_g \{g\} \times \mathcal{N}^\pm(g).$$

For generic (g,η) in one of the chambers we get a finite set of irreducible monopoles, and no reducibles and we can count them as before to obtain an integer $\mathbf{sw}^\pm_M(\sigma, g, \eta)$. Moreover $\mathbf{sw}^\pm_M(\sigma, g, \eta)$ is independent of the generic pairs (g,η) in the same chamber and thus we get two functions

$$\mathbf{sw}^\pm_M : Spin^c(M) \to \mathbb{Z}.$$

To understand the relationship between these two functions we need to pick two parameters $(g_{\pm 1}, \eta_{\pm 1}) \in \mathcal{N}^\pm$ and a suitable path

$$(g_s, \eta_s)_{s\in[-1,1]} \in \mathcal{N}$$

connecting them. We get a parametrized moduli space

$$\tilde{\mathfrak{M}} = \Big\{(s, \mathsf{C}) \in [-1,1] \times \mathcal{C}_\sigma;\ \ \mathsf{C} \text{ is a } (g_s,\eta_s) \text{ monopole}\Big\}/\mathcal{G} \subset \mathcal{B}_\sigma.$$

We assume the path $t \mapsto (g_t, \eta_t)$ crosses the wall only once in a very special fashion.

First, we assume g_s is independent of s close to $0, \pm 1$ and we set $g := g_0$. Next, we assume the path crosses the wall $\mathcal{W}_\sigma(g)$ transversally coming from the negative chamber towards the positive one

$$\frac{d}{ds}\,|_{s=0}\ \langle \eta_s, \omega_g\rangle > 0.$$

Again this is a compact metric subspace of

$$\tilde{\mathcal{B}}_\sigma = \bigcup_s \{s\} \times \mathcal{B}_\sigma(g_t).$$

To understand the local structure of $\tilde{\mathfrak{M}}_\sigma$ near (s_0, C_0) we need to introduce the parametrized Seiberg-Witten map

$$\widetilde{SW} : [-1,1] \times \mathcal{C}_\sigma \to \mathcal{C}_\sigma, \;\; (s, \mathsf{C}) = (s; \psi, A) \mapsto \Big(\mathfrak{D}_A\psi, \frac{1}{2}\boldsymbol{c}^{-1}\big(\, q(\psi)\,\big) - (*_s F_A + \mathbf{i}\eta_s)\Big)$$

and study the small solutions $(\dot{s}, \dot{\mathsf{C}})$ of the equation

$$\left\{\begin{array}{l} \widehat{SW}(s_0 + \dot{s}, \mathsf{C}_0 + \dot{\mathsf{C}}) = 0 \\ \\ \qquad \mathfrak{L}^*_{\mathsf{C}_0}\dot{\mathsf{C}} = 0 \end{array}\right. \tag{2.5}$$

We can choose the path $s \mapsto (g_s, \eta_s)$ generically such that whenever $s_0 \neq 0$ (i.e. η_{s_0} is not on the wall) a neighborhood of (s_0, C_0) in $\tilde{\mathfrak{M}}_\sigma$ is diffeomorphic to $\mathbb{R}$. Thus, away from the reducibles we can assume that the parametrized moduli space is a finite union of paths. One should think of this parametrized space as a smooth cobordism between the moduli spaces $\mathfrak{M}_\sigma(g_{\pm 1}, \eta_\pm)$. The singularities arise when η_s crosses the wall, i.e. when $s = 0$. To make further progress we need to understand how the reducible part $\tilde{\mathfrak{M}}^{red}_\sigma$ sits inside $\tilde{\mathfrak{M}}_\sigma$. We will achieve this using a little perturbation theory.

Observe first that

$$\tilde{\mathfrak{M}}^{red}_\sigma = \{0\} \times \mathfrak{M}^{red}_\sigma(g, \eta_0) \cong S^1.$$

If $\mathsf{C} \in \mathfrak{M}^{red}_\sigma$ is a regular reducible monopole then its Kuranishi map is $\equiv 0$ and thus a neighborhood of $(0, \mathsf{C}) \in \tilde{\mathfrak{M}}_\sigma$ looks like a neighborhood of a point inside the circle of reducibles. The local structure problem is therefore interesting only at the non-regular monopoles. As in [17] we can choose the path (g_s, η_s) carefully so that there are only finitely many irregular reducibles which however are only *mildly irregular*. A mildly irregular reducible monopole is by definition a monopole $(0, A)$ such that $\dim_{\mathbb{C}} \ker \mathfrak{D}_A = 1$. Here we need to more explicit because the Dirac operator $\mathfrak{D}_A$ is not $\mathcal{G}$-invariant.

First, by rescaling the metric g we can assume that ω_g is a generator of the lattice $H^1(M, \mathbb{Z})/Tors \subset H^1(M, \mathbb{R})$. The circle of reducibles $\mathfrak{M}_\sigma(g, \eta_0)$ can be described in $\mathcal{C}_\sigma$ as a path

$$(0, A_t := A_0 + \mathbf{i}t\omega + \mathbf{i}df_t), \;\; t \in [0, 4\pi], \;\; f_t \in \Omega^0(M).$$

The results of [17] state that for generic choices of the path f_t $\ker \mathfrak{D}_{A_t}$ is nontrivial only for finitely many t's and when this happens we have $\dim_{\mathbb{C}} \ker \mathfrak{D}_{A_t} = 1$. For clarity purposes we will assume that $f_t = 0$, $\forall t$. The general case requires no new ideas.

The above singular cobordism consists of finitely many real analytic paths, some of which approach a mildly irregular reducible.

By slightly perturbing the path (g_s, η_s) we can assume that for every mildly irregular monopole $(0, A)$ we have

$$\kappa := \langle \boldsymbol{c}(\mathbf{i}\omega_g)\Phi, \Phi\rangle_{L^2} \neq 0, \tag{$*$}$$

where Φ denotes an unitary spanning vector of the one-dimensional space $\ker \mathfrak{D}_A$.

To understand the local structure near a mildly irregular reducible we will use the Kuranishi deformation technique. Suppose $(0, \mathsf{C}_0)$ is a (mildly irregular) reducible. We can write the equation (2.5) in the form

$$\mathcal{F}(s, \mathsf{C}) = 0$$

where $\mathcal{F} : \mathbb{R} \times \mathcal{S}_{\mathsf{C}_0} \to \mathcal{S}_{\mathsf{C}_0}$ is given by

$$\mathcal{F}(\dot{s}, \dot{\mathsf{C}}) = \Pi\Big(\widetilde{SW}(\dot{s}, \mathsf{C}_0 + \dot{\mathsf{C}})\Big)$$

where Π is the L^2-orthogonal projection onto the slice. Its linearization $\underline{\mathcal{F}}$ at $(0, \mathsf{C}_0, 0)$ acts according to the rule

$$\underline{\mathcal{F}} \left[\begin{array}{c} \dot{s} \\ \dot{\psi} \\ \mathbf{i}\dot{a} \end{array} \right] = \left[\begin{array}{cc} \mathfrak{D}_{A_0} & 0 \\ 0 & - * d \end{array} \right] \cdot \left[\begin{array}{c} \dot{\psi} \\ \mathbf{i}\dot{a} \end{array} \right] + \left[\begin{array}{c} 0 \\ -\mathbf{i}\dot{s}\eta'(0) \end{array} \right], \;\; d^*\dot{a} = 0.$$

We assume

$$\nu := \langle \eta'(0), \omega_g \rangle_{L^2} > 0. \tag{$**$}$$

The kernel of $\underline{\mathcal{F}}$ consists of triples $(\dot{s}, \dot{\psi}, \mathbf{i}\dot{a})$ such that

$$\mathfrak{D}_{A_0}\dot{\psi} = 0$$

and

$$\begin{cases} - * d\dot{a} = \dot{s}\eta'(0) \\ \\ d^*\dot{a}^* = 0 \end{cases}$$

The first equation has a (complex) one dimensional space of solutions spanned by Φ.

Since $[*d\dot{a}] = 0$ and $[\eta'(0)] \neq 0$ (by $(**)$) we deduce $\dot{s} = 0$. In particular, this implies $\dot{a}$ is a harmonic 1-form. We deduce that any vector $\vec{v} \in \ker \underline{\mathcal{F}}$ can be uniquely written as

$$(0, z\Phi, \mathbf{i}\lambda\omega_g), \;\; z \in \mathbb{C}, \;\; \lambda \in \mathbb{R}.$$

The cokernel consists of vectors

$$(\phi, \mathbf{i}b) \in \Gamma(\mathbb{S}_\sigma) \times \mathbf{i}\Omega^1(M), \;\; d^*b = 0$$

such that

$$\langle \mathfrak{D}_{A_0}\dot{\psi}, \phi \rangle + \langle - * d\mathbf{i}\dot{a} - \mathbf{i}\dot{s}\eta'(0), \mathbf{i}b \rangle = 0 \tag{2.6}$$

$\forall \dot{s}, \dot{\psi}, \mathbf{i}\dot{a}$. We deduce

$$\phi \in \ker \mathfrak{D}_{A_0}.$$

By letting $\dot{\psi} = 0$ and $\dot{s} = 0$ in (2.6) we deduce that b must be harmonic and

$$\langle \dot{s}\eta'(0), b \rangle = 0, \;\; \forall \dot{s}.$$

Since $[\eta'(0)] \neq 0$ we conclude that $b = 0$. Thus any vector $\vec{w}$ in the cokernel of $\underline{\mathcal{F}}$ can be represented as

$$\vec{w} = (\zeta\Phi, 0), \;\; \zeta \in \mathbb{C}, \;\; v \in \mathbb{R}.$$

Denote by $\mathcal{P}$ the orthogonal projection onto the kernel of $\underline{\mathcal{F}}$ and by $\mathfrak{Q}$ the orthogonal projection onto the cokernel. Observe that

$$\mathcal{P} \left[\begin{array}{c} \dot{s} \\ \dot{\psi} \\ \mathbf{i}\dot{a} \end{array} \right] = \left[\begin{array}{c} 0 \\ \\ \langle \dot{\psi}, \Phi \rangle \Phi \\ \\ \mathbf{i}[\dot{a}] \end{array} \right]$$

Similarly, the orthogonal projection $\mathfrak{Q}$ is given by

$$\mathfrak{Q}\begin{bmatrix} \dot{\psi} \\ \mathbf{i}\dot{a} \end{bmatrix} = \begin{bmatrix} \langle \dot{\psi}, \Phi\rangle\Phi \\ \\ 0 \end{bmatrix}$$

Set $\mathcal{P}^{\perp} := 1 - \mathcal{P}$, $\mathfrak{Q}^{\perp} := 1 - \mathfrak{Q}$. Denote the vectors $(\dot{s}, \dot{\psi}, \mathbf{i}\dot{a}, \mathbf{i}f)$ in the domain of $\underline{\mathcal{F}}$ by Ξ. The equation

$$\mathcal{F}(\mathsf{C}_0 + \Xi) = 0$$

can be rewritten as

$$\underline{\mathcal{F}}(\Xi) + R(\Xi) = 0 \tag{2.7}$$

where R is the nonlinear remainder,

$$R\begin{bmatrix} \dot{s} \\ \dot{\psi} \\ \mathbf{i}\dot{a} \\ \mathbf{i}f \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\boldsymbol{c}(\mathbf{i}\dot{a})\dot{\psi} - \frac{\mathbf{i}}{2}f\dot{\psi} \\ \\ \frac{1}{2}q(\dot{\psi}) - \mathbf{i}\eta(\dot{s}) + \mathbf{i}\dot{s}\eta'(0) \\ \\ 0 \end{bmatrix}$$

Decompose

$$\Xi := \Xi_0 + \Xi^{\perp},\ \ \Xi_0 := \mathcal{P}\Xi,\ \ \Xi^{\perp} = \mathcal{P}^{\perp}\Xi.$$

The equation (2.7) is equivalent to the pair of equations

$$\underline{\mathcal{F}}(\Xi^{\perp}) + \mathfrak{Q}^{\perp}R(\Xi) = 0 \tag{2.8a}$$

$$\underline{\mathcal{F}}(\Xi^{\perp}) + \mathfrak{Q}R(\Xi) = 0 \tag{2.8b}$$

For each sufficiently small Ξ_0 we can solve the first equation for $\Xi^{\perp}$ and we obtain a smooth function

$$\Xi_0 \mapsto \Xi^{\perp}(\Xi_0).$$

Using the coordinates (z, λ) on $\ker \underline{\mathcal{F}}$ we can regard $\Xi^{\perp}$ as a function of (z, λ),

$$\Xi^{\perp} = \Xi^{\perp}(z, \lambda).$$

Note that $\Xi^{\perp}(0) = 0$ and $D\mid_{\Xi_0=0}\Xi^{\perp} = 0$. Thus $\Xi_0 = 0$ is a zero of order at least 2 of $\Xi^{\perp}$. We can extract even more precise information.

Observe that

$$\mathfrak{Q}^{\perp}R(\Xi) = \begin{bmatrix} \frac{1}{2}\boldsymbol{c}(\mathbf{i}\dot{a})\dot{\psi} - \langle\frac{1}{2}\boldsymbol{c}(\mathbf{i}\dot{a})\dot{\psi}, \Phi\rangle\Phi \\ \\ \frac{1}{2}q(\dot{\psi}) - \mathbf{i}(\eta(\dot{s}) - \dot{s}\eta'(0) - \eta(0)) \end{bmatrix}$$

so that (2.8a) can be rewritten

$$\begin{cases} \mathfrak{D}_{A_0}\dot{\psi}^{\perp} + \frac{1}{2}\boldsymbol{c}(\mathbf{i}\dot{a})\dot{\psi} - \langle\frac{1}{2}\boldsymbol{c}(\mathbf{i}\dot{a})\dot{\psi}, \Phi\rangle\Phi = 0 \\ \\ -\mathbf{i} * d\dot{a}^{\perp} - \mathbf{i}\dot{s}\eta'(0) - \mathbf{i}\big(\eta(\dot{s}) - \eta(0) - \dot{s}\eta'(0)\big) + \frac{1}{2}q(\dot{\psi}) = 0 \\ \\ d^*\dot{a}^{\perp} = 0 \end{cases} \tag{2.9}$$

$$\Xi^{\perp} = \begin{bmatrix} \dot{s} \\ \\ \dot{\psi}^{\perp} \\ \\ \mathbf{i}\dot{a}^{\perp} \end{bmatrix}.$$

When we take the inner product of the second equation in (2.9) with $\mathbf{i}\omega_g$ we get

$$2\langle \eta(\dot{s}) - \eta(0), \omega_g \rangle = \frac{1}{2}\langle q(\dot{\psi}), c(\mathbf{i}\omega_g) \rangle = \frac{1}{2}\langle \boldsymbol{c}(\mathbf{i}\omega_g)\dot{\psi}, \dot{\psi} \rangle$$

Since

$$\eta(\dot{s}) = \eta(0) + \dot{s}\eta'(0) + o(\dot{s}^2)$$

and

$$\dot{\psi} = z\Phi + \dot{\psi}^{\perp} = z\Phi + \dot{\psi}^{\perp}(z, \lambda)$$

we deduce from $(*)$ and $(**)$ that

$$2\dot{s}\nu + o(\dot{s}^2) = \frac{1}{2}\Big\langle c(\mathbf{i}\omega_g)(z\Phi + \dot{\psi}^{\perp}(z,\lambda)), z\Phi + \dot{\psi}^{\perp}(z,\lambda) \Big\rangle = \frac{1}{2}\kappa |z|^2 + O(3)$$

where $O(3)$ denotes generically a term with a zero of order at least 3 at $(z, \lambda) = (0, 0)$. Hence

$$\dot{s} = \frac{\kappa}{4\nu}|z|^2 + O(3). \tag{2.10}$$

The equations (2.8b) have the explicit form

$$F(z, \lambda) := \Big\langle \boldsymbol{c}(\mathbf{i}\lambda\omega_g + \mathbf{i}\dot{a}^{\perp})(z\Phi + \dot{\psi}^{\perp}), \Phi \Big\rangle - = 0. \tag{2.11}$$

The Kuranishi map K of $(0, \mathsf{C}_0)$ is given by

$$(z, \lambda) \mapsto F(z, \lambda, u).$$

Notice that

$$F(z, \lambda) = \kappa\lambda z + O(3). \tag{2.12}$$

The following proposition summarizes the facts established so far.

Proposition 2.6. *A neighborhood of the mildly irregular reducible $(0, \mathsf{C}_0)$ inside the parameterized moduli space $\tilde{\mathfrak{M}}_\sigma$ is homeomorphic to*

$$\Big\{(z, \lambda);\ F(\lambda, z) = 0\Big\}/S^1.$$

where S^1 acts on the component z by complex multiplication.

Using the estimate (2.12) we deduce that a neighborhood of a mildly irregular reducible in $\tilde{\mathfrak{M}}_\sigma$ looks like a neighborhood of 0 in $(\mathbb{C} \times \mathbb{R})/S^1$ where S^1 acts on the first component by complex multiplication. The singular cobordism $\tilde{\mathfrak{M}}_\sigma$ then looks roughly like in Figure 1.

To understand the difference $\mathbf{sw}^+ - \mathbf{sw}^-$ we need to understand the orientation of the above singular cobordism. The irreducible part of the parametrized moduli space $\tilde{\mathfrak{M}}_\sigma$ consists of two types of arcs.

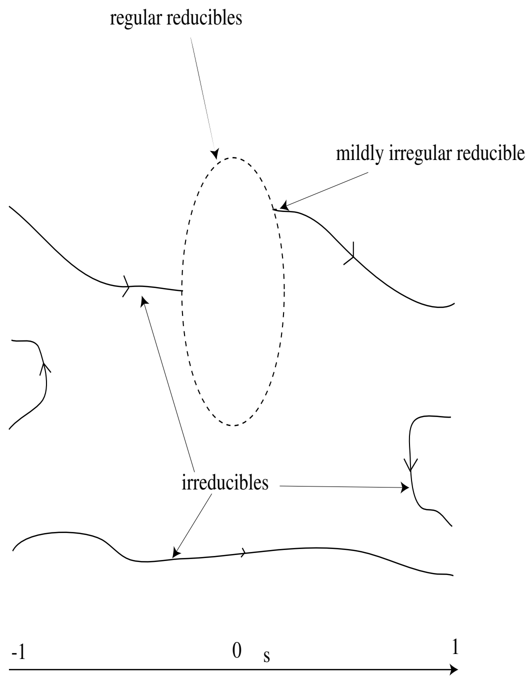

Figure 1: *A singular one dimensional cobordism in the case* $b_1 = 1$

• **Good arcs**, i.e. arcs $[-1,1] \ni t \mapsto \big(s_j(t), \mathsf{C}_j(t)\big)$ $1 \le j \le n$ not approaching the reducibles.
• **Bad arcs**, i.e. arcs $[-1,0] \ni t \mapsto (s_k(t), \mathsf{C}_k(t))$ $n < k \le m+n$ approaching a mildly irregular reducible as $t \to 1$. (Only the last monopole $\mathsf{C}_k(1)$ is reducible.)

For every good arc we have $s_j(-1) = \pm 1$ and $s_j(1) = \pm 1$. Note that

$$\mathbf{sw}^+_M(\sigma) - \mathbf{sw}^-_M(\sigma) = \sum_{j=1}^{n} \Big\{ s_j(-1)\epsilon\big(\mathsf{C}_j(-1)\big) + s_j(1)\epsilon\big(\mathsf{C}_j(1)\big)\Big\} + \sum_{k=n+1}^{n+m} s_k(-1)\epsilon\big(C_k(-1)\big).$$

As in [26, Sec. 2.3] we can show that the first sum is zero so that

$$\mathbf{sw}^+_M(\sigma) - \mathbf{sw}^-_M(\sigma) = \sum_{k=n+1}^{n+m} s_k(-1)\epsilon\big(C_k(-1)\big). \tag{2.13}$$

Suppose $t \mapsto (s(t), \mathsf{C}_t) \in [-1,1] \times \mathcal{C}_\sigma$, $|t| \le 1$ is a bad arc path such that $(s(0), \mathsf{C}_0) = (0, \mathsf{C}_0)$. is a mildly irregular reducible. We assume that for t close to zero the configuration C_t is in the local slice at C_0. Set $\epsilon := \epsilon(\mathsf{C}_{-1}) = \pm 1$. As in [26, Sec. 2.3], the sign ε is given by the orientation transport along the path of Fredholm selfadjoint operators

$$[0,1] \ni \tau \mapsto \mathcal{T}_\tau := \mathcal{T}_{\mathsf{C}_{-\tau}}.$$

Our assumptions guarantee that the only contribution to the orientation transport occur at $\tau = 0$ and thus we need to understand $\mathcal{T}_\tau$ for very small τ. We write

$$s = \dot{s}t + \ddot{s}t^2 + o(t^2), \;\; \mathsf{C} = \mathsf{C}_0 + t\dot{\mathsf{C}} + t^2\ddot{\mathsf{C}} + o(t^2),$$

or more explicitly,

$$\psi_t = t\dot{\psi} + t^2\ddot{\psi} + o(t^2), \;\; A = A_0 + t\mathbf{i}\dot{a} + t^2\mathbf{i}\ddot{a} + o(t^2).$$

From the estimate (2.10) we deduce that s is quadratic in t. We assume the parametrization $t \mapsto (s(t), \mathsf{C}_t)$ is nondegenerate at $t = 0$, i.e.

$$\ddot{s} \neq 0. \tag{$***$}$$

Observe that

$$\mathfrak{T}_\tau \begin{bmatrix} \mathbf{i}\underline{\psi} \\ \mathbf{i}\underline{a} \\ \mathbf{i}\underline{f} \end{bmatrix} = \begin{bmatrix} \mathfrak{D}_{A_{-\tau}} & 0 & \\ & & \\ 0 & -*d & d \\ & & \\ 0 & d^* & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{i}\underline{\psi} \\ \mathbf{i}\underline{a} \\ \mathbf{i}\underline{f} \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\boldsymbol{c}(\mathbf{i}\underline{a})\psi_{-\tau} - \frac{\mathbf{i}}{2}\underline{f}\psi_{-\tau} \\ \\ \frac{1}{2}\dot{q}(\psi_{-\tau}, \underline{\psi}) \\ \\ \frac{\mathbf{i}}{2}\,\mathbf{Im}\,\langle \psi_{-\tau}, \underline{\psi}\rangle \end{bmatrix}$$

where

$$\dot{q}(\psi, \phi) := \frac{d}{dt}\,|_{t=0}\; q(\psi + t\phi).$$

Set $\dot{\mathfrak{T}} := \frac{d}{d\tau}\,|_{\tau=0}\,\mathfrak{T}_\tau$. We deduce that

$$\dot{\mathfrak{T}} \begin{bmatrix} \mathbf{i}\underline{\psi} \\ \\ \mathbf{i}\underline{a} \\ \\ \mathbf{i}\underline{f} \end{bmatrix} = - \begin{bmatrix} \frac{1}{2}\boldsymbol{c}(\mathbf{i}\dot{a})\underline{\psi} + \frac{1}{2}\boldsymbol{c}(\mathbf{i}\underline{a})\dot{\psi} - \frac{\mathbf{i}}{2}\underline{f}\dot{\psi} \\ \\ \frac{1}{2}\dot{q}(\dot{\psi}, \underline{\psi}) \\ \\ \frac{\mathbf{i}}{2}\,\mathbf{Im}\,\langle \dot{\psi}, \underline{\psi}\rangle \end{bmatrix}.$$

Denote by K the kernel of $\mathfrak{T}_0$

$$K \cong \ker \mathfrak{D}_{A_0} \oplus H^1(M, g) \oplus H^0(M, g),$$

and by P the orthogonal projection onto K. To understand the orientation transport we need to understand the resonance operator

$$R : K \to K, \;\; R\vec{k} = P\dot{\mathfrak{T}}\vec{k}.$$

We use the coordinates (z, λ, u) on K,

$$(z, \lambda, u) \mapsto (z\Phi, \mathbf{i}\lambda\omega_g, \mathbf{i}u).$$

We have

$$\dot{\mathfrak{T}} \begin{bmatrix} z \\ \lambda \\ u \end{bmatrix} = - \begin{bmatrix} \frac{z}{2}\boldsymbol{c}(\mathbf{i}\dot{a})\Phi + \frac{1}{2}\boldsymbol{c}(\mathbf{i}\lambda\omega)\dot{\psi} - \frac{\mathbf{i}}{2}u\dot{\psi} \\ \\ \frac{1}{2}\dot{q}(\dot{\psi}, z\Phi) \\ \\ \frac{\mathbf{i}}{2}\,\mathbf{Im}\,\langle \dot{\psi}, z\Phi\rangle \end{bmatrix}. \tag{2.14}$$

To gain some more insight we need to learn more about $(\dot{\psi}, \mathbf{i}\dot{a})$. We will achieve this using perturbation techniques.

For t close to zero the configuration C_t satisfies the equation

$$\begin{cases} \widehat{SW}(s_0 + t\dot{s} + t^2\ddot{s}, \mathsf{C}_0 + t\dot{\mathsf{C}} + t^2\ddot{\mathsf{C}} + o(t^2)) = 0 \\ \\ \mathfrak{L}^*_{\mathsf{C}_0}(t\dot{\mathsf{C}} + t^2\ddot{\mathsf{C}} + o(t^2)) = 0 \end{cases}.$$

The last equation is equivalent to

$$\mathfrak{D}_{A_0 + t\mathbf{i}\dot{a} + t^2\mathbf{i}\ddot{a} + \cdots, g}(t\dot{\psi} + t^2\ddot{\psi}) = 0,$$

and

$$*_g(F_{A_0} + \mathbf{i}td\dot{a} + \mathbf{i}t^2 d\ddot{a} + \cdots) = \frac{1}{2}q(t\dot{\psi} + t^2\ddot{\psi} + \cdots)$$

$$-\mathbf{i}\Big\{\eta(0) + \dot{s}t\eta'(0) + \Big(\frac{\dot{s}^2}{2}\eta''(0) + \ddot{s}\eta'(0)\Big)t^2 + \cdots\Big\}$$

$$\mathfrak{L}^*_{\mathsf{C}_0}\dot{\mathsf{C}} = 0 \Longleftrightarrow d^*\dot{a} = 0$$

$$\mathfrak{L}^*_{\mathsf{C}_0}\ddot{\mathsf{C}} = 0 \Longleftrightarrow d^*\ddot{a} = 0$$

where $\eta'(0)$ and $\eta''(0)$ denote the first and second s-derivatives of η_s at $s = 0$.

The first order contributions are equivalent to

$$\mathfrak{D}_{A_0}\dot{\psi} = 0. \tag{2.15a}$$

$$*d\dot{a} = -\dot{s}\eta'(0), \quad d^*\dot{a} = 0 \tag{2.15b}$$

The equality (2.15a) implies

$$\dot{\psi} = \zeta\Phi, \quad \zeta \in \mathbb{C}.$$

Note that since we assumed $\nu := \langle \eta'(0), \omega_g \rangle_{L^2} > 0$ we deduce that $[\eta'(0)] \neq 0$. Since $[*d\dot{a}] = 0$ we deduce $\dot{s} \equiv 0$. Thus, (2.15b) is equivalent to

$$d\dot{a} = 0, \quad d^*\dot{a} = 0. \tag{2.16}$$

Thus $\dot{a}$ is a harmonic 1-form and thus can be uniquely represented as $\rho\omega_g$, $\rho \in \mathbb{R}$.

The second order contributions are equivalent to

$$\mathfrak{D}_{A_0}\ddot{\psi} + \frac{1}{2}\boldsymbol{c}(\mathbf{i}\dot{a})\dot{\psi} = 0. \tag{2.17a}$$

$$\mathbf{i} * d\ddot{a} = \frac{1}{2}q(\dot{\psi}) - \mathbf{i}\ddot{s}\eta'(0), \quad d^*\ddot{a} = 0. \tag{2.17b}$$

Now take the (real) inner product of (2.17b) with $\mathbf{i}\omega_g$. We obtain

$$0 = \langle *d\ddot{a}, \dot{a} \rangle_{L^2} = \frac{1}{2}\langle \boldsymbol{c}^{-1}(q(\dot{\psi})), \mathbf{i}\dot{a} \rangle_{L^2} - \ddot{s}\langle \eta'(0), \omega_g \rangle_{L^2},$$

so that

$$\ddot{s}\nu = \ddot{s}\langle \eta'(0), \omega_g \rangle_{L^2} = \frac{1}{2}\langle \boldsymbol{c}^{-1}(q(\dot{\psi})), \mathbf{i}\omega \rangle_{L^2} = \frac{1}{4}\langle q(\dot{\psi}), \boldsymbol{c}(\mathbf{i}\omega_g) \rangle_{L^2} = \frac{1}{4}\langle \boldsymbol{c}(\mathbf{i}\omega_g)\dot{\psi}, \dot{\psi} \rangle_{L^2}.$$

so that

$$\ddot{s}\nu = \frac{|\zeta|^2}{4}\kappa. \tag{2.18}$$

Using $(***)$ we deduce that $\zeta \neq 0$. Modulo a gauge transformation in the stabilizer of C_0 we can assume $\zeta \in \mathbb{R}$.

Multiplying the equation(2.17a) by $\dot{\psi}$ and integrating over M we deduce

$$\rho \langle \boldsymbol{c}(\mathbf{i}\omega)\dot{\psi}, \dot{\psi}\rangle_{L^2} = 0$$

which in view of $(*)$ implies $\rho = 0 \Longrightarrow \dot{a} = 0$. We can now rewrite (2.14) as

$$\dot{\mathfrak{T}} \begin{bmatrix} z \\ \lambda \\ u \end{bmatrix} = - \begin{bmatrix} \frac{\zeta}{2}\boldsymbol{c}(\mathbf{i}\lambda\omega)\Phi - \frac{\mathbf{i}}{2}u\zeta\Phi \\ \\ \frac{1}{2}\dot{q}(\zeta\Phi, z\Phi) \\ \\ \frac{\mathbf{i}}{2}\,\mathbf{Im}\,\langle \zeta\Phi, z\Phi\rangle \end{bmatrix}. \tag{2.19}$$

In particular

$$P\dot{\mathfrak{T}} \begin{bmatrix} z \\ \lambda \\ u \end{bmatrix} = - \begin{bmatrix} \Big\langle \big(\frac{\zeta}{2}\boldsymbol{c}(\mathbf{i}\lambda\omega)\Phi - \frac{\mathbf{i}}{2}u\zeta\Phi\big), \Phi \Big\rangle \\ \\ \frac{1}{2}\Big\langle \dot{q}(\zeta\Phi, z\Phi), \boldsymbol{c}(\mathbf{i}\omega)\Big\rangle \\ \\ \frac{1}{2}\,\mathbf{Im}\,\langle \zeta\Phi, z\Phi\Big\rangle \end{bmatrix}.$$

To understand the resonance operator $R = P\dot{\mathfrak{T}}$ we further decompose $z = a + \mathbf{i}b$. Differentiating at $t = 0$ the identity

$$\big\langle q(\psi + t\phi), T\big\rangle = \big\langle T(\psi + t\phi), \psi + t\phi\big\rangle, \ \ \forall \psi, \phi \in \Gamma(\mathbb{S}_\sigma), \ T \in \mathrm{End}_0(\mathbb{S}_\sigma),$$

we deduce

$$\langle \dot{q}(\psi, \phi), T)\rangle = \langle T\psi, \phi\rangle + \langle T\phi, \psi\rangle = 2\,\mathbf{Re}\,\langle T\psi, \phi\rangle.$$

Thus

$$\Big\langle \boldsymbol{c}^{-1}\big(\dot{q}(\zeta\Phi, z\Phi)\big), \mathbf{i}\omega\Big\rangle = \frac{1}{2}\Big\langle \dot{q}(\zeta\Phi, z\Phi), \boldsymbol{c}(\mathbf{i}\omega)\Big\rangle$$
$$= \mathbf{Re}\,\langle \zeta\boldsymbol{c}(\mathbf{i}\omega)\Phi, z\Phi\rangle = \mathbf{Re}\,(\zeta\kappa z) = \zeta\kappa a.$$

Also

$$\mathbf{Im}\,\langle \zeta\Phi, z\Phi\rangle = -\zeta b.$$

Thus

$$R \begin{bmatrix} a \\ b \\ \lambda \\ u \end{bmatrix} = - \begin{bmatrix} 0 & 0 & \frac{\zeta\kappa}{2} & 0 \\ \\ 0 & 0 & 0 & -\frac{\zeta}{2} \\ \\ \frac{\zeta\kappa}{2} & 0 & 0 & 0 \\ \\ 0 & -\frac{\zeta}{2} & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ \lambda \\ u \end{bmatrix}.$$

We deduce that

$$\det R = \frac{\zeta^2 \kappa^2}{8} > 0.$$

Since along the path $\mathcal{T}_\tau$ we encounter kernels only for $\tau = 0$ we deduce that the orientation transport along this path is

$$\epsilon = (-1)^{\dim_{\mathbb{R}} K} \cdot \operatorname{sign} \det R = 1.$$

Using this information in (2.13) we deduce

$$\mathbf{sw}^+_M(\sigma) - \mathbf{sw}^-_M(\sigma) = \sum_{k=n+1}^{n+m} s_k(-1).$$

To understand the signs $s_k(-1)$ we need to understand the type of crossing that occurs when the mildly irregular reducible monopole appears in the singular cobordism.

Consider again the bad arc $(s(t), C_t)$. We need to understand the sign of $s(-1)$ which is the same as the sign of $s(t) - s(0)$ for small t. Since

$$s = \ddot{s}t^2 + o(t^2)$$

we deduce that this sign is given by $\ddot{s}$. Using (2.18) we deduce that this sign is also given by κ.

$$\operatorname{sign} \kappa = \operatorname{sign} s(-1).$$

On the other hand

$$\operatorname{sign} \kappa = \operatorname{sign} \langle \boldsymbol{c}(\mathbf{i}\omega)\Phi, \Phi \rangle.$$

The last quantity is precisely the spectral flow of the short path of operators

$$(-\varepsilon, \varepsilon) \ni t \mapsto \mathfrak{D}_{A_0 + t\mathbf{i}\omega}.$$

If the mildly irregular reducibles are $(0, A_k)$, $k = n+1, \cdots, n+m$ we deduce

$$\mathbf{sw}^+_M(\sigma) - \mathbf{sw}^-_M(\sigma) = \sum_{k=n+1}^{n+m} SF(\mathfrak{D}_{A_k + t\mathbf{i}\omega};\ |t| < \varepsilon).$$

We deduce that $\mathbf{sw}^+_M(\sigma) - \mathbf{sw}^-_M(\sigma)$ can be identified with the spectral flow of the loop of operators $\mathfrak{D}_A$ obtained as A runs once along the circle of reducible $\eta(0)$-monopoles. By rescaling the metric we can assume that ω generates the lattice $H^1(M, \mathbb{Z}) \subset H^1(M, \mathbb{R})$. Modulo $\mathcal{G}$ we can identify the circle of reducibles with the path

$$[0, 4\pi] \ni t \mapsto (0, A_0 + t\mathbf{i}\omega)$$

where $(0, A_0)$ is a fixed reducible $\eta(0)$-monopole. The spectral flow of the path $[0, 4\pi] \mapsto \mathfrak{D}_{A_0 + t\mathbf{i}\omega}$ is easily computed from the formula

$$SF(\mathfrak{D}_{A_0 + t\mathbf{i}\omega};\ t \in [0, 4\pi]) = -\frac{1}{8} \int_{[0,4\pi] \times M} c_1(\hat{A}) \wedge c_1(\hat{A}) = \frac{1}{32\pi^2} \int_{[0,4\pi] \times M} F_{\hat{A}} \wedge F_{\hat{A}},$$

where $\hat{A}$ is the connection on $\det\sigma \to [0,4\pi]\times M$ given by

$$\hat{A} = A_0 + \mathbf{i}t\omega.$$

Then

$$F_{\hat{A}} = F_{A_0} + \mathbf{i}dt\wedge\omega,\ \ F_{\hat{A}}\wedge F_{\hat{A}} = 2\mathbf{i}dt\wedge\omega\wedge F_{A_0}$$

$$2\int_{[0,4\pi]\times M}\mathbf{i}dt\wedge\omega\wedge F_{A_0} = 8\pi\mathbf{i}\int_M \omega\wedge F_{A_0}$$

$$= 16\pi^2\int_M \omega\wedge\frac{\mathbf{i}}{2\pi}F_{A_0} = 16\pi^2\int_M \omega\wedge c_1(\det\sigma).$$

Thus

$$SF(\mathfrak{D}_{A_0+t\mathbf{i}\omega};\ t\in[0,4\pi]) = \frac{1}{2}\int_M \omega\wedge c_1(\det\sigma).$$

We can now state the main result of this section. It was proved for the first time by Y. Lim in [17] by a slightly different approach.

Theorem 2.7. [17, Lim](Wall crossing formula)

$$\mathbf{sw}^+_M(\sigma) - \mathbf{sw}^-_M(\sigma) = \frac{1}{2}\int_M \omega\wedge c_1(\det\sigma)$$

where ω is a generator of the lattice $H^1(M,\mathbb{Z})$ which induces the orientation on $H^1(M,\mathbb{R})$ needed to define positive/negative chambers.

Example 2.8. Consider the manifold $M = S^1\times S^2$ equipped with the natural round metric of constant positive scalar curvature s. Denote by ω the harmonic 1-form $\frac{1}{2\pi}d\theta$. ω is a generator of $H^1(M,\mathbb{Z})$, and $*\omega$ is a generator of $H^2(M,\mathbb{Z})$. Choose a closed 2-form η whose L^∞-norm is much smaller than s. We assume that $\tau := \langle *\omega,\eta\rangle_{L^2} > 0$.

