Of Shapes, Differentials and Integrals

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A *n-manifold* is a space which locally "looks like" the n-dimensional Euclidean space $\mathbb{R}^n$. 1-manifolds are also called *curves*, while the 2-manifolds are also called *surfaces*. A manifold can be obtained by gluing open subsets of $\mathbb{R}^n$. Figure 1 describes how to obtain the sphere by gluing two disks in $\mathbb{R}^2$. The higher dimensional objects are not as easy to visualize, and this gluing procedure can produce many shapes.

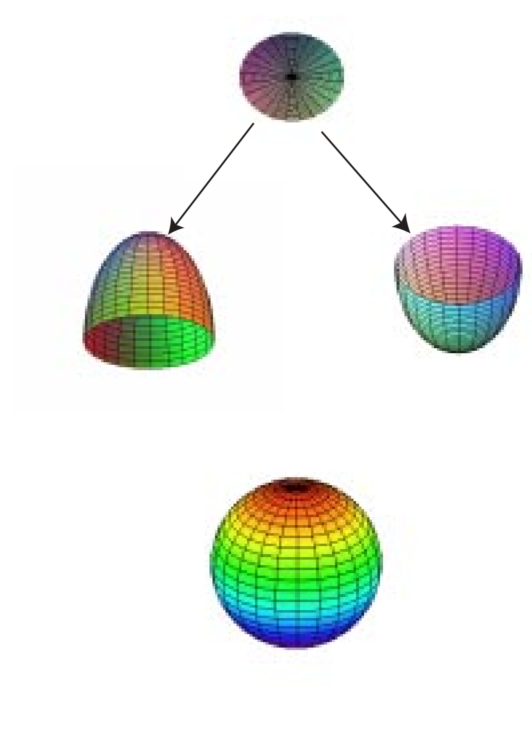

Figure 1: *Constructing a sphere by pasting two disks*

The Big Question *How many "shapes" can we produce in this fashion, and how can we distinguish them?*

The surfaces in Figure 2 look different but "aren't that different". The surfaces in Figure 3 look dramatically more different.

Figure 2: *These two surfaces aren't that different*

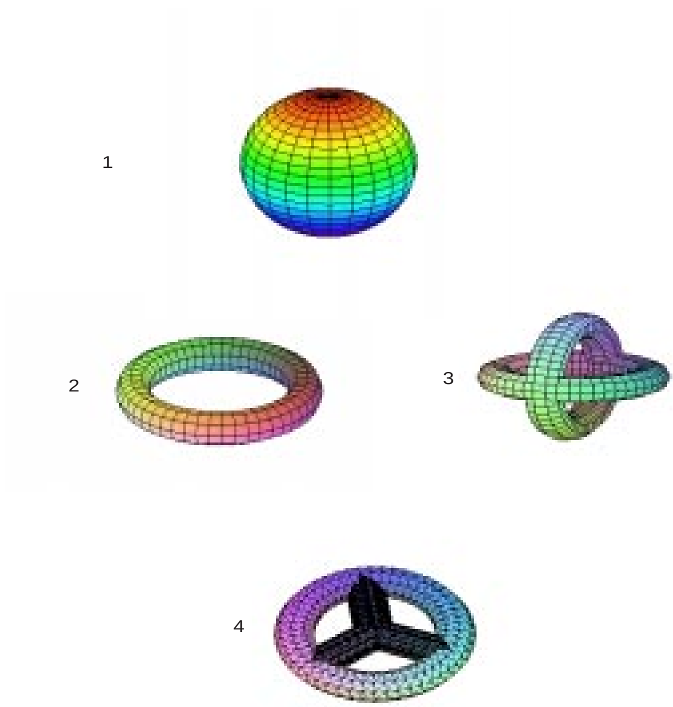

Figure 3: *Are all these surfaces distinct?*

The meaning of the word "different" has to be clarified. If the surfaces in Figure 2 were made of an elastic material then we could deform one into the other. However, if they were made of a material which is only flexible (such as canvas) then we cannot deform[1] one to the other any longer. In more mathematical terms, we say that the two surfaces in Figure 2 are *homeomorphic, but not isometric.*

The first three surfaces in Figure 3 don't seem to be homeomorphic. However, the third and the fourth surface are. (*Can you see this?*) The point of this heuristic discussion is that *looks can be deceiving*, and more robust techniques for distinguishing shapes have to be designed. There is an even stronger argument: the higher dimensional manifolds cannot be even visualized, so there is no hope of distinguishing them by looking at a nonexisting picture.

