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Schubert calculus on the Grassmannian of hermitian lagrangian spaces

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Abstract

With an eye towards index theoretic applications we describe a Schubert like stratification on the Grassmannian of hermitian lagrangian spaces in $\mathbb{C}^n \oplus \mathbb{C}^n$. This is a natural compactification of the space of hermitian $n \times n$ matrices. The closures of the strata define integral cycles, and we investigate their intersection theoretic properties. We achieve this by blending Morse theoretic ideas, with techniques from o-minimal (or tame) geometry and geometric integration theory.

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Introduction

A hermitian lagrangian subspace is a subspace L of the complex Hermitian vector space $\mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n$ satisfying $L^\perp = \boldsymbol{J}L$, where $\boldsymbol{J} : \mathbb{C}^n \oplus \mathbb{C}^n \to \mathbb{C}^n \oplus \mathbb{C}^n$ is the unitary operator with the block decomposition

$$\boldsymbol{J} = \begin{bmatrix} 0 & -\mathbb{1}_{\mathbb{C}^n} \\ \mathbb{1}_{\mathbb{C}^n} & 0 \end{bmatrix}.$$

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We denote by $\mathrm{Lag}_h(n)$ the Grassmannian of such subspaces. This space can be identified with a more familiar space.

Denote by $F^{\pm} \subset \mathbb{C}^{2n}$ the $\pm\boldsymbol{i}$ eigenspace of $\boldsymbol{J}$, $F^{\pm} = \{(\boldsymbol{e}, \mp\boldsymbol{i}\boldsymbol{e}); \boldsymbol{e} \in \mathbb{C}^n\}$. V.I. Arnold has shown in [2] that $L \subset \mathbb{C}^{2n}$ is a hermitian lagrangian subspace if and only if, when viewed as a subspace of $F^+ \oplus F^-$, it is the graph of a unitary operator $F^+ \to F^-$. Thus we have a natural diffeomorphism $\mathrm{Lag}_h(n) \to U(n)$. The space Herm_n of hermitian operators $\mathbb{C}^n \times \mathbb{C}^n$ embeds in $\mathrm{Lag}_h(n)$ via the graph map, and the restriction of the Arnold diffeomorphism to Herm_n is none other that the classical Cayley transform

$$\mathrm{Herm}_n \ni A \mapsto (1 - \boldsymbol{i}A)(1 + \boldsymbol{i}A)^{-1} \in U(n).$$

The unitary groups are arguably some of the most investigated topological spaces and much is known about their cohomology (see [14, Chap. IV], [46, VII.4, VIII.9]). One could fairly ask what else is there to say about these spaces. To answer this, we need to briefly explain the question which gave the impetus for the investigations in this paper.

Atiyah and Singer [3] have shown that a certain component $\mathcal{FS}_0$ of the space of bounded Fredholm self-adjoint operators on a separable complex Hilbert space H is a classifying space for the (complex) K-theoretic functor K^1. The graph of an operator $A \in \mathcal{FS}_0$, defines a hermitian lagrangian Γ_A in the hermitian symplectic space $H \oplus H$ which intersects the horizontal axis $H \oplus 0$ along a finite dimensional subspace. We denote by $\mathrm{Lag}_h(\infty)$ the space of such lagrangians. In [36] we have shown that the graph map $\mathcal{FS}_0 \to \mathrm{Lag}_h(\infty)$ is a homotopy equivalence.

The integral cohomology of $\mathrm{Lag}_h(\infty)$ is an exterior algebra $\Lambda(x_1, x_2, \ldots)$, where $\deg x_i = 2i - 1$. If X is a compact, oriented smooth manifold, $\dim X = n$, then the results of [36] imply that any smooth[1] family $(A_x)_{x \in X}$ Fredholm self-adjoint operators defines a smooth map $A : X \to \mathrm{Lag}_h(\infty)$. We thus obtain cohomology classes $A^* x_i \in H^{2i-1}(X, \mathbb{Z})$.

We are interested in localization formulæ, i.e., in describing *concrete geometric realizations* of cycles representing the Poincaré duals of these classes. Some of the most interesting situations arise when X is an odd dimensional sphere $X = S^{2m-1}$. In this case, the Poincaré dual of $A^* x_m$ is a 0-dimensional homology class, and we would like to produce an explicit 0-cycle representing it.

For example, in the lowest dimensional case, $X = S^1$, we have such a geometric realization because the integer $\int_{S^1} A^* x_1$ is the spectral flow of the loop of self-adjoint operators, and as is well known, in generic cases, it can be computed by counting with appropriate multiplicities the points $\theta \in S^1$ where $\ker A_\theta = 0$. Thus, the Poincaré dual of $A^* x_1$ is represented by a certain 0-dimensional degeneracy locus.

Equivalently, we could view the family $(A_\theta)_{\theta \in S^1}$ as a loop in $\mathrm{Lag}_h(\infty)$. Adopting this point of view, we can interpret the integer $\int_{S^1} A^* x_1$ as a Maslov index, and using the techniques developed by Arnold in [1] one can explicitly describe a 0-cycle dual to the class $A^* x_1$; see [32].

To the best of our knowledge there are no such degeneracy loci descriptions of the Poincaré dual of $A^* x_m$ in the higher dimensional cases $A : S^{2m-1} \to U(\infty)$, $m > 1$, and the existing descriptions of the cohomology ring of $U(n)$ do not seem to help in this respect.

With an eye towards such applications, we describe in this paper a natural, Schubert-like, Whitney regular stratification of $\mathrm{Lag}_h(n)$, and we investigate its intersection theoretic properties.

[1] We will not elaborate here on the precise meaning of differentiability of a family of possibly unbounded operators. This rather delicate issue is addressed by D. Cibotaru in his dissertation [5].

As in the case of usual Grassmannians, this Schubert-like stratification of $\mathrm{Lag}_h(n)$ has a Morse theoretic description. Many of the ideas involved are classical, going back to the pioneering work of Pontryagin, [40]. We recommend the nice presentation by Dynnikov and Vesselov in [12].

We denote by (e_i) the canonical unitary basis of $\mathbb{C}^n$, and we define the hermitian operator

$$A:\mathbb{C}^n\to\mathbb{C}^n,\qquad Ae_i=\left(i-\frac{1}{2}\right)e_i,\quad \forall i=1,\dots,n.$$

The operator A defines a function $f=f_A:U(n)\to\mathbb{R}$, $f(S)=-\mathbf{Re}\,\mathrm{tr}(AS)+\frac{n^2}{2}$. This is a Morse function with one critical point $S_I\in U(n)$ for every subset $I\subset\{1,\dots,n\}$. More precisely

$$S_Ie_i=\begin{cases} e_i, & i\in I,\\ -e_i, & i\notin I.\end{cases}$$

Its Morse index is $\mathrm{ind}(S_I)=f(S_I)=\sum_{i\in I^c}(2i-1)$, where I^c denotes the complement of I in $\{1,2,\dots,n\}$. In particular, this function is self-indexing.

We denote by $W_I^{\pm}$ the stable/unstable manifold of S_I with respect to the (negative) gradient flow Φ_t of f_A. These unstable manifolds are loci of certain Schubert-like incidence relations and they can be identified with the orbits of a real algebraic group acting on $\mathrm{Lag}_h(n)$. Basic facts of real algebraic geometry imply that the stratification given by these unstable manifolds is Whitney regular. In particular, this implies that our gradient flow satisfies the Morse–Smale transversality condition. We can thus define the Morse–Floer complex, and it turns out that the boundary operator of this complex is trivial.

The collection of strata has a natural Bruhat-like partial order given by the inclusion of a stratum in the closure of another. In Section 4 we give a purely combinatorial description of this partial order, and we relate it to the natural partial order on the critical set of a gradient Morse–Smale flow. In [38] we carry a deeper investigation of the combinatorics of this poset.

Given that the Morse–Floer complex of the gradient flow Φ_t is perfect, it is natural to ask if the unstable manifolds W_I^- define geometric (co)cycles in any reasonable way, and if so, investigate their intersection theory. Several possibilities come to mind.

One possible approach, used by Vassiliev in [44], is to produce resolutions of W_I^-, i.e., smooth maps $f:X_I\to\mathrm{Lag}_h(n)$, where X_I is a compact oriented manifold, $f(X_I)=\boldsymbol{cl}(W_I^-)$, and f is a diffeomorphism over the smooth part of $\boldsymbol{cl}(W_I^-)$. As explained in [44], this approach reduces the computation of the intersection cycles $f_*[X_I]\bullet f_*[X_J]$ to classical Schubert calculus on Grassmannians, but the combinatorial complexity seems to hide the simple geometric intuition.

M. Goresky [17] has explained how to associate (co)cycles to Whitney stratified (co)oriented objects and perform intersection theoretic computations with such objects. While the closures of W_I^- are stratified cycles in the sense of Goresky, this possible approach seems difficult to use in concrete computations due to the stringent transversality conditions needed for such a calculus.

G. Ruget [42] proposed another technique[2] of associating a cocycle to a (possibly) singular analytic subvariety. This seems the ideal approach for infinite dimensional situations, but has one small finite dimensional problem: it does not mesh well with the Poincaré duality.

Instead, we chose the most basic approach, and we looked at the integration currents defined by orienting the semi-algebraic sets W_I^-. This seems to be an ideal compromise between the

[2] U. Koschorke uses a very similar idea in [27].

approaches of Goresky [17] and Ruget [42] afforded by the elegant theory of intersection of subanalytic cycles developed by R. Hardt [18–20]. In these papers R. Hardt lays the foundations of a geometric intersection theory over the reals. The cycles are integration currents supported by subanalytic sets. A key fact in this theory is a slicing theorem which is essentially a real analytic version of the operation of pullback of complex cycles via flat holomorphic maps. For the reader's convenience, we have included in Appendix B a short survey of Hardt's main results in a topologists' friendly language.

The manifolds W_I^- are semi-algebraic, have finite volume, and carry natural orientations $\boldsymbol{or}_I$, and thus define integration currents $[W_I, \boldsymbol{or}_I]$. In Proposition 33 we show that the closure of W_I^- is a naturally oriented pseudo-manifold, i.e., it admits a stratification by smooth manifolds, with top stratum oriented, while the other strata have (relative) codimension at least 2. Using the fact that the current $[W_I^-, \boldsymbol{or}_I]$ is a subanalytic current as defined in [20], it follows that $\partial[W_I^-, \boldsymbol{or}_I] = 0$ in the sense of currents. These cycles define homology classes $\boldsymbol{\alpha}_I \in H_\bullet(U(n), \mathbb{Z})$.

The currents $[W_I^-, \boldsymbol{or}_I]$ define a perfect subcomplex of the complex of integrally flat currents. This subcomplex is isomorphic to the Morse–Floer complex of the gradient flow Φ_t, and via the finite-volume-flow technique of Harvey–Lawson [21] we conclude that the cycles $\boldsymbol{\alpha}_I$ form an integral basis of $H_\bullet(U(n), \mathbb{Z})$. This basis coincides with the basis described in [14, IV, §3], and by Vassiliev in [44].

The cycle $\boldsymbol{\alpha}_I$ has codimension $w(I) = \sum_{i \in I}(2i - 1)$. We denote by $\boldsymbol{\alpha}_I^\dagger \in H^\bullet(U(n), \mathbb{Z})$ its Poincaré dual. When I is a singleton, $I = \{i\}$, we use the simpler notation $\boldsymbol{\alpha}_i$ and $\boldsymbol{\alpha}_i^\dagger$ instead of $\boldsymbol{\alpha}_{\{i\}}$ and respectively $\boldsymbol{\alpha}_{\{i\}}^\dagger$. We call the cycles $\boldsymbol{\alpha}_i$ the *basic Arnold–Schubert cycles*. Let us point out that $\boldsymbol{\alpha}_1^\dagger$ is the hermitian analogue of the real Maslov class described in [1].

It is well known that the cohomology of $U(n)$ is related via transgression to the cohomology of its classifying space $BU(n)$. We prove that the basic class $\boldsymbol{\alpha}_i^\dagger$ is obtainable by transgression from the Chern class c_i.

More precisely, denote by E the rank n complex vector bundle over $S^1 \times U(n)$ obtained from the trivial complex vector bundle of rank n over the cylinder $[0, 1] \times U(n)$ by identifying the point $\vec{z} \in \mathbb{C}^n$ in the fiber over $(1, g) \in [0, 1] \times U(n)$ with the point $g\vec{z}$ in the fiber over $(0, g) \in [0, 1] \times U(n)$. We denote by $p : S^1 \times U(n) \to U(n)$ the natural projection, and by

$$p_! : H^\bullet\big(S^1 \times U(n), \mathbb{Z}\big) \to H^{\bullet-1}\big(U(n), \mathbb{Z}\big)$$

the induced Gysin map. The first main result of this paper is a transgression formula (Theorem 37) asserting that

$$\boldsymbol{\alpha}_i^\dagger = p_!\big(c_i(E)\big). \tag{$\dagger$}$$

We prove the above equality at the level of *currents* by explicitly representing $c_i(E)$ as a Thom–Porteous degeneracy current.

This shows that the integral cohomology ring is an exterior algebra with generators $\boldsymbol{\alpha}_i^\dagger$, $i = 1, \dots, n$, so that an integral basis of $H^\bullet(U(n), \mathbb{Z})$ is given by the exterior monomials

$$\boldsymbol{\alpha}_{i_1}^\dagger \cup \cdots \cup \boldsymbol{\alpha}_{i_k}^\dagger, \quad 1 \leqslant i_1 < \cdots < i_k \leqslant n,\ 0 \leqslant k \leqslant n.$$

The second main result of this paper, Theorem 47 gives a description of the Poincaré dual of $\boldsymbol{\alpha}^{\dagger}_{i_1} \cup \cdots \cup \boldsymbol{\alpha}^{\dagger}_{i_k}$ as a degeneracy cycle. More precisely, if $I = \{i_1 < \cdots < i_k\}$, then

$$\boldsymbol{\alpha}^{\dagger}_{I} = \boldsymbol{\alpha}^{\dagger}_{i_1} \cup \cdots \cup \boldsymbol{\alpha}^{\dagger}_{i_k}. \tag{$\ddagger$}$$

The last equality completely characterizes the intersection ring of $\mathrm{Lag}_h(n)$ in terms of the integral basis $\boldsymbol{\alpha}^{\dagger}_{I}$.

The intersection theory investigated in this paper is closely related to the traditional Schubert calculus on complex Grassmannians, but we used very little of the traditional Schubert calculus. In the complex case the difficulties have mostly a combinatorial nature, while in this case the difficulties are essentially geometric in nature. Given that the cycles involved are represented by singular real semi-algebraic objects, there are some orientation issues to deal with, and the general position arguments are considerably more delicate.

We deal with the transversality problems in a novel way. Unlike the case of complex Grassmannians, we do not move cycles in general position by using the rich symmetry of $\mathrm{Lag}_h(n)$ as a homogeneous space. Instead, we use the unitary group incarnation, $\mathrm{Lag}_h(n) \cong U(n)$, and we rely on the following simple observation: any degree 1 semi-algebraic continuous map $\Phi : S^1 \to S^1$ induces via functional calculus a semi-algebraic continuous map

$$U(n) \ni T \mapsto \Phi(T) \in U(n)$$

semi-algebraically homotopic to the identity. These maps may not even be Lipschitz continuous, but the push-forward of semi-algebraic currents by semi-algebraic maps is still a well defined operation according to the results of R. Hardt [19,20]. By using certain maps Φ which are not homeomorphisms, but still are semi-algebraic and homotopic to the identity, we can deform the Schubert–Arnold cycles to semi-algebraic cycles for which the transversality issues become trivial.

These cycles, although homologous to AS-cycles, they have a rather different description, as *spectral loci* consisting of unitary matrices whose spectra have certain types of degeneracies.

As an application of this Schubert calculus we discuss a spectral multiplicity problem closely related to the recent K-theoretic investigations by R. Douglas and J. Kaminker in [7]. In Proposition 49 we show that the locus $\Sigma_{k,n} \subset U(n)$ consisting of unitary matrices which have at least one eigenvalue with multiplicity $\geqslant k$ defines a cycle representing the homology class $-n\boldsymbol{\alpha}_{2,\dots,k}$.