Since $H^2(M,\mathbb{Z})$ has no 2-torsion we can identify each *spin*^{c} structure σ on M with its determinant $\det\sigma$. For each $n\in\mathbb{Z}$ we denote by σ_n the unique*spin*^{c} structure on M such that

$$c_1(\det\sigma_n) = 2n*\omega.$$

Since $s \gg \|\eta\|_{L^\infty} > 0$ we deduce $\mathbf{sw}_M(\sigma_n,\eta) = 0$. On the other hand, we deduce that η belongs to the positive σ_n-chamber if and only if $n\leq 0$. Hence

$$\mathbf{sw}^+_M(\sigma_n) = 0,\ \ \forall n\leq 0.$$

On the other hand

$$\mathbf{sw}^+_M(\sigma_n) = n,\ \ \forall n\geq 0.$$

If we denote by $\mathbf{sw}_M(t)$ the generating function of the sequence $\mathbf{sw}^+_M(\sigma_n)$ we deduce

$$\mathbf{sw}^+_M(t) := \sum_{n\in\mathbb{Z}}\mathbf{sw}^+_M(\sigma_n)t^n = \sum_{n>0} nt^n = \frac{t}{(1-t)^2}.$$

Suppose now that M is a 3-manifold with $H_1(M,\mathbb{Z})\cong\mathbb{Z}$ we can pick a harmonic 2-form ω generating $H^2(M,\mathbb{Z})\cong\mathbb{Z}$), and a closed 2-form η such that $\langle\omega,\eta\rangle_{L^2}$ is a very small positive number τ. We can again uniquely define $\sigma_n\in Spin^c(M)$ by requiring

$$c_1(\det\sigma_n) = 2n\omega.$$

Denote by $\mathbf{sw}_{M,\eta}(t)$ the generating function of the sequence $\mathbf{sw}_M(\sigma_n, \eta)$, and by $\mathbf{sw}_M(t)$ the generating function of the sequence $\mathbf{sw}_M^+(\sigma_n)$. The function $\mathbf{sw}_{M,\eta}(t)$ is a Laurent *polynomial* satisfying

$$\mathbf{sw}_{M,\eta}(t) = \mathbf{sw}_{M,\eta}(t^{-1}).$$

We deduce as above that

$$\mathbf{sw}_M(t) = \mathbf{sw}_{M,\eta}(t) + \frac{t}{(1-t)^2}.$$

A result of Meng-Taubes [20] states that, $\exists m \in \mathbb{Z}$ such that

$$\mathbf{sw}_M(t) = \pm t^m \frac{\Delta_M(t)}{(1-t)^2},$$

where $\Delta_M(t)$ is the Alexander polynomial of M, [28, 31]. Thus

$$\mathbf{sw}_{M,\eta}(t)(1-t)^2 + t = \pm t^m \Delta_M(t) \Longleftrightarrow \mathbf{sw}_{M,\eta}(t)(t - 2 + t^{-1}) = \pm t^{m-1}\Delta_M(t) - 1.$$

We can now remove the $\pm$ ambiguity by using the identity $\Delta_M(t) = 1$. Moreover, we can be even more specific about m if we recall that *the* Alexander polynomial is symmetric

$$\Delta_M(t) = \Delta_M(t^{-1}).$$

Hence

$$\mathbf{sw}_{M,\eta}(t)(t - 2 + t^{-1}) = \Delta_M(t) - 1.$$

This shows that $\mathbf{sw}_{M,\eta}(t)$ is a topological invariant of M, and the chamber issue is moot. Derivating the above equality twice at $t = 1$ we deduce

$$2\sum_{n\in\mathbb{Z}} \mathbf{sw}_{M,\eta}(\sigma_n) = 2\mathbf{sw}_{M,\eta}(1) = \Delta''_M(1).$$

Consider now the most general case, $b_1(M) = 1$ so that $H := H^2(M, \mathbb{Z}) \cong \mathbb{Z} \oplus G$, where G is a finite Abelian group.

Fix again a harmonic 2-form ω such that $*\omega$ defines a generator of $H^1(M, \mathbb{Z})$. In particular, we can think of ω as generating the free part of $H^2(M, \mathbb{Z})$. Choose the closed 2-form η as before, and fix a *spin*^{c} structure σ_0 such that $\det \sigma_0$ is trivial. Such a choice is always possible since M admits *spin* structures. Any other *spin*^{c} structure on M has the form $\sigma_h := \sigma_0 \otimes h$. Set

$$\omega_h = \langle \omega, h_{free} \rangle_{L^2}$$

where h_{free} denotes the free part of h. The wall crossing correction term K is the element in the ring of formal power series $\mathbb{Z}[[H]]$ is defined by

$$K := \mathcal{S}\frac{T}{(1-T)^2}, \;\; \mathcal{S} := \sum_{g\in G} g.$$

T is the formal variable corresponding to the chosen generator of the free part of H, $T := \exp(\omega)$. We have an equality

$$\sum_{h\in H} \mathbf{sw}_M^+(\sigma_h)h =: \mathbf{sw}_M(h) = \mathbf{sw}_{M,\eta}(h) + K(h) := \sum_{h\in H} \mathbf{sw}_{M,\eta}(\sigma_h)h + \mathcal{S}\frac{T}{(1-T)^2}.$$

A result of Turaev [34] refining work of Meng-Taubes implies that $\exists h \in H$ such that

$$\mathbf{sw}_M(h) = \pm h_0 \tau_M(h)$$

where $\tau_M(h)$ denotes the Reidemeister torsion of M. Define

$$\mathfrak{aug} : \mathbb{Z}[[H]] \to \mathbb{Z}[[H/G]] = \mathbb{Z}[[T, T^{-1}]]$$

by

$$\mathfrak{aug} f = \sum_{n\in\mathbb{Z}} \Bigl(\sum_{g\in G} f(gT^n) \Bigr) T^n,$$

and set

$$\overline{\mathbf{sw}}_M = \mathfrak{aug}(\mathbf{sw}_M), \;\; \overline{\mathbf{sw}}_{M,\eta} = \mathfrak{aug}(\mathbf{sw}_{M,\eta}), \;\; \bar{\tau}_M = \mathfrak{aug}(\tau_M).$$

Note that $\mathfrak{aug}(\mathcal{S}) = |G|$ and $\mathfrak{aug}(T) = T$. Then

$$\mathbf{s}\bar{\mathbf{w}}_{M,\eta} + \frac{|G|T}{(1-T)^2} = \pm \bar{\tau}_M(T) = \pm T^{k/2} \frac{\Delta_M(T)}{(1-T)^2}, \;\; k \in \mathbb{Z},$$

where $\Delta_M(T) \in \mathbb{Z}[T^{1/2}, T^{-1/2}]$ is the Alexander polynomial of M which satisfies

$$\Delta_M(1) = |G|, \;\; \Delta_M(T) = \Delta_M(T^{-1}).$$

We conclude as before that

$$\overline{\mathbf{sw}}_{M,\eta}(T)(T^{-1} - 2 + T) = \Delta_M(T) - |G|, \;\; \overline{\mathbf{sw}}_{M,\eta} = \frac{1}{2}\Delta_M''(1).$$

We conclude this example with a computation we will need later on. Suppose now that there exists a positive integer m_0 such that

$$(T^{-m_0/2} - T^{m_0/2}) \frac{\Delta_M(T)}{(1-T)} = T^{-1/2} P(T), \;\; P \in \mathbb{Z}[T^{1/2}, T^{-1/2}].$$

Observe that $P(T^{-1}) = P(T)$, and

$$P(T) = P(T^{-1}) = \Delta_M(T) \frac{(T^{m_0/2} - T^{-m_0/2})}{(T^{1/2} - T^{-1/2})} = T^{-\frac{m_0-1}{2}} \Delta_M(T)(1 + \cdots + T^{m_0-1}).$$

In particular, $P(1) = m_0|G|$. Let $r \in \{0, 1\}$ such that $m_0 - 1 - r \in 2\mathbb{Z}$. Derivating twice at $t = 1$ we get

$$P''(1) = 2\Delta_M(1) \Bigl(\sum_{r}^{\frac{m_0-1}{2}} k^2 \Bigr) + m_0 \Delta_M''(1) = \frac{2}{3} \Delta_M(1) \Bigl\{ B_3\bigl(\frac{m_0+1}{2}\bigr) - B_3\bigl(\frac{r}{2}\bigr) \Bigr\} + m_0 \Delta_M''(1),$$

where $B_3(z)$ denotes the third Bernoulli polynomial

$$B_3(z) = z^3 - \frac{3}{2}z^2 + \frac{z}{2} = \frac{z(z-1)(2z-1)}{2}.$$

Since $B_3(r/2) = 0$ we deduce

$$B_3\left(\frac{m_0+1}{2}\right) = \frac{m_0(m_0^2-1)}{8}, \quad P''(1) = \frac{m_0(m_0^2-1)}{12}\Delta_M(1) + m_0\Delta''_M(1)$$

Hence

$$\overline{\mathbf{sw}}_{M,\eta}(1) = \frac{1}{2}\Delta''_M(1) = \frac{1}{2m_0}P''(1) - \frac{m_0^2-1}{24}\Delta_M(1).$$

In particular,

$$\overline{\mathbf{sw}}_{M,\eta}(1) - \frac{\Delta_M(1)}{12} = \frac{1}{2}\Delta''_M(1) - \frac{\Delta_M(1)}{12} = \frac{1}{2m_0}P''(1) - \frac{m_0^2+1}{24}\Delta_M(1).$$

2.4 The case $b_1 = 0$

This case is the worst possible because we always have $[*\eta] = [2\pi c_\sigma] = 0$. Thus no matter how we choose the perturbation (η, w) there will always be reducible η- monopoles. In fact, since $b_1 = 0$ there will always be exactly one reducible monopole in $\mathfrak{M}_\sigma(g, \eta, \mu)$.

Since $H^1(M, \mathbb{R}) = H^2(M, \mathbb{R}) = 0$ for every closed two form ω on M there exists an unique co-closed 1-from on M such that $d\alpha = \omega$. We will denote the form α by $d^{-1}\omega$. If we fix a flat connection B on $\det\sigma$ then the equation

$$*F_A + \mathbf{i}\eta$$

has the explicit solution

$$A = A_{g,\eta} := B - \mathbf{i}d^{-1} * \eta.$$

Define as in the previous section

$$\mathcal{N} := \bigcup_g \{g\} \times \mathcal{N}(g).$$

The transversality results of [7, 17] show that there exists a dense open set $\mathcal{Z}^0 \subset \mathcal{Z}$ such that for all $(g, \eta) \in \mathcal{N}^0$ the operator $\mathfrak{D}_{A_{g,\eta}}$ has trivial kernel, i.e. the corresponding reducible monopole is regular.

For generic $(g, \eta) \in \mathcal{N}^0$ the space of irreducible monopoles consists of finitely nondegenerate points. We can define as before the signed count of irreducible monopoles which we denote by $\mathbf{sw}'(\sigma, g, \eta)$. The space of parameters is disconnected and its components are separated by the wall

$$\mathcal{N}^{(1)} := \mathcal{Z} \setminus \mathcal{Z}^0.$$

There exists a filtration

$$\mathcal{N}^{(1)} \supset \mathcal{N}^{(2)} \supset \cdots \mathcal{N}^{(j)} \supset$$

where

$$\mathcal{N}^{(j)} = \left\{(g, \eta, \mu);\ \dim_{\mathbb{C}} \ker \mathfrak{D}_{A_{g,\eta}} \geq j\right\}.$$

As explained in [7, 17] we have

$$\operatorname{codim} \mathcal{N}^{(j)} \geq j.$$

We need to understand the dependence of $\mathbf{sw}'_M(\sigma, g, \eta)$ on (g, η). Pick a smooth path $(g(s), \eta(s), \mu(s))$, $s \in [-1, 1]$ of such parameters. Assume that this path crosses the wall $\mathcal{Z}^{(1)}$ only once, at $s = 0$. Transversality arguments show that we can assume that the crossing occurs in a regular fashion, i.e.

$$\dim_{\mathbb{C}} \ker \mathfrak{D}_{A_{g(0),\eta(0)}} = 1$$

and, if we denote by $\lambda(s)$, $|s| \ll 1$ the smooth function such that $\lambda(s)$ is an eigenvalue of $\mathfrak{D}_{A_{g(s),\eta(s)}}$ and $\lambda(0) = 0$, then

$$\nu := \frac{d}{ds} |_{s=0} \lambda(s) \neq 0. \tag{$\checkmark$}$$

Fix an unitary spinor Φ spanning $\ker \mathfrak{D}_{A_{g(0),\eta(0)}}$. By slightly perturbing the path $g(s)$ near $s = 0$ we can assume that $g(s)$ is independent of s for s near zero. The condition ($\checkmark$) is then equivalent to

$$\nu := \Big\langle \boldsymbol{c}(\mathbf{i}\varphi)\Phi, \Phi \Big\rangle \neq 0. \tag{$\checkmark\checkmark$}$$

where

$$\varphi := d^{-1}\big(*\eta'(0)\big).$$

As in the previous section we can form the parametrized moduli space

$$\tilde{\mathfrak{M}}_\sigma = \{(s, \mathsf{C});\ \ \mathsf{C} \text{ is a } (g(s), \eta(s))\text{-monopole}\}/\mathcal{G}.$$

The reducible part of the parametrized moduli space is easy to describe. For each s there exists exactly one reducible monopole $\mathsf{C}_s = (0, A_s)$ and thus $\tilde{\mathfrak{M}}^{red}_\sigma$ consists of a the smooth path

$$s \mapsto (0, A_s).$$

For a generic choice of the path $(g(s), \eta(s))$ we can assume that away from $s = 0$ the space $\tilde{\mathfrak{M}}^{irr}_\sigma$ is a smooth, oriented one dimensional manifold. The compact singular cobordism $\tilde{\mathfrak{M}}_\sigma$ thus consists of of one reducible arc and several irreducible ones, some of which approaching the mildly irregular reducible $(0, A_0)$. This cobordism looks roughly as in Figure 2.

To understand the structure of the singular cobordism near the mildly irregular reducible point we follow a strategy very similar to the one used in the case $b_1 = 1$. Denote the mildly irregular reducible point of $\tilde{\mathfrak{M}}_\sigma$ by $(0, \mathsf{C}_0)$. We are looking at solutions

$$(\dot{s}, \mathsf{C}_0 + \dot{\mathsf{C}}) = (0, \mathsf{C}_0),\ \ \dot{\mathsf{C}} \in \mathcal{S}_{\mathsf{C}_0}$$

close to $(0, \mathsf{C}_0, 0)$ of the nonlinear equation

$$\mathcal{F}(s, \dot{\mathsf{C}}) = 0.$$

Its linearization at $(0, \mathsf{C}_0)$ is given by

$$\underline{\mathcal{F}} \left[\begin{array}{c} \dot{s} \\ \dot{\psi} \\ \mathbf{i}\dot{a} \end{array} \right] = \left[\begin{array}{cc} \mathfrak{D}_{A_0} & 0 \\ 0 & -*d \end{array} \right] \cdot \left[\begin{array}{c} \dot{\psi} \\ \mathbf{i}\dot{a} \end{array} \right] + \left[\begin{array}{c} 0 \\ -\mathbf{i}\dot{s}\eta'(0) \end{array} \right],\ \ d^*\dot{a} = 0.$$

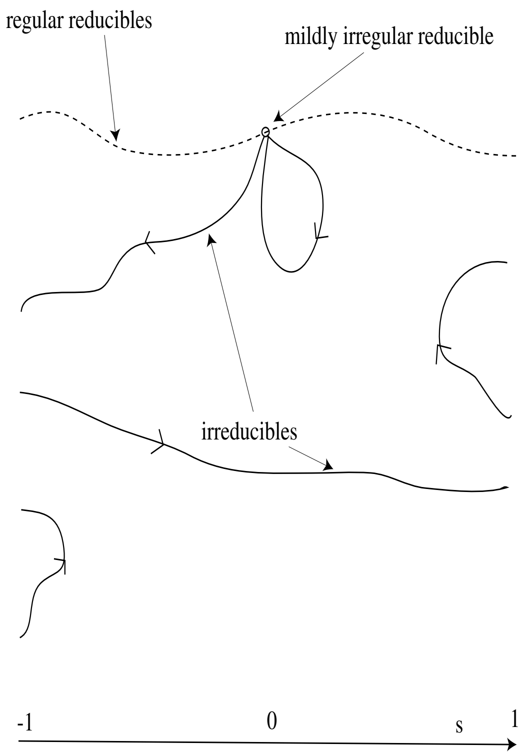

Figure 2: *Another singular one dimensional cobordism*

The kernel of $\underline{\mathfrak{F}}$ consists of triples $(\dot{s}, \dot{\psi}, \mathbf{i}\dot{a})$ such that

$$\mathfrak{D}_{A_0}\dot{\psi} = 0$$

and

$$\begin{cases} -*d\dot{a} = -\dot{s}\eta'(0) \\ \\ d^*\dot{a}^* = 0 \end{cases}$$

The first equation has a (complex) one dimensional space of solutions spanned by Φ. The second equations have an unique solution

$$\dot{a} = -\dot{s}\varphi.$$

Thus the kernel of $\underline{\mathfrak{F}}$ consists triples of the form

$$(\dot{s}_0, z\Phi, -\mathbf{i}\dot{s}_0\varphi), \quad \dot{s} \in \mathbb{R}, \quad z \in \mathbb{C}.$$

Note that $\dim_{\mathbb{R}} \ker \underline{\mathfrak{F}} = 3$. The pair $(\dot{s}_0, z)$ defines coordinates on this vector space.

The cokernel consists of vectors

$$(\phi, \mathbf{i}b) \in \Gamma(\mathbb{S}_\sigma) \times \mathbf{i}\Omega^1(M), \quad d^*b = 0$$

such that

$$\langle \mathfrak{D}_{A_0}\dot{\psi}, \phi\rangle + \langle -*d\mathbf{i}\dot{a} - \mathbf{i}\dot{s}\eta'(0), \mathbf{i}b\rangle = 0 \tag{2.20}$$

$\forall \dot{s}, \dot{\psi}, \mathbf{i}\dot{a}$. We deduce

$$\phi \in \ker \mathfrak{D}_{A_0}.$$

By letting $\dot{s}, \dot{\psi}, \dot{a} = 0$ we deduce $*db = 0$. In particular, since $b_1 = 0$ we deduce that $b = 0$. Hence the cokernel consists of pairs

$$(z\Phi, 0), \quad z \in \mathbb{C}.$$

Denote by $\mathcal{P}$ the orthogonal projection onto the kernel of $\underline{\mathcal{F}}$ and by $\mathfrak{Q}$ the orthogonal projection onto the cokernel.) Denote the vectors $(\dot{s}, \dot{\psi}, \mathbf{i}\dot{a}, \mathbf{i}f)$ in the domain of $\underline{\mathcal{F}}$ by Ξ. The equation

$$\underline{\mathcal{F}}(\mathsf{C}_0 + \Xi) = 0$$

can be rewritten as

$$\underline{\mathcal{F}}(\Xi) + R(\Xi) \tag{2.21}$$

where R is the nonlinear remainder,

$$R \begin{bmatrix} \dot{s} \\ \dot{\psi} \\ \mathbf{i}\dot{a} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\boldsymbol{c}(\mathbf{i}\dot{a})\dot{\psi} - \frac{\mathbf{i}}{2} f\dot{\psi} \\ \\ \frac{1}{2}q(\dot{\psi}) - \mathbf{i}(\eta(\dot{s}) - \mathbf{i}\dot{s}\eta'(0) - \eta(0)) \end{bmatrix}, \quad d^*\dot{a} = 0.$$

(Above we have used the simplifying assumption $g'(0) = 0$.) Decompose

$$\Xi := \Xi_0 + \Xi^{\perp}, \quad \Xi_0 := \mathcal{P}\Xi, \quad \Xi^{\perp} = \mathcal{P}^{\perp}\Xi.$$

The equation (2.21) is equivalent to the pair of equations

$$\underline{\mathcal{F}}(\Xi^{\perp}) + \mathfrak{Q}^{\perp}R(\Xi) = 0 \tag{2.22a}$$

$$\underline{\mathcal{F}}(\Xi^{\perp}) + \mathfrak{Q}R(\Xi) = 0 \tag{2.22b}$$

For each sufficiently small Ξ_0 we can solve the first equation for $\Xi^{\perp}$ and we obtain a smooth function

$$\Xi_0 \mapsto \Xi^{\perp}(\Xi_0).$$

Note that $\Xi^{\perp}(0) = 0$ and $D\mid_{\Xi_0=0} \Xi^{\perp} = 0$. Thus $\Xi_0 = 0$ is a zero of order at least 2 of $\Xi^{\perp}$. We can extract even more precise information.

Observe that

$$\mathfrak{Q}^{\perp}R(\Xi) = \begin{bmatrix} \frac{1}{2}\boldsymbol{c}(\mathbf{i}\dot{a})\dot{\psi} - \langle \frac{1}{2}\boldsymbol{c}(\mathbf{i}\dot{a})\dot{\psi}, \Phi\rangle\Phi \\ \\ \frac{1}{2}q(\dot{\psi}) - (\mathbf{i}\eta(\dot{s}) - \dot{s}\eta'(0) - \eta(0)) \\ \\ 0 \end{bmatrix}$$

so that (2.22a) can be rewritten

$$\begin{cases} \mathfrak{D}_{A_0}\dot{\psi}^{\perp} + \frac{1}{2}\boldsymbol{c}(\mathbf{i}\dot{a})\dot{\psi} - \langle \frac{1}{2}\boldsymbol{c}(\mathbf{i}\dot{a})\dot{\psi}, \Phi\rangle\Phi = 0 \\ \\ -\mathbf{i} * d\dot{a}^{\perp} - \mathbf{i}\dot{s}^{\perp}\eta'(0) - \mathbf{i}\big(\eta(\dot{s}) - \eta(0) - \dot{s}\eta'(0)\big) + \frac{1}{2}q(\dot{\psi}) = 0 \\ \\ d^*\dot{a}^{\perp} = 0 \end{cases} \tag{2.23}$$

$$\Xi^{\perp} = \begin{bmatrix} \dot{s}^{\perp} \\ \\ \dot{\psi}^{\perp} \\ \\ \mathbf{i}\dot{a}^{\perp} \end{bmatrix}, \quad \dot{s}^{\perp} = \langle \dot{a}^{\perp}, \varphi \rangle.$$

We write

$$\dot{s} = \dot{s}_0 + \dot{s}^{\perp}(\dot{s}_0, z), \quad \dot{\psi} = z\Phi + \dot{\psi}^{\perp}(\dot{s}_0, z), \quad \dot{a} = -\dot{s}_0\varphi + \dot{a}^{\perp}(\dot{s}_0, z).$$

The equation (2.22b) can be rewritten as

$$F(\dot{s}_0, z) = \langle \boldsymbol{c}(-\mathbf{i}\dot{s}_0\varphi + \mathbf{i}\dot{a}^{\perp})(z\Phi + \dot{\psi}^{\perp}), \Phi \rangle = 0 \tag{2.24}$$

Observe that

$$F(\dot{s}_0, z) = \frac{d}{dt}\,|_{t=0} \Big\langle \mathfrak{D}_{A_{g(t),\eta(t)},g(t)}\Phi, \Phi \Big\rangle \dot{s}_0 z + O(3) = -\nu\dot{s}_0 z + O(3). \tag{2.25}$$

We deduce that a neighborhood of $(0, \mathsf{C}_0)$ looks like a neighborhood of the origin in the quotient $(\mathbb{R}\times\mathbb{C})/S^1$ or equivalently, as a neighborhood of the origin in $\mathbb{R}\times\mathbb{R}_+$. In particular, only one arc of irreducibles approaches the mildly irregular reducible point.

We decompose the closure of $\tilde{\mathfrak{M}}_{\sigma}^{irr}$ into arcs. There are two types of arcs.

- A **bad arc** $[-1, 0] \ni t \mapsto (s(t), \mathsf{C}(t))$ which approaches the mildly irregular reducible.
- **Good arcs** $[-1, 1] \ni t \mapsto (s_j(t), \mathsf{C}_j(t))$, $j = 1, \cdots, n$.

As in the previous subsection we deduce

$$\mathbf{sw}'_M(\sigma, g(1), \eta(1)) - \mathbf{sw}'_M(\sigma, g(-1), \eta(-1)) = \epsilon(\mathsf{C}(-1))s(-1).$$

We assume that for t close to zero the configuration $\mathsf{C}(t)$ is in the local slice at C_0. Set $\epsilon := \epsilon(\mathsf{C}) = \pm 1$. As in [26, Sec. 2.3], the sign ε is given by the orientation transport along the path of Fredholm selfadjoint operators

$$[0, 1] \ni \tau \mapsto \mathfrak{T}_{\tau} := \mathfrak{T}_{\mathsf{C}_{-\tau}}$$

As before, only contribution to the orientation transport occurs at $\tau = 0$ but the case $b_1 = 0$ requires a bit more work. More precisely, we need to understand the small eigenvalues of the (real) operator $\mathfrak{T}_{\tau}$. We write

$$s = \dot{s}t + \ddot{s}t^2 + o(t^2), \quad \mathsf{C}(t) = \mathsf{C}_0 + t\dot{\mathsf{C}} + t^2\ddot{\mathsf{C}} + o(t^2),$$

$$\psi(t) = t\dot{\psi} + t^2\ddot{\psi} + \dddot{\psi}t^3 + o(t^3), \quad A(t) = A_0 + t\mathbf{i}\dot{a} + t^2\mathbf{i}\ddot{a} + o(t^2).$$

The estimate (2.25) shows that the parametrizations $t \mapsto s(t)$ and $t \mapsto \psi(t)$ can be chosen so that

$$\ddot{s} \neq 0, \quad \dot{\psi} \neq 0.$$

Observe that

$$\mathfrak{T}_{\tau} \begin{bmatrix} \mathbf{i}\underline{\psi} \\ \mathbf{i}\underline{a} \\ \mathbf{i}\underline{f} \end{bmatrix} = \begin{bmatrix} \mathfrak{D}_{A_{-\tau},g(s(-\tau))} & 0 & \\ \\ 0 & -*_{g(s(-\tau))}\, d & d \\ \\ 0 & d^{*_{-\tau}} & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{i}\underline{\psi} \\ \mathbf{i}\underline{a} \\ \mathbf{i}\underline{f} \end{bmatrix}$$

$$+\begin{bmatrix} \frac{1}{2}\boldsymbol{c}(\mathbf{i}\underline{a})\psi_{-\tau}+\frac{1}{2}\boldsymbol{c}(\mathbf{i}\dot{a})\underline{\psi}-\frac{\mathbf{i}}{2}\underline{f}\psi_{-\tau} \\ \\ \frac{1}{2}\dot{q}(\psi_{-\tau},\underline{\psi}) \\ \\ \frac{\mathbf{i}}{2}\,\mathbf{Im}\,\langle\psi_{-\tau},\underline{\psi}\rangle \end{bmatrix}$$

Set $\dot{\mathfrak{T}} := \frac{d}{d\tau}\mid_{\tau=0}\mathfrak{T}_\tau$, $\ddot{\mathfrak{T}} = \frac{1}{2}\frac{d^2}{d\tau^2}\mid_{\tau=0}\mathfrak{T}_\tau$ so that

$$\mathfrak{T}_\tau = \mathfrak{T}_0 + \tau\dot{\mathfrak{T}} + \tau^2\ddot{\mathfrak{T}} + o(\tau)^2.$$

Using the condition $g'(0) = g''(0) = 0$ we deduce

$$\dot{\mathfrak{T}}\begin{bmatrix} \mathbf{i}\underline{\psi} \\ \\ \mathbf{i}\underline{a} \\ \\ \mathbf{i}\underline{f} \end{bmatrix} = -\begin{bmatrix} \frac{1}{2}\boldsymbol{c}(\mathbf{i}\dot{a})\underline{\psi}+\frac{1}{2}\boldsymbol{c}(\mathbf{i}\underline{a})\dot{\psi}-\frac{\mathbf{i}}{2}\underline{f}\dot{\psi} \\ \\ \frac{1}{2}\dot{q}(\dot{\psi},\underline{\psi}) \\ \\ \frac{\mathbf{i}}{2}\,\mathbf{Im}\,\langle\dot{\psi},\underline{\psi}\rangle \end{bmatrix}$$

and

$$\ddot{\mathfrak{T}}\begin{bmatrix} \mathbf{i}\underline{\psi} \\ \\ \mathbf{i}\underline{a} \\ \\ \mathbf{i}\underline{f} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\boldsymbol{c}(\mathbf{i}\ddot{a})\underline{\psi}+\frac{1}{2}\boldsymbol{c}(\mathbf{i}\underline{a})\ddot{\psi}-\frac{\mathbf{i}}{2}\underline{f}\ddot{\psi} \\ \\ \frac{1}{2}\dot{q}(\ddot{\psi},\underline{\psi}) \\ \\ \frac{\mathbf{i}}{2}\,\mathbf{Im}\,\langle\ddot{\psi},\underline{\psi}\rangle \end{bmatrix}.$$

Again we need to learn more about $(\dot{\psi}, \mathbf{i}\dot{a})$. For t close to zero the configuration $\mathsf{C}(t)$ satisfies the equation

$$\begin{cases} \widehat{SW}(s_0 + t\dot{s} + t^2\ddot{s}, \mathsf{C}_0 + t\dot{\mathsf{C}} + t^2\ddot{\mathsf{C}} + o(t^2)) = 0 \\ \\ \mathfrak{L}^*_{\mathsf{C}_0}(t\dot{\mathsf{C}} + t^2\ddot{\mathsf{C}} + o(t^2)) = 0 \end{cases}. \tag{2.26}$$

Derivating this equation with respect to t at $t = 0$ we get

$$\mathfrak{D}_{A_0}\dot{\psi} = 0 \tag{2.27a}$$

$$*d\dot{a} = -\dot{s}\eta'(0), \;\; d^*\dot{a} = 0 \tag{2.27b}$$

We deduce

$$\dot{\psi} = \zeta\Phi, \;\; (z \neq 0) \;\; \dot{a} = -\dot{s}\varphi.$$

Taking the second t-derivative at $t = 0$ we obtain

$$\mathfrak{D}_{A_0}\ddot{\psi} + +\frac{1}{2}\boldsymbol{c}(\mathbf{i}\dot{a})\dot{\psi} = 0 \Longleftrightarrow \mathfrak{D}_{A_0}\ddot{\psi} - \frac{\dot{s}\zeta}{2}\boldsymbol{c}(\mathbf{i}\varphi)\Phi = 0 \tag{2.28a}$$

$$\mathbf{i} * d\ddot{a} = \frac{1}{2}q(\dot{\psi}) - \mathbf{i}\ddot{s}\eta'(0), \;\; d^*\ddot{a} = 0. \tag{2.28b}$$

Taking the inner product of (2.28a) with Φ we deduce

$$\dot{s}\zeta\nu = 0$$

so that

$$\dot{s} = 0, \quad \dot{a} = 0.$$

Denote by $\Omega = \Omega(\Phi)$ the co-closed 1-form such that

$$\mathbf{i} * d\Omega = \boldsymbol{c}^{-1}(q(\Phi))$$

and set

$$\chi := \frac{1}{2}\langle \boldsymbol{c}(\mathbf{i}\Omega), q(\Phi)\rangle = \langle \Omega, *d\Omega\rangle = \int_M \Omega \wedge d\Omega.$$

Note that we can solve (2.28b) explicitly

$$\ddot{a} = \frac{\zeta^2}{2}\Omega - \ddot{s}\varphi.$$

The scalar χ is intimately related to ν. Indeed, identifying the coefficients of t^3 in the first equation of (2.26) and using $\dot{a} = 0$ we deduce

$$\mathfrak{D}_{A_0}\dddot{\Psi} + \frac{1}{2}\boldsymbol{c}(\mathbf{i}\ddot{a})\dot{\psi} = 0.$$

Taking the inner product of this equation with Φ we deduce

$$0 = \zeta\langle \boldsymbol{c}(\mathbf{i}\ddot{a})\Phi, \Phi\rangle = \zeta\langle \boldsymbol{c}(\mathbf{i}a), q(\Phi)\rangle = \zeta\Big\langle \boldsymbol{c}\big(\mathbf{i}\frac{\zeta^2}{2}\Omega - \mathbf{i}\ddot{s}\varphi\big), q(\Phi)\Big\rangle = \chi\zeta^3 - \zeta\nu\ddot{s}. \tag{2.29}$$

Thus

$$\chi\zeta^2 = \nu\ddot{s}. \tag{2.30}$$

As we have explained, to understand the orientation transport along the path $\mathfrak{T}_\tau$ we need to understand the small eigenvalues of $\mathfrak{T}_\tau$, $|\tau| \ll 1$. To achieve this we use a little perturbation theory

We write

$$\mathfrak{T}_\tau = \mathfrak{T}_0 + \tau\dot{\mathfrak{T}} + t^2\ddot{\mathfrak{T}} + \cdots$$

Denote by K_0 the kernel of $\mathfrak{T}_0$

$$K_0 \cong \ker \mathfrak{D}_{A_0} \oplus H^0(M, g)$$

and by P_0 the orthogonal projection onto K. Define the (first) resonance operator

$$R_0 : K_0 \to K_0, \quad R_0\vec{k} = P_0\dot{\mathfrak{T}}\vec{k}.$$

We begin by looking for linearly small eigenvalues of $\mathfrak{T}_\tau$ i.e. eigenvalues $\lambda(\tau)$ of $\mathfrak{T}_\tau$ which behave like $c\tau$ ($c \neq 0$) as $\tau \to 0$.

Since the path $\mathfrak{T}_\tau$ is real analytic the eigenvalues and eigenvectors can be parametrized in a real analytic fashion (see [11]). Denote by $\Xi(t)$ a real analytic path of unitary eigenvectors corresponding to the linearly small eigenvalue $\lambda(t)$. We write

$$\lambda(\tau) = \lambda_1\tau + \cdots, \quad \Xi(\tau) = \Xi_0 + \tau\Xi_1 + \cdots, \quad \|\Xi_0\| = 1, \quad \lambda_1 \neq 0.$$

We deduce

$$\begin{cases} \mathfrak{T}_0\Xi_0 = 0 \\ \\ \mathfrak{T}_0\Xi_1 + \dot{\mathfrak{T}}\Xi_0 = \lambda_1\Xi_0 \end{cases}$$

Thus $\Xi_0 \in K_0$. The second equation has a solution Ξ_1 if and only if

$$R_0\Xi_0 = P_0\dot{\mathfrak{T}}\Xi_0 = \lambda_1.$$

Thus $\lambda_1 \neq 0$ is a nonzero eigenvalue of the resonance operator R_0. We use the coordinates (z, u) on K_0. We have

$$\dot{\mathfrak{T}}\begin{bmatrix} z \\ u \end{bmatrix} = -\begin{bmatrix} \frac{z}{2}\boldsymbol{c}(\mathbf{i}\dot{a})\Phi - \frac{\mathbf{i}}{2}u\dot{\psi} \\ \\ \frac{1}{2}\dot{q}(\zeta\Phi, z\Phi) \\ \\ \frac{\mathbf{i}}{2}\,\mathbf{Im}\,\langle\dot{\psi}, z\Phi\rangle \end{bmatrix} \tag{2.31}$$

so that

$$R_0\begin{bmatrix} z \\ u \end{bmatrix} = -\begin{bmatrix} -\frac{\mathbf{i}}{2}u\zeta\Phi \\ \\ \frac{\mathbf{i}}{2}\,\mathbf{Im}\,\langle\dot{\psi}, z\Phi\rangle \end{bmatrix}$$

Again, writing $z = a + \mathbf{i}b$ we deduce that R_0 has the matrix description

$$R_0\begin{bmatrix} a \\ b \\ u \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \\ 0 & 0 & \frac{\zeta}{2} \\ \\ 0 & \frac{\zeta}{2} & 0 \end{bmatrix}$$

We deduce that R_0 has two nonzero eigenvalues, $\pm\frac{\zeta}{2}$. The corresponding eigenvectors are

$$\Psi_\pm = \mathbf{i}\Phi \pm \mathbf{i} = \begin{bmatrix} 0 \\ 1 \\ \pm 1 \end{bmatrix}.$$

Thus there are only two linearly small eigenvalues of $\mathfrak{T}_\tau$, $\lambda_\pm(\tau) = \pm\frac{\zeta}{2}\tau + \cdots$ with corresponding eigenvectors

$$\Xi_\pm(t) = \frac{1}{\sqrt{2}}\Psi_\pm + \cdots.$$

The above argument shows that there exists one eigenvalue $\lambda_0(\tau)$ which is at least quadratically small. We set

$$\lambda_0(t) = \lambda_2\tau^2 + \cdots$$

and denote the corresponding family of eigenvectors

$$\Xi(\tau) = \Xi_0 + \tau\Xi_1 + \tau^2\Xi_2 + \cdots, \;\; \|\Xi_0\| = 1.$$

The equality

$$(\mathfrak{T} + \tau\dot{\mathfrak{T}} + \tau^2\ddot{\mathfrak{T}} + \cdots)(\Xi_0 + \tau\Xi_1 + \tau^2\Xi_2 + \cdots) = (\lambda_2\tau^2 + \cdots)(\Xi_0 + \tau\Xi_1 + \tau^2\Xi_2 + \cdots)$$

implies

$$\begin{cases} \mathfrak{T}\Xi_0 = 0 \\ \\ \mathfrak{T}\Xi_1 + \dot{\mathfrak{T}}\Xi_0 = 0 \\ \\ \mathfrak{T}_0\Xi_0 + \dot{\mathfrak{T}}\Xi_1 + \ddot{\mathfrak{T}}\Xi_0 = \lambda_1\Xi_0 \end{cases} \tag{2.32}$$

The first equation implies $\Xi_0 \in K_0$ while the second one implies

$$P_0\dot{\mathfrak{T}}\Xi_0 = 0 \Longleftrightarrow \Xi_0 \in \ker R_0 =: K_1.$$

The kernel of the resonance operator R_0 is spanned by

$$\Psi_0 = \Phi \oplus 0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Thus $\Xi_0 = \Psi_0$. Observe that

$$\dot{\mathfrak{T}}\Psi_0 = -\begin{bmatrix} 0 \\ \\ \frac{1}{2}\dot{q}(\zeta\Phi, \Phi) \\ \\ 0 \end{bmatrix}$$

The vector Ξ_1 is obtained by solving the equation

$$\mathfrak{T}_0\Xi_1 = \begin{bmatrix} 0 \\ \\ \frac{1}{2}\dot{q}(\zeta\Phi, \Phi) \\ \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \\ \zeta q(\Phi) \\ \\ 0 \end{bmatrix}$$

The solution set of this equation is the affine space

$$\Xi_1 \in \zeta(*d)^{-1}(q(\Phi)) + K_0 = \mathbf{i}\zeta\Omega + K_0.$$

Taking the inner product of the third equation in (2.32) with Ξ_0 we deduce

$$\lambda_0 = \langle \dot{\mathfrak{T}}\Xi_1, \Xi_0 \rangle + \langle \ddot{\mathfrak{T}}\Xi_0, \Xi_0 \rangle.$$

In the first inner product the K_0-components of Ξ_1 do not contribute because $\Xi_0 \in \ker R_0$. Now observe that since $\dot{a} = 0$

$$\dot{\mathfrak{T}}\begin{bmatrix} 0 \\ \\ \mathbf{i}\zeta\Omega \\ \\ 0 \end{bmatrix} = -\begin{bmatrix} \frac{\zeta^2}{2}\boldsymbol{c}(\mathbf{i}\Omega)\Phi \\ \\ 0 \\ \\ 0 \end{bmatrix}$$

so that

$$\langle \dot{\mathfrak{T}}\Xi_1, \Xi_0\rangle = -\frac{\zeta^2}{2}\langle \boldsymbol{c}(\mathbf{i}\Omega), q(\Phi)\rangle = -\zeta^2\chi.$$

On the other hand, using (2.29)

$$\langle \ddot{\mathfrak{T}}\Xi_0, \Xi_0\rangle = \frac{1}{2}\langle \boldsymbol{c}(\mathbf{i}\ddot{a})\Phi, \Phi\rangle = 0$$

Thus

$$\lambda_2 = -\zeta^2\chi \overset{(2.30)}{=} -\nu\ddot{s}.$$

We conclude that $\mathfrak{T}_\tau$ has a quadratically small eigenvalue $\lambda_0(\tau) = -\ddot{s}\nu\tau^2 + \cdots$ with corresponding eigenvector

$$\Xi_0(\tau) = \Psi_0 + \cdots$$

We can finally determine the orientation transport along the path $\mathfrak{T}_\tau$. Imitating the strategy used in the proof of the orientation transport formula [26, §1.5.1-Eq.(1.5.9)] we deduce that the orientation transport in this case is given by

$$\mathrm{sign}\,(-\frac{\zeta}{2}\cdot\frac{\zeta}{2}\ddot{s}\nu) = -\mathrm{sign}\,(\ddot{s}\nu).$$

Hence

$$\epsilon(\mathsf{C}(-1))s(-1) = -\mathrm{sign}\,(\ddot{s}\nu)\cdot\mathrm{sign}\,(\ddot{s}) = -\mathrm{sign}\,(\nu) = -SF\big(\mathfrak{D}_{A_{g(s),\eta(s)}}, s\in[-1,1]\big).$$

This proves that the usual monopole count $\mathbf{sw}'_M$ is not a topological invariant and satisfies a wall crossing formula

$$\mathbf{sw}'_M(\sigma, g(1), \eta(1)) - \mathbf{sw}'_M(\sigma, g(-1), \eta(-1)) = -SF\big(\mathfrak{D}_{A_{g(s),\eta(s)}}, s\in[-1,1]\big). \tag{2.33}$$

To produce a topological invariant we need to alter the above count by a quantity which will change in the opposite way. For every metric g on M and Hermitian connection A on det define the *Kreck-Stolz* invariant

$$KS(A,g) = 4\eta_{dir}(A,g) + \eta_{sign}(g)$$

where $\eta_{dir}(A,g)$ denotes the eta invariant of the Dirac operator $\mathfrak{D}_A(g)$ on $\mathbb{S}_\sigma$ determined by g and A and $\eta_{sign}(g)$ denotes the eta invariant of the signature operator $\mathbf{SIGN} = \mathbf{SIGN}(g)$ (see [1]).