You may ask whether there is a reason, other than pure intellectual curiosity, to address these types of issues. The simple answer is yes, Nature throws these issues at us under various guises. Perhaps one of the most famous such instances is H.Poincaré's research in Celestial Mechanics. One major and as yet still not completely answered questions is whether the evolution of our solar system is periodic. This is the special case of the so called "many-body problem". H. Poincaré showed that a system of several material points moving under the influence of the gravitational force can have a periodic motion provided some higher dimensional shapes are different.

We can now rephrase the Big Question in more technical terms.

☞ *Design methods for deciding whether two manifolds are homeomorphic.*

I will outline one such method relying on good old fashioned calculus. The reason that calculus is available to such an investigation is that any point in a manifold has a neighborhood which looks like an open set in $\mathbb{R}^n$. In particular, if the manifold M is obtained by *smooth* gluings of open subsets in $\mathbb{R}^n$ then all the basic calculus operations can be transplanted to M. Such manifolds are called *smooth.* To start this program I need to introduce a bit of modern language which we may have already encountered in any basic differential geometry class. I will work most of the time on the simplest n-manifold, the Euclidean space, so you do not get distracted by technical details.

The main characters in this program are the *differential forms.* On a n-manifold there are differential forms of degrees varying from 0 to n. A differential form of degree k is usually referred to as a k-form. The 0-forms are the functions of n variables, $f(x^1, \cdots, x^n)$. The 1-forms are expressions of the type

$$\alpha = f_1 dx^1 + \cdots + f_n dx^n$$

where the coefficients f_i are 0-forms. The 2-forms are expressions of the type

$$\beta = \alpha_1 \wedge dx^1 + \cdots + \alpha_n \wedge dx^n$$

where α_i are 1-forms, and the *wedge* operation "$\wedge$" between differential forms is the associative and addition distributive operation uniquely determined by the properties

$$f_i dx^i \wedge f_j dx^j = -(f_i f_j) dx^j \wedge dx^i, \;\; dx^i \wedge dx^i = 0, \;\; \forall i, j,$$

[1]This is actually the content of Theorema Egregium (Gold Theorem)a fundamental result due to Gauss.

and any functions f_i, f_j. For example the 2-forms in $\mathbb{R}^3$ can be written as

$$\beta = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy = (Pdy) \wedge dz + (Qdz) \wedge dx + (Rdx) \wedge dy$$

$$= (Qdz - Rdy) \wedge dx + (Pdy) \wedge dz$$

More generally, the k-forms are expressions of the type

$$\beta = \alpha_1 \wedge dx^1 + \cdots \alpha_n + \wedge dx^n$$

where α_i are $(k-1)$-forms. In general

$$p\text{-form} \wedge q\text{-form} = (p+q) - \text{form}.$$

The vector space of differential k-forms on the manifold M is denoted by $\Omega^k(M)$. The differential forms on have a very simple mission in life: they exist to be *differentiated* and *integrated.*

➊ The *exterior derivative* of a k-form ω is a $(k+1)$-form denoted by $d\omega$. More precisely, the exterior derivative of a function (0-form) f is the total differential

$$df = \frac{\partial f}{\partial x^1} dx^1 + \cdots + \frac{\partial f}{\partial x^n} dx^n.$$

In general, if we write

$$\omega = \omega_1 \wedge dx^1 + \cdots + \omega_n \wedge dx^n$$

where ω_i are $(k-1)$-forms then

$$d\omega = (d\omega_1) \wedge dx^1 + \cdots + (d\omega_n) \wedge dx^n.$$

For example, the exterior derivative of a 1-form $\alpha = Pdx + Qdy + Rdz$ in $\mathbb{R}^3$ is

$$d\alpha = (dP) \wedge dx + (dQ) \wedge dy + (dR) \wedge dz$$

$$= (P'_x dx + P'_y dy + P'_z dz) \wedge dx + (Q'_x dx + Q'_y dy + Q'_z dz) \wedge dy + (R'_x dx + R'_y dy + R'_z dz) \wedge dz.$$

Using the anticommutativity of "$\wedge$" we deduce

$$d\alpha = (Q'_x - P'_y) dx \wedge dy + (R'_y - Q'_z) dy \wedge dz + (P'_z - R'_x) dz \wedge dx.$$

From the classical theorem

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}, \quad \forall f, \quad \forall i, j,$$

it follows easily that the exterior derivative satisfies the fundamental identity

$$d(d\omega) = 0$$

for any differential form ω. The exterior differential operator d defines a linear map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ which is an example of *partial differential operator*, or p.d.o. for brevity.