The factor n above becomes a serious issue when $n = \infty$. The Maslov class $\boldsymbol{\alpha}^{\dagger}_{1}$ is an obstruction to renormalization in the following sense. Suppose X is a smooth compact real analytic oriented manifold, and $g : X \to U(n)$ is a smooth subanalytic map. If $g^{*}\boldsymbol{\alpha}^{\dagger}_{1} = 0$ then, under certain generic transversality conditions we can find continuous subanalytic maps

$$\lambda_1, \dots, \lambda_n : X \to \mathbb{R}$$

such that for every $x \in X$ the spectrum of g_x, including multiplicities, is $\{e^{\boldsymbol{i}\lambda_k(x)};\ 1 \leqslant k \leqslant n\}$ and $\lambda_1(x) \leqslant \cdots \leqslant \lambda_n(x)$. Then the locus

$$S_{k,n} = \big\{x \in X;\ \lambda_1(x) = \cdots = \lambda_k(x)\big\}$$

determines a closed subanalytic integration current $[S_{k,n}]$ and $n[S_{k,n}] = [g^{-1}\Sigma_{k,n}]$. When $k = 2$, the Poincaré dual of $[S_{k,n}]$ is a 3-dimensional cohomology class $\mu_2(g) \in H^3(X, \mathbb{Z})$. We believe that this is the finite dimensional analogue of the 3-dimensional class constructed by Douglas and Kaminker [7, Sect. 6]. We will investigate this in more detail elsewhere.

In Proposition 52, Corollary 53 and Proposition 57 we describe the various compatibilities between the above Schubert calculus and the operation of symplectic reduction, while in Proposition 54 we show that the symplectic reduction can be interpreted as an asymptotic limit of a natural Morse–Bott flow on $\mathrm{Lag}_h(n)$. This type of Morse–Bott flow appears under a different guise in the work of D. Quillen, [41, §5.B]. In fact, Quillen's transgression formulæ are special cases of our formulæ in Corollary 55.

The sought for localization formulæ are built in our Morse theoretic approach. More precisely, if Φ_t denotes the (downward) gradient flow of the Morse function f, then Φ_t is a *tame* flow in the sense of [37], and in particular, it is a finite volume flow in the sense of Harvey–Lawson [21]. If we denote by ϖ the (matrix valued) Maurer–Cartan form on $U(n)$, and by $B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ the Euler Beta function, then the results in [21] imply that the closed forms

$$\Theta_k(t)=(-1)^{k+1}\frac{B(k,k)}{(2\pi\boldsymbol{i})^k}\Phi_t^*\operatorname{tr}\bigl(\varpi^{\wedge(2k-1)}\bigr)\in\Omega^{2k-1}\bigl(U(n)\bigr),$$

converge as currents when $t\to-\infty$ to the currents $\boldsymbol{\alpha}_k$.

Since the basics of tame geometry do not seem to be familiar to many geometers, we felt compelled to outline the most salient features of this theory in Appendix A.

The infinite dimensional extension and the index theoretic applications of the above Schubert calculus were developed by Daniel Cibotaru in his PhD dissertation [5]. There he observes that any Hilbert basis of H determines a similar Schubert-like stratification of $\mathrm{Lag}_h(\infty)$ with strata of finite codimension. Moreover, one can associate an increasing filtration by open subsets

$$\mathrm{Lag}_h(\infty,1)\subset\mathrm{Lag}_h(\infty,2)\subset\cdots.$$

The open set $\mathrm{Lag}_h(\infty,n)$ is a finite union of Schubert strata, and the symplectic reduction process defines smooth fiber bundles $\mathcal{R}_{n,\infty}:\mathrm{Lag}_h(\infty,n)\to\mathrm{Lag}_h(n)$ with contractible fibers. The maps $\mathcal{R}_{n,\infty}$ are compatible with the stratifications, and the canonical inclusions $\mathrm{Lag}_h(n)\hookrightarrow\mathrm{Lag}_h(\infty)$ are sections of these fibrations.

If $S^{2k-1}\ni x\xmapsto{\mathcal{L}}\mathcal{L}_x\in\mathrm{Lag}_h(\infty)$ is a smooth map, then $\mathcal{L}(S^{2k-1})\subset\mathrm{Lag}_h(\infty,n)$, for all sufficiently large n. If we set $\mathcal{L}_n:=\mathcal{R}_{n,\infty}\circ\mathcal{L}:S^{2k-1}\to\mathrm{Lag}_h(n)$, then $\mathcal{L}^*x_k\in H^{2k-1}(S^{2k-1})$ is represented by a 0-dimensional co-oriented submanifold $D_k(\mathcal{L})$ (degeneracy locus) of S^{2k-1} which is the preimage via $\mathcal{L}_n$ of a certain Schubert-like variety in $\mathrm{Lag}_h(n)$ of codimension $2k-1$.

The set $D_k(\mathcal{L})$ is independent of n, but depends on a (generic) choice of a $(k-1)$-dimensional subspace $V\subset H$. In fact

$$D_k(\mathcal{L})=D_k(\mathcal{L},V)=\bigl\{x\in S^{2k-1};\ 0\neq\bigl(\mathcal{L}_x\cap(H\oplus 0)\bigr)\subset(V\oplus 0)^{\perp}\bigr\}.$$

There is an equivalent way of looking at the Grassmannian $\mathrm{Lag}_h(n)$ which has proved quite useful in K-theoretic problems, [24,33].

Denote by Cl_n the complex Clifford algebra with n-generators, i.e., the $\mathbb{C}$-algebra with generators $u_1,\dots,u_n$ and relations

$$u_iu_j+u_ju_i=-2\delta_{ij},\quad\forall 1\leqslant i,j\leqslant n.$$

A hermitian $\mathbb{C}l_n$-module is a representation $\rho : \mathbb{C}l_n \to \mathrm{End}_{\mathbb{C}}(\widehat{E})$ of $\mathbb{C}l_n$ on a hermitian vector space $\widehat{E}$, such that the endomorphisms $J_i := \rho(u_i)$ are skew-hermitian. A ($\mathbb{Z}/2$-)grading of this module is a hermitian involution R of $\widehat{E}$ which anticommutes with each J_i. Note that such a grading induces isomorphisms

$$\ker(\boldsymbol{i} - J_k) \xrightarrow{R} \ker(\boldsymbol{i} + J_k), \quad \forall 1 \leqslant k \leqslant n.$$

In the simplest case, $n = 1$, we see that a hermitian $\mathbb{C}l_1$-module is a pair $(\widehat{E}, \boldsymbol{J})$, where $\boldsymbol{J}$ is a skew-hermitian operator such that $\boldsymbol{J}^2 = -\mathbb{1}$. Such a $\mathbb{C}l_1$-module admits a grading if and only if the eigenspaces $(\boldsymbol{i} \pm \boldsymbol{J})$ have the same dimension. Moreover, a hermitian involution R of $\widehat{E}$ is a grading if and only if the fixed subspace of R, $L = \ker(\mathbb{1} - R)$, is a hermitian lagrangian subspace, i.e., $L^{\perp} = \boldsymbol{J}L$. The relationship between this point of view and the Cayley transform is nicely explained by D. Quillen in [41, §2].

In [33] we used methods borrowed from symplectic geometry to investigate the homotopy theory of the spaces of gradings of $\mathbb{C}l_n$-modules and their K-theoretic relevance. In this paper we look only at the case $n = 1$, but we have a different goal in mind: describe a natural geometric homology theory for the space of gradings of $\mathbb{C}l_1$-modules, or equivalently, a geometric homology theory for the unitary group. However, the techniques in this paper extend to arbitrary n.

Notations and conventions

- For any finite set I, we denote by $\#I$ or $|I|$ its cardinality.
- $\mathbb{I}_n := \{-n, \dots, -1, 1, \dots, n\}$, $\mathbb{I}_n^+ = \{1, \dots, n\}$.
- $\boldsymbol{i} := \sqrt{-1}$.
- For an oriented manifold M with boundary ∂M, the induced orientation on the boundary is obtained using the outer-normal first convention.
- For a fiber bundle $F \hookrightarrow E \to B$ with orientable base, fiber and total space, we orient the total space by using the fiber-first convention.
- For any subset S of a topological space X we denote by $\boldsymbol{cl}(S)$ its closure in X.
- For any complex hermitian vector space we denote by $\mathrm{End}^+(E)$ the space of hermitian linear operators $E \to E$.
- For every complex vector space E and every nonnegative integer $m \leqslant \dim_{\mathbb{C}} E$ we denote by $\mathbf{Gr}_m(E)$ (respectively $\mathbf{Gr}^m(E)$) the Grassmannian of complex subspaces of E of dimension m (respectively codimension m).
- Suppose E is a complex vector space of dimension n and

  $$\boldsymbol{Fl}_{\bullet} := \{\boldsymbol{F}_0 \subset \boldsymbol{F}_1 \subset \cdots \subset \boldsymbol{F}_n\}$$

  is a complete increasing flag of subspaces of E, i.e., $\dim \boldsymbol{F}_i = i$, $\forall i = 0, \dots, n$.
  For every integer $0 \leqslant m \leqslant n$, and every partition $\mu = \mu_1 \geqslant \mu_2 \cdots$ such that $\mu_1 \leqslant m$ and $\mu_i = 0$, for all $i > n - m$, we define the Schubert cell $\Sigma_\mu(\boldsymbol{Fl}_{\bullet})$ to be the subset of $\mathbf{Gr}^m(E)$ consisting of subspaces V satisfying the incidence relations $\dim(V \cap \boldsymbol{F}_j) = i$, $\forall i = 1, \dots, m$, $\forall j, m + i - \mu_i \leqslant j \leqslant m + i - \mu_{i+1}$.

1. Hermitian lagrangians

In this section we collect a few basic facts concerning hermitian lagrangian spaces which we will need in our study. All of the results are due to V.I. Arnold, [2]. In this section all vector spaces will be assumed finite dimensional.

Definition 1. A *hermitian symplectic space* is a pair $(\widehat{E}, J)$, where $\widehat{E}$ is a complex hermitian space, and $J : \widehat{E} \to \widehat{E}$ is a unitary operator such that

$$J^2 = -\mathbb{1}_{\widehat{E}}, \qquad \dim_{\mathbb{C}} \ker(J - \boldsymbol{i}) = \dim_{\mathbb{C}} \ker(J + \boldsymbol{i}).$$

An *isomorphism of hermitian* symplectic spaces $(\widehat{E}_i, J_i)$, $i = 0, 1$, is a unitary map $T : \widehat{E}_0 \to \widehat{E}_1$ such that $TJ_0 = J_1 T$.

If $(\widehat{E}, R, J)$ is a hermitian symplectic space, and $h(\bullet, \bullet)$ is the hermitian metric on $\widehat{E}$, then the *symplectic hermitian form* associated to this space is the form

$$\omega : \widehat{E} \times \widehat{E} \to \mathbb{C}, \qquad \omega(\boldsymbol{u}, \boldsymbol{v}) = h(J\boldsymbol{u}, \boldsymbol{v}).$$

Observe that ω is linear in the first variable and conjugate linear in the second variable. Moreover,

$$\omega(\boldsymbol{u}, \boldsymbol{v}) = -\overline{\omega(\boldsymbol{v}, \boldsymbol{u})}, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \widehat{E}.$$

The $\mathbb{R}$-bilinear map

$$q : \widehat{E} \times \widehat{E} \to \mathbb{R}, \qquad q(\boldsymbol{u}, \boldsymbol{v}) := \mathbf{Re}\, h(\boldsymbol{i} J\boldsymbol{u}, \boldsymbol{v})$$

is symmetric, nondegenerate and has signature 0. We denote by $\mathrm{Sp}_h(\widehat{E}, J)$ the subgroup of $\mathrm{GL}_{\mathbb{C}}(\widehat{E})$ consisting of complex linear automorphisms of $\widehat{E}$ which preserve ω, i.e.,

$$\omega(T\boldsymbol{u}, \boldsymbol{v}) = \omega(\boldsymbol{u}, \boldsymbol{v}), \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \widehat{E}.$$

Equivalently,

$$\mathrm{Sp}_h(\widehat{E}, J) = \{T \in \mathrm{GL}_{\mathbb{C}}(\widehat{E});\ T^* J T = J\}.$$

Observe that $\mathrm{Sp}_h(\widehat{E}, J)$ is isomorphic to the noncompact Lie group $U(n, n)$, $n = \frac{1}{2} \dim_{\mathbb{C}} \widehat{E}$. We denote by $\underline{sp}_h(\widehat{E}, J)$ its Lie algebra. We set

$$F^{\pm} := \ker(\pm \boldsymbol{i} - J).$$

We fix an *isometry* $T : F^+ \to F^-$ and we define

$$\widehat{E}^+ := \left\{\frac{1}{2}(f + Tf);\ f \in F^+\right\}, \qquad \widehat{E}^- := \left\{\frac{1}{2\boldsymbol{i}}(f - Tf);\ f \in F^+\right\}.$$

Observe that $\widehat{E}^-$ is the orthogonal complement of $\widehat{E}^+$, and the operator J induces a unitary isomorphism $\widehat{E}^+ \to \widehat{E}^-$. Thus, we can think of $\widehat{E}^\pm$ as two different copies of the same hermitian space E.

Conversely, given a hermitian space E, we form $\widehat{E} = E \oplus E$, and we define $\boldsymbol{J} : \widehat{E} \to \widehat{E}$ by with reflection

$$\boldsymbol{J} = \begin{bmatrix} 0 & -\mathbb{1}_E \\ \mathbb{1}_E & 0 \end{bmatrix}.$$

Note that

$$F^\pm = \{(\boldsymbol{i}\boldsymbol{x}, \pm\boldsymbol{x}) \in E \oplus E;\ \boldsymbol{x} \in E\},$$

and we have a canonical isometry $F^+ \ni (\boldsymbol{i}\boldsymbol{x}, \boldsymbol{x}) \overset{T}{\longmapsto} (\boldsymbol{i}\boldsymbol{x}, -\boldsymbol{x}) \in F^-$.

For this reason, in the sequel we will assume that our hermitian symplectic spaces have the standard form

$$\widehat{E} = E \oplus E, \qquad \boldsymbol{J} = \begin{bmatrix} 0 & -\mathbb{1}_E \\ \mathbb{1}_E & 0 \end{bmatrix}.$$

We set $\widehat{E}^+ := E \oplus 0$, $\widehat{E}^- := 0 \oplus E$. We say that $\widehat{E}^+$ (respectively $\widehat{E}^-$) is the *horizontal* (respectively *vertical*) component of $\widehat{E}$.

Definition 2. Suppose $(\widehat{E}, \boldsymbol{J})$ is a hermitian symplectic space. A *hermitian lagrangian subspace* of $\widehat{E}$ is a *complex* subspace $L \subset \widehat{E}$ such that $L^\perp = \boldsymbol{J}L$. We will denote by $\mathrm{Lag}_h(\widehat{E})$ the set of hermitian lagrangian subspaces of $\widehat{E}$.

Remark 3. If ω is the symplectic form associated to $(\widehat{E}, \boldsymbol{J})$, then a subspace L is hermitian lagrangian if and only if

$$L = \{\boldsymbol{u} \in \widehat{E};\ \omega(\boldsymbol{u}, \boldsymbol{x}) = 0,\ \forall \boldsymbol{x} \in L\}.$$

This shows that the group $\mathrm{Sp}_h(\widehat{E}, J)$ acts on $\mathrm{Lag}_h(\widehat{E})$, and it is not hard to prove that the action is transitive.

Observe that if $L \in \mathrm{Lag}_h(\widehat{E})$ then we have a natural isomorphism $L \oplus JL \to \widehat{E}$. It follows that $\dim_{\mathbb{C}} L = \frac{1}{2}\dim_{\mathbb{C}} \widehat{E}$. We set $2n := \dim_{\mathbb{C}} \widehat{E}$ and we deduce that $\mathrm{Lag}_h(\widehat{E})$ is a subset of the Grassmannian $\mathbf{Gr}_n(\widehat{E})$ of complex n-dimensional subspaces of $\widehat{E}$. As such, it is equipped with an induced topology.

Example 4. Suppose E is a complex hermitian space. To any linear operator $A : E \to E$ we associate its graph

$$\Gamma_A = \{(x, Ax) \in E \oplus E;\ x \in E\}.$$

Then Γ_A is a hermitian lagrangian subspace of $E \oplus E$ if and only if the operator A is self-adjoint.

More generally, if L is a lagrangian subspace in a hermitian symplectic vector space $\widehat{E}$, and $A : L \to L$ is a linear operator, then the graph of $JA : L \to JL$ viewed as a subspace in $L \oplus JL = \widehat{E}$ is a lagrangian subspace if and only if A is a hermitian operator.