If g_s (resp. A_s), $s\in[0,1]$, is a path of metrics (resp. connections) then (see [2])

$$\frac{1}{2}\Big(\eta_{dir}(A_1,g_1) + h_1 - \eta_{dir}(A_0,g_0) - h_0\big) = SF\big(\mathfrak{D}_{A_s}(g_s), s\in[0,1]\big)$$

$$+\frac{1}{8}\int_{[0,1]\times M}\big(-\frac{1}{3}p_1(\hat{\nabla}) + c_1(\hat{A})^2\big)$$

where $h_i = \dim_{\mathbb{C}}\ker\mathfrak{D}_{A_i}(g_i)$, $i=0,1$, $\hat{\nabla}$ denotes the Levi-Civita connection of the metric $ds^2 + g_s$ on $[0,1]\times M$ and $\hat{a}$ denotes the connection $ds\otimes\partial_s + A_s$ on the pullback of $\det(\sigma)$ over the cylinder $[0,1]\times M$. Similarly (see [2])

$$\eta_{sign}(g_1) - \eta_{sign}(g_0) = \frac{1}{3}\int_{[0,1]\times M} p_1(\hat{\nabla}).$$

Hence

$$KS(A_1,g_1) - KS(A_0,g_0) = 8SF\big(\mathfrak{D}_{A_s}(g_s),\ s\in[0,1]\big)$$
$$+4(h_0-h_1) + \int_{[0,1]\times M} c_1(\hat{A})^2.$$

Suppose now that $A_s = A_{g_s,\eta(s)}$ as in (2.33). Then

$$h_0 = h_1 = 0.$$

We can write $A_j = B + \mathbf{i}a_j$, $j=0,1$, where B denotes a flat connection on $\det(\sigma)$. Then

$$a_j = -d^{-1}\big(*_j\eta(j)\big),\ \ j=0,1,$$

and a simple computation shows

$$\int_{[0,1]\times M} F_{\hat{A}}^2 = \int_M (a_0\wedge da_0 - a_1\wedge da_1).$$

In particular

$$\int_{[0,1]\times M} c_1(\hat{A})^2 = -\frac{1}{4\pi^2}\int_{[0,1]\times M} F_{\hat{A}}^2 = \frac{1}{4\pi^2}\int_M (a_1\wedge da_1 - a_0\wedge da_0).$$

The quantity

$$\int_M a_j\wedge da_j$$

does not depend on the choice of flat connection B or the gauge equivalence class of A_j. We set

$$\Theta(g,\eta) := \frac{1}{4\pi^2}\int_M d^{-1}\big(*_g\eta\big)\wedge *_g\eta.$$

We deduce

$$KS(A_{g(1),\eta(1)}, g(1)) - KS(A_{g(0),\eta(0)}) = 8SF\big(\mathfrak{D}_{A_{g(s),\eta(s)}}, s\in[-1,1]\big)$$
$$+\Theta(g(1),\eta(1)) - \Theta(g(0),\eta(0)).$$

Now set

$$\mathbf{sw}_M(\sigma,g,\eta) = \frac{1}{8}\Big(KS(A_{g,\eta},g) + \Theta(g,\eta)\Big) + \mathbf{sw}'_M(\sigma,g,\eta).$$

The above observations show that $\mathbf{sw}_M(\sigma,g,\eta)$ is independent of g and η and thus it is a topological invariant of (M,σ).

3 Moduli spaces of finite energy monopoles

3.1 Finite energy monopoles on admissible 3-manifolds

Suppose $(M,\hat{g})$ is an admissible cylindrical 3-manifold with the cylindrical end isometric to $\mathbb{R}_+\times N$, where N is a disjoint union of tori equipped with flat metrics. For each $R>0$ we set

$$M_R := M\setminus(R,\infty)\times N.$$

Fix a cylindrical *spin*^{c} structure $\hat{\sigma}$ on M, set $\sigma := \partial_\infty\hat{\sigma}$ and choose a strongly cylindrical smooth reference connection $\hat{A}_0$ on $\det\hat{\sigma}$ and a compactly supported co-closed 1-form η. Arguing exactly as in the proof of [26, Prop. 4.3.2] we deduce the following result.

Lemma 3.1. *Suppose* $\hat{\mathsf{C}} = (\hat{\psi}, \hat{A})$ *is a smooth configuration. If*

$$\mathsf{C}_R := \hat{\mathsf{C}} \mid_{\partial M_R} = (\psi_R, A_R) = (\hat{\psi} \mid_{\partial M_R}, \hat{A} \mid_{\partial M_R})$$

then

$$\int_{M_R} \big(|\hat{\mathfrak{D}}_{\hat{A}} \hat{\psi}|^2 + \frac{1}{2} |\hat{\mathbf{c}}(*F_A + \mathbf{i}\eta) - \frac{1}{2} q(\hat{\psi})|^2 \big) dv(\hat{g})$$

$$= \int_{M_R} \Big(|\hat{\nabla}^{\hat{A}} \hat{\psi}|^2 + |F_{\hat{A}}|^2 + |\eta|^2 + \frac{1}{8} |q(\psi)|^2 + \frac{s}{4} |\hat{\psi}|^2 + \big\langle \hat{\mathbf{c}}(\mathbf{i}\eta), \hat{\mathbf{c}}(F_{\hat{A}}) - \frac{1}{2} q(\hat{\psi}) \big\rangle \Big) dv(\hat{g})$$

$$- \int_{\partial M_R} \langle \hat{\psi}, \mathfrak{D}_{A_R} \hat{\psi} \rangle dv(g).$$

In particular, if $\hat{\mathsf{C}}$ *is a* $(\hat{g}, \eta)$*-monopole we have*

$$2\mathfrak{E}(\mathsf{C}_R) = \int_{\partial M_R} \langle \hat{\psi}, \mathfrak{D}_{A_R} \hat{\psi} \rangle dv(g)$$

$$= \int_{M_R} \Big(|\hat{\nabla}^{\hat{A}} \hat{\psi}|^2 + |F_{\hat{A}}|^2 + |\eta|^2 + \frac{1}{8} |q(\psi)|^2 + \frac{s}{4} |\hat{\psi}|^2 + \big\langle \hat{\mathbf{c}}(\mathbf{i}\eta), \hat{\mathbf{c}}(F_{\hat{A}}) - \frac{1}{2} q(\hat{\psi}) \big\rangle \Big) dv(\hat{g})$$

where $\mathfrak{E}$ *is the energy functional discussed at the end of 1.2.*

We now define the *energy* of a configuration $\hat{\sigma}$=monopole $\hat{\mathsf{C}} = (\hat{\psi}, \hat{A})$ over a closed subset $S \subset M$ by

$$\mathbb{E}_S(\hat{\mathsf{C}}) := \int_S \Big(|\hat{\nabla}^{\hat{A}} \hat{\psi}|^2 + |F_{\hat{A}}|^2 + |\eta|^2 + \frac{1}{8} |q(\psi)|^2 + \frac{s}{4} |\hat{\psi}|^2 + \big\langle \hat{\mathbf{c}}(\mathbf{i}\eta), \hat{\mathbf{c}}(F_{\hat{A}}) - \frac{1}{2} q(\hat{\psi}) \big\rangle \Big) dv(\hat{g}).$$

A monopole $\hat{\mathsf{C}}$ is said to have *finite energy* if $\mathbb{E}_M(\hat{\mathsf{C}}) < \infty$.

To better understand the significance of the energy we will discuss in detail the special case $M = I \times T^2$, where $I \subset \mathbb{R}$ is a closed, possibly unbounded, interval, $I = [R_-, R_+]$. Suppose $\hat{\mathsf{C}} = (\hat{\psi}, \hat{A})$ is a smooth monopole on this cylinder such that $\hat{A}$ is a temporal connection, i.e. it has the form

$$\hat{A} = A_0 + \mathbf{i}a(t)$$

where A_0 is a connection on N and $[R_-, R_+] \ni t \mapsto a(t)$ is a smooth path of real 1-forms on N. In this case we can think of $\hat{\mathsf{C}}$ as a path $[R_-, R_+] \ni t \mapsto \mathsf{C}(t)$ of configurations on N and we define the *kinetic energy* of the configuration $\hat{\mathsf{C}}$ by the formula

$$\mathbb{E}_{kin}(\hat{\mathsf{C}}) = \int_{R_-}^{R_+} dt \int_N |\dot{a}(t)|^2 + |\dot{\psi}(t)|^2 dv(g) = \int_{R_-}^{R_+} dt \int_N |\dot{\mathsf{C}}(t)|^2 dv(g).$$

If $\hat{\mathsf{C}} = \mathsf{C}(t)$ is a monopole, then according to Lemma 1.1 $\mathsf{C}(t)$ defines a flow line of the gradient of $\mathfrak{E}$,

$$\dot{\mathsf{C}} = \nabla \mathfrak{E}(\mathsf{C}).$$

Thus

$$\mathbb{E}_{kin}(\hat{\mathsf{C}}) = \mathfrak{E}(\mathsf{C}(R_+)) - \mathfrak{E}(\mathsf{C}(R_-)).$$

Using Lemma 3.1 we now deduce

$$\mathbb{E}_{kin}(\hat{\mathsf{C}}) = \mathfrak{E}(\mathsf{C}(R_+)) - \mathfrak{E}(\mathsf{C}(R_-)) = \frac{1}{2} \mathbb{E}(\hat{\mathsf{C}}). \tag{3.1}$$

The following result describes one important source of finite energy monopoles.

Proposition 3.2. *Suppose X is a closed, oriented 3-manifold decomposed into two manifolds with boundary Y diffeomorphic to a disjoint union of tori. Fix a metric $\hat{g}$ on X such that a cylindrical neighborhood of Y in X is isometric to the cylinder $([-1,1]\times Y, dt^2+g)$ where g is a flat metric on M. Denote the complement of $(-1,1)\times Y$ in X by X_0.*

For $R\gg 0$ denote by $(X_R,\hat{g}_R)$ the Riemann manifold obtained from (X,hg) by replacing the cylinder $C_R := ([-1,1]\times Y, dt^2+g)$ with the longer one $([-R,R]\times Y, dt^2+g)$. Then there exists a positive constant $C>0$ such that for all $R\gg 0$ and all $\hat{g}_R$-monopole $\hat{\mathsf{C}}$ we have

$$\mathbb{E}_{C_R}(\hat{\mathsf{C}}) < C.$$

Proof Set $\hat{\mathsf{C}} = (\hat{\psi},\hat{A})$. Since the scalar curvature s_R of g_R is $O(1)$ as $R\to\infty$ we deduce that there exists $C>0$ such that

$$\|\hat{\psi}_R\|_\infty < C.$$

Using Lemma 3.1 we deduce

$$\mathbb{E}_{X_0}(\hat{\mathsf{C}}) + \mathbb{E}_{C_R}(\hat{\mathsf{C}}) = \mathbb{E}_{X_R}(\hat{\mathsf{C}}_0)$$

$$= \int_{X_R} \big(|\hat{\mathfrak{D}}_{\hat{A}}\hat{\psi}|^2 + \frac{1}{2}|\hat{\mathbf{c}}(*F_A) - \frac{1}{2}q(\hat{\psi})|^2\big)dv(\hat{g}) = 0.$$

Hence

$$\mathbb{E}_{C_R}(\hat{\mathsf{C}}) = -\mathbb{E}_{X_0}(\hat{\mathsf{C}}) \leq -\int_{X_0}\frac{s_R}{4}|\hat{\psi}|^2dv(\hat{g})$$

$$\leq \frac{1}{4}\|s_R\|_\infty\cdot\|\hat{\psi}\|_\infty^2\cdot \mathrm{vol}(X_0) \leq C. \;\blacksquare$$

The finite energy monopoles have a nice asymptotic behavior. First a bit of notation. Denote by $[\mathsf{C}]$ (resp. $[\hat{\mathsf{C}}]$) the $\mathcal{G}_N$-orbit (resp. $\widehat{\mathcal{G}}_M$-orbit) of a configuration. Denote by $\mathcal{G}_N^0$ (resp. $\widehat{\mathcal{G}}_M^0$) the identity component of $\mathcal{G}_N$ (resp. $\widehat{\mathcal{G}}_M$). We will use the notation $[\bullet]_0$ to denote $\mathcal{G}_N^0$ or $\widehat{\mathcal{G}}_M^0$-orbits.

Theorem 3.3. ([6, Carey-Marcolli-Wang] , [7, Chen]) *Consider an admissible 3-manifold $(M,\hat{g})$ with a cylindrical end isometric to $(\mathbb{R}_+\times N, dt^2+g)$ where N is a disjoint union of tori and g is a flat metric.*

Fix a cylindrical spin^c structure $\hat{\sigma}$ on M, set $\sigma := \partial_\infty\hat{\sigma}$ and pick a co-closed 1-form η on M supported away from the cylindrical end. If $\hat{\mathsf{C}}$ is a finite energy $(\hat{\sigma},\hat{g},\eta)$-monopole on $(M,\hat{g})$ then there exist a gauge transformation $\hat{\gamma}$ in the identity component of $\widehat{\mathcal{G}}_M$ and a critical point C_∞ of $\mathfrak{E}_\sigma$ such that

$$\Upsilon_\sigma(\mathsf{C}_\infty) = 0 \;\text{ and }\; \lim_{t\to\infty}\big\|(\hat{\gamma}\cdot\hat{\mathsf{C}})\mid_{t\times N} - \mathsf{C}_\infty\big\|_{L^2(N)} = 0$$

where we recall that Υ_σ is the map $\mathcal{C}_\sigma\to\mathbf{i}\Omega^0(N)$ given by (see 1.2)

$$\Upsilon_\sigma(\psi,A) = *F_A - \frac{\mathbf{i}}{4}\Big(|\psi_+|^2 - |\psi_-|^2\Big).$$

The asymptotic limit map $[\hat{\mathsf{C}}]_0\mapsto[\mathsf{C}_\infty]$ is called the asymptotic limit map and will be denoted by ∂_∞.

The above theorem shows that we need to better understand the set of critical points of $\mathfrak{E}_\sigma$ which lie on the level set $\Upsilon_\sigma^{-1}(0)$. These are known as *σ-vortices* on N.

Proposition 3.4. *A configuration $(\psi, A) = (\psi_+, \psi_-, A)$ is a σ-vortex on N if and only if*

$$c_1(\det \sigma) = 0, \;\; \psi_+ = \psi_- = 0 \;\; \textit{and} \;\; F_A = 0.$$

Proof Let us first observe that a Hermitian connection A on $\det\sigma$ induces a connection, still denoted by A, on $\mathbb{S}_\sigma$ compatible with the natural $\mathbb{Z}_2$-grading $\mathbb{S}_\sigma = \mathbb{S}_\sigma^+ \oplus \mathbb{S}_\sigma^-$. This implies that A induces connections A_+ on $\mathbb{S}_+ =: L$ and A_- on $\mathbb{S}_\sigma^- \cong K_N^* \otimes L$. Thus A_+ defines a holomorphic structure on the line bundle L and we denote by (L, A_+) the resulting holomorphic line bundle. Similarly A_- defines a holomorphic line bundle $(K_N^* \otimes L, A_-)$ and, moreover,

$$(K_N^* \otimes L, A_-) \cong_{biholo} K_N^* \otimes (L, A_+).$$

The configuration (ψ_+, ψ_-, A) is a critical point of $\mathfrak{E}_\sigma$ if and only if (see 1.2)

$$\begin{cases} \bar{\partial}_{A_+}\psi_+ = 0 \\ \bar{\partial}^*_{A_-}\psi_- = 0 \\ \psi_+ \otimes \bar{\psi}_- + \bar{\psi}_+ \otimes \psi_- = 0 \end{cases} \tag{3.2}$$

The first equation implies that ψ_+ is a holomorphic section of (L, A_+). The second equation implies that $\bar{\psi}_-$ is a holomorphic section of $(K_N^* \otimes L, A_-)^*$. Since N is a disjoint union of tori we deduce that the canonical line bundle K_N is holomorphically trivial and thus

$$(K_N^* \otimes L, A_-) \cong_{biholo} (L, A_+).$$

Thus $\bar{\psi}_-$ is a holomorphic section of $(L, A_+)^*$. The third equation in (3.2) implies that $\psi_+ \otimes \bar{\psi}_- \equiv 0$. The unique continuation principle for holomorphic objects implies that at least one of the sections ψ_+ or ψ_- is trivial. We want to show that both must be trivial. We argue by contradiction.

Suppose $\psi_+ \not\equiv 0$. (The case $\psi_- \not\equiv 0$ can be dealt with in a similar fashion.) This implies that $\psi_- = 0$ and $\deg L \geq 0$. On the other hand, using the condition $\Upsilon(\psi_+, \psi_-, A) = 0$ we deduce

$$F_A = \frac{\mathbf{i}}{4}|\psi_+|^2 dv(g)$$

so that

$$0 \leq \deg L = \frac{\mathbf{i}}{2\pi}\int_N F_A = -\frac{1}{8\pi}\int_N |\psi_+|^2 dv(g) < 0.$$

Thus $\psi_+ \equiv \psi_- \equiv 0$ and the condition $\Upsilon(\psi_+, \psi_-, A) = 0$ implies $F_A = 0$ and $c_1(\det\sigma) = 0$.

■

Denote by σ_0 the unique $spin^c$ structure on N such that $c_1(\det\sigma_0)=0$. We denote the set of σ_0-vortices by $\mathcal{Z}$. It consists of configurations $\mathsf{C}=(\psi,A)$ such that $\psi=0$, $F_A=0$. Now denote by $\mathfrak{M}$ (resp. $\mathfrak{M}^0$) the set of $\mathcal{G}_N$-orbits (resp. $\mathcal{G}_N^0$-orbits) of σ_0-vortices. The last result shows that $\mathfrak{M}$ can be identified with the set flat connections on the trivial line bundle on N modulo the action of *even* gauge transformations. Thus $\mathfrak{M}$ can be identified with the union of tori

$$H^1(N,\mathbb{R})/H^1(N,4\pi\mathbb{Z}).$$

In the remainder of this paper we will assume that N is connected, i.e. we will consider exclusively admissible manifolds with connected ends.

With this convention in place we see that we can identify $\mathfrak{M}$ with a 2-torus. We can produce angular coordinate on $\mathfrak{M}$ as follows.

A. Fix a trivialization of $L=\mathbb{S}^+_\sigma$ and denote by A_0 the associated trivial connection. (This is tantamount to fixing a *spin* structure on N.)

B. Fix a basis $\{\vec{\mu},\vec{\lambda}\}$ of $H_1(N,\mathbb{Z})$.

C. if $(0,A)$ is a σ_0-vortex then we set

$$\mathbf{i}\theta(A):=\int_{\vec{\mu}}(A-A_0)\ \text{ and }\ \mathbf{i}\varphi(A):=\int_{\vec{\lambda}}(A-A_0).$$

We can be more specific[1] about the choice **A**. On N there are four *spin* structures, $\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3$. The *spin* structure ϵ_0 is canonically determined from the Lie group trivialization of TN. Equivalently, it is the only *spin* structure on the torus N which is not *Spin*-bordant. The line bundle $\mathbb{S}^+_{\epsilon_0}$ is naturally trivialized. The Levi-Civita connection on (TN,g) is the trivial connection because g is flat. This connection induces the trivial connection on the spin bundle $\mathbb{S}_{\epsilon_0}=\mathbb{S}^+_{\epsilon_0}\oplus\mathbb{S}^-_{\epsilon_0}$ and thus induces the trivial connection A_0 on L. In the sequel, the choice **A** will always be determined by the *spin* structure ϵ_0. We want to describe a few more analytical features of this choice.

Denote by $\mathfrak{D}_0$ the complex *spin*-Dirac operator corresponding to the *spin* structure ϵ_0. This operator is none other than the Hodge-Dolbeault operator

$$\mathfrak{D}_0=\left[\begin{array}{cc}0 & \mathbf{i}\bar{\partial}^*\\ \bar{\partial} & 0\end{array}\right]:\Omega^{0,*}(N)\to\Omega^{0,*}(N).$$

Similarly we obtain flat connections A_k, $k=0,1,2$ on L and Dirac operators $\mathfrak{D}_k$, $k=1,2,3$, corresponding to the *spin* structures ϵ_k, $k=1,2,3$. The *spin* structures $\epsilon_0,\cdots,\epsilon_3$ canonically induce $spin^c$ structures, all isomorphic to the $spin^c$ structure σ_0. The Dirac operators $\mathfrak{D}_0,\cdots,\mathfrak{D}_3$ correspond to different choices of connections on σ_0. More precisely, the Dirac operator $\mathfrak{D}_i$ is obtained using the connection $A_i^{\otimes 2}$ induced by A_i on $\det\sigma_0\cong L^2$, $i=0,\cdots,3$. These operators can also be described as Hodge-Dolbeault operators coupled with the connection A_i on L. Only one of these four Dirac operators has nontrivial kernel, namely $\mathfrak{D}_0$ because only one of the four holomorphic line bundles (L,A_i), $i=0,\cdots,3$, admits (anti)-holomorphic sections.

[1] I am indebted to Stephan Stolz for clarifying this rather confusing point.

The σ-vortices are of two types: **good** and **bad**. By definition, a vortex (ψ, A) is good if and only if $\ker \mathfrak{D}_A = 0$. Otherwise the vortex is called bad. If $(0, A)$ is a bad vortex then the holomorphic line bundle (L, A_+) or $(L, A_+)^*$ admits nontrivial holomorphic sections. This is possible only when (L, A_+) is holomorphically trivial, i.e. A_+ coincides with the connection A_0 introduced above. This shows that there is only one bad point in $\mathfrak{M}$, namely the orbit of $\mathsf{C}_0 := (0, A_0^{\otimes 2})$. It has coordinates $(\mu, \lambda) = (0, 0)$.

The moduli space $\mathfrak{M}^0$ can be identified with $H_1(N, \mathbb{R})$. It covers $\mathfrak{M}$ and the unique bad point in $\mathfrak{M}$ lifts to the lattice of bad points $H^1(N, 4\pi\mathbb{Z}) \subset H^1(N, \mathbb{R})$. The role of good vortices is explained in the following refinement of Theorem 3.3.

Theorem 3.5. ([6, Carey-Marcolli-Wang], [7, Chen]) *Suppose* $\hat{\mathsf{C}} = (\hat{\psi}, \hat{A})$ *is a smooth finite energy monopole. Set* $\mathsf{C} = (0, A) := \partial_\infty \hat{\mathsf{C}}$ *and*

$$\delta(A) := \operatorname{dist}\bigl(\operatorname{spec}\mathfrak{D}_A, 0\bigr).$$

If $\delta(A) > 0$, *so that* C *is a good vortex, then there exists a gauge transformation* $\hat{\gamma} \in \widehat{\mathcal{G}}^0$ *such that*

$$\hat{\gamma} \cdot \hat{\mathsf{C}} \in L^{2,2}_{\mu,ex}, \ \ \forall 0 < \mu < \delta(A).$$

In the next subsections we will use this result to describe the local structure of the moduli space of finite energy monopoles with *good* asymptotic limit.

3.2 Local structure

In this subsection we will study in detail the set of finite energy monopoles with good asymptotic limit. We follow closely the approach in [26]

Consider an admissible 3-manifold $(M, \hat{g})$ with

$$\partial_\infty M =: N \cong T^2, \ \ g := \partial_\infty \hat{g} - \text{flat metric}.$$

Fix a cylindrical *spin*^{c} structure $\hat{\sigma}$ on M such that $\partial_\infty \hat{\sigma} = \sigma_0$ and a strongly cylindrical connection $\hat{A}_0$ such that $\partial_\infty \hat{A}_0 = A_0$.

We need a suitable functional setup. For every positive number μ and set the space of configurations $\hat{\mathsf{C}} = (\hat{\psi}, \hat{A})$ such that

$$\hat{\mathcal{C}}_{\mu,ex} = \hat{\mathcal{C}}_{\mu,ex}(\hat{\sigma}) := \Bigl\{ \hat{\mathsf{C}} = (\hat{\psi}, \hat{A}); \ \ \partial_\infty \hat{\mathsf{C}} \in \mathcal{Z} \Bigr\}$$

$$= \Bigl\{ \hat{\mathsf{C}}; \ \ (\hat{\psi}, \hat{A} - \hat{A}_0) \in L^{2,2}_{\mu}(\mathbb{S}_{\hat{\sigma}}) \oplus L^{2,2}_{\mu,ex}(\mathbf{i}\Lambda^1 T^* M), \ \ d\partial_\infty(\hat{A} - \hat{A}_0) = 0 \Bigr\}.$$

Note in particular that if $(\psi, \hat{A}) \in \hat{\mathcal{C}}_{\mu,ex}$ then the connection $\hat{A}$ is asymptotically strongly cylindrical. Define

$$\widehat{\mathcal{G}}_{\mu,ex} := L^{3,2}_{\mu,ex}(M, S^1), \ \ \widehat{\mathcal{G}}_{\mu} := L^{3,2}_{\mu}(M, S^1)$$

and set

$$\mathcal{G}^{\partial} := \partial_\infty \widehat{\mathcal{G}}_{\mu,ex} \subset \mathcal{G} := \mathcal{G}_N.$$

If $*$ is a point on $\partial_\infty M$ we denote by $\widehat{\mathcal{G}}_{\mu,ex}(*)$ (resp. $\mathcal{G}^\partial(*)$ the based versions of these groups, consisting of gauge transformations $= 1$ at $*$. The gauge group $\mathcal{G}^\partial$ consists of $L^{3,2}$-gauge transformations on N which extend to $L^{3,2}_{\mu,ex}$-gauge transformations on M. Denote by $\hat{\mathfrak{C}}$ the (discrete) group of components of $\widehat{\mathcal{G}}_{\mu,ex}$ and by $\mathfrak{C}^\partial$ the group of components of $\mathcal{G}^\partial$. Note that $\hat{\mathfrak{C}} \cong H^1(M,\mathbb{Z})$ while $\mathfrak{C}^\partial$ can be identified with the image of $H^1(M, 4\pi\mathbb{Z})$ in $H^1(\partial_\infty M, 4\pi\mathbb{Z}) \cong (4\pi\mathbb{Z})^2$.

For any configuration $\mathsf{C} = (0, A) \in \mathcal{Z}$ we have an infinitesimal action

$$\mathfrak{L}_{\mathsf{C}} : T_{\mathbf{1}}\mathcal{G}^0 \cong L^{3,2}(\mathbf{i}\Lambda^0 T^*N) \to T_{\mathsf{C}}\mathcal{C} \cong L^{2,2}(\mathbb{S} \oplus \mathbf{i}\Lambda^1 T^*N)$$

given by

$$\mathfrak{L}_{\mathsf{C}}(\mathbf{i}f) := \frac{d}{dt}\,|_{t=0}\, e^{\mathbf{i}tf} \cdot \mathsf{C} = (0, -2\mathbf{i}df).$$

Its formal adjoint is given by

$$T_{\mathsf{C}}\mathcal{C} \ni (\dot\psi, \mathbf{i}\dot a) \mapsto \mathfrak{L}^*_{\mathsf{C}}(\dot\psi, \mathbf{i}\dot a) = -2\mathbf{i}d^*\dot a \in T_{\mathbf{1}}\mathcal{G}^0.$$

Define the *slice at* C to be the closed subspace $\mathcal{S}_{\mathsf{C}_\infty} \subset T_{\mathsf{C}_\infty}\mathcal{Z}$ defined by

$$\mathcal{S}_{\mathsf{C}} := \Big\{ \dot{\mathsf{C}} \in T_{\mathsf{C}}\mathcal{C};\ \ \mathfrak{L}^*_{\mathsf{C}}\dot{\mathsf{C}} = 0 \Big\} \cong \ker\Big(\Delta : \Omega^1(N) \to \Omega^1(N)\Big).$$

Denote by $[\bullet]_\partial$ the $\mathcal{G}^\partial$-orbit of $\bullet$. We have the following standard local structure result.

Proposition 3.6. *The $L^{2,2}$-metric on $\mathcal{Z}$ induces a metric on the quotient $\mathfrak{M}^\partial := \mathcal{Z}/\mathcal{G}^\partial$ defined by*

$$\mathrm{dist}_{2,2}([\mathsf{C}_1]_\partial, [\mathsf{C}_2]) = \inf_{\gamma \in \mathcal{G}^\partial} \|\mathsf{C}_1 - \gamma \cdot \mathsf{C}_2\|_{2,2}.$$

Moreover

$$\mathfrak{M}^\partial \cong H^1(M,\mathbb{R})/\mathfrak{C}^\partial.$$

We want to study the local structure of the quotient $\hat{\mathcal{B}}_{\mu,ex} := \hat{\mathcal{C}}_{\mu,ex}/\widehat{\mathcal{G}}_{\mu,ex}$. Fix a configuration $\hat{\mathsf{C}}_0 := (\hat\psi, \hat A) \in \hat{\mathcal{C}}_{\mu,ex}$ and set $\mathsf{C}_\infty = (\psi_\infty, A_\infty) = \partial_\infty \hat{\mathsf{C}}_0 \in \mathcal{Z}$. Set

$$G_\infty = \mathbf{Stab}\,(\mathsf{C}_\infty) \cong S^1,\ \ \hat G_0 := \mathbf{Stab}(\hat{\mathsf{C}}_0)$$

and

$$\hat{\mathcal{S}}_{\hat{\mathsf{C}}_0} := \Big\{ \underline{\hat{\mathsf{C}}} \in T_{\hat{\mathsf{C}}_0}\mathcal{C}_{\mu,ex};\ \ \mathfrak{L}^{*_\mu}_{\hat{\mathsf{C}}_0}\underline{\hat{\mathsf{C}}} = 0 \Big\}$$

where $*_\mu$ denotes the L^2_μ-adjoint. Observe that since every $\underline{\hat{\mathsf{C}}} = (\underline{\hat\psi}, \mathbf{i}\underline{\hat a})$ is asymptotically *strongly* cylindrical we have

$$\partial_\infty \mathfrak{L}^*_{\hat{\mathsf{C}}_0}\underline{\hat{\mathsf{C}}} = \mathfrak{L}^*_{\partial_\infty\hat{\mathsf{C}}_0}\partial_\infty\underline{\hat{\mathsf{C}}}$$

so that

$$\partial_\infty \hat{\mathcal{S}}_{\hat{\mathsf{C}}_0} \subset \mathcal{S}_{\partial_\infty \hat{\mathsf{C}}_0}.$$

Arguing as in the proof of [26, Proposition 4.3.7] we deduce the following result.

Proposition 3.7. *(a) There exists a small $\hat{G}_0$-invariant neighborhood $\hat{V}$ of $\hat{\mathsf{C}}_0$ in $\hat{\mathsf{C}}_0 + \hat{\mathcal{S}}_{\hat{\mathsf{C}}_0}$ such that every $\widehat{\mathcal{G}}_{\mu,ex}(\mathsf{C}_\infty)$-orbit intersects $\hat{\mathsf{C}}_0 + \hat{V}$ along at most one $\hat{G}_0$-orbit.*

(b) There exists a $L^{2,2}_{\mu,ex}$-small neighborhood $\hat{U}_0$ of 0 in $\hat{\mathcal{S}}_{\hat{\mathsf{C}}_0}$ such that some neighborhood of $[\hat{\mathsf{C}}_0]$ in $\mathcal{B}_{\mu,ex}$ (equipped with the quotient topology) is homeomorphic to the quotient $\hat{U}_0/\hat{G}_0$.

(c)The based quotient $\mathcal{B}_{\mu,ex}(*) = \hat{\mathcal{C}}_{\mu,ex}/\widehat{\mathcal{G}}_{\mu,ex}(*)$ is a smooth Hilbert manifold equipped with a smooth S^1-action and a neighborhood of $\hat{\mathsf{C}}_0$ in this based quotient is S^1-equivariantly diffeomorphic to the quotient

$$(S^1 \times \hat{U}_0)/\hat{G}_0$$

where $\hat{G}_0$ acts diagonally on the above product.

Definition 3.8. *The neighborhood $\hat{U}_0$ constructed above is called a* local slice *at $\hat{\mathsf{C}}_0$.*

Denote by $\hat{\mathcal{Z}}_\mu = \hat{\mathcal{Z}}(\hat{\sigma})$ the set of finite energy monopoles $\hat{\mathsf{C}} = (\hat{\psi}, \hat{A})$ such that $\hat{\mathsf{C}} \in \hat{\mathcal{C}}_{\mu,ex}$. It can be described as the zero set of the Seiberg-Witten map[2]

$$\mathbf{SW} : \hat{\mathcal{C}}_{\mu,ex} \to L^{1,2}_\mu(\mathbb{S}_{\hat{\sigma}} \oplus \mathbf{i}T^*M), \;\; (\hat{\psi}, \hat{A}) \mapsto \big(\mathfrak{D}_{\hat{A}}\psi, \frac{1}{2}q(\hat{\psi}) - (*F_{\hat{A}} + \mathbf{i}\eta)\big).$$

More rigorously, $\mathbf{SW}$ should be regarded as a $\widehat{\mathcal{G}}_{\mu,ex}$-equivariant section of the $\widehat{\mathcal{G}}_{\mu,ex}$ equivariant Hilbert vector bundle

$$L^{1,2}_\mu(\mathbb{S}_{\hat{\sigma}} \oplus \mathbf{i}T^*M) \times \hat{\mathcal{C}}_{\mu,ex} \to \hat{\mathcal{C}}_{\mu,ex}.$$

If $K \subset \mathfrak{M}^0$ is a compact subset not containing any bad vortex then, according to Theorem 3.5, we can find a positive number $\mu = \mu(K)$ such that if $\hat{\mathsf{C}}$ is a finite energy monopole with $\partial_\infty[\hat{\mathsf{C}}]_0 \in K$ then $\hat{\mathsf{C}}$ is gauge equivalent to a monopole in $\hat{\mathcal{Z}}_\mu(\hat{\sigma})$. To describe the local structure of the moduli space $\widehat{\mathfrak{M}}_\mu := \hat{\mathcal{Z}}_\mu/\widehat{\mathcal{G}}_{\mu,ex}$ we will study as in [16] the deformation theory of a different nonlinear equation which is equivalent to the Seiberg-Witten equations. Set

$$\mathcal{X} := \Big\{\mathbf{i}f \in L^{2,2}_{\mu,ex}(M\mathbf{i}\mathbb{R});\;\; d(\partial_\infty f) = 0\Big\}.$$

Define

$$\mathcal{F} : \hat{\mathcal{C}}_{\mu,ex} \times \mathcal{X} \to L^{1,2}_\mu(\mathbb{S}_{\hat{\sigma}} \oplus \mathbf{i}T^*M)$$

by

$$(\hat{\mathsf{C}}, \mathbf{i}f) \mapsto \mathbf{SW}(\hat{\mathsf{C}}) - \frac{1}{2}\mathfrak{L}_{\hat{\mathsf{C}}}(\mathbf{i}f).$$

We let the group $\widehat{\mathcal{G}}_{\mu,ex}$ act trivially on $\mathcal{X}$ and thus we can regard $\mathcal{F}$ as a $\widehat{\mathcal{G}}_{\mu,ex}$-equivariant section of the $\widehat{\mathcal{G}}_{\mu,ex}$-equivariant bundle

$$L^{1,2}_\mu(\mathbb{S}_{\hat{\sigma}} \oplus \mathbf{i}T^*M) \times \Big(\hat{\mathcal{C}}_{\mu,ex} \times \mathcal{X}\Big) \to \hat{\mathcal{C}}_{\mu,ex} \times \mathcal{X}.$$

Notice that

$$\Big\langle \mathbf{SW}(\hat{\mathsf{C}}), \mathfrak{L}_{\hat{\mathsf{C}}}(\mathbf{i}f)\Big\rangle_{L^2(M)} = 0. \tag{3.3}$$

This implies the following result.