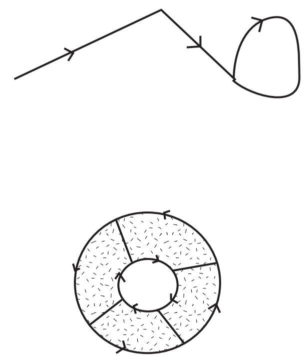

Figure 4: *A 1-chain and a 2-chain*

❷ A k-form on a smooth manifold M can be *integrated* along a k-*chain*. An *elementary k-dimensional surface* in a smooth manifold M is a smooth deformation of a k-dimensional polyhedron, and a k-chain in M is an union of elementary k-dimensional surfaces. Every k-chain c has a boundary ∂c which is a $(k-1)$-chain. If $\partial c = \emptyset$ we say that c is a k-cycle. The spheres in Figure 2 are 2-cycles in $\mathbb{R}^3$

The Fundamental Theorem of Calculus describes a relationship between these two operations on forms which usually goes by the name of Stokes' Theorem.

Stokes Theorem *For any $(k+1)$-chain c in the smooth manifold M and any k-form ω we have*

$$\int_{\partial c} \omega = \int_c d\omega.$$

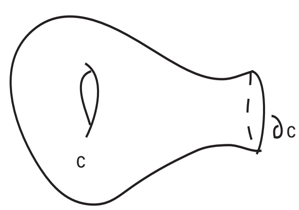

Figure 5: *A 2-chain with boundary*

I will now formulate of problem which has no apparent connection to the Big Question.

PROBLEM *Given a smooth manifold M and a k-form ω on it, find a $(k-1)$-form β such that*

$$d\beta = \omega. \tag{$*$}$$

A solution β of the above partial differential equation is called a *potential* of ω.

✍ *The equation $(*)$ need not have a solution for all ω. To see this, note that if β is a potential of ω then*

$$d\omega = d(d\beta) = 0.$$

Thus a first necessary condition for existence of a potential of ω is

$$d\omega = 0. \tag{$**$}$$

A form which admits a potential is called *exact*. A differential form ω satisfying $(**)$ is called *closed*. Thus

$$\omega \ \text{ is exact} \Longrightarrow \omega \ \text{ is closed.}$$

✍ *If $(*)$ has a solution β_0 then it has infinitely many*[2]*. Indeed, for any $(k-2)$-form α, we get a new potential $\beta_0 + d\alpha$ of ω,*

$$d(\beta_0 + d\alpha) = d\beta_0 + d(d\alpha) = \omega.$$

✍ *The condition $(**)$ may not be sufficient. Suppose c is a k-cycle, i.e. $\partial c = 0$, and β is a potential of ω. Then Stokes Theorem implies*

$$\int_c \omega = \int_c d\beta = \int_{\partial c} \beta = 0.$$

This shows that the integral of ω along any k-cycle must be zero.

The integral of a closed k-form along a k-cycle is classically known as a *period* of that form. We have thus shown that

$$\omega \ \text{ is exact} \Longrightarrow \omega \ \text{ is closed and all its periods are zero.}$$

The remarkable fact is that the two conditions on the right-hand-side are also sufficient.

DeRham Theorem. Part 1. *A form ω is exact if and only if it is closed and all its periods are zero.*

What does this all have to do with shapes? To answer this, lets us note that the k-cycles come in two flavors (see Figure 6).

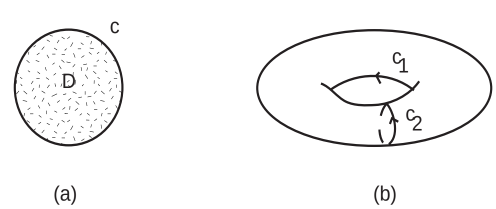

Figure 6: *Cycles which bound (a) and cycles which do not bound (b).*

• cycles c which bound, i.e. there exists $(k+1)$-chain D such that $c = \partial D$. These are called *boundaries*.

• cycles which do not bound. These "wrap around holes" in the manifold M.

[2]This is closely related to the classical fact that a function has infinitely many antiderivatives.

Note that if c is a boundary, $c = \partial D$, and ω is closed, $d\omega = 0$, the

$$\int_c \omega = \int_{\partial D} \omega = \int_D d\omega = 0.$$

Thus the periods along boundaries are automatically trivial. We can rephrase this in a different way.