The following elementary observation will play a central role in this paper. We refer to [2] for a proof.

Lemma 5 *(Arnold). Suppose E is a complex hermitian space, and $S : E \to E$ is a linear operator. Define*

$$\mathcal{L}_S := \big\{\big(\boldsymbol{i}(\mathbb{1} + S)x, (\mathbb{1} - S)x\big);\ x \in E\big\} \subset E \oplus E. \tag{1.1}$$

Then $\mathcal{L}_S \in \mathrm{Lag}_h(E \oplus E)$ if and only if S is a unitary operator.

Lemma 6. *If $L \in \mathrm{Lag}_h(\widehat{E})$ then $L \cap F^{\pm} = \{0\}$.*

Proof. Suppose $f \in F^{\pm} \cap L$. Then $Jf \in L^{\perp}$ so that $\langle Jf, f\rangle = 0$. On the other hand, $Jf = \pm \boldsymbol{i} f$ so that

$$0 = \langle Jf, f\rangle = \pm\boldsymbol{i}|f|^2 \quad\Longrightarrow\quad f = 0.$$

Using the isomorphism $\widehat{E} = F^+ \oplus F^-$ we deduce from the above lemma that L can be represented as the graph of a linear isomorphism $T_L : F^+ \to F^-$, i.e.,

$$L = \mathrm{Graph}(T_L) = \big\{f \oplus T_L f;\ f \in F^+\big\}.$$

Denote by $\mathcal{I}_{\pm} : E \to F^{\pm}$ the unitary map

$$E \ni x \mapsto \frac{1}{\sqrt{2}}(\boldsymbol{i}x, -x) \in F^{\pm}.$$

We denote by $\mathcal{S}_L : E \to E$ the linear map given by the commutative diagram

$$\begin{tikzcd} E \arrow[r, "\mathcal{S}_L"] \arrow[d, "\mathcal{I}_+"'] & E \arrow[d, "\mathcal{I}_-"] \\ F_+ \arrow[r, "T_L"'] & F^- \end{tikzcd}, \quad \text{i.e.,}\quad \mathrm{Graph}(T_L) = (\mathcal{I}_+ \times \mathcal{I}_-)\,\mathrm{Graph}(\mathcal{S}_L).$$

A simple computation shows that $L = \mathcal{L}_{\mathcal{S}_L}$. From Lemma 5 we deduce that the operator $\mathcal{S}_L$ is unitary, and that the map

$$\mathrm{Lag}_h(\widehat{E}) \ni L \xmapsto{\mathcal{S}} \mathcal{S}_L \in U(E)$$

is the inverse of the map $S \mapsto \mathcal{L}_S$. This proves the following result. $\square$

Proposition 7 *(Arnold). Suppose E is a complex hermitian space, and denote by $U(E)$ the group of unitary operators $S : E \to E$. Then the map*

$$\mathcal{L} : U(E) \to \mathrm{Lag}_h(E \oplus E), \qquad S \mapsto \mathcal{L}_S$$

is a homeomorphism. In particular, we deduce that $\mathrm{Lag}_h(\widehat{E})$ is a smooth, compact, connected, orientable real *manifold of dimension* $\dim_{\mathbb{R}} \mathrm{Lag}_h(\widehat{E}) = (\dim_{\mathbb{C}} E)^2$.

Suppose $A : E \to E$ is a self-adjoint operator. Then its graph Γ_A is a lagrangian subspace of $\widehat{E} = E \oplus E$, and thus there exists a unitary operator $S \in U(E)$ such that

$$\Gamma_A = \mathcal{L}_S = \big\{ \big(\boldsymbol{i}(\mathbb{1} + S)x, (\mathbb{1} - S)x\big);\ x \in E \big\}.$$

Note that the graph Γ_A intersects the "vertical axis" $\widehat{E}^- = 0 \oplus E$ only at the origin, so that the operator $\mathbb{1} + S$ is invertible.

Next observe that for every $x \in E$ we have $(\mathbb{1} - S)x = \boldsymbol{i}A(\mathbb{1} - S)x$ so that

$$A = -\boldsymbol{i}(\mathbb{1} - S)(\mathbb{1} + S)^{-1} = -2\boldsymbol{i}(\mathbb{1} + S)^{-1} + \boldsymbol{i}. \tag{1.2}$$

We conclude

$$S = \mathcal{S}_{\Gamma_A} = \mathcal{C}(\boldsymbol{i}A) := (\mathbb{1} + \boldsymbol{i}A)(\mathbb{1} - \boldsymbol{i}A)^{-1} = 2(\mathbb{1} + \boldsymbol{i}A)^{-1} - \mathbb{1}. \tag{1.3}$$

The expression $\mathcal{C}(\boldsymbol{i}A)$ is the so called *Cayley transform* of $\boldsymbol{i}A$. For this reason, we will refer to the inverse diffeomorphism $\mathcal{S} = \mathcal{L}^{-1} : \mathrm{Lag}_h(E \oplus E) \to U(E)$, as the Cayley transform (of a hermitian lagrangian space).

Observe that we have a left action "$*$" of $U(E) \times U(E)$ on $U(E)$ given by

$$(T_+, T_-) * S = T_- S T_+^*, \quad \forall T_+, T_-, S \in U(E).$$

To obtain a lagrangian description of this action we need to consider the *symplectic unitary group*

$$U(\widehat{E}, \boldsymbol{J}) := U(\widehat{E}) \cap \mathrm{Sp}_h(\widehat{E}, \boldsymbol{J}) = \big\{ T \in U(\widehat{E});\ T\boldsymbol{J} = \boldsymbol{J}T \big\}.$$

The subspaces $F^\pm$ are invariant subspaces of any operator $T \in U(\widehat{E}, \boldsymbol{J})$ so that we have an isomorphism $U(\widehat{E}, \boldsymbol{J}) \cong U(F^+) \times U(F^-)$. Now identify $F^\pm$ with E using the isometries

$$\frac{1}{\sqrt{2}}\mathcal{I}_\pm : E \to F^\pm, \qquad \mathcal{I}_J : E \oplus E \to F^+ \oplus F^-.$$

We obtain an isomorphism

$$U(\widehat{E}, \boldsymbol{J}) \ni T \mapsto (T_+, T_-) \in U(E) \times U(E).$$

Moreover, for any lagrangian $L \in \mathrm{Lag}_h(\widehat{E})$, and $S \in U(E)$, and any $T \in U(\widehat{E}, \boldsymbol{J})$ we have

$$\mathcal{S}_{TL} = (T_+, T_-) * \mathcal{S}_L, \qquad \mathcal{L}_{(T_+, T_-)*S} = T\mathcal{L}_S. \tag{1.4}$$

2. Morse flows on the Grassmannian of hermitian lagrangians

In this section we will describe a few properties of some nice Morse functions on the Grassmannian of complex lagrangian subspaces. The main source for all these facts is the very nice paper by I.A. Dynnikov and A.P. Vesselov [12].

Suppose E is complex hermitian space of complex dimension n. We equip the space $\widehat{E} = E \oplus E$ with the canonical complex symplectic structure. Recall that

$$\widehat{E}^+ := E \oplus 0, \qquad \widehat{E}^- := 0 \oplus E.$$

For every symmetric operator $A : \widehat{E}^+ \to \widehat{E}^+$ we denote by $\widehat{A} : \widehat{E} \to \widehat{E}$ the symmetric operator

$$\widehat{A} := \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} : \widehat{E} \to \widehat{E}.$$

Let us point out that $\widehat{A} \in \underline{sp}_h(\widehat{E}, \boldsymbol{J})$. Define

$$f_A : U\big(\widehat{E}^+\big) \to \mathbb{R}, \qquad f_A(S) := \mathbf{Re}\,\mathrm{tr}(AS),$$

and

$$\varphi_A : \mathrm{Lag}_h(\widehat{E}) \to \mathbb{R}, \qquad \varphi_A(L) = \mathbf{Re}\,\mathrm{tr}(\widehat{A}P_L),$$

where P_L denotes the orthogonal projection onto L. An elementary computation shows that

$$P_{\mathfrak{L}_S} = \frac{1}{2}\begin{bmatrix} \mathbb{1} + \frac{1}{2}(S+S^*) & \frac{i}{2}(S-S^*) \\ \frac{i}{2}(S-S^*) & \mathbb{1} - \frac{1}{2}(S+S^*) \end{bmatrix}, \tag{2.1}$$

and we deduce

$$\varphi_A(\mathfrak{L}_S) = f_A(S), \quad \forall S \in U\big(\widehat{E}^+\big).$$

The following result is classical, and it goes back to Pontryagin, [40].

Proposition 8. *If* $\ker A = \{0\}$, *then a unitary operator* $S \in U(\widehat{E}^+)$ *is a critical point of* f_A *if and only if there exists a unitary basis* $e_1, \dots, e_n$ *of* E *consisting of eigenvectors of* A *such that*

$$Se_k = \pm e_k, \quad \forall k = 1, \dots, n.$$

We can reformulate the above result by saying that when $\ker A \neq 0$, then a unitary operator S is a critical point of f_A if and only if S is an involution and both $\ker(\mathbb{1} - S)$ and $\ker(\mathbb{1} + S)$ are invariant subspaces of A. Equivalently this means

$$S = S^*, \qquad S^2 = \mathbb{1}_E, \qquad SA = AS.$$

To obtain more detailed results, we fix a unitary basis $e_1, \dots, e_n$ of E. For any $\vec{\alpha} \in \mathbb{R}^n$ such that

$$0 < \alpha_1 < \cdots < \alpha_n, \tag{2.2}$$

we denote by $A = A_{\vec{\alpha}}$ the symmetric operator $E \to E$ defined by $Ae_k = \alpha_k e_k$, $\forall k$. We set $f_{\vec{\alpha}} := f_{A_{\vec{\alpha}}}$, and we denote by $\mathbf{Cr}_{\vec{\alpha}} \subset U(\widehat{E}^+)$ the set of critical points of $f_{\vec{\alpha}}$. For every $\vec{\epsilon} \in \{\pm 1\}^n$ we define $S_{\vec{\epsilon}} \in U(\widehat{E}^+)$ by

$$S_{\vec{\epsilon}} e_k = \epsilon_k e_k, \quad k = 1, 2, \dots, n.$$

Then

$$\mathbf{Cr}_{\vec{\alpha}} = \bigl\{ S_{\vec{\epsilon}};\ \vec{\epsilon} \in \{\pm 1\}^n \bigr\}.$$

Note that this critical set is independent of the vector $\vec{\alpha}$ satisfying (2.2). For this reason we will use the simpler notation $\mathbf{Cr}_n$ when referring to this critical set.

To compute the index of $f_{\vec{\alpha}}$ at the critical point $S_{\vec{\epsilon}}$ we need to compute the Hessian

$$\mathcal{Q}_{\vec{\epsilon}}(H) := \frac{d^2}{dt^2}\bigg|_{t=0} \mathbf{Re}\,\mathrm{tr}\bigl(A S_{\vec{\epsilon}} e^{tH}\bigr), \quad H \in \underline{u}(E) = \text{the Lie algebra of } U(E).$$

We have

$$\mathcal{Q}_{\vec{\epsilon}}(H) = \mathbf{Re}\,\mathrm{tr}\bigl(A_{\vec{\alpha}} S_{\vec{\epsilon}} H^2\bigr) = -\mathbf{Re}\,\mathrm{tr}\bigl(A_{\vec{\alpha}} S_{\vec{\epsilon}} H H^*\bigr).$$

Using the basis (e_i) we can represent $H \in \underline{u}(E)$ as $H = \boldsymbol{i} Z$, where Z is a hermitian matrix $(z_{jk})_{1 \leqslant i,j \leqslant n}$, $z_{jk} = \bar{z}_{kj}$. Note that z_{jj} is a *real* number, while z_{ij} can be any complex number if $i \neq j$. Then a simple computation shows

$$\mathcal{Q}_{\vec{\epsilon}}(\boldsymbol{i} Z) = -\sum_{i,j} (\epsilon_i \alpha_i + \epsilon_j \alpha_j)|z_{ij}|^2 = -\sum_i \epsilon_i \alpha_i |z_{ii}|^2 - 2\sum_{i<j} (\epsilon_i \alpha_i + \epsilon_j \alpha_j)|z_{ij}|^2. \quad (2.3)$$

Hence, the index of $f_{\vec{\alpha}}$ at $S_{\vec{\epsilon}}$ is

$$\mu_{\vec{\alpha}}(\vec{\epsilon}) := \#\{i;\ \epsilon_i = 1\} + 2\#\bigl\{(i,j);\ i < j,\ \epsilon_i \alpha_i + \epsilon_j \alpha_j > 0\bigr\}.$$

Observe that if $i < j$ then $\epsilon_i \alpha_i + \epsilon_j \alpha_j > 0$ if and only if $\epsilon_j = 1$. Hence

$$\mu_{\vec{\alpha}}(\vec{\epsilon}) = \sum_{\epsilon_j = 1} (2j - 1).$$

In particular, we see that the index is independent of the vector $\vec{\alpha}$ satisfying the conditions (2.2).

It is convenient to introduce another parametrization of the critical set. Recall that

$$\mathbb{I}_n^+ := \{1, \dots, n\}.$$

For every subset $I \subset \mathbb{I}_n^+$ we denote by $S_I \in U(\widehat{E}^+)$ the unitary operator defined by

$$S_I e_j = \begin{cases} e_j, & j \in I, \\ -e_j, & j \notin I. \end{cases}$$

Then $\mathbf{Cr}_n := \{S_I;\, I \subset \mathbb{I}_n^+\}$, and the index of S_I is

$$\operatorname{ind}(S_I) = \sum_{i \in I} (2i-1). \tag{2.4}$$

The co-index is

$$\operatorname{coind}(S_I) = \operatorname{ind}(S_{I^c}) = n^2 - \mu_I, \tag{2.5}$$

where I^c denotes the complement of I, $I^c := \mathbb{I}_n^+ \setminus I$.

Definition 9. We define the *weight* of a finite subset $I \subset \mathbb{Z}_{>0}$ to be the integer

$$\boldsymbol{w}(I) := \begin{cases} 0, & I = \emptyset, \\ \sum_{i \in I}(2i-1), & I \neq \emptyset. \end{cases}$$

Hence $\operatorname{ind}(S_I) = \boldsymbol{w}(I)$. Let us observe a remarkable fact.

Proposition 10. *Let*

$$\vec{\xi} = \left(\frac{1}{2}, \frac{3}{2}, \ldots, \frac{2n-1}{2}\right) \in \mathbb{Q}^n,$$

and set $f_0 := f_{\vec{\xi}}$, $\varphi_0 := \varphi_{\vec{\xi}}$. Then for every $I \subset \mathbb{I}_n^+$ we have

$$\boldsymbol{w}(I) = f_0(S_I) + \frac{n^2}{2} = \varphi_0(\Lambda_I) + \frac{n^2}{2}.$$

In other words the gradient flow of f_ξ is self-indexing, i.e.,

$$f_0(S_I) - f_0(S_J) = \boldsymbol{w}(J) - \boldsymbol{w}(I).$$

Proof. We have

$$f_0(S_I) = \frac{1}{2}\big(\boldsymbol{w}(I) - \boldsymbol{w}\big(I^c\big)\big).$$

On the other hand, we have

$$\frac{1}{2}\big(\boldsymbol{w}(I) + \boldsymbol{w}\big(I^c\big)\big) = \frac{1}{2}\boldsymbol{w}\big(\mathbb{I}_n^+\big) = \frac{n^2}{2}.$$

Adding up the above equalities we obtain the desired conclusion. □

The *positive* gradient flow of the function f_A has an explicit description. More precisely, we have the following result [12, Proposition 2.1].