[2]Note that if $\hat{\mathsf{C}}$ is a $L^{2,2}_{\mu,ex}$-configuration with $\partial_\infty\hat{\mathsf{C}} \in \mathcal{Z}$ then indeed $SW(\hat{\mathsf{C}})$ is in $L^{1,2}_\mu$, i.e. $\partial_\infty SW(\hat{\mathsf{C}}) = 0$.

Proposition 3.9. *The natural map*

$$\hat{\mathcal{Z}}_\mu := \mathbf{SW}^{-1}(0) \to \mathcal{F}^{-1}(0), \quad \hat{\mathsf{C}} \mapsto (\hat{\mathsf{C}}, 0)$$

is $1-1$*. Moreover,* $\mathcal{F}(\hat{\mathsf{C}}, \mathbf{i}f) = 0$ *if and only if* $\mathbf{SW}(\hat{\mathsf{C}}) = \mathfrak{L}_{\hat{\mathsf{C}}}(\mathbf{i}f) = 0$.

The above simple observation shows that the local structure of $\mathbf{SW}^{-1}(0)/\widehat{\mathcal{G}}_{\mu,ex}$ is identical to the local structure of $\mathcal{F}^{-1}(0)/\widehat{\mathcal{G}}_{\mu,ex}$.

Definition 3.10. *The space*

$$\mathcal{F}^{-1}(0)/\widehat{\mathcal{G}}_{\mu,ex} \subset (\mathcal{C}_{\mu,ex} \times \mathcal{X})/\widehat{\mathcal{G}}_{\mu,ex}$$

is called the **extended moduli space***. We will denote it by* $\mathcal{M}_\mu$.

Fix a smooth monopole $\hat{\mathsf{C}} = (\hat{\psi}, \hat{A}) \in \hat{\mathcal{Z}}_\mu$ and set $\mathsf{C}_\infty = (0, A_\infty) := \partial_\infty \hat{\mathsf{C}} \in \mathcal{Z}$. The local structure of $\mathcal{F}^{-1}(0)/\widehat{\mathcal{G}}_{\mu,ex}$ near $\hat{\mathsf{C}}$ can be read off the deformation complex

$$0 \to \mathbf{E}^0 := T_{\mathbf{1}}\widehat{\mathcal{G}}_{\mu,ex} \xrightarrow{\mathfrak{L}_{\hat{\mathsf{C}}}\oplus 0} \mathbf{E}^1_{\hat{\mathsf{C}}} := T_{\hat{\mathsf{C}}}\mathcal{C}_{\mu,ex} \oplus \mathcal{X} \xrightarrow{D\mathcal{F}} \mathbf{E}^2_{\hat{\mathsf{C}}} := L^{1,2}_\mu(\mathbb{S}_{\hat{\sigma}} \oplus \mathbf{i}T^*M) \to 0 \qquad (\mathbf{E})$$

where $D\mathcal{F}$ denotes the linearization of $\mathcal{F}$ at $\hat{\mathsf{C}}$. To ease the presentation we will denote More precisely

$$D\mathcal{F}(\underline{\hat{\mathsf{C}}}, \mathbf{i}\dot{f}) = \underline{\mathbf{SW}}(\underline{\hat{\mathsf{C}}}) - \frac{1}{2}\mathfrak{L}_{\hat{\mathsf{C}}}(\mathbf{i}\dot{f})$$

where $\underline{\mathbf{SW}}$ denotes the linearization of $\mathbf{SW}$ at $\hat{\mathsf{C}}$. Observe that $(\mathbf{E})$ fits inside the short exact sequence of complexes

$$0 \to \mathbf{F} \to \mathbf{E} \xrightarrow{\partial_\infty} \mathbf{B} \to 0 \qquad (3.4)$$

where

$$0 \to \mathbf{F}^0_{\hat{\mathsf{C}}} := T_{\mathbf{1}}\widehat{\mathcal{G}}_\mu \xrightarrow{\mathfrak{L}_{\hat{\mathsf{C}}}\oplus 0} \mathbf{F}^1_{\hat{\mathsf{C}}} := L^{2,2}_\mu\big(\mathbb{S}_{\hat{\sigma}} \oplus \mathbf{i}(\Lambda^1 \oplus \Lambda^0)T^*M\big) \xrightarrow{D\mathcal{F}} \mathbf{F}^2_{\hat{\mathsf{C}}} := L^{1,2}_\mu(\mathbb{S}_{\hat{\sigma}} \oplus \mathbf{i}T^*M) \to 0 \qquad (\mathbf{F})$$

and

$$0 \to \mathbf{B}^0_{\mathsf{C}_\infty} := T_{\mathbf{1}}\mathcal{G}^\partial \xrightarrow{\mathfrak{L}_{\mathsf{C}_\infty}} \mathbf{B}^1_{\mathsf{C}_\infty} := T_{\mathsf{C}_\infty}\mathcal{Z} \oplus \mathbf{i}\mathbb{R} \to \mathbf{B}^2_{\mathsf{C}_\infty} := 0 \to 0. \qquad (\mathbf{B})$$

Clearly $(\mathbf{B})$ is a Fredholm complex and its cohomology is given by

$$H^0(\mathbf{B}_{\mathsf{C}_\infty}) \cong T_{\mathbf{1}}\mathbf{Stab}\,(\mathsf{C}_\infty) \cong \mathbf{i}\mathbb{R}$$

and

$$H^1(\mathbf{B}_{\mathsf{C}_\infty}) \cong \mathbf{H}^1(N, g) \oplus \mathbf{H}^0(N, g).$$

To study the Fredholm properties of $(\mathbf{F})$ we need to understand the Fredholm properties of the operator

$$\mathcal{T}_{\hat{\mathsf{C}},\mu} := D\mathcal{F} \oplus (\mathfrak{L}_{\hat{\mathsf{C}}} \oplus 0)^{*_\mu} : L^{2,2}_\mu\big(\mathbb{S}_{\hat{\sigma}} \oplus \mathbf{i}(\Lambda^1 \oplus \Lambda^0)T^*M\big) \to L^{1,2}_\mu(\mathbb{S}_{\hat{\sigma}} \oplus \mathbf{i}T^*M) \oplus L^{1,2}_\mu(M, \mathbf{i}\mathbb{R})$$

where $*_\mu$ denotes the adjoint with respect to the L^2_μ-metric. More explicitly, $\mathcal{T}_{\hat{\mathsf{C}},\mu}$ is described by

$$L^{2,2}_\mu\left(\begin{array}{c}\mathbb{S}_{\hat{\sigma}}\\ \oplus\\ \mathbf{i}\Lambda^1T^*M\\ \oplus\\ \mathbf{i}\Lambda^0T^*M)\end{array}\right)\ni\begin{bmatrix}\underline{\hat{\psi}}\\ \\ \mathbf{i}\underline{\hat{a}}\\ \\ \mathbf{i}f\end{bmatrix}\mapsto\mathcal{T}_{\hat{\mathsf{C}},\mu}\begin{bmatrix}\underline{\hat{\psi}}\\ \\ \mathbf{i}\underline{\hat{a}}\\ \\ \mathbf{i}f\end{bmatrix}$$

$$=\begin{bmatrix}\mathfrak{D}_{\hat{A}} & 0 & 0\\ \\ 0 & -\hat{*}\hat{d} & \hat{d}\\ \\ 0 & \hat{d}^{*_\mu} & 0\end{bmatrix}\cdot\begin{bmatrix}\underline{\hat{\psi}}\\ \\ \mathbf{i}\underline{\hat{a}}\\ \\ \mathbf{i}f\end{bmatrix}+\begin{bmatrix}\frac{1}{2}\hat{\mathbf{c}}(\mathbf{i}\underline{\hat{a}})\hat{\psi}-\frac{\mathbf{i}}{2}f\hat{\psi}\\ \\ \frac{1}{2}\dot{q}(\hat{\psi},\underline{\hat{\psi}})\\ \\ \frac{\mathbf{i}}{2}\,\mathbf{Im}\langle\hat{\psi},\underline{\hat{\psi}}\rangle\end{bmatrix}\in L^{1,2}_\mu\left(\begin{array}{c}\mathbb{S}_{\hat{\sigma}}\\ \oplus\\ \mathbf{i}\Lambda^1T^*M\\ \oplus\\ \mathbf{i}\Lambda^0T^*M)\end{array}\right).$$

We denote the first operator above by $\mathcal{T}^0_{\hat{\mathsf{C}},\mu}$ and the second one by $\mathcal{P}_{\hat{\mathsf{C}}}$. Observe that $\mathcal{T}^0_{\hat{\mathsf{C}},\mu}$ is precisely the direct sum $\mathfrak{D}_{\hat{A}}\oplus-\mathbf{SIGN}_\mu$, where $-\mathbf{SIGN}_\mu$ is described in detail in Appendix A. Both operator $\mathcal{T}_{\hat{\mathsf{C}},\mu}$ and $\mathcal{T}^0_{\hat{\mathsf{C}},\mu}$ are APS operators and in this case

$$\vec{\partial}_\infty\mathcal{T}_{\hat{\mathsf{C}},\mu}=\vec{\partial}_\infty\mathcal{T}^0_{\hat{\mathsf{C}},\mu}=\mathfrak{D}_{A_\infty}\oplus\vec{\partial}_\infty(-\mathbf{SIGN}_\mu).$$

We will use the nation in Appendix A, $\mathcal{H}_\mu:=\vec{\partial}_\infty\mathbf{SIGN}_\mu=\vec{\partial}_\infty(-\mathbf{SIGN}_\mu)$. The results in [18] show that $\mathcal{T}_{\hat{\mathsf{C}},\mu}$ is Fredholm if $-\mu$ is not an eigenvalue of $\mathfrak{D}_{A_\infty}\oplus\mathcal{H}_\mu$. Proposition A.1 shows that if $\mu^2<\lambda_1(N)/16$, where $\lambda_1(N)$ is the first nonzero eigenvalue of the scalar Laplacian on N, then $-\mu$ is not an eigenvalue of $\mathcal{H}_\mu$. Thus $\mathcal{T}_{\hat{\mathsf{C}},\mu}$ is a Fredholm operator provided $\mu^2<\lambda_1/16$ and $-\mu$ is not an eigenvalue of $\mathfrak{D}_{A_\infty}$.

For every $0<\delta<\frac{\sqrt{\lambda_1}}{4}$ we set

$$\mathcal{Z}_\delta:=\Big\{(0,A)\in\mathcal{Z};\ \ \delta(A)>\delta\Big\}$$

where we recall that $\delta(A)$ denotes the spectral gap of the operator $\mathfrak{D}_A$ defined in Theorem 3.5.

Proposition 3.11. *If $0<\mu<\delta<\frac{\sqrt{\lambda_1}}{4}$ then for any finite energy monopole $\hat{\mathsf{C}}\in\partial_\infty^{-1}(\mathcal{Z}_\delta)$ the associated complex ($\mathbf{E}$) is Fredholm and its Euler characteristic satisfies*

$$\chi(\mathbf{E})=\chi(\mathbf{F})+\chi(\mathbf{B})=-\mathrm{ind}\mathcal{T}_{\hat{\mathsf{C}},\mu}-1.$$

Set

$$\widehat{\mathfrak{M}}_{\mu,\delta}:=\partial_\infty^{-1}(\mathcal{Z}_\delta)\cap\hat{\mathcal{Z}}_\mu/\hat{\mathcal{G}}_{\mu,ex}.$$

Observe that moduli space of finite energy monopoles with **good** asymptotic limit is covered by the open pieces $(\widehat{\mathfrak{M}}_{\mu,\delta})_{\delta\searrow0}$. If $\hat{\mathsf{C}}_0\in\hat{\mathcal{Z}}_{\mu,\delta}:=\partial_\infty^{-1}(\mathcal{Z}_\delta)\cap\hat{\mathcal{Z}}_\mu$ a neighborhood of $[\hat{\mathsf{C}}_0]$ is described by the usual Kuranishi picture. More precisely, if $\hat{G}_0=\mathbf{Stab}(\hat{\mathsf{C}}_0)$ then there exist a $\hat{G}_0$-invariant neighborhood $\mathcal{U}$ of $0\in H^1(\mathbf{E}_{\hat{\mathsf{C}}_0})$ and a real analytic, $\hat{G}_0$-equivariant map

$$\kappa:\mathcal{U}\to H^2(\mathbf{E}_{\hat{\mathsf{C}}_0})$$

such that $\kappa^{-1}(0)/\hat{G}_0$ is homeomorphic to a neighborhood of $[\hat{\mathsf{C}}_0]$ in $\widehat{\mathfrak{M}}_{\mu,\delta}$.

Definition 3.12. *A monopole* $\hat{\mathsf{C}} \in \hat{\mathcal{Z}}_{\mu,\delta}$ *is called* **regular** *if* $H^2(\mathbf{E}_{\hat{\mathsf{C}}}) = 0$.

Observe that the asymptotic trace map

$$\partial_\infty : L^2_{ex}(\mathbb{S}_{\hat{\sigma}} \oplus \mathbf{i}(\Lambda^1 \oplus \Lambda^0)T^*M) \to L^2(\mathbb{S} \oplus \mathbf{i}(\Lambda^1 \oplus \Lambda^0 \oplus \Lambda^0)T^*N)$$

splits into four components

$$L^2_{ex}(\mathbb{S}_{\hat{\sigma}} \oplus \mathbf{i}(\Lambda^1 \oplus \Lambda^0)T^*M) \ni \Xi := (\underline{\hat{\psi}}, \mathbf{i}\underline{\hat{a}}, \mathbf{i}f)$$

$$\mapsto \partial^\psi_\infty \Xi \oplus \partial^1_\infty \Xi \oplus \partial^0_\infty \Xi \oplus \partial^f_\infty \Xi := \partial_\infty \underline{\hat{\psi}} \oplus \mathbf{i}\partial_\infty(\underline{\hat{a}} - \lrcorner_t \underline{\hat{a}} dt) \oplus \mathbf{i}\partial_\infty(\lrcorner_t \underline{\hat{a}}) \oplus \mathbf{i}\partial_\infty f.$$

Set $K_\mu = \ker_{ex} \mathcal{T}_{\hat{\mathsf{C}},\mu}$, $\mathcal{T}_{\hat{\mathsf{C}}} := \mathcal{T}_{\hat{\mathsf{C}},\mu=0}$, $K_0 := \ker_{ex} \mathcal{T}_{\hat{\mathsf{C}}}$. Arguing as in [26, §4.3.2] we deduce the following results.

Lemma 3.13. *There exists a short exact sequence*

$$0 \to U_0 \to K_\mu \to H^1(\mathbf{E}_{\hat{\mathsf{C}}}) \to 0$$

where $U_0 \cong \operatorname{coker}(\partial_\infty : T_1\hat{G} \to \mathbf{i}\mathbb{R}) \cong \ker\big(H^1(\mathbf{F}_{\hat{\mathsf{C}}}) \to H^1_{\hat{\mathsf{C}}}(\mathbf{E}_{\hat{\mathsf{C}}})\big)$[3] *and* $\hat{G} = \mathbf{Stab}(\hat{\mathsf{C}})$.

Remark 3.14. Let us observe that when $\hat{\mathsf{C}}$ is irreducible, so that $\hat{G} = \{1\}$, we have $\dim U_0 = 1$ and the image of U_0 in K_μ is spanned by the vector

$$\Big(\mathfrak{L}_{\hat{\mathsf{C}}}\big(\mathbf{i}(1-\varphi_0)\big), 0\Big)$$

where φ_0 is the unique solution of the equation

$$\Delta_{\hat{\mathsf{C}},\mu}(\mathbf{i}\varphi_0) = \Delta_{\hat{\mathsf{C}},\mu}\mathbf{i}, \;\; \varphi_0 \in L^{3,2}_\mu.$$

We refer to [26, Remark 4.3.26] for a proof of this fact.

Proposition 3.15. *There exists a natural short exact sequence*

$$0 \to H^1(\mathbf{E}_{\hat{\mathsf{C}}}) \to K_0 \xrightarrow{\partial^0_\infty} U_0 \to 0. \tag{H_1}$$

In particular, we have an isomorphism

$$K_\mu \cong K_0.$$

Corollary 3.16.

$$\ker_{ex} \mathbf{SIGN}_\mu \cong \ker_{ex} \mathbf{SIGN} \cong H^1(M,\mathbb{R}) \oplus H^0(M,\mathbb{R}).$$

[3] The second isomorphism follows from the long exact sequence associated to (3.4).

Assuming $\partial_\infty \hat{\mathsf{C}} = (0, A_\infty)$ is a good vortex, we deduce that $\partial_\infty \ker_{ex} \mathfrak{T}_{\hat{\mathsf{C}}}$ is a lagrangian subspace in

$$\ker \mathcal{H} \cong H^1(M,g) \oplus \mathbb{R}^2$$

equipped with the complex structure

$$J := \begin{bmatrix} * & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Here is a consequence of the above result.

Corollary 3.17. *(a) If $\hat{\mathsf{C}}$ is irreducible then $\partial_\infty \ker_{ex} \mathfrak{T}_{\hat{\mathsf{C}}}$ is a lagrangian subspace in $\ker \mathcal{H}$ of the form*

$$\partial^1_\infty \ker_{ex} \mathfrak{T}_{\hat{\mathsf{C}}} \oplus \partial^0_\infty \ker_{ex} \mathfrak{T}_{\hat{\mathsf{C}}} \cong L_{\hat{\mathsf{C}}} \oplus U_0$$

where $L_{\hat{\mathsf{C}}} = \partial^1_\infty \ker_{ex} \mathfrak{T}_{\hat{\mathsf{C}}} \subset H^1(N,g)$ is a lagrangian subspace with respect to the complex structure given by the Hodge operator $*$. Moreover, $L_{\hat{\mathsf{C}}}$ coincides with the image of $H^1(\mathbf{E}_{\hat{\mathsf{C}}})$ in $H^1(\mathbf{B}_{\hat{\mathsf{C}}})$.
(b) If $\hat{\mathsf{C}}$ is reducible then $\mathfrak{T}_{\hat{\mathsf{C}}} = \mathfrak{D}_{\hat{A}} \oplus \mathbf{SIGN}$ and

$$\partial_\infty \ker_{ex} \mathfrak{T}_{\hat{\mathsf{C}}} \cong \partial^1_\infty \ker_{ex} \mathfrak{T}_{\hat{\mathsf{C}}} \oplus \partial^f_\infty \ker_{ex} \mathfrak{T}_{\hat{\mathsf{C}}} \cong L_{top} \oplus T_1 G_\infty,$$

where

$$L_{top} \cong \mathrm{Range}\big(H^1(M,\mathbb{R}) \to H^1(N,\mathbb{R}) \big).$$

In particular, L_{top} coincides with the image of $H^1(\mathbf{E}_{\hat{\mathsf{C}}})$ in $H^1(\mathbf{B}_{\hat{\mathsf{C}}})$ and

$$\ker_{ex} \mathfrak{T}_{\hat{\mathsf{C}}} \cong \ker_{ex} \mathfrak{D}_{\hat{A}} \oplus \mathrm{Range}\big(H^1(M,N;\mathbb{R}) \to H^1(M,\mathbb{R}) \big) \oplus L_{top} \oplus H^0(M).$$

Arguing as in the proof of [26, Prop. 4.3.30] we deduce the following result.

Proposition 3.18. *There exist a natural isomorphism*

$$H^1(\mathbf{F}_{\hat{\mathsf{C}}}) \cong \ker_\mu \mathfrak{T}_{\hat{\mathsf{C}},\mu}$$

and a short exact sequence

$$0 \to H^2(\mathbf{F}_{\hat{\mathsf{C}}}) \to K_0 \xrightarrow{\partial^f_\infty} U_0^\perp \to 0 \tag{H_2}$$

where $U_0^\perp \subset T_1 G_\infty$ can be identified with the image of $T_1 \hat{G}$ in $T_1 G_\infty$ via ∂_∞. The isomorphism $H^2(\mathbf{F}_{\hat{\mathsf{C}}}) \to \ker\big(K_0 \to U_0^\perp\big)$ is given by the map

$$H^2(\mathbf{F}_{\hat{\mathsf{C}}}) \cong \ker_\mu \underline{\mathbf{SW}}^{*\mu} \cap \ker_\mu \mathfrak{L}^{*\mu}_{\hat{\mathsf{C}}} \ni \Psi \mapsto (\mathbf{m}_{2\mu}\Psi, 0) \in \ker_{ex} \mathfrak{T}_{\hat{\mathsf{C}}}.$$

Proposition 3.18 and Corollary 3.17 imply the following consequence.

Corollary 3.19. *Suppose $\hat{\mathsf{C}}$ is a reducible monopoles with good asymptotic limit C_∞. Then we have the isomorphisms*

$$H^1(\mathbf{E}_{\hat{\mathsf{C}}}) \cong \ker_{L^2} \mathfrak{D}_{\hat{A}} \oplus H^1(M,\mathbb{R}) \oplus H^0(M,\mathbb{R}),$$

$$H^2(\mathbf{F}_{\hat{\mathsf{C}}}) \cong \ker_{L^2} \mathfrak{D}_{\hat{A}} \oplus H^1(M,\mathbb{R}),$$

$$H^1(\mathbf{B}_{\mathsf{C}_\infty}) = H^1(N,\mathbb{R}) \oplus H^0(N,\mathbb{R}).$$

The natural map $H^1(\mathbf{E}_{\hat{\mathsf{C}}}) \to H^1(\mathbf{B})$ coincides with the natural map

$$i^* : (H^1 \oplus H^0)(M,\mathbb{R}) \to (H^1 \oplus H^0)(N,\mathbb{R}),$$

where $i : N \to M$ denotes the natural inclusion. If additionally $H^1(M,\mathbb{R}) \cong \mathbb{R}$ then the connecting isomorphism

$$\partial : H^1(\mathbf{B}_{\mathsf{C}_\infty}) \to H^2(\mathbf{F}_{\hat{\mathsf{C}}})$$

can be described as follows.

Decompose $H^1(N,\mathbb{R}) = L_{top} \oplus *L_{top}$. For every $u \in L_{top}$ there exists a unique $E(u)$ in $H^1(M;\mathbb{R})$ such that $u = i^*E(u)$. If

$$u \oplus v \oplus c \in L_{top} \oplus *L_{top} \oplus H^0(N,\mathbb{R}) \cong H^1(\mathbf{B}_{\mathsf{C}_\infty})$$

then

$$\partial(u \oplus v \oplus c) = 0 \oplus E(*v) \in \ker_{L^2} \mathfrak{D}_{\hat{A}} \oplus H^1(M,\mathbb{R}) \cong H^2(\mathbf{F}_{\hat{\mathsf{C}}}).$$

The virtual dimension of the moduli space $\widehat{\mathfrak{M}}_\mu$ at a finite energy monopole $\hat{\mathsf{C}}$ is by definition the integer

$$d(\hat{\mathsf{C}}) := -\chi(\mathbf{E}_{\hat{\mathsf{C}}}).$$

Note that

$$d(\hat{\mathsf{C}}) = -\chi(\mathbf{F}_{\hat{\mathsf{C}}}) - \chi(\mathbf{B}_{\hat{\mathsf{C}}}) = \mathrm{ind}_{\mathbb{R}}\mathfrak{T}_{\hat{\mathsf{C}},\mu} + 2.$$

Arguing as in [26, §4.3.3] we deduce

$$\mathrm{ind}_{\mathbb{R}}\mathfrak{T}_{\hat{\mathsf{C}},\mu} = I_{APS}\mathfrak{T}_{\hat{\mathsf{C}}} + 1$$

so that

$$d(\hat{\mathsf{C}}) = I_{APS}(\mathfrak{T}_{\hat{\mathsf{C}}}) + 3.$$

To compute the index in the right-hand side we use the Atiyah-Patodi-Singer theorem which simplifies substantially in two ways. First, the local index density of $\mathfrak{T}_{\hat{\mathsf{C}}}$ is zero since $\mathfrak{T}_{\hat{\mathsf{C}}}$ is a formally selfadjoint operator on and odd dimensional manifold (see [10, §1.8.1]) and furthermore, the eta invariant of $\mathcal{H}$ is zero because the spectrum of $\mathcal{H}$ is symmetric with respect to the origin. Thus

$$I_{APS} = -\frac{1}{2}\dim_{\mathbb{R}}(\ker \mathfrak{D}_{A_\infty} \oplus \mathcal{H}).$$

Here is a first consequence of the above considerations.

Corollary 3.20. *Suppose $\hat{\mathsf{C}} = (\hat{\psi}, \hat{A}) \in \hat{\mathcal{Z}}_{\mu,\delta}$ is a regular finite energy monopole. Then there exists a small open neighborhood of $[\hat{\mathsf{C}}]$ in $\widehat{\mathfrak{M}}_{\mu,\delta}$ homeomorphic to $\mathbb{R}$.*

3.3 Reducibles

Before we begin describing the set of reducible finite energy monopoles we want to show that they occur quite naturally.

Proposition 3.21. (Key Estimate) *Suppose $(M, \hat{g})$ is an admissible 3-manifold such that the scalar curvature $s = s_{\hat{g}}$ is nonnegative and somewhere positive. If η is a compactly supported 1-form then any finite energy monopole $\hat{\mathsf{C}} = (\hat{\psi}, \hat{A})$ satisfies the L^∞-estimate*

$$\|\hat{\psi}\|_\infty^2 \leq \max\Big\{0, 2\sup_{x\in M}\big(2\sqrt{2}|\eta(x)| - s(x)\big)\Big\}.$$

In particular, if

$$2\sqrt{2}|\eta(x)| \leq s(x), \ \ \forall x \in M,$$

then any finite energy monopole is reducible.

Proof Using the Kato inequality and the identities (1.1) we deduce exactly as in [14] that

$$\Delta_M|\hat{\psi}|^2 \leq -\frac{s}{2}|\hat{\psi}|^2 - \frac{1}{4}|\hat{\psi}|^4 + \sqrt{2}|\eta||\hat{\psi}|^2$$

We set $u := |\hat{\psi}|^2$ so that u is a nonnegative function satisfying the differential inequality

$$\Delta_M u + \frac{1}{4}u^2 + \frac{s - 2\sqrt{2}|\eta|}{2}u \leq 0.$$

Since $\lim_{x\to\infty} u = 0$ we deduce that u achieves its maximum at a point x_0 somewhere inside M. At this point $\Delta_M u(x_0) \geq 0$ so that at this point

$$u(x_0)\Big(u(x_0) + 2\big(s - 2\sqrt{2}|\eta|\big)\Big).$$

Thus

$$\sup_{x\in M}|\hat{\psi}(x)|^2 \leq \max\Big\{0, 2\sup_{x\in M}\big(2\sqrt{2}|\eta(x)| - s(x)\big)\Big\}. \ \blacksquare$$

Suppose now that $b_1(M) = 1$ and η is a co-closed, compactly supported 1-form and $\hat{\mathsf{C}} = (\hat{\psi}, \hat{A})$ is a finite energy, reducible monopole, i.e. $\hat{\psi} = 0$ and $F_{\hat{A}} = -\mathbf{i}\hat{*}\eta$. The compactly supported closed 2-form $*\eta$ is *exact* since $H^2(M, \mathbb{R}) = 0$ so that there *will always exist* finite energy reducible monopoles.

To understand the role of the reducibles we begin by studying them separately, independently of the monopole equation. Consider a new configuration space $\hat{\mathcal{A}}_{\mu,ex} = \hat{\mathcal{A}}_{\mu,ex}(\hat{\sigma})$ consisting of pairs $\mathfrak{p} = (\hat{A}, \mathbf{i}f)$ where $\mathbf{i}f \in \mathcal{X}$ and $\hat{A}$ is an asymptotically strongly cylindrical $L^{2,2}_{\mu,ex}$-connections $\hat{A}$ on $\det(\hat{\sigma})$ such that $\partial_\infty\hat{A}$ is flat. The group $\widehat{\mathcal{G}}_{\mu,ex}$ acts on $\hat{\mathcal{A}}_{\mu,ex}$ by

$$\hat{\gamma}\cdot(\hat{A}, k\mathbf{i}f) = (\hat{A} - \frac{2d\hat{\gamma}}{\hat{\gamma}}, \mathbf{i}f).$$

Denote by $\mathcal{R}_M$ the space of $\hat{\mathcal{G}}_{\mu,ex}$-orbits of pairs $(\hat{A}, \mathbf{i}c)$ where c is a real constant and $\hat{A}$ is a connection with curvature

$$F_{\hat{A}} = -\mathbf{i} * \eta.$$

Fix such a pair $\mathfrak{p} = (\hat{A}, \mathbf{i}c)$. The local structure of $\mathcal{R}_M$ near $\mathfrak{p}$ is encoded by the deformation complex

$$0 \to E^0_{\mathfrak{p}} := T_1 \widehat{\mathcal{G}}_{\mu,ex} \stackrel{\hat{*}\hat{d}\oplus 0}{\longrightarrow} E^1_{\mathfrak{p}} := T_{\hat{A}} \hat{\mathcal{A}}_{\mu,ex} \oplus \mathcal{X} \stackrel{*\hat{d}-\hat{d}}{\longrightarrow} E^2_{\mathfrak{p}} := L^{1,2}_{\mu}(\mathbf{i}T^*M) \to 0.$$

As in the previous section we can include this complex in a short exact sequence

$$0 \to F^*_{\mathfrak{p}} \to E^*_{\mathfrak{p}} \stackrel{\partial_\infty}{\to} B^*_{\mathfrak{p}} \to 0. \tag{$\checkmark$}$$

We deduce

$$H^1(B_{\mathfrak{p}}) \cong H^1(N, \mathbb{R}) \oplus H^0(N, \mathbb{R})$$

$$H^1(E_{\mathfrak{p}}) \cong \ker_{ex} \mathbf{SIGN} \cong H^1(M, \mathbb{R}) \oplus H^0(M, \mathbb{R}), \;\; \partial_\infty H^1(E_{\mathfrak{p}}) \cong L_{top} \oplus \mathbb{R}$$

and

$$H^2(F_{\mathfrak{p}}) \cong \ker\big(\, \partial^f_\infty : \ker_{ex} \mathbf{SIGN} \to H^0(N, \mathbb{R})\,\big) \cong H^1(M, \mathbb{R}).$$

Using the long exact sequence determined by ($\checkmark$) we deduce

$$H^1(E_{\mathfrak{p}}) \stackrel{\partial_\infty}{\to} H^1(B_{\mathfrak{p}}) \to H^2(F_{\mathfrak{p}}) \to H^2(E_{\mathfrak{p}}) \to 0$$

or equivalently

$$0 \to L_{top} \oplus H^0(N, \mathbb{R}) \to H^1(N, \mathbb{R}) \oplus H^0(N, \mathbb{R}) \to H^2(F_{\mathfrak{p}}) \to H^2(E_{\mathfrak{p}}) \to 0.$$

Thus

$$\dim H^2(E_{\mathfrak{p}}) = \dim H^1(M, \mathbb{R}) - \dim H^1(N, \mathbb{R}) + \dim L_{top} = \dim H^1(M, \mathbb{R}) - 1.$$

The above computation leads to the following conclusion.

Corollary 3.22. *If $\dim H^1(M, \mathbb{R}) = 1$ then for every co-closed, compactly supported 1-form η there will exist reducible finite energy η-monopoles and $\mathcal{R}_M$ is diffeomorphic to the cylinder*

$$H^1(M, \mathbb{R})/H^1(M, 4\pi\mathbb{Z}) \times \mathbb{R}.$$

In particular, in this case, the space of $\widehat{\mathcal{G}}_{\mu,ex}$-orbits of such monopoles is homeomorphic to the circle

$$H^1(M, \mathbb{R})/H^1(M, 4\pi\mathbb{Z}).$$

Example 3.23. Suppose M is diffeomorphic to the complement of a tubular neighborhood of a knot K in a rational homology sphere X. Then $H^1(M, \mathbb{R}) \cong \mathbb{R}$ and (see [28])

$$H_1(M, \mathbb{Z}) \cong \Big\{ (\alpha, c) \in \mathbb{Q} \times H_1(X, \mathbb{Z});\;\; \alpha \equiv \mathbf{lk}_X(c, K) \mod \mathbb{Z} \Big\} \tag{3.5}$$

where $\mathbf{lk}_X$ denotes the linking pairing

$$\mathbf{lk}_X : H_1(X, \mathbb{Z}) \times H_1(X, \mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}.$$

Moreover, (see [28])

$$H_2(M,\mathbb{Z}) \cong H_1(M,\partial M;\mathbb{Z}) \cong H_1(X,\mathbb{Z})/\langle K\rangle$$

where $\langle K\rangle$ denotes the cyclic group generated by the homology class of K.

Any representation $\rho : H_1(M,\mathbb{Z}) \to S^1$ determines a line bundle equipped with a flat connection. We denote it by L_ρ. The space of representations $H_1(N,\mathbb{Z}) \to S^1$ is a disjoint union of circles parametrized by $\mathrm{Hom}(H_1(N;\mathbb{Z})^\tau, S^1)$, where the superscript τ indicates the torsion part. The universal coefficients theorem provides a natural isomorphism

$$\mathrm{Hom}(H_1(M;\mathbb{Z})^\tau, S^1) \cong \mathrm{Ext}\,(H_1(M;\mathbb{Z})^\tau, \mathbb{Z}) \cong H^2(M,\mathbb{Z}).$$

The above isomorphism $\mathrm{Hom}(H_1(M;\mathbb{Z})^\tau, S^1) \to H^2(M,\mathbb{Z})$ is precisely described by the correspondence

$$\mathrm{Hom}(H_1(M;\mathbb{Z})^\tau, S^1) \ni \rho \mapsto c_1(L_\rho) \in H^2(M,\mathbb{Z}).$$

If $\hat{\sigma}$ is a *spin*^{c} structure on M then the space of finite energy reducible $\hat{\sigma}$-monopoles on M can be identified to the nontrivial double cover of the component of

$$\mathrm{Hom}(H_1(N,\mathbb{Z}) \to S^1)$$

labeled by $c_1(\det\hat{\sigma})$.

We see that the reducibles cannot always be avoided and we would now like to understand their relative position inside the moduli space of all finite energy monopoles. We will concentrate exclusively on the situation discussed in Example 3.23, when M is diffeomorphic to the complement of a knot inside a rational homology sphere.

Fix a cylindrical *spin*^{c} structure $\hat{\sigma}$ on M. For simplicity, we assume that the perturbation parameter η (co-closed 1-form) is trivial. Suppose $\hat{\mathsf{C}} = (0, A)$ is a finite energy reducible $\hat{\sigma}$-monopole. We know that the asymptotic limit $\mathsf{C}_\infty = (0, A_\infty) =: \partial_\infty\hat{\mathsf{C}}$ is a good vortex so that modulo a gauge transformation we can assume $\hat{\mathsf{C}} \in \hat{\mathcal{Z}}_{\mu,\delta}$ for some sufficiently small $0 < \mu < \delta$. The local structure of the *extended moduli space* $\mathcal{M}_\mu$ near $(\hat{\mathsf{C}}, 0)$ is given by the Kuranishi deformation picture. The deformation complex at $\hat{\mathsf{C}}$ has cohomology

$$H^0(\mathbf{E}_{\hat{\mathsf{C}}}) \cong \mathbf{i}\mathbb{R}, \;\; H^1(\mathbf{E}_{\hat{\mathsf{C}}}) \cong \ker_{ex}(\mathfrak{D}_{\hat{A}} \oplus \mathbf{SIGN}) \cong \ker_{ex}\mathfrak{D}_{\hat{A}} \oplus H^1(M,\mathbb{R}) \oplus H^0(M,\mathbb{R})$$

Fix a harmonic form $\omega_0 \in L^2_{ex}(T^*M)$ which spans $H^1(M,\mathbb{R})$. Then we can identify $H^1(\mathbf{E}_{\hat{\mathsf{C}}})$ with the subspace of $T_{\hat{\mathsf{C}}}\mathcal{C}_{\mu,ex} \times \mathcal{X}$ given by

$$\Big\{(\Phi, x\mathbf{i}\omega_0, \mathbf{i}c);\;\; \Phi \in \ker_{ex}\mathfrak{D}_{\hat{A}},\;\; x, c \in \mathbb{R}\Big\},$$

We also have an isomorphism

$$H^2(\mathbf{F}_{\hat{\mathsf{C}}}) \cong \ker_{ex}\mathfrak{D}_{\hat{A}} \oplus H^1(M,\mathbb{R}).$$

More precisely, according to Proposition 3.18, $H^2(\mathbf{F}_{\hat{\mathsf{C}}})$ can be identified with the subspace of $L^{1,2}_\mu(\mathbb{S}_{\hat{\sigma}} \oplus \mathbf{i}(\Lambda^1 \oplus \Lambda^0)T^*M)$ spanned by

$$\mathbf{m}_{-2\mu} \cdot \ker\big(\partial^f_\infty : K_0 \to H^0(N,\mathbb{R})\big).$$

Observe that $\hat{\mathsf{C}}$ *is regular if and only if* $\ker_{ex}\mathfrak{D}_{\hat{A}} = 0$. A regular reducible $\hat{\mathsf{C}}$ defines a smooth point $(\hat{\mathsf{C}}, 0)$ of $\mathcal{M}_\mu$. Moreover, a neighborhood of this point inside the extended moduli space $\mathcal{M}_\mu$ is homeomorphic to an open disk in $\mathbb{R}^2$.