Corollary *If there exists a closed form with nontrivial periods, then there must exist cycles which do not bound. These cycles "surround holes " in the manifold M.*

Thus the closed forms with nontrivial periods are indicators of "holes" in the manifold. Remarkably, all the "holes" can be detected in this fashion!!!

DeRham Theorem. Part 2. *If c is a k-cycle in the smooth manifold M which does not bound, then there exists a closed k-form ω whose period along c is nontrivial.*

This result provides a way of distinguishing two manifolds: if one manifolds has "more holes" then another, then these two manifolds must be different. The above result shows that in order to "count the holes" it suffices to count the closed forms with nontrivial periods. Naturally, we need to have a systematic way of producing all such forms. Let us introduce a bit more terminology.

We say that two closed forms ω_1 and ω_2 of identical degrees are equivalent, if they have identical periods. We write this $\omega_1 \sim \omega_2$. DeRham Theorem implies

$$\omega_1 \sim \omega_2 \Longleftrightarrow \text{all the periods of } \omega_1 - \omega_2 \text{ are trivial} \Longleftrightarrow \omega_1 - \omega_2 \text{ is exact.}$$

Denote by Z^k the vector space of all closed k-forms. For any $\omega \in Z^k$ we denote by V_ω the space of closed k-forms equivalent to ω. V_ω is an affine subspace of the infinite dimensional space Z^k (see Figure 7). The following question immediately comes to mind.

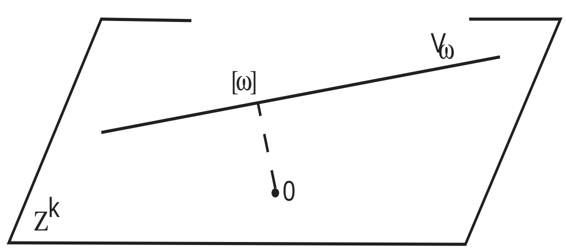

Figure 7: *The space of closed k-forms.*

Is there a natural way of selecting an element in V_ω?

Here is a possible strategy. Suppose we have a way of measuring the distance[3] between two points in Z^k. Then a natural candidate would be the point in V_ω closest to the origin.

[3]This can be achieved by fixing a Riemann metric on the manifold M.

This intuitive approach which goes back to Riemann is marred by several difficulties[4] which all have their origin in the infinite dimensionality of both Z^k and V_ω. Dealing with these difficulties requires the use of deep analytic results in the theory of elliptic partial differential equations. We have the following fundamental result.

Hodge Theorem *Fix a Riemann metric on the* **compact** *manifold M.*

(a) *For every $\omega \in Z^k$ there* **exists** *a* **unique** *element $[\omega] \in V_\omega$ closest to the origin of Z^k. This element is the unique solution inside V_ω of a first order partial differential equation*

$$D_k \eta = 0. \qquad (***)$$

(b) $\ker D_k$ *is finite dimensional.*

The differential forms satisfying $(***)$ are called *harmonic*. The dimension of $\ker D_k$ is called the *k-th Betti number of M*, and is denoted by $b_k(M)$. We can think of $b_k(M)$ as the number of k-dimensional holes inside M. We have the following consequence.

Corollary *If two manifolds have different Betti numbers then they are not homeomorphic.*

For example the first Betti number of the 2-sphere is zero, while the first Betti number of the 2-torus is 2 so that these two surfaces cannot be homeomorphic. The first Betti numbers of the 3rd and 4th surface in Figure 3 are both equal to 6 and we can conclude that these two surfaces are homeomorphic. However, we cannot conclude in general that if two manifolds have identical Betti numbers then they are homeomorphic, and distinguishing then requires additional work.

What next?

Think of the above questions and partial answers as part of an ongoing story where you get the chance to add a new chapters and characters, and where you are limited only by your own curiosity. The points of view I have outlined have wide ranging applications and they continue to produce remarkable discoveries. If you are curious about these kinds of issues you could stop by my office for a math chat, or open any of the references below to get a more in depth look at this subject.

References

[1] R.Bott, L.Tu: *Differential Forms in Algebraic Topology*, Springer-Verlag, 1982.

[2] Th. Frankel: *The Geometry of Physics. An Introduction*, Cambridge Univ. Press, 1997.

[3] L. I. Nicolaescu: *Lectures on the Geometry of Manifolds*, World Scientific, 1996.

[4]These problems were immediately noticed by K. Weierstrass who severely criticized this approach and discouraged further investigations. Decades later, during the famous Paris Congress in 1900, D. Hilbert rehabilitated Riemann's insight and asked in his 19th and 20th problems to set rigorous foundations for Riemann's program.