Proposition 11. *Suppose $A = A_{\vec{\alpha}}$ where $\vec{\alpha} \in \mathbb{R}^n$ satisfies* (2.2). *We equip $U(\widehat{E}^+)$ with the left invariant metric induced from the inclusion in the Euclidean space* $\mathrm{End}_{\mathbb{C}}(E)$ *equipped with the inner product*

$$\langle X, Y\rangle = \mathbf{Re}\,\mathrm{tr}\big(XY^*\big).$$

We denote by ∇f_A the gradient of $f_A : U(\widehat{E}^+) \to \mathbb{R}$ with respect to this metric, and we denote by

$$\Phi_A : \mathbb{R} \times U\big(\widehat{E}^+\big) \to U\big(\widehat{E}^+\big), \qquad S \mapsto \Phi_A^t(S),$$

the flow defined by ∇f_A, i.e., the flow associated to the o.d.e. $\dot{S} = \nabla f_A(S)$. Then, for any $S \in U(\widehat{E}^+)$ and any $t \in \mathbb{R}$, we have

$$\Phi_A^t(S) = \big(\sinh(tA) + \cosh(tA)S\big)\big(\cosh(tA) + \sinh(tA)S\big)^{-1}. \tag{2.6}$$

It is convenient to have a lagrangian description of the above results via the diffeomorphism $\mathcal{L} : U(\widehat{E}^+) \to \mathrm{Lag}_h(\widehat{E})$. First, we use this isomorphism to transport isometrically the metric on $U(\widehat{E}^+)$. Next, for every $I \subset \mathbb{I}_n^+$ we set $\Lambda_I := \mathcal{L}_{S_I}$. For every $i \in \mathbb{I}_n^+$ we define

$$\boldsymbol{e}_i := e_i \oplus 0 \in E \oplus E, \qquad \boldsymbol{f}_i = 0 \oplus e_i \in E \oplus E.$$

Then

$$\Lambda_I = \ker(\mathbb{1} - S_I) \oplus \ker(\mathbb{1} + S_I) = \mathrm{span}\{\boldsymbol{e}_i;\ i \in I\} + \mathrm{span}\big\{\boldsymbol{f}_j;\ j \in I^c\big\}.$$

The lagrangians Λ_I are the critical points of the function $\varphi_A : \mathrm{Lag}_h(\widehat{E}) \to \mathbb{R}$. Using (1.1) and (2.6) we deduce that for every $S \in U(\widehat{E}^+)$ we have

$$\mathcal{L}_{\Phi^t(S)} = e^{t\widehat{A}}\mathcal{L}_S. \tag{2.7}$$

The above equality describes the (positive) gradient flow of φ_A on $\mathrm{Lag}_h(\widehat{E})$. We denote this flow by Ψ_A^t.

We can use the lagrangians Λ_I to produce the *Arnold atlas* as in [1]. Define

$$\mathrm{Lag}_h(\widehat{E})_I := \big\{L \in \mathrm{Lag}_h(\widehat{E});\ L \cap \Lambda_I^\perp = 0\big\}.$$

Then $\mathrm{Lag}_h(\widehat{E})_I$ is an open subset of $\mathrm{Lag}_h(\widehat{E})$, and

$$\mathrm{Lag}_h(\widehat{E}) = \bigcup_I \mathrm{Lag}_h(\widehat{E})_I.$$

Denote by $\mathrm{End}^+_{\mathbb{C}}(\Lambda_I)$ the space of self-adjoint endomorphisms of Λ_I. We have a diffeomorphism

$$\mathrm{End}^+_{\mathbb{C}}(\Lambda_I) \to \mathrm{Lag}_h(\widehat{E})_I,$$

which associates to each symmetric operator $T : \Lambda_I \to \Lambda_I$ the graph $\Gamma_{\boldsymbol{J}T}$ of the operator $\boldsymbol{J}T : \Lambda_I \to \Lambda_I^\perp$ regarded as a subspace in $\Lambda_I \oplus \Lambda_I^\perp \cong \widehat{E}$.

More precisely, if the operator T is described in the orthonormal basis $\{\boldsymbol{e}_i, \boldsymbol{f}_j;\ i\in I,\ j\in I^c\}$ by the hermitian matrix $(t_{ij})_{1\leqslant i,j\leqslant n}$, then the graph of $\boldsymbol{J}T$ is spanned by the vectors

$$\boldsymbol{e}_i(T) := \boldsymbol{e}_i + \sum_{i'\in I} t_{i'i}\boldsymbol{f}_{i'} - \sum_{j\in I^c} t_{ji}\boldsymbol{e}_j, \quad i\in I, \tag{2.8a}$$

$$\boldsymbol{f}_j(T) := \boldsymbol{f}_j + \sum_{i\in I} t_{ij}\boldsymbol{f}_i - \sum_{j'\in I^c} t_{j'j}\boldsymbol{e}_{j'}, \quad j\in I^c. \tag{2.8b}$$

We will refer to the inverse map $\mathcal{A}_I : \mathrm{Lag}_h(\widehat{E})_I \to \mathrm{End}^+_{\mathbb{C}}(\Lambda_I)$ as the *Arnold coordinatization map* on $\mathrm{Lag}_h(\widehat{E})_I$.

Let $I\subset \mathbb{I}^+_n$. If $L\in \mathrm{Lag}_h(\widehat{E})_I$ has Arnold coordinates $\mathcal{A}_I(L)=T$, i.e., T is a symmetric operator $T:\Lambda_I\to\Lambda_I$, and $L=\Gamma_{\boldsymbol{J}T}$, then $\Psi^t_A L = e^{t\widehat{A}}\Gamma_{\boldsymbol{J}T}$ is spanned by the vectors

$$e^{t\alpha_i}\boldsymbol{e}_i + \sum_{i'\in I} t_{i'i}e^{-t\alpha_{i'}}\boldsymbol{f}_{i'} - \sum_{j\in I^c} t_{ji}e^{t\alpha_j}\boldsymbol{e}_j, \quad i\in I,$$

$$e^{-t\alpha_j}\boldsymbol{f}_j + \sum_{i\in I} t_{ij}e^{-t\alpha_i}\boldsymbol{f}_i - \sum_{j'\in I^c} t_{j'j}e^{t\alpha_{j'}}\boldsymbol{e}_{j'}, \quad j\in I^c,$$

or, equivalently, by the vectors

$$\boldsymbol{e}_i + \sum_{i'\in I} t_{i'i}e^{-t(\alpha_{i'}+\alpha_i)}\boldsymbol{f}_{i'} - \sum_{j\in I^c} t_{ji}e^{t(\alpha_j-\alpha_i)}\boldsymbol{e}_j, \quad i\in I, \tag{2.9a}$$

$$\boldsymbol{f}_j + \sum_{i\in I} t_{ij}e^{t(\alpha_i-\alpha_j)}\boldsymbol{f}_i - \sum_{j'\in I^c} t_{j'j}e^{t(\alpha_{j'}+\alpha_j)}\boldsymbol{e}_{j'}, \quad j\in I^c. \tag{2.9b}$$

This shows that $e^{t\widehat{A}}\Gamma_{\boldsymbol{J}T}\in \mathrm{Lag}_h(\widehat{E})_I$, so that $\mathrm{Lag}_h(\widehat{E})_I$ is invariant under the flow Ψ^t_A.

If we denote by A_I the restriction of $\widehat{A}$ to Λ_I, and we regard A_I as a symmetric operator $\Lambda_I\to\Lambda_I$, then we deduce from the above equalities that

$$e^{t\widehat{A}}\Gamma_{\boldsymbol{J}T} = \Gamma_{\boldsymbol{J}e^{tA_I}Te^{tA_I}}.$$

We can rewrite the above equality in terms of Arnold coordinates as

$$\mathcal{A}_I\big(\Psi^t L\big) = e^{tA_I}\mathcal{A}_I(L)e^{tA_I}, \quad \forall L\in \mathrm{Lag}_h(\widehat{E})_I. \tag{2.10}$$

3. Unstable manifolds

The unstable manifolds of the positive gradient flow of φ_A have many similarities with the Schubert cells of complex Grassmannians, and we want to investigate these similarities in great detail.

The stable/unstable variety of Λ_I *with respect to the positive gradient flow* Ψ^t_A is defined by

$$W^{\pm}_I := \Big\{L\in \mathrm{Lag}_h(\widehat{E});\ \lim_{t\to\infty} e^{\pm t\widehat{A}}L = \Lambda_I\Big\} \stackrel{(2.10)}{=} \Big\{L\in \mathrm{Lag}_h(\widehat{E})_I;\ \lim_{t\to\pm\infty} e^{tA_I}\mathcal{A}_I(L)e^{tA_I} = 0\Big\}.$$

If $\mathcal{A}_I(L) = (t_{ij})_{1\leqslant i,j\leqslant n}$, then the equalities (2.9a) and (2.9b) imply that

$$\lim_{t\to-\infty} e^{tA_I}\mathcal{A}_I(L)e^{tA_I} = 0 \quad\iff\quad t_{ij} = 0, \quad \text{if } i, j \in I, \text{ or } j \in I^c,\ i \in I \text{ and } j < i.$$

We can rewrite the last system of equalities in the more compact form

$$W_I^- = \{T \in \operatorname{End}^+(\Lambda_I);\ t_{ji} = 0,\ \forall 1 \leqslant j \leqslant i,\ i \in I\}. \tag{3.1}$$

This shows that W_I^- has real codimension $\sum_{i\in I}(2i-1)$. This agrees with our previous computation (2.5) of the index of Λ_I. Thus

$$\operatorname{codim}_{\mathbb{R}} W_I^- = \boldsymbol{w}(I), \qquad \dim_{\mathbb{R}} W_I^- = n^2 - \boldsymbol{w}(I) = \frac{1}{2}\dim_{\mathbb{R}} \operatorname{Lag}_h(\widehat{E}) - \varphi_0(\Lambda_I).$$

For any $L \in \operatorname{Lag}_h(\widehat{E})$ we set

$$L^\pm := L \cap \widehat{E}^\pm.$$

The dimension of L^+ is called the *depth* of L, and will be denoted by $\delta(L)$. From the description (3.1) of the unstable variety W_I^-, $\#I = k$ we deduce the following result.[3]

Proposition 12. *Let $L \in \operatorname{Lag}_h(\widehat{E})$, $I \subset \{1,\dots,n\}$, $k = \#I$. We denote by $S \in U(\widehat{E}^+)$ the unitary operator corresponding to L. The following statements are equivalent.*

(a) $L \in W_I^-$.
(b) $L \in \operatorname{Lag}_h(\widehat{E})_I$ *and* $\lim_{t\to\infty} e^{-tA}L^+ = \Lambda_I^+$.
(c) $\dim L^+ = k$ *and* $\lim_{t\to\infty} e^{-tA}L^+ = \Lambda_I^+$.
(d) $\dim\ker(\mathbb{1} - S) = k$ *and* $\lim_{t\to\infty} e^{-tA}\ker(\mathbb{1} - S) = \Lambda_I^+$.

Proof. The description (3.1) shows that (a) $\Rightarrow$ (b) $\Rightarrow$ (c). Suppose that L satisfies (c) and let $\Lambda_J = \lim_{t\to\infty} e^{-t\widehat{A}}L$, i.e., $L \in W_J^-$. Then using the implication (a) $\Rightarrow$ (b) for the unstable manifold W_J^- we deduce $\lim_{t\to\infty} e^{tA}L^+ = \Lambda_J^+$.

On the other hand, since L satisfies (c) we have $\lim_{t\to\infty} e^{tA}L^+ = \Lambda_I^+$. This implies $I = J$ which proves the implication (c) $\Rightarrow$ (a). Finally, observe that (d) is a reformulation of (c) via the Cayley diffeomorphism $\mathcal{S} : \operatorname{Lag}_h(\widehat{E}) \to U(\widehat{E}^+)$. $\square$

The condition $\lim_{t\to\infty} e^{-tA}L^+ = \Lambda_I^+$ can be rephrased as an incidence condition. We arrange the elements of I in decreasing order, $I = \{\nu_1 > \dots > \nu_k\}$. Then $\lim_{t\to\infty} e^{-tA}L^+ = \Lambda_I^+$ if and only if L^+ is the graph of a linear map

$$X : \Lambda_I^+ \to \Lambda_{I^c}^+, \qquad X\boldsymbol{e}_i = \sum_{j\in I^c} x_i^j \boldsymbol{e}_j, \quad \forall i \in I,$$

such that $x_i^j = 0$, $\forall i \in I$, $j \in I^c$, $j < i$.

[3] The characterization in Proposition 12 depends essentially on the fact that the eigenvalues of A satisfy the inequalities $0 < \alpha_1 < \dots < \alpha_n$. This corresponds to a choice of a Weyl chamber for the unitary group.

We consider the complete decreasing flag $\boldsymbol{Fl}^\bullet = \{\widehat{E}^+ = \boldsymbol{F}^0 \supset \boldsymbol{F}^1 \supset \cdots \supset \boldsymbol{F}^n = 0\}$ of subspaces of $\widehat{E}^+$,

$$\boldsymbol{F}^\ell := \operatorname{span}_{\mathbb{C}}\{\boldsymbol{e}_i;\, i > \ell\},$$

and we form the associated *increasing flag* $\boldsymbol{Fl}_\bullet = \{\boldsymbol{F}_0 \subset \cdots \subset \boldsymbol{F}_n\}$, $\boldsymbol{F}_j := \boldsymbol{F}^{n-j}$. Then $\lim_{t\to\infty} e^{-tA}L^+ \to \Lambda_I^+$ if and only if

$$\forall i = 0, 1, \ldots, k:\ \dim_{\mathbb{C}}\big(L^+ \cap \boldsymbol{F}^\ell\big) = i, \quad \forall \ell,\ \nu_{i+1} \leqslant \ell < \nu_i,$$

or, equivalently,

$$\forall i = 0, 1, \ldots, k:\ \dim_{\mathbb{C}}\big(L^+ \cap \boldsymbol{F}_\nu\big) = i, \quad \forall \nu,\ n+1-\nu_i \leqslant \nu \leqslant n - \nu_{i+1}, \quad \nu_0 = n+1.$$

We define μ_i so that

$$n - k + i - \mu_i = n + 1 - \nu_i \quad \Longleftrightarrow \quad \mu_i = \nu_i - (k + 1 - i),$$

and we obtain a partition $\mu_I = (\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_k \geqslant 0)$. We deduce that $L^+ \in \Sigma_{\mu_I}(\boldsymbol{Fl}_\bullet)$, where $\Sigma_{\mu_I}(\boldsymbol{Fl}_\bullet) \subset \mathbf{Gr}_k(\widehat{E}^+)$ denotes the Schubert cell associated to the partition μ, and the increasing flag $\boldsymbol{Fl}_\bullet$.

Remark 13. The partition $(\mu_1, \ldots, \mu_k)$ can be given a very simple intuitive interpretation. We describe the set I by placing $\bullet$'s on the positions $i \in I$, and $\circ$'s on the positions $j \in I^c$. If $I = \{\nu_1 > \cdots > \nu_k\}$, then μ_i is equal to the number of $\circ$'s situated to the left of the $\bullet$ located on the position ν_i. Thus

$$\mu_{\{k\}} = (k-1, 0, \ldots, 0), \qquad \mu_{\{1,\ldots,k-1,k+1,\ldots,n\}} = 1^{n-k} = \underbrace{(1, \ldots, 1)}_{n-k}.$$

A critical lagrangian Λ_I is completely characterized by its depth $k = \delta(\Lambda_I) = \#I$, and the associated partition μ. More precisely,

$$I = \{\mu_1 + k > \mu_2 + k - 1 > \cdots > \mu_k + 1\}. \tag{3.2}$$

The Ferres diagram of the partition μ_I fits inside a $k \times (n-k)$ rectangle. We denote by $\mathcal{C}_n$ the set

$$\mathcal{C}_n = \big\{(k, \mu);\ k \in \{0, \ldots, n\},\ \mu \in \mathcal{P}_{k,n-m}\big\},$$

where $\mathcal{P}_{k,n-k}$ is the set of partitions whose Ferres diagrams fit inside a $k \times (n-k)$ rectangle. We have a bijection

$$\mathbb{I}_n^+ \supset I \mapsto \pi_I = (\#I, \mu_I) \in \mathcal{C}_n.$$

For every $\pi = (m, \mu) \in \mathcal{C}_n$ there exists a unique $I \subset \mathbb{I}_n^+$ such that $\pi_I = (m, \mu)$. We set

$$\Lambda_{(m,\mu)} := \Lambda_I, \qquad W^{\pm}_{(m,\mu)} := W^{\pm}_I.$$

Observe that

$$\operatorname{codim}_{\mathbb{R}} W^-_{(m,\mu)} = m^2 + 2|\mu|, \quad \text{where } |\mu| := \sum_i \mu_i, \tag{3.3}$$

and

$$\dim_{\mathbb{R}} W^-_{(m,\mu)} = n^2 - m^2 - 2|\mu| = (n-m)^2 + \dim_{\mathbb{R}} \Sigma_\mu. \tag{3.4}$$

The involution $I \mapsto I^c$ on the collection of subsets of $\mathbb{I}_n^+$ is mapped to the involution

$$\mathcal{C}_n \ni \pi = (m, \mu) \mapsto \pi^* := \big(n-m, \mu^*\big) \in \mathcal{C}_n,$$

where μ^* is the transpose of the complement of μ in the $k \times (n-k)$ rectangle. In other words, $\pi_{I^c} = \pi_I^*$.