Definition 3.24. *The reducible finite energy monopole* $\hat{\mathsf{C}} := (0, \hat{A})$ *is called* **mildly irregular** *if*

- $\ker_{ex} \mathfrak{D}_{\hat{A}} \cong \mathbb{C}$. **Fix** *a spinor* $\Phi \in \ker_{ex} \mathfrak{D}_{\hat{A}} = \ker_{\mu} \mathfrak{D}_{\hat{A}}$ *such that* $\|\Phi\|_{L^2_{-\mu}} = 1$.
- $\kappa := \langle \hat{\boldsymbol{c}}(\mathbf{i}\omega_0)\Phi, \Phi\rangle_{L^2} \neq 0$. *We set* $\epsilon(\hat{\mathsf{C}}) := \operatorname{sign} \kappa$.

Assume now that $\hat{\mathsf{C}}$ is mildly irregular. The long exact sequence associated to (3.4) leads to

$$0 \to L_{top} \oplus \mathbb{R} \to H^2(\mathbf{F}_{\hat{\mathsf{C}}}) \cong \ker_{ex} \mathfrak{D}_{\hat{A}} \oplus H^1(M, \mathbb{R}) \to H^2(\mathbf{E}_{\hat{\mathsf{C}}}) \to 0.$$

This shows that $H^2(\mathbf{E})$ is a complex 1-dimensional space generated by $\mathbf{m}_{-2\mu}\Phi$.

To describe the Kuranishi picture we use deformation theory. More precisely we look for $L^{2,2}_{\mu,ex}$-small solutions $(\underline{\hat{\mathsf{C}}}, \mathbf{i}f)$ of the system

$$(\hat{\mathsf{C}} + \underline{\hat{\mathsf{C}}}, \mathbf{i}f) \in \mathcal{C}_{\mu,ex} \times \mathcal{X}, \quad \begin{cases} \mathcal{F}(\hat{\mathsf{C}} + \underline{\hat{\mathsf{C}}}) = 0 \\ \\ \mathfrak{L}^*_{\hat{\mathsf{C}}}(\underline{\hat{\mathsf{C}}}) = 0 \end{cases}. \tag{3.6}$$

To ease the presentation we set

$$\mathbf{X} := T_{\hat{\mathsf{C}}}\mathcal{C}_{\mu,ex} \times \mathcal{X}, \quad \mathfrak{L}^*_{\mathsf{C}_\infty}\partial_\infty \underline{\hat{\mathsf{C}}} = 0,$$

$$\mathbf{Y} := L^{1,2}_{\mu}(\mathbb{S}_{\hat{\sigma}} \oplus \mathbf{i}(\Lambda^1 \oplus \Lambda^0)).$$

Observe that any solution of (3.6) automatically belongs to $\mathbf{X}$. The solutions of (3.6) as precisely the zeros of the nonlinear map

$$\mathcal{N} : \mathbf{X} \to \mathbf{Y}, \quad (\underline{\hat{\mathsf{C}}}, \mathbf{i}f) \mapsto (\mathcal{F}(\underline{\hat{\mathsf{C}}}), \mathfrak{L}^{*\mu}_{\hat{\mathsf{C}}}\underline{\hat{\mathsf{C}}}).$$

Observe that $H^1(\mathbf{E}_{\hat{\mathsf{C}}}) \subset \mathbf{X}$ and $H^2(\mathbf{E}_{\hat{\mathsf{C}}}) \subset \mathbf{Y}$. We denote the orthogonal complement of $H^1(\mathbf{E}_{\hat{\mathsf{C}}})$ with respect to the $L^2_{\mu,ex}$-metric by $\mathbf{X}^\perp$ and the orthogonal complement of $H^2(\mathbf{E}_{\hat{\mathsf{C}}})$ with respect to the L^2_μ-metric by $\mathbf{Y}^\perp$. Denote by $\underline{\mathcal{N}}$ the linearization of $\mathcal{N}$ at $(0,0) \in \mathbf{X}$. Observe that

$$\underline{\mathcal{N}} = \mathcal{T}_{\hat{\mathsf{C}},\mu}$$

and

$$\mathcal{R}(\underline{\hat{\psi}}, \mathbf{i}f) := \mathcal{N}(\underline{\hat{\psi}}, \mathbf{i}\underline{\hat{a}}, \mathbf{i}f) - \underline{\mathcal{N}}(\underline{\hat{\mathsf{C}}}, \mathbf{i}\underline{\hat{a}}, \mathbf{i}f) = \begin{bmatrix} \frac{1}{2}\hat{\boldsymbol{c}}(\mathbf{i}\underline{\hat{a}})\underline{\hat{\psi}} - \frac{\mathbf{i}}{2}f\underline{\hat{\psi}} \\ \\ \frac{1}{2}q(\underline{\hat{\psi}}) \\ \\ 0 \end{bmatrix} \in \mathbf{Y}. \tag{3.7}$$

Denote by $\mathcal{P} : \mathbf{Y} \to \mathbf{Y}$ the L^2_μ-orthogonal projection onto $H^2(\mathbf{E}_{\hat{\mathsf{C}}})$. For every $\mathbf{x} \in \mathbf{X}$ we denote by $\mathbf{x}^0 \oplus \mathbf{x}^\perp$ its decomposition determined by the direct sum $\mathbf{X} = H^1(\mathbf{E}_{\hat{\mathsf{C}}}) \oplus \mathbf{X}^\perp$.

The equation $\mathcal{N}(\mathbf{x}) = 0$ is equivalent to the pair of equations

$$(1 - \mathcal{P})(\mathbf{x}^0 + \mathbf{x}^\perp) = 0, \tag{3.8a}$$

$$\mathcal{P}\mathcal{N}(\mathbf{x}^0 + \mathbf{x}^\perp) = 0. \tag{3.8b}$$

The equation (3.8a) has a unique small solution $\mathbf{x}^\perp = \mathbf{x}^\perp(\mathbf{x}^0)$ for all sufficiently small $\mathbf{x}^0$. Moreover,

$$\|\mathbf{x}^\perp\|_{L^2_{\mu,ex}} = O(\|\mathbf{x}^0\|^2_{L^{2,2}_{\mu,ex}}).$$

We can be much more precise than this. We write

$$\mathbf{x} =: (\underline{\hat{\psi}}, \mathbf{i}\underline{\hat{a}}, \mathbf{i}f), \;\; \mathbf{x}^0 =: (\underline{\hat{\psi}}^0, \mathbf{i}\underline{\hat{a}}^0, \mathbf{i}f^0), \;\; \mathbf{x}^\perp =: (\underline{\hat{\psi}}^\perp, \mathbf{i}\underline{\hat{a}}^\perp, \mathbf{i}f^\perp).$$

The collection $\{\Phi, \mathbf{i}\omega_0, \mathbf{i}\}$ is a basis of $H^1(\mathbf{E}_{\hat{\mathsf{C}}})$. We write

$$\mathbf{x}^0 = (z\Phi, \mathbf{i}x\omega_0, \mathbf{i}c), \;\; z \in \mathbb{C}, \;\; x, c \in \mathbb{R}.$$

The Kuranishi map at $(\hat{\mathsf{C}}, 0)$ is given by

$$F : H^1(\mathbf{E}_{\hat{\mathsf{C}}}) \to \mathbb{C}, \;\; \mathbf{x}^0 \mapsto \big\langle \mathcal{R}(\mathbf{x}^0 + \mathbf{x}^\perp(\mathbf{x}^0)), \Phi \big\rangle_{L^2}.$$

We regard F as a S^1-equivariant function of the variables (z, x, c), where S^1 acts by complex multiplication on z. We take the L^2-inner product of first component of $\mathcal{R}$ in (3.7) with Φ and we obtain the estimate

$$F(z, x, c) = \frac{z}{2}(\kappa x + \mathbf{i}c) + O(3),$$

where the nonzero constant κ was introduced in Definition 3.24. A neighborhood of $(\hat{\mathsf{C}}, 0) \in \mathcal{M}_\mu$ looks like a neighborhood of 0 in the quotient

$$\big\{F(z, x, c) = 0\big\}/S^1.$$

Since the regular moduli space is defined by the additional constraint $c = 0$ we obtain the following result.

Proposition 3.25. *A neighborhood of a mildly iregular reducible monopole $\hat{\mathsf{C}}$ in $\widehat{\mathfrak{M}}_\mu$ is homeomorphic to the real algbraic variety*

$$\big\{xz = 0\big\}/S^1 \cong \mathbb{R} \times \mathbb{R}_+$$

where the branch $\{0\} \times \mathbb{R}_+$ correspond to irreducible monopoles approaching the reducible $\hat{\mathsf{C}}$.

3.4 Global structure

It is now time to put together the results established so far and provide a global picture of the moduli space of finite energy monopoles.

We first define carefully the setting. $(M, \hat{g})$ is admissible 3-manifold diffeomorphic to the complement of a tubular neighborhood of a knot in a rational homology sphere. Fix a cylindrical *spin*^{c}-structure $\hat{\sigma}$ on M such that $\partial_\infty\hat{\sigma} = \sigma_0$, where σ_0 is the canonical *spin*^{c}-structure on N. The moduli space of vortices on N is a 2-torus $\mathfrak{M}$ with an unique bad point $\mathsf{C}_0 = (0, A_0)$. We fix a basis $(\vec{\mu}, \vec{\lambda})$ of $H_1(T^2; \mathbb{Z})$ such that $\vec{\mu}$ is the meridian of the knot K oriented by the right-hand-rule. We obtain in this fashion angular coordinates (θ, φ) on $\mathfrak{M}$ such that $(\theta(A_0), \varphi(A_0)) = (0, 0)$. Pick a small positive number r and define

$$\mathcal{Z}(r) := \{(0, A) \in \mathcal{Z};\;\; \theta(A)^2 + \varphi(A)^2 \geq r^2\}, \;\; \mathfrak{M}(r) := \mathcal{Z}(r)/\mathcal{G}, \;\; \mathfrak{M}^\partial(r) := \mathcal{Z}(r)/\mathcal{G}^\partial.$$

Set

$$\delta(r) := \inf_{(0,A)\in\mathcal{Z}(r)} \operatorname{dist}(0, \operatorname{spec}(\mathfrak{D}_A)).$$

Note that $\delta(r) > 0$ and $\delta(r) \searrow 0$ as $r \searrow 0$. We can choose r small enough so that $\delta(r) < \frac{\lambda_1}{4}$. Fix a positive number $\mu < \delta(r)$ and set

$$\hat{\mathcal{Z}}_\mu(r) := \{\hat{\mathsf{C}} \in \hat{\mathcal{Z}}_{\mu,ex};\ \ \partial_\infty\hat{\mathsf{C}} \in \mathcal{Z}(r)\},$$

$$\widehat{\mathfrak{M}}_\mu(r) := \hat{\mathcal{Z}}_\mu(r)/\widehat{\mathcal{G}}_{\mu,ex}.$$

Observe the following fact.

Proposition 3.26. *For all sufficiently small, compactly supported perturbation parameters (η, w) any reducible $(\hat{\sigma}, \eta, w)$-monopole is gauge equivalent to a monopole in $\hat{\mathcal{Z}}_\mu(r)$.*

We have the following genericity result.

Theorem 3.27. ([16, Lim]) *Fix $r > 0$ small and $\mu < \delta(r)$. Then we can generically choose the compactly supported parameters (η, w) such that the following hold.*
(i) η are w are small enough so that all the $(\hat{\sigma}, \eta, w)$-reducibles are gauge equivalent to configurations in $\hat{\mathcal{Z}}_\mu(r)$.
(ii) Any irreducible monopole $\hat{\mathsf{C}} \in \hat{\mathcal{Z}}_\mu(r, \eta, w)$ is regular.
(iii) Any reducible monopole $\hat{\mathsf{C}} \in \hat{\mathcal{Z}}_\mu(r, \eta, w)$ is either regular or only mildly irregular.
(iv) The map $\partial_\infty : \widehat{\mathfrak{M}}_\mu(r, \eta, w) \to \mathfrak{M}^\partial$ is an immersion.

In the sequel we will exclusively work with parameters (η, w) satisfying the conditions in Theorem 3.27. To simplify the presentation we will, most of the time, assume they are both equal to zero.

The finite energy condition and the lack of nontrivial tunnelings, i.e. finite energy monopoles on $\mathbb{R} \times T^2$ can be used as in [26, §4.4.2] to show that $\widehat{\mathfrak{M}}_\mu(r, \eta, w)$ is compact. The structure of $\widehat{\mathfrak{M}}_\mu(r)$ can now be easily described. It consists of

- a circle of reducible monopoles,
- a finite collection of circles consisting of regular irreducible monopoles,
- a finite number of disjoint smooth arcs with one end on the reducible part and the other an irreducible monopole whose asymptotic limit lies on the boundary of $\mathfrak{M}^\partial(r)$
- a finite number of disjoint smooth arcs beginning and ending on the circle of reducibles, (see Figure 3).

The last issue we address in this section is that of orientation. The family $\{H^*(\mathbf{B}_\mathsf{C})\}_{\mathsf{C}\in\mathcal{Z}(r)}$ is constant,

$$H^0(\mathbf{B}_\mathsf{C}) = H^0(N, \mathbb{R}) = \mathbb{R},\ \ H^1(\mathbf{B}_\mathsf{C}) = H^1(N, \mathbb{R}),$$

and we can fix an orientation by fixing an orientation on $H^1(T^2, \mathbb{R})$. This is equivalent to choosing an orientation on N. We will work with the orientation of N as boundary of M. Using the short exact sequence (3.4) we now see that the orientability of the family $\{H^*(\mathbf{E}_{\hat{\mathsf{C}}})\}_{\hat{\mathsf{C}}\in\hat{\mathcal{Z}}_\mu(r)}$ is decided by the orientability of the family $\{H^*(\mathbf{F}_{\hat{\mathsf{C}}})\}_{\hat{\mathsf{C}}\in\hat{\mathcal{Z}}_\mu(r)}$.

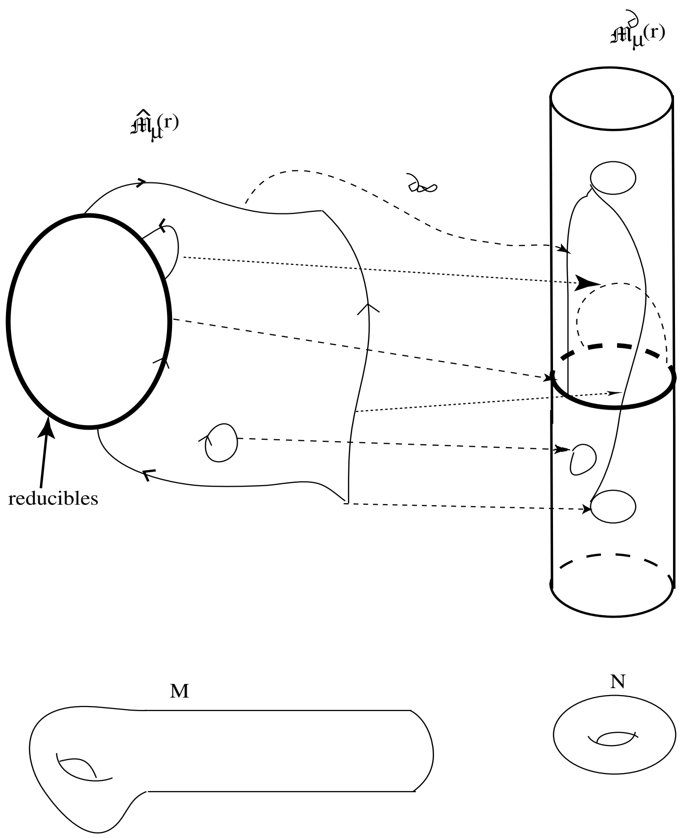

Figure 3: *The moduli space of finite energy monopoles*

The orientability of the family $\{H^*(\mathbf{F}_{\hat{\mathsf{C}}})\}_{\hat{\mathsf{C}}\in\hat{\mathcal{Z}}_\mu(r)}$ is equivalent to the orientability of the determinant line bundle of the family of Fredholm operators

$$\Big\{\mathcal{T}_{\hat{\mathsf{C}},\mu} : L^{1,2}_\mu\big(\mathbb{S}_{\hat{\sigma}}\oplus\mathbf{i}(\Lambda^0\oplus\Lambda^1)T^*M\big)\to L^2_\mu\big(\mathbb{S}_{\hat{\sigma}}\oplus\mathbf{i}(\Lambda^0\oplus\Lambda^1)T^*M\big);\ \ \hat{\mathsf{C}}\in\hat{\mathcal{Z}}_\mu(r)\Big\}.$$

For every $\hat{\mathsf{C}} = (\hat{\psi},\hat{A})\in\hat{\mathcal{Z}}_\mu(r)$ the operator $\mathcal{T}_{\hat{\mathsf{C}},\mu}$ can be written as a sum (see 3.2)

$$\mathcal{T}_{\hat{\mathsf{C}},\mu} = \mathcal{T}^0_{\hat{\mathsf{C}},\mu} + \mathcal{P}_{\hat{\mathsf{C}}},\ \ \mathcal{T}^0_{\hat{\mathsf{C}},\mu} = \mathfrak{D}_{\hat{A}}\oplus -\mathbf{SIGN}_\mu,$$

where $\mathcal{P}_{\hat{\mathsf{C}}}$ is a zero order term decaying exponentially to zero along the cylindrical neck. We deduce that for each $s\in[0,1]$ the differential operator

$$\mathcal{T}^s_{\hat{\mathsf{C}},\mu} := \mathcal{T}^0_{\hat{\mathsf{C}},\mu} + s\mathcal{P}_{\hat{\mathsf{C}}}$$

defines a Fredholm operator $L^{1,2}_\mu\to L^2_\mu$. Using the deformation $s\mapsto\mathcal{T}^s_{\bullet,\mu}$ we can transfer the orientability problem to the family $\{\mathcal{T}^0_{\hat{\mathsf{C}},\mu}\}_{\hat{\mathsf{C}}\in\hat{\mathcal{Z}}_\mu(r)}$. The the determinant line of the family $\{\mathfrak{D}_{\hat{A}}\}_{(\hat{\psi},\hat{A})\in\hat{\mathcal{Z}}_\mu(r)}$ is naturally oriented as these operators are complex. On the other hand, the operator $-\mathbf{SIGN}_\mu$ is independent of $\hat{\mathsf{C}}$. We have thus reached the following conclusion.

Proposition 3.28. *The moduli space $\widehat{\mathfrak{M}}_\mu(r)$ is orientable and an orientation can be specified by specifying an orientation on*

$$\det \operatorname{ind}\Big(\mathbf{SIGN}_\mu : L^{1,2}_\mu\big(\,(\Lambda^1 \oplus \Lambda^0)T^*M\big) \to L^2_\mu\big(\,(\Lambda^1 \oplus \Lambda^0)T^*M\big)\Big).$$

Observe that

$$\ker_\mu \mathbf{SIGN}_\mu = 0, \;\; \ker_\mu \mathbf{SIGN}^{*_\mu}_\mu \cong \ker_{ex} \mathbf{SIGN}^*_\mu \cong H^1(M, \mathbb{R}).$$

We have thus obtained the following consequence.

Corollary 3.29. *An orientation on $H^1(M, \mathbb{R})$ canonically specifies an orientation on $\widehat{\mathfrak{M}}_\mu(r)$.*

We illustrate the orientation rules on a simple example which will be useful later on.

Example 3.30. Fix an orientation on $H^1(M, \mathbb{R})$ and a nonzero harmonic 1-form $\omega_0 \in L^2_{ex}$ which defines a positively oriented basis of $H^1(M, \mathbb{R})$. Suppose $\hat{\mathsf{C}}_0 = (0, \hat{A}_0)$ is a mildly irregular reducible monopole. We know that a neighborhood of $\hat{\mathsf{C}}_0 \in \widehat{\mathfrak{M}}_\mu(r)$ is homeomorphic to a neigborhood of 0 in the $\perp$-shaped region

$$\{(x, r) \in \mathbb{R} \times \mathbb{R}_+; \;\; \rho x = 0\}.$$

The horizontal part $\rho = 0$ describes a neighborhood of $\hat{\mathsf{C}}_0$ inside the moduli space *reducible* monopoles. The vertical part $x = 0$ describes a branch of irreducible monopoles bifurcating at $\hat{\mathsf{C}}_0$ (see Figure 4). Because the moduli space is oriented we can attach arrows to these two branches. We want to explain how.

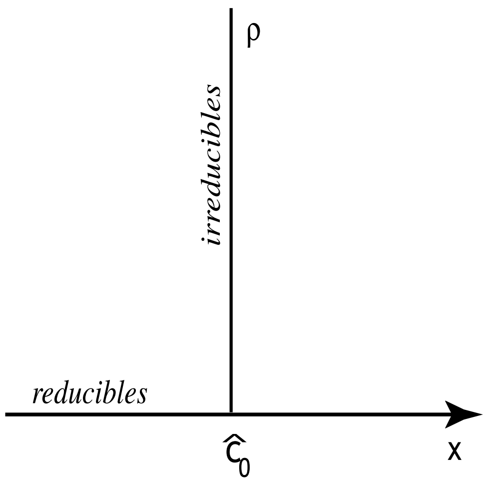

Figure 4: *The branching behaviour near a mildly irregular reducible*

The reducible branch In this case $\mathfrak{T}_{\hat{\mathsf{C}}_0} = \mathfrak{D}_{\hat{A}_0} \oplus \mathbf{SIGN}$. Fix oriented bases in $H^*(\mathbf{F}_{\hat{\mathsf{C}}_0})$ and $H^*(\mathbf{B}_{\hat{\mathsf{C}}_0})$. For orientation purposes we can neglect the spinorial components. The long exact sequence determined by (3.4) has the form

$$0 \to (H^1 \oplus H^0)(M, \mathbb{R}) \cong H^1(\mathbf{E}_{\hat{\mathsf{C}}_0}) \xrightarrow{i^*} (H^1 \oplus H^0)(N, \mathbb{R}) \cong H^1(\mathbf{B}_{\hat{\mathsf{C}}})$$

$$\xrightarrow{\partial} H^2(\mathbf{F}_{\hat{\mathsf{C}}_0}) \cong H^1(M,\mathbb{R}) \to 0$$

Set $\omega_\infty := \partial_\infty \omega_0 = i^*\omega_0$. The space $H^1(\mathbf{B}_{\hat{\mathsf{C}}})$ is oriented by the basis $\mathbf{c}_1 = \{\omega_\infty, *\omega_\infty, 1\}$, while $H^2(\mathbf{F}_{\hat{\mathsf{C}}_0})$ is oriented by the basis $\mathbf{c}_0 = \{\omega_0\}$. Fix the basis $\mathbf{c}_2 = \{\omega_0, 1\}$ of $H^1(\mathbf{E}_{\hat{\mathsf{C}}})$. We can now regard the above short exact sequence as an acyclic chain complex of **based** vector spaces. The chosen basis of $H^1(\mathbf{E}_{\hat{\mathsf{C}}})$ is positively oriented if and only if the torsion of this acyclic based complex is positive. In this case the torsion is very easy to compute. Define an algebraic contraction of this chain complex

$$\chi : H^1(\mathbf{B}_{\hat{\mathsf{C}}_0}) \to H^1(\mathbf{E}_{\hat{\mathsf{C}}_0}), \;\; \chi(\omega_\infty) = \omega_0, \;\; \chi(*\omega_\infty) = 0, \;\; \chi(1) = 1.$$

We deduce (see [28]) that the torsion is given by the determinant of the map

$$\chi \oplus \partial : H^1(\mathbf{B}_{\hat{\mathsf{C}}_0}) \to H^1(\mathbf{E}_{\hat{\mathsf{C}}_0}) \oplus H^2(\mathbf{F}_{\hat{\mathsf{C}}_0})$$

Using Corollary 3.19 we deduce that $\partial\omega_\infty = 0$ and $\partial * \omega_\infty = -\omega_0$. It follows that with respect to the bases $\mathbf{c}_1$ on $H^1(\mathbf{B}_{\hat{\mathsf{C}}_0})$ and $\mathbf{c}_2 \cup \mathbf{c}_0$ in $H^1(\mathbf{E}_{\hat{\mathsf{C}}_0}) \oplus H^2(\mathbf{F}_{\hat{\mathsf{C}}_0})$ the above operator has the matrix description

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

The determinant of this operator is 1 which implies that $\mathbf{c}_2$ is a positively oriented basis. In particular, it follows that the map

$$\mathbb{R} \ni x \mapsto x\omega_0 \in H^1(M,\mathbb{R})$$

is an orientation preserving map between the horizontal branch in Figure 4 and the oriented space $H^1(M,\mathbb{R})$.

The irreducible branch Suppose $\hat{\mathsf{C}}_0 = (0, \hat{A}_0)$ is a mildly irregular reducible monopole. The local structure of the *extended moduli space* $\mathcal{M}_\mu$ near $(\hat{\mathsf{C}}, 0)$ is given be the Kuranishi deformation picture. The deformation complex at $\hat{\mathsf{C}}_0$ has cohomology

$$H^0(\mathbf{E}_{\hat{\mathsf{C}}_0}) \cong \mathbf{i}\mathbb{R}, \;\; H^1(\mathbf{E}_{\hat{\mathsf{C}}_0}) \cong \ker_{ex}(\mathfrak{D}_{\hat{A}_0} \oplus \mathbf{SIGN}) \cong \ker_{ex} \mathfrak{D}_{\hat{A}_0} \oplus H^1(M,\mathbb{R}) \oplus H^0(M,\mathbb{R})$$

More precisely, we can identify $H^1(\mathbf{E}_{\hat{\mathsf{C}}_0})$ with the subspace of $T_{\hat{\mathsf{C}}_0}\mathcal{C}_{\mu,ex} \times \mathcal{X}$ given by

$$\Big\{(z\Phi, x\mathbf{i}\omega_0, \mathbf{i}c); \;\; z \in \mathbb{C}, x, c \in \mathbb{R}\}$$

where Φ is a spinor spanning $\ker_{ex} \mathfrak{D}_{\hat{A}_0}$ such that

$$\|\Phi\|_{L^2_{-\mu}} = 1.$$

We assume

$$\|\omega_0\|_{L^2_{-\mu}} = 1, \;\; \kappa := \langle \hat{\mathbf{c}}(\mathbf{i}\omega_0)\Phi, \Phi\rangle_{L^2} \neq 0 \tag{3.9}$$

We also have an isomorphism

$$H^2(\mathbf{F}_{\hat{\mathsf{C}}_0}) \cong \ker_{ex} \mathfrak{D}_{\hat{A}_0} \oplus H^1(M,\mathbb{R}).$$

More precisely, according to Proposition 3.18, $H^2(\mathbf{F}_{\hat{\mathsf{C}}_0})$ can be identified with the subspace of $L^{1,2}_\mu(\mathbb{S}_{\hat{\sigma}} \oplus \mathbf{i}(\Lambda^1 \oplus \Lambda^0)T^*M)$ spanned by $\{\mathbf{m}_{-2\mu} \cdot \mathbf{i}\omega_0, \mathbf{m}_{-2\mu} \cdot \Phi\}$.

According to the computations in 3.3, we can approximate the irreducible branch monopoles approaching $\hat{\mathsf{C}}_0$ by real analytic path

$$[0, \varepsilon) \ni s \mapsto \hat{\mathsf{C}}_s = \hat{\mathsf{C}}_0 + s(\Phi, 0) + O(s^2).$$

Note that

$$H^2(\mathbf{F}_{\hat{\mathsf{C}}_s}) \cong \ker_{ex} \mathfrak{T}_{\hat{\mathsf{C}}_s}$$

and

$$H^0(\mathbf{F}_{\hat{\mathsf{C}}_s}) = 0 = H^0(\mathbf{E}_{\hat{\mathsf{C}}_s}).$$

Using Corollary 3.17 we deduce that the image of $H^1(\mathbf{E}_{\hat{\mathsf{C}}_s})$ in $H^1(\mathbf{B}_{\hat{\mathsf{C}}_s})$ is a lagrangian subspace $L_{\hat{\mathsf{C}}_s}$ of $H^1(N, \mathbb{R})$. The exact sequence

$$H^1(\mathbf{E}_{\hat{\mathsf{C}}_s}) \to H^1(\mathbf{B}_{\hat{\mathsf{C}}_s}) \to H^2(\mathbf{F}_{\hat{\mathsf{C}}_s}) \to 0$$

now implies that $\dim H^2(\mathbf{F}_{\hat{\mathsf{C}}_s}) = 2 = \dim \ker_{ex} \mathfrak{T}_{\hat{\mathsf{C}}_s}$. In particular this shows that

$$\ker_\mu \mathfrak{T}_{\hat{\mathsf{C}}_s} = 0$$

and

$$\dim H^1(\mathbf{F}_{\hat{\mathsf{C}}_s}) = H^2(\mathbf{F}_{\hat{\mathsf{C}}_s}) + \mathrm{ind}\mathfrak{T}_{\hat{\mathsf{C}}_s,\mu} = 1.$$

More precisely $H^1(\mathbf{F}_{\hat{\mathsf{C}}_s}) \subset L^{2,2}_\mu$ is generated by $\mathfrak{L}_{\hat{\mathsf{C}}_s}(\mathbf{i}\varphi_s)$, where $\varphi_s \in L^{2,2}_{\mu,ex}$ is the unique solution of the boundary value problem

$$\Delta_{\hat{\mathsf{C}}_s,\mu}\varphi_s = 0, \;\; \partial_\infty \varphi_s = 1.$$

We regard the long exact sequence associated to (3.4)

$$0 \to H^0(\mathbf{B}_{\hat{\mathsf{C}}_s}) \to H^1(\mathbf{F}_{\hat{\mathsf{C}}_s}) \to H^1(\mathbf{E}_{\hat{\mathsf{C}}_s}) \to H^1(\mathbf{B}_{\hat{\mathsf{C}}_s}) \to H^2(\mathbf{F}_{\hat{\mathsf{C}}_s}) \to 0 \tag{3.10}$$

as an acyclic complex. To find the orientation of $H^1(\mathbf{E}_{\hat{\mathsf{C}}_s})$ we need to find oriented bases on $H^1(\mathbf{F}_{\hat{\mathsf{C}}_s})$ and $H^2(\mathbf{F}_{\hat{\mathsf{C}}_s})$ which induce the fixed orientation on $\det \mathrm{ind}\, \mathfrak{T}_{\hat{\mathsf{C}}_s,\mu}$, where s is very small. Then the orientation on $H^1(\mathbf{E}_{\hat{\mathsf{C}}_s})$ is determined so that the torsion of (3.10) is positive. We arbitrarily fix an orientation on $H^2(\mathbf{F}_{\hat{\mathsf{C}}_s}) \cong \mathrm{coker}\, \mathfrak{T}_{\hat{\mathsf{C}}_s,\mu}$ so that we reduce the problem to finding an orientation on $H^1(\mathbf{F}_{\hat{\mathsf{C}}_s}) \cong \ker_\mu \mathfrak{T}_{\hat{\mathsf{C}}_s,u}$.

Recall that the symplectic vector space $\ker \mathcal{H}$ admits a natural decomposition

$$\ker \mathcal{H} = H^1(N, \mathbb{R}) \oplus \big(dt \wedge H^0(N, \mathbb{R}) \oplus H^0(N, \mathbb{R})\big) \cong H^1(N, \mathbb{R}) \oplus (dt \wedge \mathbb{R} \oplus \mathbb{R}).$$

For simplicity we denote the component $dt \wedge \mathbb{R}$ by U_0 and the component $\mathbb{R}$ by U_f. U_0 has a canonical basis $\{u_0\}$ and U_f has a canonical basis $\{u_f\}$. The symplectic structure is given by

$$J = \begin{bmatrix} * & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Since $\ker_\mu \mathcal{T}_{\hat{\mathsf{C}}_s} = 0$ we can identify $\ker_{ex} \mathcal{T}_{\hat{\mathsf{C}}_s}$ with its image in $\ker \mathcal{H}$ via ∂_∞. This image is

$$\partial_\infty\big(H^2(\mathbf{F}_{\hat{\mathsf{C}}_s})\big) = L_{\hat{\mathsf{C}}_s} \oplus U_0.$$

Fix a basis h_s of $L_{\hat{\mathsf{C}}_s}$. There exist $\hat{h}_s, \Xi_s \in \ker_{ex} \mathcal{T}_{\hat{\mathsf{C}}_s}$ such that

$$\partial_\infty \hat{h}_s = h_s \ \text{ and } \ \Xi_s = u_0.$$

The vector $\hat{h}_s$ spans the tangent space to the irreducible branch at the configuration $\hat{\mathsf{C}}_s$. For s sufficiently small this branch is well approximated by the curve $(s \mapsto s\Phi, \hat{A}_0)$. Thus the tangent space can be well approximated by the real line spanned by Φ.

• *In the sequel we assume that h_s is chosen such that the oriented real line $\langle \hat{h}_s\rangle$ converges to the oriented real line $\langle\Phi\rangle$.*

We can be more specific about Ξ_s as well. More precisely, $\Xi_s = \mathfrak{L}_{\hat{\mathsf{C}}_s}(\hat{v}_s + t)$, where $\hat{v}_s$ is the unique solution of the problem

$$\hat{v}_s \in L^{3,2}_{\mu,ex}, \ \ \Delta_{\hat{\mathsf{C}}_s}\hat{v}_s = -\Delta_{\hat{\mathsf{C}}_s}(t), \ \ \partial_\infty \hat{v}_0 = \text{const}, \ \ 0 < s \ll 1.$$

We orient $H^2(\mathbf{F}_{\hat{\mathsf{C}}_s})$ using the basis $\mathbf{m}_{-2\mu}(\hat{h}_s, \Xi_s)$. We identify $H^1(\mathbf{B}_{\hat{\mathsf{C}}_s})$ with the subspace $H^1(N, \mathbb{R}) \oplus U_f \subset \ker \mathcal{H}$. The space $H^1(\mathbf{F}_{\hat{\mathsf{C}}_s})$ is generated by $\Upsilon_s := \mathfrak{L}_{\hat{\mathsf{C}}_s}(\mathbf{i}\varphi_s)$. As we mentioned earlier, we have an orientation on $\det \operatorname{ind} \mathcal{T}_{\hat{\mathsf{C}}_s,\mu}$ and thus we can represent an oriented basis as $\epsilon\Upsilon_s$, $\epsilon = \pm 1$. The connecting morphism

$$\delta : H^1(\mathbf{B}_{\hat{\mathsf{C}}_s}) = \operatorname{span}\big\{h_s, *h_s, u_f\big\} \to H^2(\mathbf{F}_{\hat{\mathsf{C}}_s})$$

is given by

$$h_s \to 0, \ \ (*h_s) \mapsto \partial_\infty^{-1}(J(*h_s)) = -\mathbf{m}_{-2\mu}\hat{h}_s, \ \ u_f \mapsto \partial_\infty^{-1}(Ju_f) = -\mathbf{m}_{-2\mu}\Xi_s.$$

To decide whether the basis $\hat{h}_s$ of $H^1(\mathbf{E}_{\hat{\mathsf{C}}_s})$ is positively oriented we need analyze the torsion of the acyclic complex of *based oriented*

$$0 \to H^0(\mathbf{B}_{\hat{\mathsf{C}}_s}) = \langle u_f\rangle \to H^1(\mathbf{F}_{\hat{\mathsf{C}}_s}) = \langle \epsilon\Upsilon_s\rangle \to H^1(\mathbf{E}_{\hat{\mathsf{C}}_s}) = \langle \hat{h}_s\rangle$$

$$\to H^1(\mathbf{B}_{\hat{\mathsf{C}}_s}) = \langle h_s, *h_s, u_f\rangle \to H^2(\mathbf{F}_{\hat{\mathsf{C}}_s}) = \langle \mathbf{m}_{-2\mu}\hat{h}_s, \mathbf{m}_{-2\mu}\Xi_s\rangle \to 0.$$

This is given by the determinant of the map

$$T : H^1(\mathbf{F}_{\hat{\mathsf{C}}_s}) \oplus H^1(\mathbf{B}_{\hat{\mathsf{C}}_s}) = \langle \epsilon\Upsilon_s, h_s, *h_s, u_f\rangle$$

$$\to H^0(\mathbf{B}_{\hat{\mathsf{C}}_s}) \oplus H^1(\mathbf{E}_{\hat{\mathsf{C}}_s}) \oplus H^2(\mathbf{F}_{\hat{\mathsf{C}}_s}) = \langle u_f, \hat{h}_s, \mathbf{m}_{-2\mu}\hat{h}_s, \mathbf{m}_{-2\mu}\Xi_s\rangle$$

described by

$$\Upsilon_s \mapsto u_f, \ \ h_s \mapsto \hat{h}_s, \ \ (*h_s) \mapsto -\mathbf{m}_{-2\mu}\hat{h}_s, \ \ u_f \mapsto -\mathbf{m}_{-2\mu}\Xi_s.$$

This determinant is equal to ϵ. Thus we need to compare the canonical orientation on $\det \mathcal{T}_{\hat{\mathsf{C}}_s,\mu}$ with the orientation induced by

$$\Upsilon_s \otimes (\mathbf{m}_{-2\mu}\hat{h}_s \wedge \mathbf{m}_{-2\mu}\Xi_s)^*.$$

We follow the principles outlined in Appendix C to which we refer for details and notations.

We need to pick an oriented stabilizer for the family $[0,\varepsilon] \ni s \mapsto \mathcal{T}_{\hat{\mathsf{C}}_s,\mu}$. The natural choice is the cokernel of $\mathcal{T}_{\hat{\mathsf{C}}_0,\mu} : L^{1,2}_\mu \to L^{1,2}_\mu$ which is precisely $V_0 := H^2(\mathbf{F}_{\hat{\mathsf{C}}_0})$. More precisely, we have a natural *real* basis of V_0,

$$V_0 = \langle \mathbf{m}_{-2\mu}\Phi, \mathbf{i}\mathbf{m}_{-2\mu}\Phi, \mathbf{i}\mathbf{m}_{-2\mu}\omega_0\rangle.$$

The choice ω_0 defines an orientation on V_0. More generally, set

$$V_s := \ker \mathcal{T}^{*_\mu}_{\hat{\mathsf{C}}_s,\mu} \cong H^2(\mathbf{F}_{\hat{\mathsf{C}}_s}).$$

Now form the operators

$$\mathcal{O}'_s := \mathcal{T}_{\hat{\mathsf{C}}_s,\mu} \uplus V_0 : L^{1,2}_\mu\big(\mathbb{S}_{\hat{\sigma}} \oplus \mathbf{i}(\Lambda_1 \oplus \Lambda_0)T^*M\big) \oplus V_0 \to L^2_\mu(\mathbb{S}_{\hat{\sigma}} \oplus \mathbf{i}(\Lambda_1 \oplus \Lambda_0)T^*M),$$

$$(\underline{\hat{\psi}}, \mathbf{i}\underline{\hat{a}}, \mathbf{i}f; v) \mapsto \mathcal{T}_{\hat{\mathsf{C}}_s,\mu}(\underline{\hat{\psi}}, \mathbf{i}\underline{\hat{a}}, \mathbf{i}f) + v$$

and similarly $\mathcal{O}''_s = \mathcal{T}_{\hat{\mathsf{C}}_s,\mu} \uplus V_s$, $s \in (0,\varepsilon]$. Set

$$K_{V_0}(s) = \ker \mathcal{O}'_s, \;\; K_{V_s} = \ker \mathcal{O}''_s.$$

The spaces $K_{V_0}(s)$ form a bundle over $[0,\varepsilon]$ and for s sufficiently small we have the short exact sequences,

$$0 \to V_s \to V_0 \to V_0/\mathbf{proj}_{V_0}(V_s) \to 0, \tag{3.11a}$$

$$0 \to H^1(\mathbf{F}_{\hat{\mathsf{C}}_s}) \to K_{V_s} \to 0, \tag{3.11b}$$

$$0 \to K_{V_s} \to K_{V_0}(s) \xrightarrow{F} V_0/\mathbf{proj}_{V_0}(V_s) \to 0. \tag{3.11c}$$

We analyze successively the above short exact sequences. As $s \searrow 0$ the 2-plane $\mathbf{proj}_{V_0}(V_s)$ converges to the oriented 2-plane $\langle \mathbf{m}_{-2\mu}\Phi, \mathbf{m}_{-2\mu}\mathbf{i}\Phi\rangle$ so that we can identify V_s with this oriented 2-plane and $V_0/\mathbf{proj}_{V_0}(V_s)$ with the oriented line spanned by $\mathbf{m}_{-2\mu}\mathbf{i}\omega_0$.

Using (3.11b) we deduce that

$$K_{V_s} \approx \big(H_1(\mathbf{F}_{\hat{\mathsf{C}}_s}) \oplus 0\big) \subset L^{1,2}_\mu\big(\mathbb{S}_{\hat{\sigma}} \oplus \mathbf{i}(\Lambda_1 \oplus \Lambda_0)T^*M\big) \oplus V_0$$

and is oriented by the basis $\Big\{\epsilon\Upsilon_s\Big\}$. Using the sequence (3.11c) we obtain an orientation on $K_{V_0}(s)$, $0 < s \ll 1$.

$$\mathbf{or}\,(K_{V_0}(s)) = \mathbf{or}\,(K_{V_s}) \wedge \langle F^{-1}(\mathbf{m}_{-2\mu}\mathbf{i}\omega_0)\rangle$$

As $s \searrow 0$ the oriented line spanned by Υ_s converges to the oriented line spanned by $\mathbf{i}\Phi$. The sign ϵ is determined by the requirement that as $s \searrow 0$ the *oriented* 2-plane $K_{V_0}(s)$ converges to the *oriented* 2-plane

$$\langle \Phi, \mathbf{i}\Phi\rangle \cong K_{V_0}(s=0).$$

We need to determine $\Theta_s \in L^{1,2}_\mu\big(\mathbb{S}_{\hat{\sigma}} \oplus \mathbf{i}(\Lambda_1 \oplus \Lambda_0)T^*M\big)$ such that

$$\mathcal{T}_{\hat{\mathsf{C}}_s}\Theta_s + \mathbf{m}_{-2\mu}\mathbf{i}\omega_0 = 0,$$

i.e. $\Theta_s \oplus \mathbf{m}_{-2\mu}\mathbf{i}\omega_0 = F^{-1}(\mathbf{m}_{-2\mu}\mathbf{i}\omega_0)$, and then study the behavior of the oriented line spanned by $\Theta_s \oplus \mathbf{m}_{-2\mu}\mathbf{i}\omega_0$ as $s \searrow 0$. We write

$$\Theta_s = \underline{\hat{\psi}}_s \oplus \mathbf{i}\underline{\hat{a}}_s \oplus \mathbf{i}f_s.$$

Observe that $\hat{\mathsf{C}}_s = \hat{\mathsf{C}}_0 + s(\Phi, 0) + O(s^2)$ so that

$$\mathcal{T}_{\hat{\mathsf{C}}_s,\mu} = \mathcal{T}^0_{\hat{\mathsf{C}}_s,\mu} + \mathcal{P}_{\hat{\mathsf{C}}_s} = \mathcal{T}_{\hat{\mathsf{C}}_0,\mu} + \mathcal{P}_{\hat{\mathsf{C}}_s}$$

where

$$\mathcal{P}_{\hat{\mathsf{C}}_s}\Theta_s = s\begin{bmatrix} \frac{1}{2}\hat{\mathbf{c}}(\mathbf{i}\underline{\hat{a}})\Phi - \frac{\mathbf{i}}{2}f_s\Phi \\ \\ \frac{1}{2}\dot{q}(\Phi, \underline{\hat{\psi}}_s) \\ \\ \frac{\mathbf{i}}{2}\,\mathbf{Im}\langle\Phi, \underline{\hat{\psi}}_s\rangle \end{bmatrix} + O(s^2)\Theta_s =: s\big(\mathfrak{R} + O(s)\big)\Theta_s.$$

Observe now that

$$\mathcal{T}_{\hat{\mathsf{C}}_0,\mu}\Theta_s + s\mathfrak{R}\Theta_s + \mathbf{m}_{-2\mu}\mathbf{i}\omega_0 = O(s^2)\Theta_s.$$

Denote by Θ_s^0 the L^2_μ-orthogonal projection of Θ_s to the kernel of $\mathcal{T}_{\hat{\mathsf{C}}_0,\mu}$ and set $\Theta_s^\perp = \Theta_s - \Theta_s^0$. Arguing as in the proof of [26, Lemma 1.5.13] we deduce that there exists

$$Z_0 = z_0\Phi \in \ker_\mu \mathcal{T}_{\hat{\mathsf{C}}_0,\mu},\ \ z_0 \in \mathbb{C},$$

such that

$$\lim_{s\searrow 0}\|s\Theta_s - Z_0\|_{L^2_\mu} = \lim_{s\searrow 0}\|s\Theta_s^0 - Z_0\|_{L^2_\mu} = 0.$$

Take the L^2_μ-inner product of the last equality with $\mathbf{m}_{-2\mu}\mathbf{i}\omega_0$. Using the normalization $\|\omega_0\|_{L^2_{-\mu}} = 1$ we e deduce

$$s\big\langle\mathfrak{R}\Theta_s, \mathbf{i}\omega_0\big\rangle_{L^2} + 1 = O(s^2)\big\langle\Theta_s, \mathbf{i}\omega_0\big\rangle_{L^2_{-\mu}}.$$

If we let $s \searrow 0$ we conclude

$$-1 = \big\langle\mathfrak{R}(z_0\Phi), \mathbf{i}\omega_0\big\rangle = \frac{1}{2}\,\mathbf{Re}\ z_0\big\langle q(\Phi), \hat{\mathbf{c}}(\mathbf{i}\omega_0)\big\rangle_{L^2}.$$

Using the normalization condition (3.9) we deduce that $r_0 := \mathbf{Re}\,(z_0) \neq 0$ and more precisely, $r_0 \cdot \kappa < 0$. This shows that oriented plane $K_{V_0}(s)$ converges to the oriented plane

$$\big\langle\epsilon\mathbf{i}\Phi, -\kappa\Phi\big\rangle.$$

Since we require that this plane has the same orientation as $\big\langle\Phi, \mathbf{i}\Phi\big\rangle$ we deduce $\epsilon\cdot\kappa > 0$. We conclude that when $\kappa > 0$ the irreducible branch leaves the reducible locus while when $\kappa < 0$ the irreducible branch is directed towards the reducible locus.

Remark 3.31. For every cylindrical *spin*^{c} structure $\hat{\sigma}$ and every choice of perturbations (η, w) obe can form an integer $SF(\hat{\sigma}, \eta, w))$ defined as the spectral flow of the family of Dirac operators on $\mathbb{S}_{\hat{\sigma}}$,

$$(\mathfrak{D}_A)_{(0,A)\in\widehat{\mathfrak{M}}^{red}_\mu(r)}$$

parametrized by the space of gauge equivalence classes of reducible $(\hat{\sigma}, \eta, w)$-monopoles. This spectral flow is independent of (η, w) and we will denote it by $SF(\hat{\sigma})$. One can choose the parameters (η, w) such that

• If $SF(\hat{\sigma}) = 0$ then $\ker \mathfrak{D}_A = 0$ for any operator $\mathfrak{D}_A$ in the above family.
• If $SF(\hat{\sigma}) \neq 0$ then $\dim_{\mathbb{C}} \ker \mathfrak{D}_A \leq 1$ for all operators in the above family with equality for only finitely many of them. Moreover $\mathfrak{D}_A$ is any of these operators with nontrivial kernel spanned by a spinor Φ, then

$$SF(\hat{\sigma}) \cdot \langle \boldsymbol{c}(\mathbf{i}\omega_0)\Phi, \Phi \rangle > 0.$$

In less rigorous but more intuitive terms, the above condition signifies that the (possibly nonexistent) eigenvalues of the above family of Dirac operators cross the 0-value transversally, exactly $|SF(\hat{\sigma})|$ times, and always in the same direction.

We will refer to a perturbation (η, w) as above as a *monotone perturbation*. For monotone perturbations, there are no irreducible branches of $\widehat{\mathfrak{M}}_\mu(r)$ which begin and end on the reducible branch.

4 Gluing results

4.1 Dehn surgery and *spin*^{c} structures

Suppose M is a compact, oriented 3-manifold with boundary ∂M such that $b_1(M) = 1$ and $\chi(M) = 0$. It follows that ∂M is diffeomorphic to a torus. We will think of M as an admissible 3-manifold with a *fixed* cylindrical structure along the end. Set $T := \partial_\infty M$, and denote by $\mathbf{j}$ the inclusion $T \hookrightarrow M$. The kernel of $\mathbf{j}_* : H_1(T, \mathbb{Z}) \to H_1(M, \mathbb{Z})$ is a rank 1-free Abelian group. We fix a longitude $\lambda \in H_1(T, \mathbb{Z})$, i.e. a generator of $\ker \mathbf{j}$, and denote by m_0 its multiplicity. This is a positive integer such that $\lambda = m_0\lambda_0$ where $\lambda_0 \in H_1(T, \mathbb{Z})$ is a primitive element. The cycle $\mathbf{j}_*\lambda$-bounds a chain $\Lambda \subset M$ which generates $H_2(M, \partial M; \mathbb{Z})$.

We equip T with the orientation as boundary of M. This orientation defines a nondegenerate (symplectic) intersection pairing on $H_1(T, \mathbb{Z})$. We obtain a symplectic lattice Θ. Fix $\mu_0 \in \Theta$ such that $\lambda_0 \cdot \mu_0 = 1$. The dual of Θ is the symplectic lattice $\Theta^\sharp$ is defined by

$$\Theta^\sharp = \mathrm{Hom}_{\mathbb{Z}}(\Theta, \mathbb{Z}) \cong H^1(\partial M, \mathbb{Z}).$$

The intersection pairing on Θ defines an element $\lambda^\sharp \in \Theta^*$ uniquely determined by the requirements

$$\langle \lambda^\sharp, u \rangle = \lambda \cdot u, \ \ \forall u \in \Theta.$$

In particular, $\langle \lambda^\sharp, \lambda \rangle = 0$. $\lambda^\sharp$ is a generator of Θ^*_M, the range of the morphism

$$\mathbf{j}^* : H^1(M, \mathbb{Z}) \to H^1(\partial M, \mathbb{Z}).$$

The group $H_1(M, \partial M; \mathbb{Z})$ is finite and the universal coefficients theorem coupled with the Poincaré-Lefschetz duality imply that we have natural isomorphisms

$$H^2(M; \mathbb{Z}) \cong H_1(M, \partial M; \mathbb{Z}) \cong \mathrm{Hom}(H_1(N)^\tau, \mathbb{Z}).$$

In particular, the restriction map $\mathbf{j}^* : H^2(M, \mathbb{Z}) \to H^2(T, \mathbb{Z})$ is trivial. In particular, all the *spin*^{c} structures on M are admissible.

Denote by $\mathrm{Pic}^\infty(M, \partial M)$ the space of ismorphisms classes of pairs (L, ϕ) where L is a complex line bundle over M and $\phi : L\mid_{\partial M} \to \underline{\mathbb{C}}$ is a trivialization along the boundary. The realtive Chern class produces an isomorphism

$$c_1^{rel} : \mathrm{Pic}^\infty(M, \partial M) \to H^2(M, \partial M; \mathbb{Z}).$$

The group $\Theta^\sharp$ acts on $\mathrm{Pic}^\infty(M, \partial M)$ as follows. If $(L, \phi) \in \mathrm{Pic}^\infty(M)$ and $c \in \Theta^\sharp$ then we define

$$c(L, \phi) = (L, c\phi) := (L, \gamma\phi)$$

where $\gamma : \partial M \to S^1$ is a gauge transformation such that the closed 1-form $\frac{1}{2\pi\mathbf{i}}$ is harmonic and represents the element $c \in H^1(\partial M, \mathbb{Z})$. We have the equality[4]

$$c_1^{rel}(L, c\phi) = c_1^{rel}(L, \phi) + \delta_M c,$$

where $\delta_M : H^1(\partial M, \mathbb{Z}) \to H^2 M, \partial M; \mathbb{Z})$ is the connecting morphism of the pair $(M, \partial M)$. In particular this shows that the stabilizer of this action is the subgroup $\Theta^\sharp_M$.

The group $\mathrm{Pic}^\infty(M, \partial M)$ acts freely and transitively on $Spin^c_{cyl}(M)$. Fix a cylindrical $spin^c$ structure on M. By doing so we provide an identification

$$Spin^c_{cyl}(M) \cong \mathrm{Pic}^\infty(M, \partial M) \cong H^2(M, \partial M; \mathbb{Z}).$$

We can now think of cylindrical $spin^c$ structures over M as complex line bundles over M equipped with a trivialization along ∂M. In particular we have an action of $\Theta^\sharp$ on $Spin^c_{cyl}(M, \partial M)$

$$H^1(\partial M, \mathbb{Z}) \times Spin^c_{cyl}(M) \ni (c, \hat{\sigma}) \mapsto c \cdot \hat{\sigma}$$

and

$$\det(c \cdot \hat{\sigma}) = 2\delta_M c + \det \hat{\sigma}.$$

Consider the solid torus $S^1 \times D^2$. We obtain an admissible manifold X equipped with a cylindrical structure by attaching the cylinder $\mathbb{R}_+ \times S^1 \times \partial D^2$. Denote by $\mathfrak{m}_0 \in H_1(\partial X, \mathbb{Z})$ the homology class carried by $\{1\} \times \partial D^2$ and by $\mathbf{k}_0 \in H_1(\partial X, \mathbb{Z})$ carried by $S^1 \times \{pt\}$. We orient ∂X as boundary of X so that $\mathbf{m}_0 \cdot \mathbf{k}_0 = 1$. Fix an orientation *reversing* diffeomorphism $\Gamma : \partial X \to T$ such that $\Gamma(\mathbf{k}_0) = \lambda_0$, $\mu_0 := \Gamma(\mathbf{m}_0)$. Denote by $\mu_0^\sharp \in H^1 \partial M, \mathbb{Z})$ the element defined by

$$\langle \mu_0^\sharp, x\rangle = \mu_0 \cdot x, \quad \forall x \in \Theta.$$

For each orientation *preserving* diffeomorphism $\varphi : \partial X \to X$ we denote by Y_φ the manifold obtained by attaching X to M via the orientation reversing diffeomorphism $\Gamma_\varphi := \Gamma \circ \varphi$. Alternatively, we can think of φ as changing the cylindrical structure of Y so that we get a cylindrical manifold X_φ where the cylindrical structure allong the neck is given by

$$\mathbf{1}_{\mathbb{R}_+} \times \varphi : \mathbb{R}_+ \times \partial X \to \mathbb{R}_+ \times \partial X.$$

The manifold Y_φ can also be thought of gluing the cylindrical manifold M with X_φ using the gluing map Γ.

[4]A proof of this identity is described later in this section.

The diffeomorphism type of Y_φ depends only on the attaching curve $c = \Gamma_\varphi(\mathbf{m}_0)$. Using the basis $\{\mathbf{m}_0, \mathbf{k}_0\}$ of $H_1(\partial X, \mathbb{Z})$ we can represent φ as a matrix

$$\varphi = \begin{bmatrix} p & \alpha \\ q & \beta \end{bmatrix} \in SL(2, \mathbb{Z}),$$

so that $c = p\mu_0 + q\lambda_0$, and will also write $Y_{p/q}$ instead of Y_φ. The operation we have just described is called *Dehn surgery along c*

Remark 4.1. Since we are interested in monopoles on Y_φ we have to specify a metric on this manifold. The metric we will work with will have a cylindrical neck of length $r \gg 1$, and we assume the gluing of ∂X and ∂M via φ takes place in the middle of this neck. We also assume that ∂M is equipped with a flat metric $g_{\partial M}$, and X is equipped with a metric of nonnegative scalar curvature such that φ is an isometry between $g_{\partial X}$ and $g_{\partial M}$. Using the construction in Appendix B we can explicitly produce such metrics on solid tori, by attaching to the solid torus $S^1 \times D^2$, a cylinder $[0, r] \times T^2$ with a nonnegative scalar curvature metric which interpolates between the canonical flat metric g_0 on T^2, and the flat metric $\varphi^* g_0$.

We would like to describe the basic topological invariants of $Y_{p/q}$ in terms of (p, q) and the invariants of Y. For more details and proofs we refer to [28]. We will distinguish two cases.

A. $p = c \cdot \lambda_0 \neq 0$. In this case Y_c is a rational homology sphere and we have a short exact sequence

$$0 \to \mathbb{Z}\langle \mathbf{j}_* c \rangle \to H_1(M, \mathbb{Z}) \to H_1(Y_c, \mathbb{Z}) \to 0.$$

We set $K_c := \Gamma_0 \circ \varphi(\mathbf{k}_0)$ and we continue to denote by K_c the image of $\mathbf{j}_* K_c$ in $H_1(Y_c, \mathbb{Z})$. We denote the linking form of Y_c by $\mathbf{lk}_c$. The above short exact sequence defines an element in $\mathrm{Ext}\,(H_1(Y_c), \mathbb{Z}) \cong \mathrm{Hom}(H_1(Y_c, \mathbb{Z}), \mathbb{Q}/\mathbb{Z})$ which can be canonically identified with the character $\mathbf{lk}_c(K_c, \bullet)$ of $H_1(Y_c, \mathbb{Z})$. Moreover, the torsion part of $H_1(M, \mathbb{Z})$ is naturally isomorphic to the kernel of this character.

By passing to Poincaré duals we obtain the short exact sequence

$$0 \to \mathbb{Z}\langle c^\sharp \rangle \xrightarrow{\delta_M} H^2(M, T; \mathbb{Z}) \to H^2(Y_c, \mathbb{Z}) \to 0. \tag{4.1}$$

We can use this short exact sequence to explain how to glue cylindrical $spin^c$ structures on M and X to obtain all the $spin^c$ structures on Y_c.

The cylindrical manifold X is equipped with a canonical $spin^c$ structure. To describe it, we use Turaev's description of $spin^c$-structures in terms of smooth Euler structures, [28, 33, 35].

A cylindrical $spin^c$ structure on X can be described by indicating a nowhere vanishing vector field V on X pointing outwards along the boundary.[5] The determinant line bundle of the $spin^c$ structure σ_V determined by V is the oriented real 2-plane bundle $\langle V^\perp \rangle \to X$. Along ∂X we have $\langle V^\perp \rangle \equiv T\partial X$. The tangent bundle of ∂X has a canonical trivialization,

[5]For technical purposes, in all the computations to come, we will slightly deform such vector fields to get vector fields coinciding with ∂_t in a small neighborhood of the boundary, where t denotes the outgoing normal coordinate near the boundary.

uniquely defined up to a homotopy. More precisely, an oriented basis of $H_1(\partial X;\mathbb{Z})$ induces a trivialization of $T\partial X$. Different bases lead to homotopic trivializations.

We use the basis $\{\mathbf{m}_0, \mathbf{k}_0\}$ of $H_1(\partial X, \mathbb{Z})$ to produce a trivialization of $T\partial X$. The canonical *spin*^{c} structure on $S^1 \times D^2$ is described by the vector field

$$\mathbf{V}_0 = \cos(\frac{\pi r}{2})\mathbf{k}_0 + \sin(\frac{\pi r}{2})\partial_r, \;\; r \in [0,1],$$

where we denote by r the radial coordinate along the disk D^2. We will denote this cylindrical *spin*^{c} structure by *core*. Alternatively, we have an injection

$$Spin^c_{cyl}(X) \to H^2(X, \partial X; \mathbb{Z}) \cong \mathbb{Z}, \;\; \sigma \mapsto c_1^{rel}(\det \sigma).$$

Its image consists of the odd elements of $H^2(X_\varphi, \partial X_\varphi; \mathbb{Z}) \cong H_1(X;\mathbb{Z})$. We choose the core $S^1 \times \{0\}$ as generator of $H_1(X, \mathbb{Z})$. The canonical cylindrical *spin*^{c} structure is uniquely determined by

$$c_1^{rel}(core) = c_1(\langle \mathbf{V}_0 \rangle^\perp) = 1 \in H^2(X, \partial X; \mathbb{Z}).$$

We denote by $L_0 \in \mathrm{Pic}^\infty(X, \partial X)$ the complex line bundle $\langle \mathbf{V}_0\rangle^\perp$ equipped with the canonical trivialization along ∂X. We can obtain all the other cylindrical *spin*^{c}-structures by twisting $L_0 \mid_{\partial X}$ by a homotopically nontrivial gauge transformation. More precisely, given a gauge transformation $\gamma : \partial X \to S^1$ we denote by $[\gamma] = \gamma^*\big(\frac{1}{2\pi\mathbf{i}} d\gamma/\gamma\big) \in H^1(\partial X, \mathbb{Z})$ the cohomology class it determines. If we change the canonical trivialization of $L_0 \mid_{\partial X}$ to $\gamma L_0 \mid_{\partial X}$ then

$$c_1(\gamma \cdot core) = 1 + 2\delta_X[\gamma] \tag{4.2}$$

where $\delta_X : H^1(\partial X, \mathbb{Z}) \to H^2(X, \partial X; \mathbb{Z})$ is the connecting morphism of the pair $(X, \partial X)$. To see this, it suffices to pick a connection a A_0 of L_0 trivial near ∂X with respect to the canonical trivialization and a connection A_γ trivial near ∂X with respect to the trivialization $\gamma \cdot L_0$. Then, near ∂X we have

$$A_\gamma = \gamma A_0 \gamma_{-1} = A_0 - 2d\gamma/\gamma.$$

Choose a smooth cut-off function $c(r)$ such that $c(r) \equiv 0$ $r \approx 0$, and $c(r) \equiv 1$ for $r \approx 1$. Then we can define $A_\gamma := A_0 - 2c(r)d\gamma/\gamma$. We deduce

$$F_{A_\gamma} = F_{A_0} - 2c'(r)dr \wedge d\gamma/\gamma.$$

The compactly supported cohomology class $\delta_X[\gamma]$ is represented by

$$d(\frac{c(r)}{2\pi\mathbf{i}} d\gamma/\gamma) = \frac{c'(r)}{2\pi\mathbf{i}} dr \wedge d\gamma/\gamma.$$

Thus

$$c_1(A_\gamma) = c_1(A_0) + 2\frac{c'(r)}{2\pi\mathbf{i}} dr \wedge d\gamma/\gamma$$

from which the equality (4.2) is obvious. Note also that

$$\delta_X \mathbf{m}_0^\sharp = 0$$

so that twists by gauge trasformations which are homotopically trivial along the meridian $\mathbf{m}_0$ do not change the cylindrical structure. The twisted cylindrical structure $\gamma \cdot core$ is also described by the vector field

$$\mathbf{V}_\gamma = \cos\big(\frac{\pi r}{2}\big)\gamma \mathbf{k}_0 + \sin\big(\frac{\pi r}{2}\big)\partial_r.$$

We can now provide the following interpretation to the diagram (4.1). We begin by defining a gluing operation

$$\# : Spin^c_{cyl}(M, \partial M) \times Spin^c(X_\varphi) \to Spin^c(Y_\varphi).$$

Start with $spin^c$ structures $\hat{\sigma}_M$ and $\hat{\sigma}_X$ on X represented by vector fields V_M and V_X which are $\equiv \partial_t$ near the boundary. Observe that V_X is φ-invariant. Consider the cylinder $C_0 := [-1, 1] \times S^1 \times S^1$ with the vector field $\mathbf{T}_0$ given by

$$\mathbf{T}_0(t, \theta^1, \theta^2) = \cos\big(\frac{t+1}{2}\pi\big)\partial_t + \sin\big(\frac{t+1}{2}\pi\big)\partial_{\theta^1}.$$

This vector field points inwards along the boundary of C_0. If we now think of Y_φ as an union

$$Y_\varphi = M \cup C_0 \cup_{\varphi : \partial X \to \{1\} \times T^2} X$$

then we get a nowhere vanishing vector field on Y_φ equal to V_M on M, $\mathbf{T}_0$ on C_0 and V_X on X. We call this vector field $V_M \#_\varphi V_X$ and denote the corresponding $spin^c$ structure by $\hat{\sigma}_M \# \varphi \hat{\sigma}_X$.

If we fix a cylindrical $spin^c$ structure $\hat{\sigma}_0$ on M then any other cylindrical $spin^c$ structure on M is obtained in an unique way from σ_0 by twisting with a $L \in \mathrm{Pic}^\infty(M, \partial M)$,

$$(\hat{\sigma}_0, L) \mapsto \hat{\sigma}_0 \otimes L.$$

If we set

$$\hat{\sigma}_0(\varphi) := \hat{\sigma}_0 \#_\varphi core$$

then, for every $L_M \in \mathrm{Pic}^\infty(M, \partial M)$, $L_X \in \mathrm{Pic}^\infty(X, \partial X)$ we have

$$(\hat{\sigma}_0 \otimes L_M)\#_\varphi(core \otimes L_X) = \sigma_0(\varphi) \otimes (L_M \# L_X).$$

As we have explained, we have an identification

$$c_1^{rel} : \mathrm{Pic}^\infty(X, \partial X) \to \mathbb{Z}, \;\; L \mapsto c_1(L) \in H^2(X, \partial X; \mathbb{Z}) \cong \mathbb{Z},$$

and we denote by $L_n \in \mathrm{Pic}^\infty(X, \partial X)$ the cylindrical line bundle such that $c_1^{rel}(L_n) = n$. We also set

$$core_n := core \otimes L_n.$$

The gluing map φ induces by pullback a morphism

$$\varphi^* : \mathcal{G}_{\partial M} = \mathrm{Map}\,(\partial M, S^1) \to \mathcal{G}_{\partial X} = \mathrm{Map}\,(\partial X, S^1).$$

gauge group $\mathcal{G}_{\partial M}$ (resp. $\mathcal{G}_{\partial X}$) acts on $\mathrm{Pic}^\infty(M, \partial M)$ (resp. $\mathrm{Pic}^\infty(X, \partial X)$) by twisting the trivialization along the boundary. For every $\gamma \in \mathcal{G}_{\partial M}$ we have

$$\hat{\sigma}_0 \otimes (\gamma L_M \#_\varphi L_n) = \hat{\sigma}_0 \otimes (L_M \#_\varphi \Gamma^*_\varphi(1/\gamma) L_n)$$

$$= \hat{\sigma}_0 \otimes \big(L_M \#_\varphi L_{n-\delta_X \Gamma_\varphi^*[\gamma]}\big) = \hat{\sigma}_0 \otimes \big(L_M \#_\varphi L_{n-\langle[\gamma], \Gamma_\varphi \mathbf{m}_0\rangle}\big).$$

Recalling that $c := \Gamma_\varphi(\mathbf{m}_0)$, we deduce $c^\sharp = (\Gamma_\varphi^*)^{-1}\mathbf{m}_0^\sharp$, and if $[\gamma] = c^\sharp$ we deduce

$$\sigma_0(\varphi) \otimes (\gamma L_M \# \varphi L_n) = \sigma_0(\varphi) \otimes (L_M \#_\varphi L_n).$$

We can now provide the promised interpretation of (4.1).

Proposition 4.2. *Every $spin^c$ structure $\hat{\sigma}$ on Y_φ can be written as*

$$\hat{\sigma} = \hat{\sigma}_M \#_\varphi core$$

for some $\hat{\sigma}_M \in Spin^c_{cyl}(M)$. Moreover if $\hat{\sigma}_1, \hat{\sigma}_2 \in Spin^c_{cyl}(M)$ then

$$\hat{\sigma}_1 \#_\varphi core \cong \hat{\sigma}_2 \#_\varphi core \Longleftrightarrow \exists n \in \mathbb{Z} \;\; \text{such that} \;\; \hat{\sigma}_2 = nc^\sharp \hat{\sigma}_1.$$

B. $p = 0, q = 1 \Longleftrightarrow c = \lambda_0$. Set $Y_0 := Y_{\lambda_0}$. In this case we also have a short exact sequence

$$0 \to \mathbb{Z}\langle \lambda_0 \rangle \xrightarrow{\mathbf{j}_*} H_1(M; \mathbb{Z}) \to H_1(Y_0, \mathbb{Z}) \to 0.$$

The cycle $\mathbf{j}_*\lambda_0$ has order m_0 in $H_1(M, \mathbb{Z})$. We deduce that for every $spin^c$ structure σ on Y_0 there exist exactly m_0 cylindrical $spin^c$ structures σ on M with the property

$$\hat{\sigma} = \hat{\sigma}' \#_0 core.$$

We summarize the facts proved so far.

For every $\varphi \in SL(2, \mathbb{Z})$ there exists a natural surjection

$$\pi_\varphi : Spin^c_{cyl}(M) \to Spin^c(Y_\varphi), \;\; \hat{\sigma} \mapsto \hat{\sigma} \#_\varphi core.$$

Moreover,

$$\pi_\varphi(\hat{\sigma}_1) = \pi_\varphi(\hat{\sigma}_2) \Longleftrightarrow \exists n \in \mathbb{Z} : \;\; \hat{\sigma}_2 = (nc^\sharp)\hat{\sigma}_1.$$

Denote by $\Theta^\sharp_\varphi$ the image of $H^1(X, \mathbb{Z})$ in $\Theta^\sharp$ via $(\varphi^* \circ \Gamma^*)^{-1}$. Note that $\Theta^\sharp_\varphi$ is the group generated by $c^\sharp$. The results we have established show that the map

$$\pi_\varphi : Spin^c_{cyl}(M) \to Spin^c(Y_\varphi)$$

is $\Theta^\sharp_M + \Theta^\sharp_\varphi$-invariant. In particular, it descends to a map

$$\pi_\varphi : Spin^c_{cyl}(M)/(\Theta^\sharp_M + \Theta^\sharp_\varphi) \to Spin^c(Y_\varphi).$$

Set $G_\varphi := \Theta^\sharp/(\Theta^\sharp_M + \Theta^\sharp_\varphi)$. The quotient $Spin^c_{cyl}(M)/(\Theta^\sharp_M + \Theta^\sharp_\varphi)$ is equipped with a residual G_φ action. We can use π_φ to transport it to a G_φ-action on $Spin^c(Y_\varphi)$. On the other hand, we have isomorphisms

$$Spin^c(M) \cong Spin^c_{cyl}(M)/\Theta^\sharp \cong \Big(Spin^c_{cyl}(M)/(\Theta^\sharp_M + \Theta^\sharp_\varphi)\Big)/G_\varphi.$$

We conclude that π_φ induces a map

$$\pi_\varphi : Spin^c(M) \to Spin^c(Y_\varphi)/G_\varphi.$$

Remark 4.3. The group G_φ can be given a different geometric interpretation. The closed curve $\Gamma \circ \varphi(\mathbf{k}_0)$ determines a homology class $K_\varphi \in H_1(Y_\varphi, \mathbb{Z})$. Then G_φ is isomorphic to the cyclic group generated K_φ. For 0/1-surgery this is an infinite cyclic group. In this case the generator of G_φ may not be a primitive class. It has the form $m_0 \times$ *primitive class*. For the other surgeries the group G_φ is finite. In this case the extension

$$0 \to \mathbb{Z}^2 \cong \Theta^\sharp_M + \Theta^\sharp_\varphi \hookrightarrow \Theta^\sharp \cong \mathbb{Z}^2 \to G_\varphi \to 0$$

defines an element $\chi \in \mathrm{Ext}(G_\varphi, \mathbb{Z}^2) = \mathrm{Hom}(G_\varphi, \mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z})$, and thus can be identified with a pair (χ_1, χ_2) of characters of G_φ. More precisely

$$\chi_1(K_\varphi) = \frac{1}{r},\;\; r := |G_\varphi|,\;\; \chi_2(K_\varphi) = -\mathbf{lk}_{Y_\varphi}(K_\varphi, K_\varphi),$$

where $\mathbf{lk}_{Y_\varphi}$ denotes the linking form on Y_φ. We refer to [28] for proofs and details.

4.2 A relative "invariant"

We continue to use the set-up and notations described in 4.1. Fix a sufficiently small positive exponential weight μ. Set

$$\widehat{\mathcal{G}}_M := \widehat{\mathcal{G}}_{\mu,ex},\;\; \mathcal{G}^M_T = \partial_\infty \widehat{\mathcal{G}} \subset \mathcal{G}_T := L^{3,2}(T, S^1).$$

The group of components of $\mathcal{G}$ is $\Theta^\sharp$ while the group of components of $\mathcal{G}^{\partial M}$ is the subgroup of $\Theta^\sharp$ generated by $\lambda^\sharp$.

We identify the trivial complex line bundle over T with the (holomorphic) tangent bundle of T equipped with the canonical trivialization. Denote by $\mathbf{B}_0$ the trivial connection on the trivial complex line bundle over T. Recall that $\mathfrak{M}_T$ denotes the spaces of flat connections on the trivial line bundle over T modulo even gauge transformations in $\mathcal{G}_T$. We have a homeomorphism

$$\mathfrak{M}_T \to H^1(T, \mathbb{R})/2\Theta^\sharp$$

defined by

$$(\mathbf{B}_0 + \mathbf{i}a) \mod \mathcal{G}_T \mapsto \frac{1}{2\pi}[a] \mod 2\Theta^\sharp,$$

where $[a] \in H^1(T, \mathbb{R})$ denotes the harmonic part of the closed 1-form a. Denote by $\mathfrak{M}^M_T$ the space of flat connections on the trivial bundle on T modulo even gauge transformations in $\mathcal{G}^M_T$. This space is homemorphic to the cylinder

$$H^1(T, \mathbb{R})/2\mathbb{Z}\lambda^\sharp.$$

The space $\mathfrak{M}^M_T$ is a $\mathbb{Z}$-cover of $\mathfrak{M}_T$. We denote by $\widetilde{\mathfrak{M}}_T$ the universal cover of $\mathfrak{M}_T$,

$$\widetilde{\mathfrak{M}}_T \cong H^1(T, \mathbb{R}).$$

The unique bad reducible in $\mathfrak{M}_T$ lifts to a lattice of bad points in $\widetilde{\mathfrak{M}}_T$ which can be identified with $2\Theta^\sharp$.

For all $0 < r \ll 1$ denote by $\mathfrak{M}_T(r)$ the complement in $\mathfrak{M}_T$ of an open disk of radius r centered at $[\mathbf{B}_0] \in \mathfrak{M}_T$. Topologically, $\mathfrak{M}_T(r)$ is a torus with a small disk removed. Denote

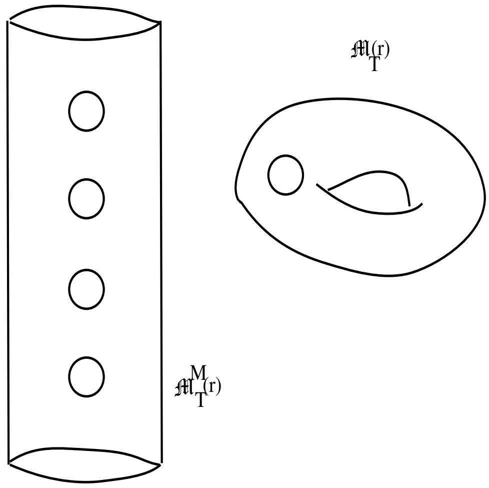

Figure 5: *Spaces of gauge equivalences classes of flat connections on a torus*

by $\mathfrak{M}_T^M(r)$ the preimage of $\mathfrak{M}_T(r)$ in $\mathfrak{M}_T^M$. Topologically, $\mathfrak{M}_T^M(r)$ is an infinite cylinder with a sequence of holes in it; see Figure 5. We define $\widetilde{\mathfrak{M}}_T(r)$ in a similar way. It is a plane with small holes centered at the bad points.