Remark 14. There is a remarkable involution in this story. More precisely, the operator $\boldsymbol{J} : \widehat{E} \to \widehat{E}$ defines a diffeomorphism $\boldsymbol{J} : \mathrm{Lag}_h(\widehat{E}) \to \mathrm{Lag}_h(\widehat{E})$, $L \mapsto \boldsymbol{J}L$.

If we use the depth-partition labelling, then to every pair $\pi = (k, \mu) \in \mathcal{C}_n$ we can associate a Lagrangian $\Lambda_{k,\mu}$ and we have $\boldsymbol{J}\Lambda_\pi = \Lambda_{\pi^*}$. We list some of the properties of this involution.

- $f_A(\boldsymbol{J}L) = -f_A(L)$, $\forall L \in \mathrm{Lag}_h(\widehat{E})$, because $P_{\boldsymbol{J}L} = \mathbb{1}_{\widehat{E}} - P_L$ and $\widehat{A}$ and $\operatorname{tr}\widehat{A} = 0$.
- $e^{t\widehat{A}}\boldsymbol{J} = \boldsymbol{J}e^{-t\widehat{A}}$ because $\boldsymbol{J}\widehat{A} = -\widehat{A}\boldsymbol{J}$.
- $\boldsymbol{J}L^\pm = (\boldsymbol{J}L)^\mp$, $\forall L \in \mathrm{Lag}_h(\widehat{E})$.
- $\boldsymbol{J}\Lambda_I = \Lambda_{I^c}$, $\forall I \subset \mathbb{I}_n^+$.
- $\boldsymbol{J}W_I^\pm = W_{I^c}^\mp$, $\forall I \subset \{1, \dots, n\}$.

The involution is transported by the diffeomorphism $\mathcal{S} : \mathrm{Lag}_h(\widehat{E}) \to U(E)$ to the involution $S \mapsto -S$ on $U(E)$.

Proposition 12 can be rephrased as follows.

Corollary 15. *Let* $L \in \mathrm{Lag}_h(\widehat{E})$ *and set* $S := \mathcal{S}(L) \in U(E)$. *Then the following hold.*

(a) $L \in W^-_{(m,\mu)}$ *if and only if* $\dim\ker(\mathbb{1} - S) = m$ *and* $\ker(\mathbb{1} - S) \in \Sigma_\mu(\boldsymbol{Fl_\bullet}) \subset \mathbf{Gr}_m(E)$.
(b) $L \in W^+_{(k,\lambda)}$ *if and only if* $\dim\ker(\mathbb{1} + S) = n-k$ *and* $\ker(\mathbb{1} + S) \in \Sigma_{\lambda^*}(\boldsymbol{Fl_\bullet}) \subset \mathbf{Gr}^k(E)$.

Finally, we can give an invariant theoretic description of the unstable manifolds W_I^-.

Definition 16.

(a) We define the *symplectic annihilator* of a subspace $U \subset \widehat{E}$ to be the subspace $U^\dagger := \boldsymbol{J}U^\perp$, where $U^\perp$ denotes the orthogonal complement.
(b) A subspace $U \subset \widehat{E}$ is called *isotropic* (respectively *coisotropic*) if $U \subset U^\dagger$ (respectively $U^\dagger \subset U$). (Observe that a lagrangian subspace is a maximal isotropic space.)

(c) A *decreasing isotropic flag* of $\widehat{E}$ is a collection of isotropic subspaces

$$\mathcal{I}^0 \supset \mathcal{I}^1 \supset \cdots \supset \mathcal{I}^n = 0, \qquad \dim \mathcal{I}^k = n-k, \qquad n = \frac{1}{2}\dim_{\mathbb{C}} \widehat{E}.$$

The top space $\mathcal{I}^0$ is called the *lagrangian subspace associated to* $\mathcal{I}^\bullet$.

Consider the decreasing isotropic flag $\mathcal{I}^\bullet$ given by $\mathcal{I}^\ell = \operatorname{span}\{\boldsymbol{e}_i;\ i > \ell\}$. If

$$I = \{\nu_k < \cdots < \nu_1\},$$

then

$$L \in W_I^- \quad \Leftrightarrow \quad \forall i = 0, 1, \ldots, k:\ \dim_{\mathbb{C}}\big(L \cap \mathcal{I}^r\big) = i, \quad \forall r,\ \nu_{i+1} \leqslant r < \nu_i, \quad \nu_0 = n+1. \tag{3.5}$$

Define the (real) Borel subgroup

$$\mathcal{B} = \mathcal{B}\big(\mathcal{I}^\bullet\big) := \big\{T \in \mathrm{Sp}_h(\widehat{E}, \boldsymbol{J});\ T\mathcal{I}^\ell \subset \mathcal{I}^\ell,\ \forall 1 \leqslant \ell \leqslant n\big\}.$$

Proposition 17. *The unstable manifold W_I^- coincides with the $\mathcal{B}$-orbit of Λ_I.*

Proof. Observe that W_I^- is $\mathcal{B}$-invariant so that W_I^- contains the $\mathcal{B}$-orbit of Λ_I. To prove the converse, we need a better understanding of $\mathcal{B}$.

Using the unitary basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n, \boldsymbol{f}_1, \ldots, \boldsymbol{f}_n$ we can identify $\mathcal{B}$ with the group of $(2n) \times (2n)$ matrices $\mathcal{T}$ which, with respect to the direct sum decomposition $\widehat{E}^+ \oplus \widehat{E}^-$, have the block description

$$\mathcal{T} = \begin{bmatrix} T & TS \\ 0 & (T^*)^{-1} \end{bmatrix},$$

where T is a lower triangular invertible $n \times n$ matrix, and S is a hermitian $n \times n$ matrix. The Lie algebra of $\mathcal{B}$ is the vector space $\mathcal{X}$ consisting of matrices X of the form

$$X = \begin{bmatrix} \dot{T} & \dot{S} \\ 0 & -\dot{T}^* \end{bmatrix},$$

where $\dot{T}$ is lower triangular, and $\dot{S}$ is hermitian. In particular, we deduce that

$$\dim_{\mathbb{R}} \mathcal{B} = n(n+1) + n^2 = 2n^2 + n.$$

Observe that the matrix $\widehat{A}$ defining the Morse flow Ψ^t on $\mathrm{Lag}_h(\widehat{E})$ belongs to the Lie algebra of $\mathcal{B}$, and for any open neighborhood $\mathcal{N}$ of Λ_I in W_I^- we have

$$W_I^- = \bigcup_{t \in \mathbb{R}} \Psi^t(\mathcal{N}).$$

Thus, to prove that $\mathcal{B}\Lambda_I = W_I^-$, it suffices to show that the orbit $\mathcal{B}\Lambda_I$ contains a tiny open neighborhood of Λ_I in W_I^-. To achieve this we look at the smooth map

$$\mathcal{B} \to W_I^-, \qquad g \mapsto g \cdot \Lambda_I,$$

and it suffices to show that its differential at $1 \in \mathcal{B}$ is surjective.

The kernel of this differential is the Lie algebra of the stabilizer of Λ_I with respect to the action of $\mathcal{B}$. Thus, if we denote by $\mathbf{St}_I$ this stabilizer, it suffices to show that

$$\dim \mathcal{B} - \dim \mathbf{St}_I = \dim W_I^- = \boldsymbol{w}\big(I^c\big).$$

Observe that X belongs to the Lie algebra of $\mathbf{St}_I$ if and only if the subspace $X\Lambda_I$ is contained in Λ_I, or equivalently, any vector in $\Lambda_I^\perp = \Lambda_{I^c}$ is orthogonal to $X\Lambda_I$. If we denote by $\langle \bullet, \bullet \rangle$ the hermitian inner product on $\widehat{E}$ we deduce that X belongs to the Lie algebra of $\mathbf{St}_I$ if and only if

$$\langle \boldsymbol{e}_j, X\boldsymbol{e}_i \rangle = \langle \boldsymbol{f}_{i'}, X\boldsymbol{e}_i \rangle = \langle \boldsymbol{f}_i, X\boldsymbol{f}_{j'} \rangle = \langle \boldsymbol{e}_{j'}, X\boldsymbol{f}_j \rangle = 0, \quad \forall i, i' \in I,\ j, j' \in I^c.$$

If we write X in bloc form

$$\begin{bmatrix} \dot{T} & \dot{S} \\ 0 & -\dot{T}^* \end{bmatrix}, \quad \dot{T} = \big(\dot{t}_i^j\big)_{1 \leqslant i,j \leqslant n}, \ \dot{S} = \big(\dot{s}_i^j\big)_{1 \leqslant i,j \leqslant n},$$

then we deduce that X is in the Lie algebra of $\mathbf{St}_I$ if and only if

$$\dot{t}_i^j = \dot{s}_{j'}^j = 0, \quad \forall i \in I,\ j, j' \in I^c.$$

Suppose $I = \{i_k < \cdots < i_1\}$. The equalities $\dot{s}_{j'}^j = 0$, $j, j' \in I^c$ impose $(n-k)^2$ *real* constraints on the matrix $\dot{S}$. For an $i_\ell \in I$, the equalities $\dot{t}_{i_\ell}^j = 0$, $j \in I^c$, $i_\ell < j$ impose $(n - i_\ell - \ell + 1)$ *complex* constraints on $\dot{T}$. The vector space of lower triangular complex $n \times n$ matrices has real dimension $n(n+1)$ so that the Lie algebra of $\mathbf{St}_I$ has real dimension

$$\begin{aligned} n(n+1) - 2\sum_{\ell=1}^k (n - i_\ell - \ell + 1) + n^2 - (n-k)^2 &= n^2 + n - k^2 + 2\sum_{\ell=1}^k (i_\ell + \ell - 1) \\ &= n^2 + n - k + 2\sum_{\ell_1}^k i_\ell = n^2 + n + \boldsymbol{w}(I). \end{aligned}$$

We deduce that $\dim_{\mathbb{R}} \mathcal{B} - \dim_{\mathbb{R}} \mathbf{St}_I = n^2 - \boldsymbol{w}(I) = \dim W_I^-$. $\square$

Corollary 18. *The collection of unstable manifolds $(W_I^-)_{I \subset \mathbb{I}_n^+}$ defines a Whitney regular stratification of $\mathrm{Lag}_h(\widehat{E})$. In particular, the flow Ψ^t satisfies the Morse–Smale transversality condition.*

Proof. The statement about the Whitney regularity follows immediately from Proposition 17 and the results of Lander [28]. For the reader's convenience, we include an alternate argument.

The unstable varieties W_I^- are the orbits of a smooth, semi-algebraic action of the semi-algebraic group $\mathcal{B}$ on $\mathrm{Lag}_h(\widehat{E})$. If $W_J^- \subset \boldsymbol{cl}(W_I^-)$ then, according to the results of

C.T.C. Wall [45], the set $\mathcal{R}$ of points in W_J^- where the pair (W_I^-, W_J^-) is Whitney regular is nonempty. Since $\mathcal{B}$ acts by diffeomorphisms of $\mathrm{Lag}_h(\widehat{E})$ the set $\mathcal{R}$ is a $\mathcal{B}$-invariant subset of W_J^-, so it must coincide with W_J^-.

Since the stratification by the unstable manifolds of the flow Ψ^t satisfies the Whitney regularity condition we deduce from [37, Thm. 8.1] that the flow satisfies the Morse–Smale transversality condition. $\square$

4. Tunnellings

The main problem we want to investigate in this section is the structure of *tunnellings* of the flow $\Psi^t = e^{t\widehat{A}}$ on $\mathrm{Lag}_h(\widehat{E})$. Given $M, K \subset \mathbb{I}_n^+$, then a *tunnelling* from Λ_M to Λ_K is a gradient trajectory

$$t \mapsto \Psi_A^t L = e^{t\widehat{A}} L, \quad L \in \mathrm{Lag}_h(\widehat{E}),$$

such that

$$\lim_{t\to\infty} \Psi_A^{-t} L = \Lambda_M, \qquad \lim_{t\to\infty} \Psi_A^t L = \Lambda_K.$$

We denote by $\mathcal{T}(M, K)$ the set of tunnellings from Λ_M to Λ_K, and we say that M *covers* K, and write this $K \prec M$, if $\mathcal{T}(M, K) \neq \emptyset$. Equivalently, $K \prec M$ if and only if $W_M^- \cap W_K^+ \neq \emptyset$. Observe that

$$L \in W_K^+ \iff \boldsymbol{J} L \in W_{K^c}^-.$$

Hence

$$W_M^- \cap W_K^+ = W_M^- \cap \boldsymbol{J} W_{K^c}^-,$$

so that

$$K \prec M \iff W_M^- \cap \boldsymbol{J} W_{K^c}^- \neq \emptyset.$$

Let us observe that, although the flow Ψ_A^t depends on the choice of the hermitian operator $A : \widehat{E}^+ \to \widehat{E}^+$, the equality (3.1) shows that the unstable manifolds W_I^- *are independent of the choice of* A. Thus, we can choose A such that

$$A\boldsymbol{e}_i = \frac{1}{2}(2i-1)\boldsymbol{e}_i, \quad \forall i = 1, \ldots, n.$$

Using Proposition 10 on self-indexing we obtain the following result.

Proposition 19. *If $J \prec I$, then $\boldsymbol{w}(J) > \boldsymbol{w}(I)$ so that* $\dim W_J^- < \dim W_I^-$.

Definition 20.

(a) For any nonempty set $K \subset \mathbb{I}_n^+$ of cardinality k we denote by ν_K the unique strictly decreasing function $\nu_K : \{1, \dots, k\} \to \mathbb{I}_n^+$ whose range is K, i.e.,

$$K = \{\nu_K(k) < \cdots < \nu_K(1)\}.$$

(b) We define a partial order $\lhd$ on the collection of subsets of $\mathbb{I}_n^+$ by declaring $J \lhd I$ if either $I = \emptyset$, or $\#J \geqslant \#I$, and for every $1 \leqslant \ell \leqslant \#I$ we have $\nu_I(\ell) \leqslant \nu_J(\ell)$.

We have the following elementary fact whose proof is left to the reader.

Lemma 21. *Let $K, M \subset \mathbb{I}_n^+$. Then the following statements are equivalent.*

(a) $K \lhd M$.
(b) *For ever $\ell \in \mathbb{I}_n^+$ we have $\#(K \cap [\ell, n]) \geqslant \#(M \cap [\ell, n])$.*
(c) $M^c \lhd K^c$.

Proposition 22. *Suppose $K, M \subset \mathbb{I}_n^+$. Then $K \prec M$ if and only if $K \lhd M$.*

Proof. Suppose $L \in W_M^-$. Then $(\boldsymbol{J}L)^+ = \boldsymbol{J}L^-$, and we deduce that

$$L \in W_M^- \cap W_K^+ \iff \lim_{t\to\infty} e^{-tA}L^+ = \Lambda_M^+ \quad \text{and} \quad \lim_{t\to\infty} e^{-tA}\boldsymbol{J}L^- = \Lambda_{K^c}^+.$$

In other words,

$$L \in W_M^- \cap W_K^+ \iff L^+ \in \Sigma_M(\boldsymbol{Fl}^*), \qquad \boldsymbol{J}L^- \in \Sigma_{K^c}(\boldsymbol{Fl}^*).$$

We denote by $U^+ = U_L^+$ the orthogonal complement of L^+ in $\widehat{E}^+$, and by $T = (t_{ij})_{1\leqslant i,j}$ the Arnold coordinates of L in the chart $\mathrm{Lag}_h(\widehat{E})_M$.