Fix a cylindrical *spin*^{c} structure $\hat{\sigma}$ on M, a very small r, and perturbations parameters (η, w) on M in the generic way explained in 3.4. We get a moduli space $\widehat{\mathfrak{M}}_{M,\hat{\sigma}}(r) := \widehat{\mathfrak{M}}_{\mu}(r)$. When no confusion is possible we will write $\widehat{\mathfrak{M}}_M(r)$ instead of $\widehat{\mathfrak{M}}_{M,\hat{\sigma}}(r)$. The image $\partial_\infty \widehat{\mathfrak{M}}_{M,\hat{\sigma}}$ is a compact immersed curve in $\mathfrak{M}_T^M$. Before we describe it we consider a special case which will play an important role in the sequel.

Example 4.4. Suppose M is the solid torus $S^1 \times D^2$. We think of it as a cylindrical manifold equipped an admissible metric of nonnegative scalar curvature as explained in Appendix B. Then all the finite energy monopoles corresponding to the canonical cylindrical *spin*^{c} structure $\hat{\sigma} = \mathit{core}$ are reducible. It consists of $\widehat{\mathcal{G}}_M$ equivalence classes of flat connections on L_0, and the moduli space $\widehat{\mathfrak{M}}_M$ is diffeomorphic to a circle. We would like to understand the image $\partial_\infty \widehat{\mathfrak{M}}_M$. We want to show that $\mathbf{B}_0 \notin \partial_\infty \widehat{\mathfrak{M}}_M$, and then describe the position of $\mathbf{B}_0$ relative to $\partial_\infty \widehat{\mathfrak{M}}_M$.

First let us emphasize one subtlety. The map ∂_∞ is not simply a restriction-to-the-boundary map. The bundle L_0 is equipped with a canonical isomorphism

$$\vartheta : L_0 \mid_T \rightarrow \underline{\mathbb{C}}_T.$$

If $\hat{B}$ is a flat connection on L_0 then

$$\partial_\infty \hat{B} = \vartheta \hat{B} |_T \, \vartheta^{-1}.$$

Then

$$\partial_\infty \hat{B} = \mathbf{B}_0 + \mathbf{i} b, \ \ b \in \Omega^1(T), \ \ db = 0.$$

Denote by r the radial coordinate along D^2, by θ the angular coordinate on D^2, and by φ the angular coordinate along the core S^1. As in 4.1 we set $\mathbf{m}_0 := \{1\} \times \partial D^2 \in \Theta$, $\mathbf{k}_0 = S^1 \times 1 \in \Theta$. Then

$$\mathbf{m}_0^\sharp = \frac{1}{2\pi} d\varphi, \ \ \mathbf{k}_0^\sharp = -\frac{1}{2\pi} d\theta.$$

Choose a smooth, nonnegative, nonincreasing cut-off function $c(r)$ such that $c(r) \equiv 1$ for $r \approx 1$ and $r \equiv 0$ for $r \approx 1/2$. Set $\hat{B}' := \hat{B} - \mathbf{i}c(r)b$

$$\partial_\infty \hat{B}' = \mathbf{B}_0$$

and

$$\frac{\mathbf{i}}{2\pi} F_{\hat{B}'} = \frac{1}{2\pi} \delta_M b.$$

The compactly supported 2-form $\frac{\mathbf{i}}{2\pi} F_{\hat{B}'}$ represents the relative Chern class of L_0 which by was chosen to be the canonical generator of $H^2(M, \partial M; \mathbb{Z})$. Equivalently, this means

$$1 = \frac{1}{2\pi} \int_{\{1\} \times D^2} \delta_M b = \frac{1}{2\pi} \int_{\partial D^2} b.$$

Thus

$$\frac{1}{2\pi} b \in -\mathbf{k}_0^\sharp + \mathbb{Z}\mathbf{m}_0^\sharp.$$

Observe that $\mathfrak{M}_T^M = \Theta^\sharp / 2\mathbf{m}_0^\sharp \mathbb{Z}$. The image $\partial_\infty \widehat{\mathfrak{M}}_M$ in $\mathfrak{M}_T^M$ is $(-\mathbf{k}_0^\sharp + \mathbb{R}\mathbf{m}_0^\sharp)/2\mathbb{Z}\mathbf{m}_0^\sharp$, and it looks like in Figure 6.

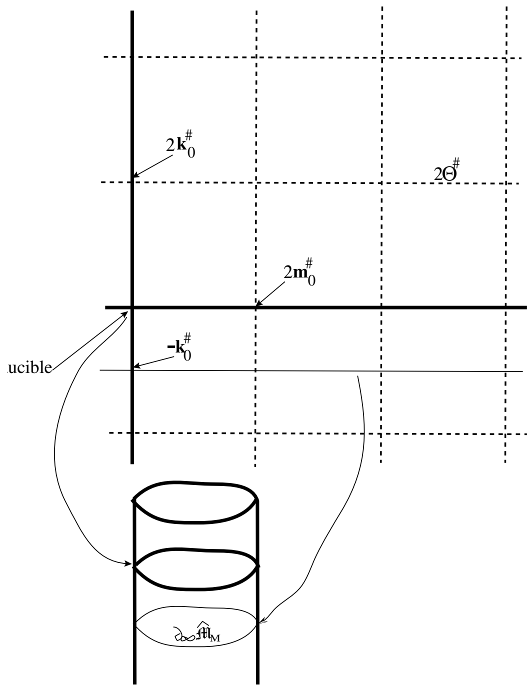

Figure 6: *The traces left by the monopoles on a solid torus*

We now consider the general case. We decompose

$$\widehat{\mathfrak{M}}_{M,\hat{\sigma}}(r) = \widehat{\mathfrak{M}}^{red}_{M,\hat{\sigma}}(r) \cup \widehat{\mathfrak{M}}^{irr}_{M,\hat{\sigma}}(r).$$

We want to analyze first $\partial_\infty \widehat{\mathfrak{M}}^{red}_{M,\hat{\sigma}}(r)$. Recall that

$$\widehat{\mathfrak{M}}^{red}_{M,\hat{\sigma}}(r) = \Big\{\hat{A};\ \ F_{\hat{A}} + *\mathbf{i}\eta = 0\Big\}/2\mathbf{i}\mathbb{Z}\omega_0$$

where $\omega_0 \in L^2_{ex}$ is a harmonic 1-form which defines a positive[6] generator of $H^1(M,\mathbb{Z})$. *We will always choose this positive generator to be the Poincaré dual of the cycle* $\Lambda \in H_2(M,\partial M;\mathbb{Z})$ *which bounds the longitude* λ.

Fix a flat connection $\hat{B}_0$ on $\det(\hat{\sigma})$ and a smooth 1-form $b_0 \in L^2_{ex}$ such that $*\eta = \hat{d}b_0$. Then the finite energy reducible monopoles have the form

$$\hat{A} = \hat{B}_0 + \mathbf{i}\hat{b}_0 + \mathbf{i}t\omega_0.$$

Let us observe that since η is compactly supported the restriction of b_0 to T is a closed 1-form. We will denote it by b_0 The restriction of ω_0 to T generates the image of the morhism

$$H^1(M,\mathbb{R}) \to H^1(T,\mathbb{R})$$

By passing to Poincaré duals we deduce that $\omega_0\,|_T = \lambda^\sharp$.

The cylindrical *spin*^{c} structure $\hat{\sigma}$ carries additional topological information. First, the absolute Chern class $c_1(\det\hat{\sigma}) \in H^2(M,\mathbb{Z})$. This is completely characterized by the holonomy representation defined by the flat connection $\hat{B}_0$

$$\mathrm{hol}_{\hat{B}_0} : H_1(M,\mathbb{Z}) \to S^1.$$

The second more refined information is the relative Chern class $c_1^{rel}(\det\hat{\sigma}) \in H^2(M,\partial M;\mathbb{Z})$, and is due to the cylindrical structure. The absolute Chern class is the image of the relative Chern class via the natural morphism

$$H^2(M,\partial M;\mathbb{Z}) \to H^2(M,\mathbb{Z}).$$

As in Example 4.4 we have

$$\partial_\infty \hat{A} = \hat{B}_0\,|_T + \mathbf{i}b_0 + \mathbf{i}t\lambda^\sharp + \mathbf{i}b$$

b is a closed 1-form such that $\frac{1}{2\pi}b$ is integral, and its presence is due to the cylindrical structure. We now write

$$\hat{B}_0\,|_T = \mathbf{B}_0 + \mathbf{i}\chi_0.$$

The closed 1-form $\frac{1}{2\pi}\chi_0$ is uniquely determined mod $\mathbb{Z}\lambda^\sharp$, and satisfies the holonomy conditions,

$$\mathrm{hol}_{\hat{B}_0}(\mathbf{j}_* c) = \exp\big(-\mathbf{i}\langle\chi_0, c\rangle\big),\ \ \forall c \in \Theta.$$

In particular, $\frac{1}{2\pi}\langle\chi_0,\lambda\rangle \in \mathbb{Z}$. Arguing exactly as in Example 4.4 we deduce

$$\frac{1}{2\pi}\langle b + \chi_0, \lambda\rangle = \deg(\hat{\sigma}) := \langle c_1^{rel}(\det\hat{\sigma}), \Lambda\rangle \in \mathbb{Z}.$$

[6]The positivity assumption refers to the chosen orientation of $H^1(M,\mathbb{R})$ required to orient the moduli spaces.

We write

$$\frac{1}{2\pi}b := u\lambda_0^\sharp + v\mu_0^\sharp, \quad u, v \in \mathbb{Z}$$

$$\frac{1}{2\pi}\chi_0 := x_0\lambda_0^\sharp + y_0\mu_0^\sharp, \quad x, y \in \mathbb{Q}.$$

We deduce

$$vm_0 = -\deg(\hat{\sigma}), \quad m_0 y \in \mathbb{Z},$$

If we define $\alpha_{\hat{\sigma}}, \beta_{\hat{\sigma}} \in \mathbb{Q} \cap [0,1)$ by the equalities

$$\exp(2\pi\mathbf{i}\beta_{\hat{\sigma}}) = \mathrm{hol}_{\hat{B}_0}(\mathbf{j}_*\lambda_0), \quad \exp(-2\pi\mathbf{i}\alpha_{\hat{\sigma}}) = \mathrm{hol}_{\hat{B}_0}(\mathbf{j}_*\mu_0).$$

We deduce

$$x_0 - \alpha_{\hat{\sigma}} \in \mathbb{Z}, \quad y_0 - \beta_{\hat{\sigma}} \in \mathbb{Z}$$

The ambiguity in y_0 is due to the fact that there is no canonical identification[7] between cylindrical *spin*^{c}-structure and cylindrical line bundles. The image of $\widehat{\mathfrak{M}}_M^{red}$ in $\mathfrak{M}_T^M \cong H^1(T;\mathbb{R})/2\mathbb{Z}\lambda^\sharp$ is a circle. Its lift to the universal cover is the line

$$t \mapsto b_0 + \Big(\beta_{\hat{\sigma}} - \frac{\deg(\hat{\sigma})}{m_0}\Big)\mu_0^\sharp + t\lambda_0^\sharp$$

depicted in Figure 7.

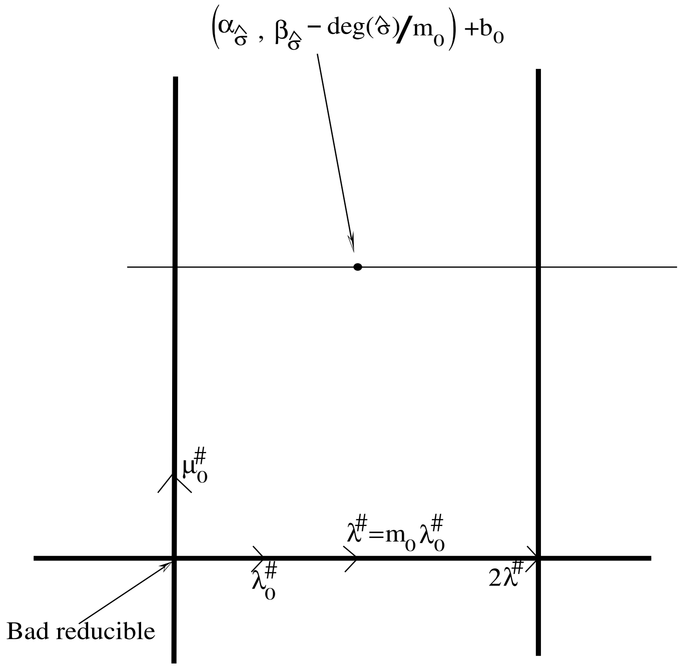

Figure 7: *The boundary trace left by the reducible monopoles on a knot complement.*

[7]When M is the complement of a knot in an *integral* homology sphere then there is a canonical way of associating a cylindrical line bundle to a *spin*^{c} structure.

Suppose now that we perform Dehn surgery on M by attaching a solid torus $X = S^1 \times D^2$ using a gluing map $\varphi : H_1(\partial X, \mathbb{Z}) \to H_1(\partial X, \mathbb{Z})$. As explained in 4.1 this means attaching ∂X to ∂M using the gluing map

$$\Gamma \circ \varphi : \partial X \xrightarrow{\varphi} \partial X \xrightarrow{\Gamma} \partial M.$$

We have an oriented basis $\mathbf{m}_0$, $\mathbf{k}_0$ of $H_1(\partial X, \mathbb{Z})$ where we orient ∂X as boundary of X and an oriented basis $\{\lambda_0, \mu_0\}$ of $H_1(\partial M, \mathbb{Z}) = \Theta$, where ∂M is oriented as boundary of M. With respect to the chosen bases Γ has the matric description

$$\Gamma := \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right] \Longleftrightarrow \mathbf{m}_0 \mapsto \mu_0, , \;\; \mathbf{k}_0 \mapsto \lambda_0.$$

The attaching map φ can be identified with a matrix in $SL(2, \mathbb{Z})$

$$\varphi = \left[\begin{array}{cc} p & \alpha \\ q & \beta \end{array}\right] \in SL(2, \mathbb{Z}),$$

Then

$$\Gamma_\varphi := \Gamma \circ \varphi = \left[\begin{array}{cc} q & \beta \\ p & \alpha \end{array}\right] \Longrightarrow \mathbf{m}_0 \mapsto p\mu_0 + q\lambda_0.$$

We will denote the above matrix by $\Gamma_{p/q}$. The map Γ_φ^{-1} induces by pullback a map

$$(\Gamma_\varphi)_\sharp : H^1(\partial X, \mathbb{Z}) \to H^1(\partial M, \mathbb{Z}) = \Theta^*.$$

With respect to the bases $\{\mathbf{m}_0^\sharp, \mathbf{k}_0^\sharp\}$ and $\{\lambda_0^\sharp, \mu_0^\sharp\}$ it has the matrix description

$$(\Gamma_{p/q})_\sharp = \left[\begin{array}{cc} \alpha & -\beta \\ -p & q \end{array}\right]^{-1} = -\left[\begin{array}{cc} q & \beta \\ p & \alpha \end{array}\right] = -\Gamma_{p/q}.$$

Consider now the horizontal *oriented* line

$$\mathfrak{T} := \big\{(-\mathbf{k}_0^\sharp + t\mathbf{m}_0^\sharp; \;\; t \in \mathbb{R}\big\}_{n \in \mathbb{Z}} \subset H^1(\partial X, \mathbb{R}).$$

This line is depicted in Figure 6 and it is the lift to $H^1(\partial X, \mathbb{R})$ of the curve

$$\partial_\infty \widehat{\mathfrak{M}}_{X,core} \subset \mathfrak{M}^X_{\partial X} \cong H^1(\partial X, \mathbb{R})/2\mathbb{Z}\mathbf{m}_0^\sharp.$$

The image of $\mathfrak{T}$ via $(\Gamma_\varphi)_\sharp$ is the line

$$\mathfrak{T}_\varphi = \mathfrak{T}_{p/q} := \Big\{-(\beta\lambda_0^\sharp + \alpha\mu_0^\sharp) - tc^\sharp; \;\; c^\sharp = q\lambda_0^\sharp + p\mu_0^\sharp, \;\; t \in \mathbb{R}\Big\} \subset H^1(\partial M, \mathbb{R}).$$

Note that $\mathfrak{T}$ stays away from the lattice of bad points $H^1(\partial X, 2\mathbb{Z})$. Since $\Gamma_{p/q}$ maps bad points to bad points we conclude that the line $\mathfrak{T}_{p/q}$ will also stay away from the lattice of bad points $2\Theta^\sharp$. Set

$$C_{\hat{\sigma}}(r) := \partial_\infty \widehat{\mathfrak{M}}^{irr}_{M,\hat{\sigma}}(r) \subset \mathfrak{M}^M_T, \;\; \chi_{\hat{\sigma}} := \partial_\infty \widehat{\mathfrak{M}}^{red}_{M,\hat{\sigma}} \subset \mathfrak{M}^M_T$$

and denote by $\bullet \mapsto [\bullet]$ either one of the natural projections

$$H^1(\partial M, \mathbb{R}) \to \mathfrak{M}_T, \;\; \mathfrak{M}_T^M \to \mathfrak{M}_T.$$

$[\mathfrak{T}_{p,q}]$ is a closed curve on $\mathfrak{M}_T$ which does not pass through the unique bad point $[\mathbf{B}_0]$. $[\chi_{\hat{\sigma}}]$is a closed curve in $\mathfrak{M}_T$, the image mod $2\Theta^\sharp$ of the line

$$t \mapsto b_0 + \Big(\beta_{\hat{\sigma}} - \frac{\deg(\hat{\sigma})}{m_0}\Big)\mu_0^\sharp + t\lambda_0^\sharp \subset H^1(\partial M, \mathbb{R}).$$

$[C_{\hat{\sigma}}(r)]$ consists of

(i) Immersed closed curves on the torus $\mathfrak{M}_T$ away from the unique bad point $\mathbf{B}_0$.
(ii) Immersed closed curves on the torus $\mathfrak{M}_T$ with boundaries on a small circle centered at the unique bad point of $\mathfrak{M}_T$.
(iii) Immersed curves with both boundary points on $[\chi_{\hat{\sigma}}]$.
(iv) Immersed curves with one boundary point on $[\chi_{\hat{\sigma}}]$, and another on the small circle centered at $[\mathbf{B}_0]$.

We conclude that $[C_{\hat{\sigma}}(r)]$ is partitioned into two parts

$$[C_{\hat{\sigma}}(r)] = A_{\hat{\sigma}}(r) \cup B_{\hat{\sigma}}(r)$$

where A consists only of closed curves while B consists only of curves with boundary. The closed curves $[\chi_{\hat{\sigma}}]$ and A carry multiplicity

As we have explained in 4.1, the orbit of $\hat{\sigma}$ in $Spin^c_{cyl}(M)$ modulo the twisting action of $H^1(\partial M, \mathbb{Z})$ can be identified with the *spin*^{c} structure underlying $\hat{\sigma}$. We will denote both this orbit, and the underlying *spin*^{c}-structure by $[\hat{\sigma}]$. The gluing operation # associates to the orbit $[\hat{\sigma}]$ (on M) a G_φ-orbit of *spin*^{c}-structures on Y_φ. We will denote this orbit by $[\hat{\sigma}]_\varphi = [\hat{\sigma}]_{p/q} \subset Spin^c(Y_{p/q})$. The rational number $\beta_{\hat{\sigma}}$ is essentially the level invariant defined in [30, Sec. 17].

The gluing results of [26, Sec.45] imply that we have a bijection

$$[C_{\hat{\sigma}}(r)] \cap [\mathfrak{T}_{p,q}] \longleftrightarrow \bigcup_{\sigma \in [\hat{\sigma}]_{p/q}} \widehat{\mathfrak{M}}(Y_{p/q}, \sigma).$$

The cohomology classes $2\lambda_0^\sharp$ and $2\mu_0^\sharp$ are natural generators of $H_1(\mathfrak{M}_T, \mathbb{Z})$.

A The odd signature operator on admissible 3-manifolds

Suppose M is an admissible 3-manifold with a cylindrical neck $\mathbb{R}_+ \times T^2$, where T^2 is equipped with a flat metric. The odd signature operator on M is the first order formally selfadjoint operator

$$\mathbf{SIGN} : (\Omega^1 \oplus \Omega^0)(M) \longrightarrow (\Omega^1 \oplus \Omega^0)$$

given by the block decomposition

$$\mathbf{SIGN} = \begin{bmatrix} \hat{*}\hat{d} & -\hat{d} \\ -\hat{d}^* & 0 \end{bmatrix}.$$

Along the neck any 1-form α has a decomposition

$$\alpha = dt \wedge \alpha_0(t) + \alpha_1(t), \quad \alpha_0 := \lrcorner_t \alpha, \quad \lrcorner_t := \partial_t \lrcorner, \quad \alpha_1 \in \Omega^1(\partial_\infty M).$$

Similarly, a 2-form has a decomposition

$$\omega = dt \wedge \omega_1 + \omega_2.$$

Then

$$\hat{d}\alpha = dt \wedge (\dot{\alpha}_1 - d\alpha_0) + d\alpha_1$$

and

$$\hat{*}\hat{d}\alpha = *(\dot{\alpha}_1 - d\alpha_0) + dt \wedge *d\alpha_1.$$

Using the equality $-\hat{d}^* = \hat{*}\hat{d}\hat{*}$ we deduce

$$-\hat{d}^* = \hat{*}\hat{d}(*\alpha_0 - dt \wedge *\alpha_1) = \hat{*}(dt \wedge *\dot{\alpha}_0 + dt \wedge d * \alpha_1)$$

$$= \dot{\alpha}_0 + *d * \alpha_1 = \dot{\alpha}_0 - d^*\alpha_1.$$

For $f \in \Omega^0(M)$ we have along the neck

$$\hat{d} = dt \wedge df + df.$$

Thus

$$\mathbf{SIGN} \begin{bmatrix} \alpha_1 \\ dt \wedge \alpha_0 \\ f \end{bmatrix} = \begin{bmatrix} *\dot{\alpha}_1 - *d\alpha_0 - df \\ dt \wedge (-\dot{f} + *d\alpha_1) \\ \dot{\alpha}_0 - d^*\alpha_1 \end{bmatrix}$$

Thus, along the neck we can regard $\mathbf{SIGN}$ as an operator on $\Omega^0 \oplus \Omega^1 \oplus \Omega^0(\partial_\infty M)$ given by

$$\mathbf{SIGN} \begin{bmatrix} \alpha_1 \\ \alpha_0 \\ f \end{bmatrix} = \left(\begin{bmatrix} * & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \partial_t + \begin{bmatrix} 0 & -*d & -d \\ *d & 0 & 0 \\ -d^* & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} \alpha_1 \\ \alpha_0 \\ f \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -1 \\ 0 & * & 0 \\ 1 & 0 & 0 \end{bmatrix} \left(\partial_t - \begin{bmatrix} 0 & d & -*d \\ d^* & 0 & 0 \\ *d & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ f \end{bmatrix}$$

This shows that **SIGN** is an APS operator in the sense of [26] and

$$\vec{\partial}_\infty \mathbf{SIGN} = \begin{bmatrix} 0 & d & -*d \\ d^* & 0 & 0 \\ *d & 0 & 0 \end{bmatrix} =: \mathcal{H}.$$

The kernel of $\mathcal{H}$ consists of triples $(\alpha_0, \alpha_1, f) \in (\Omega^0 \oplus \Omega^1 \oplus \Omega^0)(T^2)$ such that

$$\begin{cases} d^*\alpha_1 = 0, \;\; *d\alpha_1 = 0 \\ \\ \qquad d\alpha_0 = *df \end{cases}.$$

This shows that α_1 is harmonic and α_0 and f are constants. $\ker \mathcal{H}$ is equipped with symplectic structure induced by the metric on $\ker h$ and the compatible almost complex structure

$$J := \begin{bmatrix} * & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The L^2-kernel of **SIGN** consists of L^2-harmonic 1-forms and thus

$$\ker_{L^2} \mathbf{SIGN} \cong \operatorname{Range}\big(H^1(M, \partial M) \to H^1(M)\big).$$

The extended kernel of **SIGN** consists of pairs (α, f) where f is a real constant and α is an extended L^2-harmonic 1-form. Observe that if we identify $\alpha = dt \wedge \alpha_0 + \alpha_1$ with $\alpha_0 \oplus \alpha_1$ then

$$\partial_\infty \alpha = \partial_\infty \alpha_0 \oplus \partial_\infty \alpha_1.$$

We set $\partial^0_\infty \alpha := \partial_\infty \alpha_0$ and $\partial^1_\infty \alpha := \partial_\infty \alpha_1$. The subspace $L_{an} := \partial_\infty \ker_{ex} \mathbf{SIGN} \subset \ker \mathcal{H}$ is Lagrangian so that

$$J(\partial_\infty(\alpha, f)) \perp L, \;\; \forall (\alpha, f) \in \ker_{ex} \mathbf{SIGN}.$$

In particular,

$$J(\alpha, 0) \perp (0, \partial_\infty f), \;\; \forall (\alpha, f) \in \ker_{ex} \mathbf{SIGN}.$$

This implies that

$$\partial^0_\infty \alpha = 0, \;\; \forall (\alpha, f) \in \ker_{ex} \mathbf{SIGN}.$$

Using the results of [26, Example 4.1.21] we can identify L_{an} with L_{top}, the image of $(H^1 \oplus H^0)(M)$ in $(H^1 \oplus H^0)(T^2) \subset \ker \mathcal{H}$. By comparing the short exact sequences

$$0 \to \ker_{L^2} \mathbf{SIGN} \to \ker_{ex} \mathbf{SIGN} \to L_{an} \to 0$$

and

$$0 \to \ker_{L^2} \mathbf{SIGN} \to (H^1 \oplus H^0)(M) \to L_{top} \to 0$$

we deduce

$$\ker_{ex} \mathbf{SIGN} \cong (H^1 \oplus H^0)(M), \;\; L_{an} = L_{top}.$$

In this work we also use the weighted odd signature operator $\mathbf{SIGN}_\mu$, $\mu > 0$, given by

$$\mathbf{SIGN}_\mu := \begin{bmatrix} \hat{*}\hat{d} & -\hat{d} \\ -\hat{d}^{*_\mu} & 0 \end{bmatrix} = \begin{bmatrix} \hat{*}\hat{d} & -\hat{d} \\ -\mathbf{m}_{-2\mu}\hat{d}^*\mathbf{m}_{2\mu} & 0 \end{bmatrix}$$

Along the neck it has the form

$$\mathbf{SIGN}_\mu \begin{bmatrix} \alpha_1 \\ \alpha_0 \\ f \end{bmatrix} = \left(\begin{bmatrix} * & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \partial_t + \begin{bmatrix} 0 & -*d & -d \\ *d & 0 & 0 \\ -d^* & 2\mu & 0 \end{bmatrix} \right) \begin{bmatrix} \alpha_1 \\ \alpha_0 \\ f \end{bmatrix}$$

$$= \begin{bmatrix} * & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \left(\partial_t - \begin{bmatrix} 0 & d & -*d \\ d^* & -2\mu & 0 \\ *d & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} \alpha_1 \\ \alpha_0 \\ f \end{bmatrix}$$

so that $\mathbf{SIGN}_\mu$ is an APS operator as well with

$$\vec{\partial}_\infty \mathbf{SIGN}_\mu = \begin{bmatrix} 0 & d & -*d \\ d^* & -2\mu & 0 \\ *d & 0 & 0 \end{bmatrix} =: \mathcal{H}_\mu.$$

Denote by Δ_k the Laplacian on k-forms on T^2 equipped with a flat metric. Let $\lambda_1 > 0$ be the smallest positive eigenvalue of Δ_0. We want to determine the eigenvalues ν of $\mathcal{H}_\mu$ such that $\nu^2 < \frac{\lambda_1}{4}$ assuming that μ is sufficiently small, $\mu^2 < \frac{\lambda_1}{16}$. Observe that if ν is an eigenvalue of $\mathcal{H}_\mu$ then ν^2 is an eigenvalue of $\mathcal{H}_\mu^2$. Next, observe that

$$\mathcal{H}_\mu^2 = \begin{bmatrix} \Delta_1 & -2\mu d & 0 \\ -2\mu d^* & \Delta_0 + 4\mu^2 & 0 \\ 0 & 0 & \Delta_0 \end{bmatrix}.$$

Thus if (α_1, α_0, f) is a nonzero eigenvector of $\mathcal{H}_\mu$ corresponding to the small eigenvalue ν we deduce

$$\begin{cases} d^*\alpha_1 = (2\mu + \nu)\alpha_0 \\ d\alpha_0 = *df + \nu\alpha_1 \\ *d\alpha_1 = \nu f \end{cases} \tag{A.3}$$

and

$$\begin{cases} (\Delta_0 + 4\mu^2)\alpha_0 = 2\mu d^*\alpha_1 + \nu^2\alpha_0 \\ \Delta_1\alpha_1 = 2\mu d\alpha_0 + \nu^2\alpha_1 \\ \Delta_0 f = \nu^2 f \end{cases}. \tag{A.4}$$

From the second equation in (A.3) we deduce

$$\Delta\alpha_0 = d^*d\alpha_0 = \nu d^*\alpha_1 = \nu(2\mu + \nu)\alpha_0. \tag{A.5}$$

If $\nu = 0$ then we deduce that both α_0 and f are constants and $d^*\alpha_1 = 2\mu\alpha_0$. The only constant which is a divergence is 0 so that $\alpha_0 = 0$. Hence, for $\mu > 0$ we have

$$\ker \mathcal{H}_\mu = \Big\{(\alpha_0, \alpha_1, f);\ \ \alpha_0 = 0,\ \ f = \text{const},\ \ d\alpha_1 = d^*\alpha_1 = 0\Big\} \Longrightarrow \dim_{\mathbb{R}} \mathcal{H}_\mu = 3. \tag{A.6}$$

If $0 < \nu^2 < \frac{\lambda_1}{4}$ we deduce from the third equation in (A.4) that $f = 0$. If $\alpha_0 = 0$ as well then we deduce from (A.3) that $\alpha_1 = 0$. Thus $\alpha_0 \neq 0$ and we deduce from (A.5) that $\nu(2\mu + \nu)$ is an eigenvalue λ of Δ_0. Notice that

$$\mu^2 < \frac{\lambda_1}{16}, \nu^2 < \frac{\lambda_1}{4} \Longrightarrow \nu(2\mu + \nu) < \frac{\lambda_1}{2}.$$

Hence $\lambda = 0$ so that $\nu = -2\mu$ and $\alpha_0 \in \ker \Delta_0$ which means that α_0 is a constant. Using this information in the second equation of (A.3) we deduce that $\alpha_1 = 0$. We have thus proved the following result.

Proposition A.1. *For small μ, $\mu^2 < \frac{\lambda_1}{16}$, the operator $\mathcal{H}_\mu$ has only one nonzero eigenvalue in the interval $(-\frac{\sqrt{\lambda_1}}{2}, \frac{\sqrt{\lambda_1}}{2})$. This eigenvalue is -2μ and the corresponding eigenspace is spanned by the vector*

$$(\alpha_0, \alpha_1, f) = (1, 0, 0).$$

B Nonnegative scalar curvature metrics on cylinders

We denote by θ^1, θ^2 the angular coordinates on $T^2 := S^1 \times S^1$ so that

$$\int_{T^2} d\theta^1 \wedge d\theta^2 = 4\pi^2.$$

A *diagonal metric* on T^2 is a (flat) metric of the form

$$g := k_1(d\theta^1)^2 + k_2(d\theta^2)^2$$

where k_1 and k_2 are positive constants. We will prove the following result.

Proposition B.1. *Suppose $A \in SL_2(\mathbb{Z})$ and $\varepsilon > 0$ is a very small number. Denote by g_0 the flat metric on T^2 described by*

$$g_0 := A^*\Big((d\theta^1)^2 + (d\theta^2)^2\Big).$$

(In other words, g_0 is the pullback by A of the canonical metric on T^2.) Then there exists a constant $\delta > 0$ and a smooth path $g(t)$ of flat metrics on T^2 such that

(i) $g(t) \equiv \frac{1}{\delta^2} g_0$, $\forall t \leq \varepsilon$,
(ii) $g_1 := g(1)$ is a diagonal metric,
(iii) $g(t) = g_1$, $\forall t \geq 1 - \varepsilon$,
(iv) and the scalar curvature of the metric $\hat{g} := dt^2 + g(t)$ on $\mathbb{R} \times T^2$ is nonnegative.

Proof Set

$$\delta^2 := g_0(\partial_{\theta^1}, \partial_{\theta_1})$$

and, only for the ease of notation, reset

$$g_0 := \frac{1}{\delta^2} g_0.$$

Then, ∂_{θ^1} is an unit vector with respect to this metric and we can complete it to an oriented orthonormal frame of g_0. We denote its dual coframe by $\{\varphi^1,\ \varphi^2\} \subset \Omega^1(T^2)$. This coframe is related to the original one, $d\theta^1, d\theta^2$ via the equalities

$$\left\{ \begin{array}{lcr} \varphi^1 & = & d\theta^1 + a_0 d\theta^2 \\ \varphi^2 & = & k\theta^2 \end{array} \right.$$

where k is a positive constant. The path $g(t)$ will be described by indicating by a path of coframes $\{\varphi^1(t), \varphi^2(t)\}$ which we declare to be orthonormal with respect to $g(t)$.

We seek coframes of the form

$$\left\{ \begin{array}{rcr} \varphi^1 & = & d\theta^1 + a(t)d\theta^2 \\ \varphi^2 & = & k\theta^2 \end{array} \right. \tag{B.7}$$

where $a(t)$ is a smooth function such that

$$a(t) \equiv 0, \ \ \forall t \geq 1 - \varepsilon \tag{B.8a}$$

$$a(t) = a_0, \ \ \forall t \leq \varepsilon. \tag{B.8b}$$

Clearly the conditions (i)-(iii) are satisfied for all choices of $a(t)$ as above. We only need to prove that we could choose $a(t)$ constrained by (B.8a) and (B.8b) such that (iv) is satisfied as well. We will use E. Cartan's moving frame technique.

Set $\varphi^0 := dt$. Then $\{\varphi^0, \varphi^1, \varphi^2\}$ is an orthonormal coframe for $\hat{g}$ on $X := \mathbb{R} \times T^2$. This defines an orthonormal frame of X,

$$\{e_0, e_1, e_2\}$$

with respect to which the Levi-Civita is described by an $\underline{so}(3)$-valued 1-from on X

$$\Gamma := \left[\begin{array}{ccc} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{array} \right], \ \ x, y, z \in \Omega^1(X).$$

Γ is determined by Cartan's structural equations

$$d\vec{\varphi} = \Gamma \wedge \vec{\varphi}, \ \ \vec{\varphi} := \left[\begin{array}{c} \varphi^0 \\ \varphi^1 \\ \varphi^2 \end{array} \right].$$

Using (B.7) we deduce

$$d\varphi^0 = d\varphi^2 = 0, \ \ d\varphi^1 = \frac{\dot{a}}{k}\varphi^0 \wedge \varphi^2$$

where the dot denotes t-derivatives. We deduce

$$0 = x \wedge \varphi^1 + y \wedge \varphi^2 \tag{B.9a}$$

$$\frac{\dot{a}}{k}\varphi^0 \wedge \varphi^2 = -x \wedge \varphi^0 + z \wedge \varphi^2 \tag{B.9b}$$

$$0 = -y \wedge \varphi^0 - z \wedge \varphi^1 \tag{B.9c}$$

Set

$$x = \sum_i x_i \varphi^i, \ \ y = \sum_j y_j \varphi^j, \ \ z = \sum_k z_k \varphi^k, \ \ x_i, y_j, z_k \in C^\infty(X).$$

Then

$$(B.9a) \Longrightarrow x_0 = y_0 = 0, \ \ x_2 = y_1$$

$$(B.9b) \Longrightarrow x_1 = z_1 = 0, \;\; x_2 + z_0 = \frac{\dot{a}}{k}$$

$$(B.9c) \Longrightarrow y_2 = z_2 = 0, \;\; y_1 = z_0.$$

We conclude that

$$x = \frac{\dot{a}}{2k}\varphi^2, \;\; \frac{\dot{a}}{2k}\varphi^1, \;\; z = \frac{\dot{a}}{2k}\varphi^0,$$

so that

$$\Gamma := \frac{\dot{a}}{2k}\begin{bmatrix} 0 & \varphi^2 & \varphi^1 \\ -\varphi^2 & 0 & \varphi^0 \\ -\varphi^1 & -\varphi_0 & 0 \end{bmatrix}.$$

The curvature of the Levi-Civita equation, which we regard as a $\underline{so}(3)$-valued 2-form Ω, is given by

$$\Omega := d\Gamma + \Gamma \wedge \Gamma.$$

The scalar curvature of $\hat{g}$ is given by

$$\hat{s} = \sigma(\Omega)$$

where, for any $\underline{so}(3)$-valued 2-form Ω on X, we set

$$\sigma(\Omega) := \sum_{i \neq j} \langle \Omega(e_i, e_j)e_j, e_i \rangle = 2 \sum_{0 \leq i < j \leq 2} \langle \Omega(e_i, e_j)e_j, e_i \rangle$$

$$= 2\Big(\langle \Omega(e_0, e_1)e_1, e_0 \rangle + \langle \Omega(e_0, e_2)e_2, e_0 \rangle + \langle \Omega(e_1, e_2)e_2, e_1 \rangle \Big) = 2 \sum_{0 \leq i < j \leq 2} \Omega_i^j(e_i, e_j).$$

Thus

$$\hat{s} := \sigma(d\Gamma) + \sigma(\Gamma \wedge \Gamma).$$

Now observe that

$$\Gamma \wedge \Gamma = \left(\frac{\dot{a}}{2k}\right)^2 \begin{bmatrix} 0 & \varphi^0 \wedge \varphi^1 & \varphi^2 \wedge \varphi^0 \\ -\varphi^0 \wedge \varphi^1 & 0 & \varphi^1 \wedge \varphi^2 \\ \varphi^0 \wedge \varphi^2 & -\varphi^1 \wedge \varphi^2 & 0 \end{bmatrix}.$$

We deduce that

$$\sigma(\Gamma \wedge \Gamma) = 2\left(\frac{\dot{a}}{2k}\right)^2.$$

Next, observe that

$$d\Gamma = \frac{\ddot{a}}{2k}\varphi^0 \wedge \begin{bmatrix} 0 & \varphi^2 & \varphi^1 \\ -\varphi^2 & 0 & \varphi^0 \\ -\varphi^1 & -\varphi_0 & 0 \end{bmatrix} + \frac{\dot{a}}{2k}\begin{bmatrix} 0 & 0 & \frac{\dot{a}}{k}\varphi^0 \wedge \varphi^2 \\ 0 & 0 & 0 \\ -\frac{\dot{a}}{k}\varphi^0\varphi^2 & 0 & 0 \end{bmatrix} =: A_1 + A_2.$$

Clearly, $\sigma(A_1) = 0$ while

$$\sigma(A_2) = \left(\frac{\dot{a}}{k}\right)^2.$$

Thus $\hat{s} = \sigma(\Omega) \geq 0$ for any choice of $a(t)$ constrained by (B.8a) and (B.8b). ■

C Continuous families of Fredholm operators

This appendix is a quick survey and an expansion of the ideas introduced in [26, §1.5.1].