Observe that U^+ contains $\boldsymbol{J}L^-$, the subspace L^+ is spanned by the vectors

$$\boldsymbol{v}_i = \boldsymbol{e}_i - \sum_{j\in M^c,\, j>i} t_{ji}\boldsymbol{e}_j, \quad i \in M,$$

and U^+ is spanned by the vectors

$$\boldsymbol{u}_j = \boldsymbol{e}_j + \sum_{i\in M,\, i<j} t_{ij}\boldsymbol{e}_i = \boldsymbol{e}_j + \sum_{i\in M,\, i<j} \bar{t}_{ji}\boldsymbol{e}_i, \quad j \in M^c.$$

If we write

$$M^c = \{j_{n-m} < \cdots < j_1\}, \qquad K^c = \{\ell_{n-k} < \cdots < \ell_1\},$$

then the condition $\boldsymbol{J}L^- \in \Sigma_{K^c}$ is equivalent with the existence of linearly independent vectors of the form

$$\boldsymbol{w}_k = \boldsymbol{e}_k + \sum_{s>k} a_{sk}\boldsymbol{e}_s, \quad k \in K^c, \tag{4.1}$$

which span $\boldsymbol{J}L^-$. Arguing exactly as in [29, §3.2.2] we deduce that the inclusion

$$\boldsymbol{J}L^- = \operatorname{span}\{\boldsymbol{w}_k;\ k \in K^c\} \subset U^+ = \operatorname{span}\{\boldsymbol{u}_j;\ j \in M^c\}$$

can happen only if

$$n - m = \dim U^+ \geqslant \dim L^- = n - k \quad \text{and} \quad j_i \geqslant \ell_i, \quad \forall i = 1, \dots, n-k, \tag{4.2}$$

i.e., $M^c \lhd K^c$, so that $K \lhd M$.

Conversely, if (4.2) holds, then arguing as in [29, §3.2.2] we can find vectors $\boldsymbol{w}_k$ as in (4.1) and complex numbers τ_{ij} $i \in M$, $j \in M^c$, $i < j$, such that

$$\operatorname{span}\{\boldsymbol{w}_k;\ k \in K^c\} \subset \operatorname{span}\Bigg\{\boldsymbol{e}_j + \sum_{i \in M,\, i<j} \tau_{ij}\boldsymbol{e}_i;\ j \in M^c\Bigg\}.$$

Next complete the collection (τ_{ij}) to a collection $(t_{ij})_{1 \leqslant i,j \leqslant n}$ such that $t_{ij} = \bar{t}_{ji}$, $\forall i, j$, and $t_{ij} = 0$ if $i \in M$ and $j < i$. The collection (t_{ij}) can be viewed as the Arnold coordinates in the chart $\mathrm{Lag}_h(\widehat{E})_M$ of a Lagrangian $L \in W_M^- \cap W_K^+$. □

Remark 23. Proposition 22 implies that if $K \prec M$ and $M \prec N$ then $K \prec N$, so that $\prec$ is a partial order relation. This fact has an interesting consequence.

If $K_0, K_1, \dots, K_\nu \subset \mathbb{I}_n^+$ are such that for every $i = 1, \dots, \nu$ there exists tunnelling from $\Lambda_{K_{i-1}}$ to Λ_{K_i}, then there must exist tunnelling from Λ_{K_0} to Λ_{K_ν}.

Proposition 24. *Suppose $M, K \subset \mathbb{I}_n^+$. The following statements are equivalent.*

(a) $K \prec M$.
(b) $W_K^- \subset \boldsymbol{cl}(W_M^-)$.

Proof. The implication (b) ⇒ (a) follows from the above remark. Conversely assume $K \prec M$. Then we deduce $\Lambda_K \subset \boldsymbol{cl}(W_M^-)$. Since $\boldsymbol{cl}(W_M^-)$ is $\mathcal{B}$ invariant, where $\mathcal{B}$ is the real Borel group defined at the end of Section 3, we deduce that $\mathcal{B}\Lambda_K \subset \boldsymbol{cl}(W_M^-)$. We now conclude by invoking Proposition 17. □

Corollary 25. *For any $M \subset \mathbb{I}_N^+$ we have $\boldsymbol{cl}(W_M^-) = \bigsqcup_{K \preccurlyeq M} W_K^-$.*

Corollary 26. *Let $K, M \subset \mathbb{I}_n^+$, and set $k = \#K$, $m = \#M$. The following statements are equivalent.*

- $W_K^- \subset \boldsymbol{cl}(W_M^-)$ *and* $\dim W_K^- = \dim W_M^- - 1$.
- $\{1\} \in K$ *and* $M = K \setminus \{1\}$.

Remark 27. The poset defined by $\prec$ has many beautiful combinatorial properties which makes it resemble the Bruhat poset of Schubert varieties of a complex Grassmannian. For more details we refer to [38].

5. Arnold–Schubert cells, varieties and cycles

We want to use the results we have proved so far to describe a very useful collection of subsets of $\mathrm{Lag}_h(\widehat{E})$. We begin by describing this collection using the identification $\mathrm{Lag}_h(\widehat{E}) \cong U(\widehat{E}^+)$.

For every complete decreasing flag $\boldsymbol{Fl}^\bullet = \{\widehat{E}^+ = \boldsymbol{F}^0 \supset \boldsymbol{F}^1 \supset \cdots \supset \boldsymbol{F}^n = 0\}$ of $\widehat{E}^+$, and for every subset $I = \{\nu_k < \cdots < \nu_1\} \subset \mathbb{I}_n^+$, we set $\nu_0 = n+1$, $\nu_{k+1} = 0$, and we denote by $\mathcal{W}_I^-(\boldsymbol{Fl}^\bullet)$ the set

$$= \bigl\{ \boldsymbol{g} \in U\bigl(\widehat{E}^+\bigr);\ \dim_{\mathbb{C}} \boldsymbol{F}^\ell \cap \ker(\mathbb{1} - \boldsymbol{g}) = j,\ \nu_{j+1} \leqslant \ell < \nu_j,\ j = 0, \dots, k \bigr\}.$$

We say that $\mathcal{W}_I(\boldsymbol{Fl}_\bullet)$ is the *Arnold–Schubert (AS) cell of type I associated to the flag* $\boldsymbol{Fl}^\bullet$. Its closure, denoted by $\mathcal{X}_I(\boldsymbol{Fl}^\bullet)$ is called the *AS variety of type I, associated to the flag* $\boldsymbol{Fl}_\bullet$. We want to point out that

$$\boldsymbol{g} \in \mathcal{W}_I^-\bigl(\boldsymbol{Fl}^\bullet\bigr) \quad \Longrightarrow \quad \dim_{\mathbb{C}} \ker(\mathbb{1} - \boldsymbol{g}) = \#I.$$

If we fix a unitary basis $\underline{\boldsymbol{e}} = \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ of $\widehat{E}^+$ we obtain a decreasing flag

$$\boldsymbol{Fl}^\bullet(\underline{\boldsymbol{e}}),\ \boldsymbol{Fl}^\nu(\underline{\boldsymbol{e}}) := \mathrm{span}_{\mathbb{C}}\{\boldsymbol{e}_j;\ j > \nu\}.$$

We set

$$\mathcal{W}_I^-(\underline{\boldsymbol{e}}) := \mathcal{W}_I^-\bigl(\boldsymbol{Fl}^\bullet(\underline{\boldsymbol{e}})\bigr).$$

As we know, the unitary symplectic group $U(\widehat{E}, \boldsymbol{J}) \cong U(\widehat{E}^+) \times U(\widehat{E}^+)$ acts on $U(\widehat{E}^+)$, by

$$(\boldsymbol{g}_+, \boldsymbol{g}_-) * \boldsymbol{h} = U_- S U_+^*,$$

and we set

$$\mathcal{W}_I^-\bigl(\boldsymbol{Fl}^\bullet, \boldsymbol{g}_+, \boldsymbol{g}_-\bigr) := (\boldsymbol{g}_+, \boldsymbol{g}_-) * \mathcal{W}_I^-(\boldsymbol{Fl}_\bullet).$$

We denote by $\mathcal{X}_I(\boldsymbol{Fl}^\bullet, \boldsymbol{g}_+, \boldsymbol{g}_-)$ the closure of $\mathcal{W}_I^-(\boldsymbol{Fl}^\bullet, \boldsymbol{g}_+, \boldsymbol{g}_-)$. When I is a singleton, $I = \{\nu\}$, we will use the simpler notation $\mathcal{W}_\nu^-$ and $\mathcal{X}_\nu$ instead of $\mathcal{W}_{\{\nu\}}^-$ and $\mathcal{X}_{\{\nu\}}$. For every unit complex number ρ we set

$$\begin{aligned} \mathcal{W}_I^-\bigl(\boldsymbol{Fl}^\bullet, \rho\bigr) &:= \mathcal{W}_I^-\bigl(\boldsymbol{Fl}^\bullet, \bar{\rho}\mathbb{1}, \mathbb{1}\bigr) = \mathcal{W}_I^-\bigl(\boldsymbol{Fl}^\bullet, \mathbb{1}, \rho\mathbb{1}\bigr) \\ &= \bigl\{ \boldsymbol{g} \in U\bigl(\widehat{E}^+\bigr);\ \dim_{\mathbb{C}} \boldsymbol{F}^\ell \cap \ker(\rho - \boldsymbol{g}) = j,\ \nu_{j+1} \leqslant \ell < \nu_j,\ j = 0, \dots, n \bigr\}. \end{aligned}$$

When $\boldsymbol{Fl}^\bullet = \boldsymbol{Fl}^\bullet(\underline{\boldsymbol{e}})$ we will use the alternative notation

$$\mathcal{W}_I^-(\underline{\boldsymbol{e}}, \rho) := \mathcal{W}_I^-\bigl(Fl^\bullet(\underline{\boldsymbol{e}}, \rho)\bigr), \qquad \mathcal{X}_I(\underline{\boldsymbol{e}}, \rho) = \mathcal{X}_I\bigl(Fl^\bullet(\underline{\boldsymbol{e}}, \rho)\bigr). \tag{5.1}$$

Example 28. Suppose $\underline{e} = \{e_1, \dots, e_n\}$ is an orthonormal basis of $\widehat{E}^+$. For every $\nu \in \mathbb{I}_n^+$ and every unit complex number ρ we have

$$\mathcal{W}_\nu^-(\underline{e}, \rho) = \left\{ g \in U\big(\widehat{E}^+\big);\ \exists z_{\nu+1}, \dots, z_n \in \mathbb{C}\colon \ker(\rho - g) = \operatorname{span}\left\{ e_\nu + \sum_{j>\nu} z_j e_j \right\} \right\}.$$

Moreover

$$\mathcal{X}_\nu(\underline{e}, \rho) = \big\{ g \in U\big(\widehat{E}^+\big);\ \ker(\rho - g) \cap \operatorname{span}\{e_\nu, \dots, e_n\} \neq 0 \big\}.$$

Definition 29. Let $I \subset \mathbb{I}_n^+$. We say that a subset $\Sigma \subset \mathrm{Lag}_h(\widehat{E}) = U(E)$ is an *Arnold–Schubert (AS) cell, respectively variety, of type I* if there exists a flag $\boldsymbol{Fl}^\bullet$ of E, and $g_\pm \in U(E)$ such that $\Sigma = \mathcal{W}_I^-(\boldsymbol{Fl}^\bullet, g_+, g_-)$, respectively $\Sigma = \mathcal{X}_I(\boldsymbol{Fl}^\bullet, g_+, g_-)$. We will refer to $\mathcal{X}_\nu$ as the *basic AS* varieties.

Note that an AS cell of type I is a non-closed, smooth, semi-algebraic submanifold of $\mathrm{Lag}_h(\widehat{E})$, semi-algebraically diffeomorphic to $\mathbb{R}^{n^2 - \boldsymbol{w}(I)}$. The AS cells can be given a description as incidence loci of lagrangian subspaces of $\widehat{E}$.

We denote by $\mathbf{FLAG}_{\mathrm{iso}}(\widehat{E})$ the collection of isotropic flags of $\widehat{E}$. The unitary symplectic group

$$U(\widehat{E}, \boldsymbol{J}) = \big\{ T \in U(\widehat{E});\ T\boldsymbol{J} = \boldsymbol{J}T \big\},$$

maps isotropic subspaces to isotropic subspaces and thus acts on $\mathbf{FLAG}_{iso}$. It is easily seen that this action is transitive.

For any decreasing isotropic flag $\mathcal{I}^\bullet \in \mathbf{FLAG}_{\mathrm{iso}}$, and any subset $I = \{\nu_1 > \cdots > \nu_k\} \subset \mathbb{I}_n^+$ we set $\nu_0 := n+1$, $\nu_{k+1} := 0$, and we define

$$W_I^-\big(\mathcal{I}^\bullet\big) := \big\{ L \in \mathrm{Lag}_h(\widehat{E});\ \dim L \cap \mathcal{I}^\nu = i,\ \forall i,\ \nu\nu_{i+1} \leqslant \nu < \nu_i \big\}.$$

If we choose a complete decreasing flag $\boldsymbol{Fl}^\bullet$ of $\widehat{E}^+$, then $\boldsymbol{Fl}^\bullet$ is also a decreasing isotropic flag, and we observe that the diffeomorphism $\mathcal{L} : U(\widehat{E}^+) \to \mathrm{Lag}_h(\widehat{E})$ sends $\mathcal{W}_I^-(\boldsymbol{Fl}^\bullet)$ to $W_I^-(\boldsymbol{Fl}^\bullet)$. If $\underline{e}$ is a unitary basis, then we will write

$$W_I^-(\underline{e}) := W_I^-\big(\boldsymbol{Fl}^\bullet(\underline{e})\big).$$

As we explained earlier, the unitary symplectic group $U(\widehat{E}, \boldsymbol{J})$ is isomorphic to $U(\widehat{E}^+) \times U(\widehat{E}^+)$, so that every $T \in U(\widehat{E}, \boldsymbol{J})$ can be identified with a pair $(T_+, T_-) \in U(\widehat{E}^+) \times U(\widehat{E}^+)$, such that for every $S \in U(\widehat{E}^+)$ we have $T\mathcal{L}_S = \mathcal{L}_{T_- S T_+^*}$. We deduce that

$$W_I^-\big(\boldsymbol{Fl}^\bullet, T_+, T_-\big) \stackrel{\mathcal{L}}{\longleftrightarrow} T W_I^-\big(\boldsymbol{Fl}^\bullet\big).$$

Since $U(\widehat{E}, \boldsymbol{J})$ acts transitively on $\mathbf{FLAG}_{\mathrm{iso}}$ we conclude that any AS cell is of the form $\mathcal{W}_I(\mathcal{I}^\bullet)$ for some flag $\mathcal{I}^\bullet \in \mathbf{FLAG}_{\mathrm{iso}}$.

✍ *In the sequel we will use the notation* $\mathcal{W}_I^-$ *when referring to AS cells viewed as subsets of the unitary group* $U(\widehat{E}^+)$, *and the notation* W_I^- *when referring to AS cells viewed as subsets of the Grassmannian* $\mathrm{Lag}_h(\widehat{E})$.

Remark 30. We have a real algebraic version of Kleiman's transversality result, [26]. Suppose X is a smooth manifold and $f : X \to \mathrm{Lag}_h(\widehat{E})$ is a smooth map. Then there exists an isotropic flag $\mathcal{J}^\bullet \in \mathbf{FLAG}_{\mathrm{iso}}$ such that f is transversal to $W_I^-(\mathcal{J}^\bullet)$, for all I.

To prove this, we fix an isotropic flag $\mathcal{J}^\bullet$ and we consider the smooth map

$$F : \mathrm{Sp}_h(\widehat{E}, J) \times X \to \mathrm{Sp}_h(\widehat{E}, J) \times \mathrm{Lag}_h(\widehat{E}), \quad \mathrm{Sp}_h(\widehat{E}, J) \times X \ni (g, x) \mapsto \big(g, g \cdot f(x)\big).$$

The transitivity of the action $\mathrm{Sp}_h(\widehat{E}, J)$ on $\mathbf{FLAG}_{\mathrm{iso}}$ implies that for every I the map F is transversal to the submanifold $\mathrm{Sp}_h(\widehat{E}, J) \times W_I^-(\mathcal{J}^\bullet)$ of $\mathrm{Sp}_h(\widehat{E}, J) \times \mathrm{Lag}_h(\widehat{E})$. Thus, the set $\mathcal{Z}_I = F^{-1}(\mathrm{Sp}_h(\widehat{E}, J) \times W_I^-)$ is a smooth submanifold of $\mathrm{Sp}_h(\widehat{E}, J) \times \mathrm{Lag}_h(\widehat{E})$. A generic $g \in \mathrm{Sp}_h(\widehat{E}, J)$ is a regular value of the natural map

$$\pi : \mathcal{Z}_I \subset \mathrm{Sp}_h(\widehat{E}, J) \times \mathrm{Lag}_h(\widehat{E}) \to \mathrm{Sp}_h(\widehat{E}, J).$$

For such a g the map

$$f_g : X \to \mathrm{Lag}_h(\widehat{E}), \quad x \mapsto g \cdot f(x)$$

is transversal to $W_I^-(\mathcal{J}^\bullet)$ and thus f is transversal to $W_I^-(g^{-1}\mathcal{J}^\bullet)$.