Suppose X is a compact smooth manifold and

$$X \ni x \mapsto T_x \in \mathbf{Fred}(H_0, H_1)$$

is a smooth family of bounded, Fredholm operators between two Hilbert spaces H_0 and H_1.

A sub-bundle V of the trivial bundle $\underline{H}_1 := (H_1 \times X \to X)$ is called a *stabilizer* for the family $T_\bullet$ if for every $x \in X$ the operator

$$T_x \uplus V_x : \; H_0 \oplus V_x \to H_1, \;\; (h_0, v) \mapsto T_x h_0 + v \in H_1$$

is onto. In this case the family of vector spaces

$$\mathcal{K}_V(T_\bullet) := \ker(T_\bullet \uplus V)$$

can be organized as a smooth vector bundle over X. We can think of V as defining a resolution of $T_\bullet$ in the sense that we have a short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \underline{H}_0 \arrow[r, "i"] \arrow[d, "T_\bullet"'] & \underline{H}_0 \oplus V \arrow[r] \arrow[d, "T_\bullet \uplus V"'] & V \arrow[r] \arrow[d, "0"'] & 0 \\
0 \arrow[r, "0"] & \underline{H}_1 \arrow[r, "1"] & \underline{H}_1 \arrow[r, "0"] & 0 \arrow[r] & 0
\end{tikzcd}$$

where the last two vertical arrows are onto.

If $V_1 \subset V_2$ are two stabilizers then we have a short exact sequence

$$0 \longrightarrow \mathcal{K}_{V_1}(T_\bullet) \to \mathcal{K}_{V_2}(T_\bullet) \to V_2/V_1 \to 0 \tag{$K_{V_1 \hookrightarrow V_2}$}$$

where we automatically identify V_2/V_1 with the orthogonal complement of V_1 in V_2 and map $\mathcal{K}_{V_2}(T) \to V_2/V_1$ is induced by the orthogonal projection

$$H_0 \oplus V_2 \to V_2/V_1.$$

It is often convenient to regard $(K_{V_1 \hookrightarrow V_2})$ as describing an acyclic chain complex. Taking the direct sum of $(K_{V_1 \hookrightarrow V_2})$ with the acyclic complex

$$0 \to V_1^* \to V_2^* \to (V_2/V_1)^* \to 0 \tag{$C_{V_1 \hookrightarrow V_2}$}$$

we obtain the short exact sequence

$$0 \to \mathcal{K}_{V_1}(T) \oplus V_1^* \to \mathcal{K}_{V_2}(T) \oplus V_2^* \to (V_2/V_1) \oplus (V_2/V_1)^* \to 0.$$

All vector spaces in the above acyclic complex are equipped with scalar products and thus, as explained in [28], this chain complex is equipped with a natural algebraic contraction. The torsion of this acyclic complex + algebraic contraction is a natural isomorphism

$$I_{V_2/V_1} : \det \mathcal{K}_{V_1}(T) \otimes \det V_1^* \otimes \det(V_2/V_1) \otimes \det(V_2/V_1)^* \to \det \mathcal{K}_{V_2}(T) \otimes V_2^*.$$

Since we have a canonical isomorphism $\det U \otimes \det U^* \cong \mathbb{R}$ for every vector space U we will regard I_{V_2/V_1} as an isomorphism

$$I_{V_2/V_1} : \mathfrak{L}_{V_1} \to \mathfrak{L}_{V_2}$$

where for every stabilizer V we set

$$\mathfrak{L}_V := \det \mathcal{K}_V(T) \otimes \det V^*.$$

In [26] we proved that

$$I_{V_3/V_1} = I_{V_3/V_2} \circ I_{V_2/V_1}.$$

An *orientation* of the family $(T_\bullet)$ is a collection of isomorphisms

$$\varphi_V : \underline{\mathbb{R}} \to \mathfrak{L}_V, \;\; V \;\text{ stabilizer}$$

such that for every $V_1 \hookrightarrow V_2$ the diagram below is homotopically commutative

$$\begin{tikzcd} \mathfrak{L}_{V_1} \arrow[rr, "I_{V_2/V_1}"] & & \mathfrak{L}_{V_2} \\ & \underline{\mathbb{R}} \arrow[ul, "\varphi_{V_1}"] \arrow[ur, "\varphi_{V_2}"'] & \end{tikzcd}$$

A family is called *orientable* if it admits an orientation. Notice that an orientation induces an orientation on each of the lines $\det\ker T_x \otimes \det\ker T_x^*$, $x \in X$. More precisely, if V is a stabilizer then we have a short exact sequences

$$0 \to \ker T_x \to K_{V_x}(T_x) \to V_x/\mathbf{proj}_{V_x}(\ker T_x^*) \to 0$$

$$0 \to \ker T_x^* \stackrel{\mathbf{proj}_{V_x}}{\longrightarrow} V_x \to V_x/\mathbf{proj}_{V_x}(\ker T_x^*) \to 0 \quad .$$

The above considerations imply the following result.

Proposition C.1. *A family $T_\bullet$ is orientable if and only if there exists an orientable stabilizer V such that the bundle $K_V(T_\bullet)$ is orientable.*

We can organize the collection of smooth families of Fredholm operators parameterized by X as an additive category. The morphisms between two families $(S_\bullet)$, $(T_\bullet)$ are smooth families (L_0, L_1) of bounded operators

$$X \ni x \mapsto L_i(x) \in B(H_i), \;\; i = 0, 1$$

such that the diagram below is commutative for every $x \in X$,

$$\begin{tikzcd} H_0 \arrow[r, "L_0(x)"] \arrow[d, "S_x"'] & H_0 \arrow[d, "T_x"] \\ H_1 \arrow[r, "L_1(x)"] & H_1 \end{tikzcd} .$$

We can talk about short exact sequences of Fredholm families. We have the following result.

Proposition C.2. *Suppose*

$$0 \to S_\bullet \stackrel{(f_0,f_1)}{\longrightarrow} T_\bullet \stackrel{(g_0,g_1)}{\longrightarrow} U_\bullet \to 0 \tag{C.10}$$

is a short exact sequence of Fredholm families. If two of the families are orientable then so is the third.

Proof We can find a trivial vector sub-bundle $V \subset \underline{H}_1$ which is a stabilizer for all three families. We then have a short exact sequence

$$0 \to S_\bullet \uplus V \stackrel{(f_0 \oplus 1_V, f_1)}{\longrightarrow} T_\bullet \oplus V \stackrel{(g_0 \oplus 1_V, g_1)}{\longrightarrow} U_\bullet \uplus V \to 0$$

and thus a short exact sequence

$$0 \to K_V(S) \to K_V(T) \to K_V(U) \to 0.$$

The proposition is now obvious. ■

The short exact sequence (C.10) induces for each $x \in X$ a long exact sequence relating the cohomology spaces of the complexes S_x, T_x and U_x. Suppose for exemplification that we have chosen oriented bases in the cohomology of S_x and T_x. We know that there is an induced orientation on $\det H^*(U_x)$. In all concrete computations one has to address the following effectivity issue. How do we *effectively* produce bases of $H^*(U_x)$ inducing the same orientation on $H^*(U_x)$ as the orientation induced by the short exact sequence (C.10)?

The recipe is very simple. Fix an arbitrary basis of $H^*(U_x)$. We can now regard the long exact sequence derived from (C.10) as a *based* acyclic complex. The basis we chose on $H^*(U_x)$ produces the desired orientation if and only if the torsion of this based acyclic complex is positive.

D Gluing formulæ for the eta invariants

We have included here for the reader's convenience a survey of the basic facts concerning surgery formulæ for eta invariants. We follow closely the elegant presentation in [12].

The selfadjoint operators with compact resolvents behave in many respects as common finite-dimensional symmetric matrices and we will refer to such operators as *excellent*. The eta invariant extends the notion of signature from finite-dimensional symmetric matrices to an important subclass of excellent operators.

The signature of a finite-dimensional symmetric matrix A is defined as

$$\mathrm{sign}\,(A) = \text{number of positive eigenvalues} - \text{number of negative eigenvalues}.$$

This definition however does not extend to infinite dimensions since the above terms are infinite. One could try to "regularize" the definition. For each $s \in \mathbb{C}$ we set

$$\eta_A(s) = \sum_{\lambda \in \sigma^*(A)} \frac{\dim \ker(A-\lambda)}{\lambda |\lambda|^{s-1}} = \sum_{\lambda > 0} \frac{\dim \ker(A-\lambda) - \dim \ker(A+\lambda)}{\lambda^s} \tag{D.11}$$

where $\sigma^*(A) = \mathrm{spec}\,(A) \setminus \{0\}$. Then one can define

$$\mathrm{sign}\,(A) = \eta_A(0).$$

The advantage of this new definition is that it is admirably suited for infinite-dimensional extensions. Assuming for simplicity that A is invertible we can define

$$\eta_A(s) = \mathrm{tr}\,(A \cdot |A|^{-(s+1)}), \quad |A| = (A^2)^{1/2}.$$

Using the classical integral

$$\Gamma(\alpha)x^{-\alpha} = \int_0^\infty t^{\alpha-1}e^{-tx}dt, \;\; x > 0, \; \alpha > 1,$$

we get $(x \mapsto A^2, \;\; \alpha \mapsto (s+1)/2)$

$$\eta_A(s) = \frac{1}{\Gamma(\,(s+1)/2\,)}\int_0^\infty t^{(s-1)/2}\mathrm{tr}\,(Ae^{-tA^2})dt.$$

The right-hand side of the above expression has two advantages. First of all, it makes sense even when A is not invertible and on the other hand, it extends to infinite dimensions. We will denote the trace of an infinite-dimensional operator (when it exists) by "Tr" while "tr" is reserved for finite-dimensional operators. We have the following result.

Proposition D.1. *(a) Consider a closed, oriented Riemannian manifold (M, g) of dimension ν, $E \to M$ a Hermitian vector bundle and*

$$D : C^\infty(E) \to C^\infty(E)$$

a first order selfadjoint elliptic operator. Then

$$\eta_D(s) := \frac{1}{\Gamma(\,(s+1)/2\,)}\int_0^\infty t^{(s-1)/2}\mathrm{Tr}\,(De^{-tD^2})dt \tag{D.12}$$

is well defined for all $\mathbf{Re}\,s \gg 0$ *and extends to a meromorphic function on $\mathbb{C}$ which is described by the Dirichlet series (D.11) for $|s| \gg 0$. Its poles are all simple and can be located only at $s = (\nu + 1 - k)/2$, $k = 0, 1, 2, \cdots$.*
(b) If ν is odd then the residue of $\eta_D(s)$ at $s = 0$ is zero so that $s = 0$ is a regular point.

For a proof of this nontrivial result we refer to [3, 4, 10]. When d is odd we define the eta invariant of A by

$$\eta(D) := \eta_D(0).$$

Another important source of excellent operators arises from elliptic selfadjoint boundary value problems. For more details on the properties of such problems we refer to the clear presentation in [4].

Suppose $(\hat{N}, \hat{g})$ is a $(2n+1)$-dimensional, compact Riemannian manifold with boundary $N = \partial\hat{N}$ such that a tubular neighborhood of N is isometric to the cylinder $(0, 1] \times N, dt^2 + g)$ where g is a Riemann metric on N and t denotes the outgoing normal coordinate. For each $R \in (0, \infty]$ we denote by $\hat{N}_R$ the Riemann manifold obtained from $\hat{N}$ by attaching the cylinder $[0, R] \times N$.

We assume $\hat{E} \to \hat{N}$ is a hermitian vector bundle on $\hat{N}$. Set $E := \hat{E} \mid_{\partial \hat{N}}$. Suppose we are given a formally selfadjoint Dirac type operator

$$\hat{D} : V^{\infty}(\hat{E}) \to C^{\infty}(\hat{E})$$

which near the boundary has the APS form

$$\hat{D} = J(\partial_t - D), \quad J = \hat{\boldsymbol{c}}(dt),$$

$$D^* = D, \quad JD + DJ = 0, \tag{D.13}$$

where $\hat{\boldsymbol{c}}$ denotes the Clifford multiplication on $\hat{E}$ induced by $\hat{D}$ and $D : C^{\infty}(E) \to C^{\infty})(E)$ is a formally selfadjoint Dirac type operator on N. The operator J induces a symplectic structure on $L^2(E)$,

$$\omega(u, v) = \int_N \langle Ju, v \rangle dv_g.$$

Given a closed subspace $S \subset L^2(E)$ we can define a closed densely defined operator $\hat{D}_S$ to be $\hat{D}$ acting on the domain

$$Dom\,(\hat{D}_S) = \Bigl\{ u \in L^{1,2}(\hat{E}); \;\; u \mid_N \in S \Bigr\}.$$

Denote by $\mathcal{H}_D^{\pm}$ the closed subspace of $L^2(E)$ spanned by the eigenvectors of D corresponding to positive/negative eigenvalues. We denote by $\Pi_D^{\pm}$ the orthogonal projection onto $\mathcal{H}_D^{\pm}$. It is known that $\Pi_D^{\pm}$ is a zeroth order pseudodifferential operator. Its principal symbol is completely determined by the principal symbol of D. Consider the following family of closed subspaces of $L^2(E)$

$$\mathcal{L} = \mathcal{L}_{\hat{D}} = \Bigl\{ \Lambda \subset L^2(E); \;\; \Lambda^{\perp} = J\Lambda, \;\; \dim(\mathcal{H}_D^{+} \cap \Lambda) < \infty \Bigr\}$$

The condition $\Lambda^{\perp} = J\Lambda$ means that Λ is a Lagrangian subspace. Define now a subfamily $\mathcal{L}^{\infty}$ consisting of those Lagrangian subspaces such that

- The orthogonal projection P_{Λ} onto Λ is a zeroth order pseudodifferential operator.
- The operator $P_{\Lambda} - \Pi_D^{-}$ is a smoothing operator.

The space $\mathcal{L}$ is not empty and if fact it contains two remarkable elements.

The Calderon subspace Define the *Cauchy-data space*

$$\Lambda(\hat{D}) = \overline{\Bigl\{ u \mid_N; \;\; u \in \ker \hat{D} \cap L^{1/2,2}(\hat{E}) \Bigr\}}^{L^2} \subset L^2(E).$$

Then according to [4] we have $J\Lambda(\hat{D}) \in \mathcal{L}_{\hat{D}}^{\infty}$. We will refer to $J\Lambda(\hat{D})$ as the *Calderon subspace.*

The Atiyah-Patodi-Singer subspace Set $L_{\infty}(\hat{D}) := \partial_{\infty} \ker_{ex} \hat{D} \subset \ker D$. $L_{\infty}(\hat{D})$ is a Lagrangian subspace of $\ker D$. The Atiyah-Patodi-Singer (or APS) Lagrangian is

$$\Lambda_{aps}(\hat{D}) := L^{\infty}(\hat{D})^{\perp} \oplus \mathcal{H}_D^{-} = JL^{\infty}(\hat{D}) \oplus \mathcal{H}_D^{-}.$$

We have the following result (see [4, 37]) for details).

Theorem D.2. *(a) There exists a natural topology on $\mathcal{L}^\infty_{\hat{D}}$ (described in [29]) such that $\mathcal{L}^\infty_{\hat{D}}$ is a homogenous space for the group $\mathcal{U}^\infty$ of unitary operators $U : L^2(E) \to L^2(E)$ such that $1 - U$ is a smoothing. The stabilizer of $\Lambda \in \mathcal{L}^\infty_{\hat{D}}$ can be identified with the subgroup $\mathcal{O}^\infty \subset \mathcal{U}^\infty$ consisting of those unitary transformations which commute with the orrthogonal reflection through Λ. $\mathcal{L}^\infty_{\hat{D}}$ can be viewed as a smooth manifold with the tangent space at Λ canonically identified with $T_1\mathcal{U}^\infty/T_1\mathcal{O}^\infty$.*

(b) For every $\Lambda \in \mathcal{L}^\infty_{\hat{D}}$ the operator $\hat{D}_\Lambda$ is excellent. In particular, the correspondence $\Lambda \mapsto \hat{D}_\Lambda$ is smooth.

(c) For every $\Lambda \in \mathcal{L}^\infty_{\hat{D}}$ the Dirichlet series (D.11) associated to $\hat{D}_\Lambda$ converges for $|s| \gg 0$ and extends to a meromorphic function $\eta_{\hat{D}_\Lambda}$ on $\mathbb{C}$ with $s = 0$ a regular point.

We set $\eta(\hat{D}, \Lambda) := \eta_{\hat{D}_\Lambda}(0)$ and define the *reduced eta invariant* of $\hat{D}$ by the equality

$$\xi(\hat{D}, \Lambda) = \frac{1}{2}\big(\eta(\hat{D}, \Lambda) + \dim_{\mathbb{C}} \ker \hat{D}_\Lambda\big).$$

$\xi(\hat{D}_\Lambda)$ does not depend continuously on $\hat{D}$ but $\exp(2\pi\mathbf{i}\xi(\hat{D}_\Lambda)$ does. Moreover if

$$[0,1] \ni t \mapsto \Lambda_t \in \mathcal{L}^\infty_{\hat{D}}$$

is a smooth path then

$$\xi(\hat{D}, \Lambda_1) - \xi(\hat{D}, \Lambda_0) = SF(\hat{D}_{\Lambda_t}; 0 \leq t \leq 1)$$

$$+\frac{1}{2\pi\mathbf{i}}\int_0^1 \exp\big(-2\pi\mathbf{i}\xi(\hat{D}, \Lambda)\big)\frac{d}{dt}\exp\big(2\pi\mathbf{i}\xi(\hat{D}, \Lambda)\big)dt.$$

The integrand in the above equality is called the *infinitesimal variation of* ξ, and is denoted by

$$\frac{d}{dt}\xi(\hat{D}, \Lambda_t).$$

In the case when $\Lambda_t = \exp(\mathbf{i}tH)\Lambda_0$ where H is a selfadjoint smoothing operator, so that $\mathbf{i}H \in T_1\mathcal{U}^\infty$, the infinitesimal has the more explicit description

$$\frac{d}{dt}|_{t=0}\, \xi(\hat{D}, \Lambda_t) := \frac{1}{2}\mathrm{Tr}\,(H).$$

This can be given a more conceptual description as follows. Consider the Fredholm determinant map

$$\det : \mathcal{U}^\infty \to S^1,\;\; U \mapsto \det U.$$

Then, define

$$\varpi := -(\det{}^2)^*\big(\frac{1}{2\pi\mathbf{i}}d\theta\big) \in \Omega^1(\mathcal{U}^\infty).$$

This is a left invariant 1-form on $\mathcal{U}^\infty$ and for every $\mathbf{i}H \in T_1\mathcal{U}^\infty$ we have

$$\varpi(\mathbf{i}H) = -\frac{1}{\pi}\mathrm{Tr}\,(H).$$

Since the map $\det^2$ is $\mathcal{O}^\infty$ invariant, the form ϖ descends to 1-form on $\mathcal{L}^\infty_{\hat{D}}$ which we continue to denote by ϖ. Then

$$\frac{d}{dt}\xi(\hat{D},\Lambda_t)=\varpi(\dot{\Lambda}_t) \tag{D.14}$$

where $\dot{\Lambda}_t\in T_{\Lambda_t}\mathcal{L}^\infty_{\hat{D}}$ is the tangent vector to the path $t\mapsto\Lambda_t$. Let us point out that if $\gamma:=t\mapsto\Lambda_t$ is a closed loop in $\mathcal{L}^\infty_{\hat{D}}$ then the Maslov index of the loop

$$t\mapsto(\gamma(t),\Lambda(\hat{D})\,)$$

is given by

$$\mu(\gamma,\Lambda(\hat{D}))=-\oint_\gamma\varpi=-\mu(\Lambda(\hat{D}),\gamma(t)).$$

Example D.3. Consider the operator $\hat{D}$ on $L^2([0,R],\mathbb{C})$ domain

$$\Bigl\{u\in L^{1,2}([0,r],\mathbb{C});\ \ \arg u(0)=0,\ \ \arg u(r)=\pi/3\Bigr\}$$

defined by

$$\hat{D}u=-\mathbf{i}\frac{du}{dt}+\pi au$$

where a is a real number. This operator is selfadjoint has compact resolvent and the spectrum consists of simple eigenvalues $\lambda\in\mathbb{R}$ such that

$$\lambda-\pi ra\in\frac{\pi}{3}+\mathbb{Z}\pi.$$

Thus

$$\operatorname{spec}(\hat{D})=\Bigl\{\pi(ra+\frac{1}{3}+k);\ k\in\mathbb{Z}\Bigr\}.$$

Set $\alpha_r:=ra+\frac{1}{3}-\lfloor ra+\frac{1}{3}\rfloor$. Assume $\alpha_r\neq 0$. Then

$$\eta_{\hat{D}}(s)=\sum_{k\in\mathbb{Z}}\frac{\operatorname{sign}(\alpha_r+k)}{|\alpha_r+k|^s}=\sum_{k\geq 0}\frac{1}{|\alpha_r+k|^s}-\sum_{k\geq 0}\frac{1}{|k+1-\alpha_r|^s}$$

$$=\zeta(s,\alpha_r)-\zeta(s,1-\alpha_r)$$

where $\zeta(s,c)$ denotes the Riemann-Hurwitz function

$$\zeta(s,c):=\sum_{k\geq 0}\frac{1}{(k+c)^s}.$$

Using the computations in [36, Sec.13.21] we deduce

$$\eta_{\hat{D}}(0)=\zeta(0,\alpha_r)-\zeta(0,1-\alpha_r)=\frac{1}{2}-\alpha_r-(\frac{1}{2}-(1-\alpha_r))=1-2\alpha_r.$$

The eta invariant depends on the length of the interval, but observe that our operator violates the compatibility condition (D.13).

Example D.4. For $\theta \in [0, \pi)$ consider $\hat{D}_\theta$ the operator on $L^2([0, R], \mathbb{R}^2)$ with domain

$$\Big\{ u \in L^{1,2}([0, r], \mathbb{R}^2); \;\; u(0) \in L_0, \;\; u(r) \in L_\theta \Big\}$$

where L_0 is the line $y = 0$, L_θ is the line $y = \tan(\theta)x$, and $\hat{D}$ acts according to

$$\hat{D}_\theta u = J\frac{d}{dt}$$

where

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The eigenvalues of $\hat{D}_\theta$ are obtained by solving the equation

$$\exp(-r\lambda J)L_0 = L_\theta$$

so that

$$\operatorname{spec}(\hat{D}_\theta) = \frac{\pi}{r}\Big(\frac{\theta}{\pi} + \mathbb{Z}\Big).$$

Arguing as in the previous example we deduce

$$\eta_\theta := \eta_{\hat{D}_\theta}(0) = 1 - 2\frac{\theta}{\pi}, \;\; \xi_\theta := \xi_{\hat{D}_\theta}(0) = \frac{1}{2} - \frac{\theta}{\pi}.$$

Note that this eta invariant is independent of the length r. Next observe that

$$2(\xi_\varepsilon - \xi_{-\varepsilon}) = \eta_\varepsilon - \eta_{-\varepsilon} = \eta_\varepsilon - \eta_{\pi-\varepsilon} = -2\frac{\varepsilon}{\pi} + 2\frac{\pi - \varepsilon}{\pi} = 2 - 4\frac{\varepsilon}{\pi}.$$

In this case the space of lagrangian boundary conditions can be identified with the space of 1-dimensional subspaces of $\mathbb{R}^2$ and the form ϖ on $\mathbb{RP}^1 \cong S^1$ is $\varpi = -\frac{1}{\pi}d\theta$. The variational formula for the eta invariants predicts

$$\xi_\varepsilon - \xi_{-\varepsilon} = SF(\hat{D}_\theta; \;\; -\varepsilon \leq \theta \leq \varepsilon) - \frac{1}{\pi}\int_{-\varepsilon}^{\varepsilon} d\theta = 1 - \frac{2\varepsilon}{\pi}$$

which agrees with the above direct computation. Notice that

$$SF(\hat{D}_\theta; \;\; -\varepsilon \leq \theta \leq \varepsilon) = \mu(\,(L_\theta, L_0), -\varepsilon \leq \theta \leq \varepsilon).$$

Consider the more general situation where $\hat{D}_\theta$ has the same domain as above but acts according to

$$\hat{D}_\theta u = J(\frac{d}{dt} - A)u$$

where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Using the variational formula we deduce

$$\xi(\hat{D}_\theta) - \xi(\hat{D}_{\pi/2}) = SF(\hat{D}_t; t \in [\pi/2, \theta]) - \frac{\theta - \pi/2}{\pi}$$

$$= \mu\big((L_t \oplus L_0, \Gamma_{e^{rA}}); t \in [\pi/2, \theta]\big) + \frac{1}{2} - \frac{\theta}{\pi}$$

where $\Gamma_{e^{rA}} \subset \mathbb{R}^2 \oplus \mathbb{R}^2$ is the graph of the symplectic map e^{rA}. Thus we only need to compute $\xi(\hat{D}_{\pi/2})$. Observe that $\lambda \in \operatorname{spec}(\hat{D}_\theta)$ if and only if

$$\exp(r(A - \lambda J))L_0 = L_\theta.$$

To compute this exponential observe that the two matrices A and J anti-commute and $J^2 = -1 = -A^2$ so that

$$(rA - r\lambda J)^2 = r^2A^2 - r^2\lambda^2.$$

Now observe that $(rA - r\lambda J)$ commutes with $r^2A^2 - \lambda^2$ so that

$$(rA - r\lambda J)^{2k} = (r^2 - r^2\lambda^2)^k, \;\; (rA - r\lambda J)^{2k+1} = (r^2 - r^2\lambda^2)^k(rA - r\lambda J).$$

Thus

$$\exp(r(A - \lambda J)) = \sum_{k=0}^{\infty}(r^2 - r^2\lambda^2)^k\Big(\frac{1}{(2k)!} + \frac{1}{(2k+1)!}(rA - r\lambda J)\Big)$$

Now observe that

$$\exp(r(A - \lambda J))\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{1}{(2k)!}(r^2 - r^2\lambda^2)^k \\ \\ -r\lambda \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(r^2 - r^2\lambda^2)^k \end{bmatrix}$$

In the special case when $\theta = \pi/2$, so that $L_{\pi/2}$ is the y-axis of $\mathbb{R}^2$, we deduce from the above computation that the spectrum of A is symmetric with respect to the the involution $\lambda \longleftrightarrow -\lambda$ so that the eta invariant in this case is zero. Observe that $0 \notin \operatorname{spec}(\hat{D}_{\pi/2})$ so that $\xi(\hat{D}_{\pi/2}) = 0$. In general $0 \in \operatorname{spec}(\hat{D}_\theta)$ if and only if $e^{rA}L_0 = L_\theta$. This never happens since $e^{rA}L_0 = L_0$. Thus, in this general case we also have

$$\xi(\hat{D}_\theta) = \frac{1}{2} - \frac{\theta}{\pi}.$$

We have the following fundamental result due to P.Kirk and M.Lesch, [12].

Theorem D.5. (Surgery formula for eta invariants) *Suppose $(\hat{N}, \hat{g})$ is an odd dimensional manifold decomposed into two parts $\hat{N}_\pm$ by an oriented hypersurface N such that a tubular neighborhood of N is isometric to the cylinder $[-1,1] \times N$ equipped with the metric $dt^2 + g$, $g := \hat{g} \mid_N$. Denote by $\hat{N}_r$ the manifold obtained from $\hat{N}$ by replacing the neck $[-1,1] \times N$ with the longer one $[-r,r] \times N$. We get similarly two manifolds with boundary $\hat{N}_r^\pm$ (see Figure 8).*

Suppose $\hat{D}$ is a selfadjoint Dirac-type operator on $\hat{N}$ such that along the neck has the form

$$\hat{D} = (\partial_t - D), \;\; J := \hat{\mathbf{c}}(dt), \;\; JD + DJ = 0, \;\; D^* = D.$$

Denote by $\hat{D}_r$ the obvious extension of $\hat{D}$ to $\hat{N}_r$ and set $\hat{D}_r^\pm = \hat{D}_r \mid_{\hat{N}_r^\pm}$. Then

$$\xi(\hat{D}_r) = \xi\big(\,\hat{D}_r^+, J\Lambda(\hat{D}_r^+)\,\big) + \xi\big(\,\hat{D}_r^-, \Lambda(\hat{D}_r^+)\,\big).$$

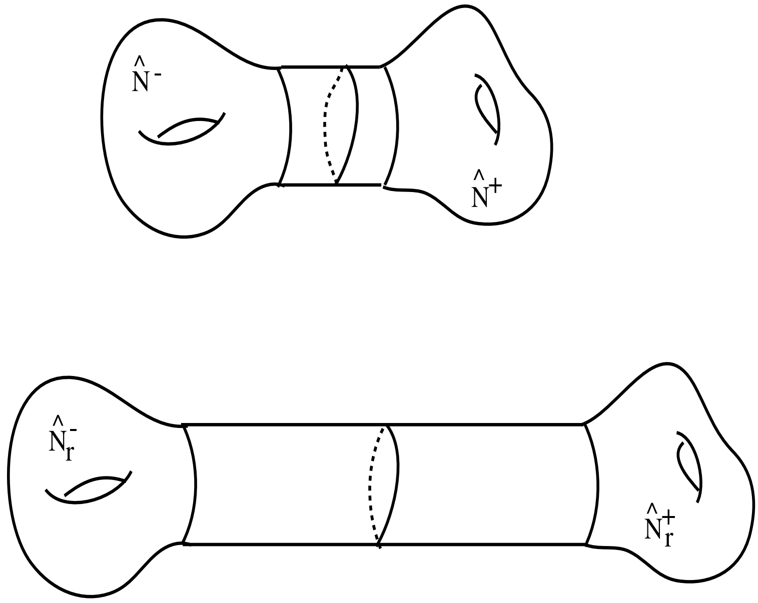

Figure 8: *Adiabatic splitting of a manifold*

More generally, if $\Lambda^{\pm} \in \mathcal{L}^{\infty}_{\hat{D}^{\pm}_r}$ *and*

$$[0,1] \ni t \mapsto \Lambda^{\pm}_t \in \mathcal{L}^{\infty}_{\hat{D}^{\pm}_r}$$

is a pair of smooth paths such that Λ^+_t *connects* $J\Lambda(\hat{D}^+_r)$ *to* Λ^+ *and* Λ^-_t *connects* $\Lambda(\hat{D}^+_r)$ *to* Λ^- *then*

$$\xi(\hat{D}_r) - \xi(\hat{D}^+_r, \Lambda^+) - \xi(\hat{D}^-_r, \Lambda^-) = SF(\hat{D}^+_r, \Lambda^+_t) + SF(\hat{D}^-_r, \Lambda^-_t) + \int_0^1 \big\{\varpi_+(\dot{\Lambda}^+_t) + \varpi_-(\dot{\Lambda}^-_t)\big\} dt$$

$$= \mu(\Lambda^+_t, \Lambda(\hat{D}^+_r)) + \mu(\Lambda^-_t, \Lambda(\hat{D}^-_r)) + \int_0^1 \big\{\varpi_+(\dot{\Lambda}^+_t) + \varpi_-(\dot{\Lambda}^-_t)\big\} dt$$

where $\mu(\bullet, \bullet)$ *denotes the Maslov index of a pair of paths of Fredholm lagrangians.*

References

[1] M.F. Atiyah, V.K. Patodi, I.M. Singer: *Spectral asymmetry and Riemannian geometry I*, Math. Proc. Cambridge Philos. Soc. **77**(1975), 43-69.

[2] M.F. Atiyah, V.K. Patodi, I.M. Singer: *Spectral asymmetry and Riemannian geometry II*, Math. Proc. Cambridge Philos. Soc. **78**(1975), 405-432.

[3] M.F. Atiyah, V.K. Patodi, I.M. Singer: *Spectral asymmetry and Riemannian geometry III*, Math. Proc. Cambridge Philos. Soc. **79**(1976), 71-99.

[4] B. Booss, K. Wojciechowski: *Elliptic Boundary Problems for Dirac Operators*, Birkhauser, 1993.

[5] S.E. Cappel, R.Lee, E.Y. Miller: *Surgery formulæ for analytical invariants of manifolds*, preprint.

[6] A.L. Carey, M. Marcolli, B.L. Wang: *Exact triangles in Seiberg-Witten-Floer theory. Part I: the geometric triangle.* math.DG/9907065.

[7] W. Chen: *The Seiberg-Witten theory of homology* 3*-spheres*, dg-ga/9703009.

[8] M. Daniel: *An extension of a theorem of Nicolaescu on spectral flow and Maslov index*, Proc. Amer. Math. Soc., **128**(2000), 611-619.

[9] M. Daniel, P. Kirk: *A general splitting formula for the spectral flow. With an appendix by K. Wojciechovski*, Mich. Math. J., **46**(1999), 589-617.

[10] P.B. Gilkey: *Invariance Theory, the Heat Equation and the Atiyah-Singer Index Theorem*, 2nd Edition, CRC Press, Boca Raton, 1995.

[11] T. Kato: *Perturbation Theory for Linear Operators*, Springer Verlag, 1984.

[12] P. Kirk, M. Lesch: *The η-invariant, and spectral flow for Dirac-type operators on manifolds with boundary*, math.DG/0012123.

[13] M. Kreck, S. Stolz: *Non connected moduli spaces of positive sectional curvature metrics*, J. Amer. Math. Soc., **6**(1993), 825-850.

[14] P. Kronheimer, T. Mrowka: *The genus of embedded surfaces in the projective plane*, Math. Res. Letters **1**(1994), 797-808.

[15] Y. Lim: *The equivalence of Seiberg-Witten and Casson invariants for homology 3-spheres*, Math. Res. Letters, **6**(1999), 631-644.

[16] Y. Lim: *Seiberg-Witten moduli spaces for* 3*-manifolds with cylindrical end* $T^2 \times \mathbb{R}_+$, preprint.

[17] Y. Lim: *Seiberg-Witten invariants for 3-manifolds in the case* $b_1 = 0$ *or* 1, Pacific J. of Math., **195**(2000), 179-204.

[18] R.B. Lockhart, R.C. McOwen: *Elliptic differential equations on noncompact manifolds*, Annali di Scuola Norm. Sup. di Pisa, **12**(1985), 409-448.

[19] R. Lockhart: *Fredholm, Hodge and Liouville theorems on noncompact manifolds*, Trans. Amer. Math. Soc., **301**(1987), 1-35.

[20] G. Meng, C.H. Taubes: $\underline{SW}$ = *Milnor torsion*, Math. Res. Letters, **3**1996, 661-674.

[21] J.W. Morgan, T. Mrowka, D. Ruberman: *The L^2-Moduli Space and a Vanishing Theorem for Donaldson Polynomial Invariants*, International Press, 1994.

[22] T. Mrowka: *A local Mayer-Vietoris principle for Yang-Mills moduli spaces*, PhD Thesis, 1988.

[23] L.I. Nicolaescu: *The Maslov index, the spectral flow and decompositions of manifolds*, Duke Math. J., **80**(1995), 485-533.

[24] L.I. Nicolaescu: *Lectures on the Geometry of Manifolds*, World Sci. Pub. Co. 1996.

[25] L.I. Nicolaescu: *Adiabatic limits of the Seiberg-Witten equations on Seifert manifolds*, Comm. in Anal. and Geom., **6**(1998), 331-392. math.DG/9601107

[26] L.I. Nicolaescu: *Notes on Seiberg-Witten Theory*, Graduate Studies in Mathematics, vol. 28, Amer. Math. Soc., 2000.

[27] L.I. Nicolaescu: *Seiberg-Witten theoretic invariants of lens spaces*, math.DG/9901071.

[28] L.I. Nicolaescu: *Reidemeister Torsion*, notes available at

`http://www.nd.edu/~lnicolae/`

[29] L.I. Nicolaescu: *On the space of Fredholm selfadjoint operators*, mathDG/0005089.

[30] P. Ozsváth, Z. Szabó: *The Theta divisor and the Casson-Walker invariant*, math.GT/0006194.

[31] V.G. Turaev: *Reidemeister torsion and the Alexander polynomial*, Math. Sbornik, **101**(1976), 252-270.

[32] V.G. Turaev: *Euler structures, nonsingular vector fields and torsions of Reidemeister type*, Izvestyia Akad. Nauk, **53**(1989); (English version- Math. USSR Izvestyia, **34**(1990), 627-662.

[33] V.G. Turaev: *Torsion invariants of $Spin^c$-structures on 3-manifolds*, Math. Res. Letters, **4**(1997), 679-695.

[34] V.G. Turaev: *A combinatorial formulation for the Seiberg-Witten invariants of 3-manifolds*, Math. Res. Letters, **5**(1998), 583-598.

[35] V.G. Turaev: *Surgery formula for torsions and Seiberg-Witten invariants of 3-manifolds*, math.GT/0101108.

[36] E.T. Whittaker, G.N. Watson: *A course of Modern Analysis*, Cambridge Univ. Press, 1996.

[37] K.P. Wojciechowski: *The ζ-determinant and the additivity of the η-invariant on the smooth self-adjoint Grassmannian*, Comm. Math. Phys., **201**(1999), 423-444.