We would like to associate cycles to the AS cells and, to do so, we must first fix some orientation conventions. First, we need to fix an orientation on $U(\widehat{E}^+)$. Via the Cayley map we then transport this orientation to $\mathrm{Lag}_h(\widehat{E})$.

To fix an orientation on $U(\widehat{E})$ it suffices to pick an orientation on the Lie algebra $\underline{u}(\widehat{E}^+) = T_{\mathbb{1}}U(\widehat{E}^+)$. This induces an orientation on each tangent space $T_S U(\widehat{E})$ via the *left translation isomorphism*

$$T_{\mathbb{1}}U\big(\widehat{E}^+\big) \to T_S U\big(\widehat{E}^+\big), \quad T_{\mathbb{1}}U\big(\widehat{E}^+\big) \ni X \mapsto SX \in T_S U\big(\widehat{E}^+\big).$$

To produce such an orientation we first choose a unitary basis of $\widehat{E}^+$,

$$\underline{\boldsymbol{e}} := \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}.$$

We can then describe any $X \in \underline{u}(\widehat{E}^+)$ as a skew-hermitian matrix

$$X = (x_{ij})_{1 \leqslant i,j \leqslant n}.$$

We identify $\underline{u}(\widehat{E}^+)$ with the space of hermitian operators $\widehat{E}^+ \to \widehat{E}^+$, by associating to the skew-hermitian operator X the hermitian operator $Z = -\boldsymbol{i}X$. Hence $X = \boldsymbol{i}Z$, and we write

$$z_{ij}(X) := -\boldsymbol{i}x_{ij}.$$

Note that $z_{ii} \in \mathbb{R}$, $\forall i$, but z_{ij} are not real if $i \neq j$. The functions $(z_{ij})_{1 \leqslant i \leqslant j}$ define linear coordinates on $\underline{u}(\widehat{E}^+)$ which via the exponential map define coordinates on an open neighborhood of $\mathbb{1}$ in $U(\widehat{E}^+)$. More precisely, to any sufficiently small hermitian matrix $Z = (z_{ij})_{1 \leqslant i,j \leqslant n}$ one associates the unitary operator $e^{\boldsymbol{i}Z}$.

Using the above linear coordinates we obtain a decomposition of $\underline{u}(\widehat{E}^+)$, as a direct sum of a n-dimensional real vector space with coordinates z_{ii}, $1 \leqslant i \leqslant n$, and a $\binom{n}{2}$-dimensional complex vector space with complex coordinates z_{ij}, $1 \leqslant i < j \leqslant n$. The complex summand has a canonical orientation, and we orient the real summand using the *ordered* basis

$$\partial_{z_{11}}, \ldots, \partial_{z_{nn}}.$$

Equivalently, if we set

$$\theta^{ij} := \mathbf{Re}\, z_{ij}, \qquad \varphi^{ij} := \mathbf{Im}\, z_{ij}, \qquad \theta^{i} := z_{ii},$$

then the linear functions $\theta^i, \theta^{ij}, \varphi^{ij} : \underline{u}(\widehat{E}^+) \to \mathbb{R}$ form a basis of the real dual of $\underline{u}(\widehat{E}^+)$. The functions $z_{ij} : \underline{u}(\widehat{E}^+) \to \mathbb{C}$ are $\mathbb{R}$-linear and we have

$$\theta^{ij} \wedge \varphi^{ij} = \frac{1}{2\boldsymbol{i}} z_{ij} \wedge \bar{z}_{ij} = \frac{1}{2\boldsymbol{i}} z_{ij} \wedge z_{ji}.$$

The above orientation of $\underline{u}(\widehat{E}^+)$ is described by the volume form

$$\boldsymbol{\Omega}_n = \left(\bigwedge_{i=1}^{n} \theta^i \right) \wedge \left(\bigwedge_{1 \leqslant i < j \leqslant n} \theta^{ij} \wedge \varphi^{ij} \right).$$

The volume form $\boldsymbol{\Omega}_n$ on $\underline{u}(\widehat{E}^+)$ is uniquely determined by the unitary basis $\underline{\boldsymbol{e}}$, and depends continuously on $\underline{\boldsymbol{e}}$. Since the set of unitary bases is connected, we deduce that the orientation determined by $\boldsymbol{\Omega}_n$ is independent of the choice of the unitary basis $\underline{\boldsymbol{e}}$. We will refer to this as the *canonical orientation on the group* $U(\widehat{E}^+)$. Note that when $\dim_{\mathbb{C}} \widehat{E}^+ = 1$, the canonical orientation of $U(1) \cong S^1$ coincides with the counterclockwise orientation on the unit circle in the plane.

Using the Cayley diffeomorphism we transport the above orientation to an orientation on $\mathrm{Lag}_h(\widehat{E})$. We will need to have a description of this orientation in terms of Arnold coordinates. For a lagrangian $\Lambda \in \mathrm{Lag}_h(\widehat{E})$ we denote by $\mathrm{Lag}_h(\widehat{E})_\Lambda$ the Arnold chart

$$\mathrm{Lag}_h(\widehat{E})_\Lambda := \{ L \in \mathrm{Lag}_h(\widehat{E});\ L \cap \Lambda^\perp = 0 \}.$$

The Arnold coordinates identify this open set with the space $\mathrm{End}^+_{\mathbb{C}}(\Lambda)$ of hermitian operators $\Lambda \to \Lambda$. By choosing a unitary basis of Λ we can identify such an operator A with a hermitian matrix $(a_{ij})_{1 \leqslant i,j \leqslant n}$, and we can coordinatize $\mathrm{End}^+_{\mathbb{C}}(\Lambda)$ using the functions $(a_{ij})_{1 \leqslant i \leqslant j \leqslant n}$. We want to prove that the orientation of $\mathrm{Lag}_h(\widehat{E})$ is described on $\mathrm{Lag}_h(\widehat{E})_\Lambda$ by the form

$$(-1)^{n^2} \left(\bigwedge_{i=1}^{n} da_{ii} \right) \wedge \left(\bigwedge_{1 \leqslant i < j \leqslant n} \frac{1}{2\boldsymbol{i}} da_{ij} \wedge da_{ji} \right). \tag{+}$$

The relationship between the Arnold coordinates on the chart $\mathrm{Lag}_h(\widehat{E})_{\widehat{E}^+} = \mathrm{Lag}_h(\widehat{E})_{\mathbb{I}_n^+}$ and the above coordinates on $U(\widehat{E}^+)$ is given by the Cayley transform. More precisely, if $\boldsymbol{g} = e^{\boldsymbol{i}Z}$, Z hermitian matrix, and A are the Arnold coordinates of the associated lagrangian $\mathcal{L}_S$, the according to (1.2) we have

$$\mathbb{1} + \boldsymbol{g} = 2(1 + \boldsymbol{i}A)^{-1} \iff \boldsymbol{i}A = 2(\mathbb{1} + \boldsymbol{g})^{-1} - \mathbb{1}.$$

To see whether this correspondence is orientation preserving we compute its differential at $\boldsymbol{g} = \mathbb{1}$, i.e., $A = 0$. We set

$$\boldsymbol{g}_t := e^{t\boldsymbol{i}Z}, \quad \boldsymbol{i}A_t = 2(\mathbb{1} + \boldsymbol{g}_t)^{-1} - \mathbb{1}$$

we deduce upon differentiation at $t = 0$ that $\dot{A}_0 = -2Z$.

Thus, the differential at $\mathbb{1}$ of the Cayley transform is represented by a *negative* multiple of identity matrix in our choice of coordinates. This shows that the orientation $(+)$ on the chart $\mathrm{Lag}_h(\widehat{E})_{\widehat{E}^+}$ agrees with the canonical orientation on the group $U(\widehat{E}^+)$.

To show that this happens for any chart $\mathrm{Lag}_h(\widehat{E})_\Lambda$ we choose $T = (T_+, T_-) \in U(\widehat{E}, \boldsymbol{J})$ such that $T\widehat{E} = \Lambda$. Then

$$\mathrm{Lag}_h(\widehat{E})_\Lambda = T\,\mathrm{Lag}_h(\widehat{E})_{\widehat{E}^+}.$$

We fix a unitary basis $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ of $\widehat{E}^+$ and we obtain unitary basis $\boldsymbol{e}_i' = \boldsymbol{T}\boldsymbol{e}_i$ of Λ. Using these bases we obtain the Arnold coordinates

$$\mathcal{A} : \mathrm{Lag}_h(\widehat{E})_{\widehat{E}^+} \to \mathrm{End}^+_{\mathbb{C}}(\mathbb{C}^n), \qquad \mathcal{A}' : \mathrm{Lag}_h(\widehat{E})_\Lambda \to \mathrm{End}^+_{\mathbb{C}}(\mathbb{C}^n).$$

Let $L \in \mathrm{Lag}_h(\widehat{E})_{\widehat{E}^+} \cap \mathrm{Lag}_h(\widehat{E})_\Lambda$. The Arnold coordinates of L in the chart $\mathrm{Lag}_h(\widehat{E})_\Lambda$ are equal to the Arnold coordinates of $L' = T^{-1}L$ in the chart $\mathrm{Lag}_h(\widehat{E})_{\widehat{E}^+}$, i.e.,

$$\mathcal{A}'(L) = \mathcal{A}\big(T^{-1}L\big).$$

Using (1.4) we deduce

$$\mathcal{S}_{T^{-1}L} = T_-^*\mathcal{S}_L T_+ = T^{-1} * \mathcal{S}_L.$$

From (1.2) and (1.3) we deduce

$$\mathcal{S}_L = \mathcal{C}_{\boldsymbol{i}}\big(\mathcal{A}(L)\big) := \big(\mathbb{1} - \boldsymbol{i}\mathcal{A}(L)\big)\big(\mathbb{1} + \boldsymbol{i}\mathcal{A}(L)\big)^{-1},$$

$$\boldsymbol{i}\mathcal{A}'(L) = \boldsymbol{i}\mathcal{A}\big(T^{-1}L\big) = 2\big(\mathbb{1} + T_-^*\mathcal{S}_L T_+\big)^{-1} - \mathbb{1} = \mathcal{C}_{\boldsymbol{i}}^{-1}\big(T_-^*\mathcal{S}_L T_+\big).$$

We seen that the transition map

$$\mathrm{End}^+_{\mathbb{C}}(\mathbb{C}^n) \ni \mathcal{A}(L) \mapsto \mathcal{A}'(L) \in \mathrm{End}^+_{\mathbb{C}}(\mathbb{C}^n)$$

is the composition of the maps

$$\mathrm{End}^+_{\mathbb{C}}(\mathbb{C}^n) \xrightarrow{\mathcal{C}_i} U(n) \xrightarrow{T*} U(n) \xrightarrow{\mathcal{C}_i^{-1}} \mathrm{End}^+_{\mathbb{C}}(\mathbb{C}^n).$$

This composition is orientation preserving if and only if the map $\boldsymbol{g} \mapsto T * \boldsymbol{g}$ is such. Now we remark that the latter is indeed orientation preserving because it is homotopic to the identity map since $U(\widehat{E}^+)$ is connected.

Fix $I = \{\nu_k < \cdots < \nu_1\} \subset \mathbb{I}_n^+$, and a unitary basis $\underline{\boldsymbol{e}} = \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ of $\widehat{E}^+$. We want to describe a canonical orientation on the AS cell $W_I^- = W_I^-(\underline{\boldsymbol{e}})$. We will achieve this by describing a canonical co-orientation.

The cell W_I^- is contained in the Arnold chart $\mathrm{Lag}_h(\widehat{E})_I$, and it is described in the Arnold coordinates $(t_{pq})_{1 \leqslant p \leqslant q \leqslant n}$ on this chart by the system of linearly independent equations

$$t_{ji} = 0, \quad i \in I, \ j \leqslant i.$$

We set

$$u_{pq} = \mathbf{Re}\, t_{pq}, \quad v_{pq} = \mathbf{Im}\, t_{pq}, \quad \forall 1 \leqslant p < q.$$

The *conormal bundle* $T^*_{W_I^-} \mathrm{Lag}_h(\widehat{E})$ of $W_I^- \subset \mathrm{Lag}_h(\widehat{E})$ is the kernel of the natural restriction map $T^* \mathrm{Lag}_h(\widehat{E})|_{W_I^-} \to T^* W_I^-$. This bundle morphism is surjective and thus we have a short exact sequence of bundles over W_I^-,

$$0 \to T^*_{W_I^-} \mathrm{Lag}_h(\widehat{E}) \longrightarrow T^* \mathrm{Lag}_h(\widehat{E})|_{W_I^-} \longrightarrow T^* W_I^- \to 0. \tag{5.2}$$

The 1-forms du_{ji}, dv_{ji}, dt_{ii}, $i \in I$, $j < i$, trivialize the conormal bundle. We can orient the conormal bundle $T^*_{W_I^-} \mathrm{Lag}_h(\widehat{E})$ using the form

$$\boldsymbol{\omega}_I = (-1)^{\boldsymbol{w}(I)} d\boldsymbol{t}_I \wedge \left(\bigwedge_{j<i,\, i \in I} du_{ji} \wedge dv_{ji} \right), \tag{5.3}$$

where $d\boldsymbol{t}_I$ denotes the wedge product of the 1-forms dt_{ii}, $i \in I$, written in increasing order,

$$d\boldsymbol{t}_I = dt_{\nu_k \nu_k} \wedge \cdots \wedge dt_{\nu_1 \nu_1}.$$

We denote by $\boldsymbol{or}_I^\perp$ this co-orientation, and we will refer to it as the *canonical co-orientation*. As explained in Appendix B this co-orientation induces a canonical orientation $\boldsymbol{or}_I$ on W_I^-. We denote by $[W_I^-, \boldsymbol{or}_I^\perp]$ the current of integration thus obtained.

To understand how to detect this co-orientation in the unitary picture we need to give a unitary description of the Arnold coordinates on $\mathcal{W}_I^-(\underline{\boldsymbol{e}}) \subset U(\widehat{E}^+)$.

Definition 31. Fix a unitary basis $\underline{\boldsymbol{e}}$ of $\widehat{E}^+$. For every subset $I \subset \mathbb{I}_n^+$ we define $\mathcal{U}_I \in U(\widehat{E}, \boldsymbol{J})$ to be the symplectic unitary operator defined by

$$\mathcal{U}_I(\boldsymbol{e}_k) = \begin{cases} \boldsymbol{e}_k, & k \in I, \\ \boldsymbol{J}\boldsymbol{e}_k, & k \notin I, \end{cases} \qquad \mathcal{U}_I(\boldsymbol{f}_k) = \begin{cases} \boldsymbol{f}_k, & k \in I, \\ \boldsymbol{J}\boldsymbol{f}_k, & k \notin I. \end{cases}$$

Via the diffeomorphism

$$U(\widehat{E}, \boldsymbol{J}) \ni T \mapsto (T_+, T_-) \in U\big(\widehat{E}^+\big) \times U\big(\widehat{E}^-\big),$$

the operator $\mathcal{U}_I$ corresponds to the pair of unitary operators $\mathcal{U}_I^+ = \mathcal{T}_I$, $\mathcal{U}_I^- = \mathcal{T}_I^*$, where

$$\mathcal{T}_I(\boldsymbol{e}_k) = \begin{cases} \boldsymbol{e}_k, & k \in I, \\ \boldsymbol{i}\boldsymbol{e}_k, & k \notin I. \end{cases} \tag{5.4}$$

Observe that $\mathcal{U}_I \widehat{E}^+ = \Lambda_I$, and that the Arnold coordinates $\mathcal{A}_I$ on $\mathrm{Lag}_h(\widehat{E})_I$ are related to the Arnold coordinates $\mathcal{A}$ on $\mathrm{Lag}_h(\widehat{E})_{\widehat{E}^+}$ via the equality $\mathcal{A}_I = \mathcal{A} \circ \mathcal{U}_I^{-1}$. We deduce that if $S \in U(\widehat{E}^+)$ is such that $\mathcal{L}_S \in \mathrm{Lag}_h(\widehat{E}^+)_I$, then

$$\mathcal{A}_I(\mathcal{L}_S) = \mathcal{C}_{\boldsymbol{i}}\big(\mathcal{U}_I^{-1} * S\big) = \mathcal{C}_{\boldsymbol{i}}(\mathcal{T}_I S \mathcal{T}_I).$$

Example 32. Let us describe the orientation of $\mathcal{W}_I^- \subset U(\widehat{E}^+)$ at certain special points. To any map $\vec{\rho} : I^c \to S^1 \setminus \{1\}$, $j \mapsto \rho_j$, we associate the diagonal unitary operator $D = D_{\vec{\rho}} \in \mathcal{W}_I^-$ defined by

$$D\boldsymbol{e}_j = \begin{cases} 1, & j \in I, \\ \rho_j, & j \in I^c. \end{cases}$$

Every tangent vector $\dot{\boldsymbol{g}} \in T_D U(\widehat{E}^+)$ can be written as $\dot{\boldsymbol{g}} = \boldsymbol{i} DZ$, Z hermitian matrix, so that

$$Z = -\boldsymbol{i} D^{-1} \dot{\boldsymbol{g}}.$$

The cotangent space $T_D^* U(\widehat{E}^+)$ has a natural basis given by the $\mathbb{R}$-linear forms

$$\theta^p, \theta^{pq}, \varphi^{pq} : T_D U(\widehat{E}) \to \mathbb{R}, \qquad \theta^p(Z) = (Z\boldsymbol{e}_p, \boldsymbol{e}_p),$$
$$\theta^{pq}(Z) = \mathbf{Re}(Z\boldsymbol{e}_q, \boldsymbol{e}_p), \qquad \varphi^{pq} = \mathbf{Im}(Z\boldsymbol{e}_q, \boldsymbol{e}_p).$$

To describe the orientation of the conormal bundle $T_{S_I} U(\widehat{E}^+)$ we use the above prescription. The Arnold coordinates on $\mathcal{W}_I^-$ are given by

$$\mathcal{W}_I^- \ni \boldsymbol{g} \mapsto \mathcal{A}_I(\boldsymbol{g}) := \mathcal{C}_{\boldsymbol{i}}(\mathcal{T}_I \boldsymbol{g} \mathcal{T}_I) = -\boldsymbol{i}(\mathbb{1} - \mathcal{T}_I \boldsymbol{g} \mathcal{T}_I)(\mathbb{1} + \mathcal{T}_I \boldsymbol{g} \mathcal{T}_I)^{-1} \in \mathrm{End}^+\big(\widehat{E}^+\big).$$

Using the equality

$$\mathcal{C}_{\boldsymbol{i}}(\mathcal{T}_I \boldsymbol{g} \mathcal{T}_I) = -2\boldsymbol{i}(\mathbb{1} + \mathcal{T}_I \boldsymbol{g} \mathcal{T}_I)^{-1} + \boldsymbol{i}\mathbb{1}$$

we deduce

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \mathcal{A}_I\big(De^{\boldsymbol{i}tZ}\big) &= -2\boldsymbol{i} \frac{d}{dt}\Big|_{t=0} \big(\mathbb{1} + \mathcal{T}_I De^{\boldsymbol{i}tZ} \mathcal{T}_I\big)^{-1} = -2\boldsymbol{i} \frac{d}{dt}\Big|_{t=0} \big(\mathbb{1} + \mathcal{T}_I D(\mathbb{1} + \boldsymbol{i}tZ)\mathcal{T}_I\big)^{-1} \\ &= -2\boldsymbol{i}(\mathbb{1} + \mathcal{T}_I D \mathcal{T}_I)^{-1} \frac{d}{dt}\Big|_{t=0} \big(\mathbb{1} + t\boldsymbol{i}\mathcal{T}_I DZ\mathcal{T}_I(\mathbb{1} + \mathcal{T}_I D\mathcal{T}_I)^{-1}\big)^{-1} \\ &= -2(\mathbb{1} + \mathcal{T}_I D\mathcal{T}_I)^{-1} \mathcal{T}_I DZ\mathcal{T}_I(\mathbb{1} + \mathcal{T}_I D\mathcal{T}_I)^{-1} = \dot{A}. \end{aligned}$$

Hence

$$Z = -\frac{1}{2} D^* \mathcal{T}_I^* (\mathbb{1} + \mathcal{T}_I D \mathcal{T}_I) \dot{A} (\mathbb{1} + \mathcal{T}_I D \mathcal{T}_I) = -\frac{1}{2} D^* \mathcal{T}_I^* \big(\mathbb{1} + \mathcal{T}_I^2 D\big) \dot{A} \big(\mathbb{1} + \mathcal{T}_I^2 D\big).$$

Note that

$$\mathcal{T}_I^2 \boldsymbol{e}_j = \begin{cases} \boldsymbol{e}_j, & j \in I, \\ -\boldsymbol{e}_j, & j \in I^c. \end{cases}$$

If $i \in I$, then for every $j \leqslant i$ we have

$$\langle \dot{Z} \boldsymbol{e}_j, \boldsymbol{e}_i \rangle = -\frac{1}{2} \big(\dot{A}\big(\mathbb{1} + \mathcal{T}_I^2 D\big)\boldsymbol{e}_j, \mathcal{T}_I D\big(\mathbb{1} + \mathcal{T}_I^2 D\big)\boldsymbol{e}_i \big),$$

$$-\big(\dot{A}\big(\mathbb{1} + \mathcal{T}_I^2 D\big)\boldsymbol{e}_j, \boldsymbol{e}_i\big) = \begin{cases} -(\dot{A}\boldsymbol{e}_j, \boldsymbol{e}_i), & j \in I, \\ \frac{1}{2}(\rho_j - 1)(\dot{A}\boldsymbol{e}_j, \boldsymbol{e}_i). & j \in I^c. \end{cases}$$

For $i \in I$, $j \in I^c$, $j < i$ we set

$$u^i(\dot{A}) := (\dot{A}\boldsymbol{e}_i, \boldsymbol{e}_i), \qquad u^{ij}(\dot{A}) := \mathbf{Re}(\dot{A}\boldsymbol{e}_j, \boldsymbol{e}_i), \qquad v^{ij}(\dot{A}) := \mathbf{Im}(\dot{A}\boldsymbol{e}_j, \boldsymbol{e}_i).$$

We deduce

$$u^i = -\varphi^i, \qquad u^{ij} \wedge v^{ij} = k_j \theta^{ij} \wedge \varphi^{ij},$$

where k_j is the *positive* constant

$$k_j = \begin{cases} 1, & j \in I, \\ \frac{1}{4}|\rho_j - 1|^2, & j \in I^c. \end{cases}$$

Using (5.3) we conclude that the conormal bundle to $\mathcal{W}_I^-$ is oriented at D by the exterior monomial

$$\theta^I \wedge \bigg(\bigwedge_{j<i, i \in I} \theta^{ji} \wedge \varphi^{ji} \bigg),$$

where θ^I denotes the wedge product of $\{\theta^i\}_{i \in I}$ written in increasing order.

In particular, if $I = \{\nu\}$ and $D = S_\nu$, i.e., $\rho_j = -1$, $\forall j \in I^c$, then the conormal orientation of $\mathcal{W}_\nu^-$ is given at $S_\nu = S_{\{\nu\}}$ by the exterior monomial

$$\boldsymbol{\omega}^\perp = \theta^\nu \wedge \big(\theta^{1\nu} \wedge \varphi^{1\nu}\big) \wedge \cdots \wedge \big(\theta^{\nu-1,\nu} \wedge \varphi^{\nu-1,\nu}\big).$$

The tangent space $T_{S_{\{\nu\}}} \mathcal{W}_{\{\nu\}}^-$ is oriented by the exterior monomial

$$\boldsymbol{\omega}_\top = (-1)^{\nu-1} \theta^1 \wedge \cdots \wedge \theta^{\nu-1} \wedge \theta^{\nu+1} \wedge \cdots \wedge \theta^n \wedge \bigg(\bigwedge_{j<k,\, k \neq \nu} \theta^{jk} \wedge \varphi^{jk} \bigg), \tag{5.5}$$

because

$$\boldsymbol{\omega}^{\perp} \wedge \boldsymbol{\omega}_{\top} = \boldsymbol{\Omega}_n = \left(\bigwedge_{i=1}^{n} \theta^i \right) \wedge \left(\bigwedge_{1 \leqslant i < j \leqslant n} \theta^{ij} \wedge \varphi^{ij} \right).$$

Proposition 33. *We have an equality of currents*

$$\partial \left[W_I^-, \boldsymbol{or}_I \right] = 0.$$

In other words, using the terminology of Definition 67*, the pair* $(W_I^-, \boldsymbol{or}_I^{\perp})$ *is an* elementary cycle*.*

Proof. The proof relies on the theory of subanalytic currents developed by R. Hardt [20]. For the reader's convenience we have gathered in Appendix B the basic properties of such currents.

Here is our strategy. We will prove that there exists an oriented, smooth, subanalytic submanifold $\mathcal{Y}_I$ of $\mathrm{Lag}_h(\widehat{E})$ with the following properties.

(a) $W_I^- \subset \mathcal{Y}_I \subset \boldsymbol{cl}(W_I^-) = \mathcal{X}_I$.
(b) $\dim(\mathcal{X}_I \setminus \mathcal{Y}_I) < \dim W_I^- - 1$.
(c) The orientation on $\mathcal{Y}_I$ restricts to the orientation $\boldsymbol{or}_I$ on W_I^-.

Assuming the existence of such a $\mathcal{Y}_I$ we observe first that, $\dim \mathcal{Y}_I = \dim W_I^-$, and that we have an equality of currents $[W_I^-, \boldsymbol{or}_I] = [\mathcal{Y}_I, \boldsymbol{or}_I]$. Moreover

$$\operatorname{supp} \partial [\mathcal{Y}_I, \boldsymbol{or}_I] \subset \boldsymbol{cl}(\mathcal{Y}_I) \setminus \mathcal{Y}_I = \mathcal{X}_I \setminus \mathcal{Y}_I,$$

so that,

$$\dim \operatorname{supp} \partial [\mathcal{Y}_I, \boldsymbol{or}_I] < \dim \mathcal{Y}_I - 1.$$

This proves that

$$\partial \left[W_I^-, \boldsymbol{or}_I \right] = \partial [\mathcal{Y}_I, \boldsymbol{or}_I] = 0.$$

To prove the existence of an $\mathcal{Y}_I$ with the above properties we recall that we have a stratification of $\mathcal{X}_I$ (see Corollary 25)

$$\mathcal{X}_I = \bigsqcup_{J \prec I} W_J^-, \tag{5.6}$$

where

$$\dim W_J^- = \dim W_I^- + \boldsymbol{w}(I) - \boldsymbol{w}(J).$$

We distinguish two cases.
A. $1 \in I$. In this case, using Corollary 26 we deduce that all the lower strata in the above stratification have codimension at least 2. Thus, we can choose $\mathcal{Y}_I = W_I^-$, and the properties (a)–(c) above are trivially satisfied.

B. $1 \notin I$. In this case, Corollary 26 implies that the stratification (5.6) had a unique codimension 1-stratum, $W^-_{I_*}$, where $I_* := \{1\} \cup I$. We set

$$\mathcal{Y}_I := W^-_I \cup W^-_{I_*}.$$

We have to prove that this $\mathcal{Y}_I$ has all the desired properties. Clearly (a) and (b) are trivially satisfied. The rest of the properties follow from our next result.

Lemma 34. *The set $\mathcal{Y}_I$ is a smooth, subanalytic, orientable manifold.*

Proof. Consider the Arnold chart $\operatorname{Lag}_h(\widehat{E})_{I^*}$. For any $L \in \operatorname{Lag}_h(\widehat{E})_{I^*}$ we denote by $t_{ij}(L)$ its Arnold coordinates. This means that $t_{ij} = \bar{t}_{ji}$ and that L is spanned by the vectors

$$\boldsymbol{e}_i(L) = \boldsymbol{e}_i + \sum_{i' \in I_*} t_{i'i} \boldsymbol{f}_{i'} - \sum_{j \in I^c_*} t_{ji} \boldsymbol{e}_{ji}, \quad i \in I_*,$$

$$\boldsymbol{f}_j(L) = \boldsymbol{f}_j + \sum_{i \in I_*} t_{ji} \boldsymbol{f}_i - \sum_{j' \in I^c_*} t_{j'j} \boldsymbol{e}_{j'}, \quad j \in I^c_*.$$

The AS cell $W^-_{I^*}$ is described by the equations

$$t_{ji} = 0, \quad \forall i \in I_*,\ j \leqslant i.$$

We will prove that

$$W^-_I \cap \operatorname{Lag}_h(\widehat{E})_{I_*} = \Omega := \{L \in \operatorname{Lag}_h(\widehat{E})_{I^*};\ t_{ji}(L) = 0,\ \forall i \in I,\ j \leqslant i,\ t_{11}(L) \neq 0\}. \tag{5.7}$$

Denote by $A \in \operatorname{End}^+_{\mathbb{C}}(\widehat{E}^+)$ the hermitian operator defined by $A\boldsymbol{e}_i = \alpha_i$, $\forall i \in \mathbb{I}^+_n$, where the real numbers α_i satisfy $0 < \alpha_1 < \cdots < \alpha_n$.

Extend A to $\widehat{A} : \widehat{E} \to \widehat{E}$ by setting $\widehat{A}\boldsymbol{f}_i = -\alpha_i \boldsymbol{f}_i$. Note that $L \in W^-_I$ if and only if

$$\lim_{t \to \infty} e^{-t\widehat{A}} L = \Lambda_I.$$

Clearly, if $L \in \Omega$, then $L_t = e^{-t\widehat{A}} L$ is spanned by the vectors

$$\boldsymbol{e}_1(L) = \boldsymbol{e}_1 + t_{11} e^{2\alpha_1 t} \boldsymbol{f}_1 - \sum_{j \in I^c,\ j \neq 1} t_{j1} e^{t(\alpha_1 - \alpha_j)} \boldsymbol{e}_j,$$

$$\boldsymbol{e}_i(L_t) = \boldsymbol{e}_i - \sum_{j \in I^c_*,\ j > i} t_{ji} e^{t(\alpha_i - \alpha_j)} \boldsymbol{e}_i, \quad i \in I,$$

$$\boldsymbol{f}_j(L_t) = \boldsymbol{f}_j + \sum_{i \in I_*, i < j} t_{ij} e^{t(\alpha_i - \alpha_j)} \boldsymbol{f}_i - \sum_{j' \in I^c_*} t_{j'j} e^{-t(\alpha_j + \alpha_{j'})} \boldsymbol{e}_{j'}, \quad j \in I^*_c.$$

We note that as $t \to \infty$ we have

$$\operatorname{span}\big\{\boldsymbol{e}_1(L_t)\big\} \to \operatorname{span}\{\boldsymbol{f}_1\}, \qquad \operatorname{span}\big\{\boldsymbol{e}_i(L_t)\big\} \to \operatorname{span}\{\boldsymbol{e}_i\}, \quad \forall i \in I,$$
$$\operatorname{span}\big\{\boldsymbol{f}_j(L_t)\big\} \to \operatorname{span}\{\boldsymbol{f}_j\}, \quad \forall j \in I_*^c.$$

This proves $L_t \to \Lambda_I$ so that $\Omega \subset W_I^- \cap \operatorname{Lag}_h(\widehat{E})_{I_*}$.

Conversely, let $L \in W_I^- \cap \operatorname{Lag}_h(\widehat{E})_{I_*}$. Then

$$\lim_{t\to\infty} L_t = \Lambda_I, \quad \text{where } L_t = e^{-t\widehat{A}}L.$$

The space L_t is spanned by the vectors

$$\boldsymbol{e}_1(L_t) = \boldsymbol{e}_1 + t_{11}e^{2\alpha_1 t}\boldsymbol{f}_1 + \sum_{i\in I} t_{i1}e^{t(\alpha_1+\alpha_i)}\boldsymbol{f}_i - \sum_{j\in I_*^c} t_{j1}e^{t(\alpha_1-\alpha_j)}\boldsymbol{e}_j,$$
$$\boldsymbol{e}_i(L_t) = \boldsymbol{e}_i + t_{1i}e^{t(\alpha_i+\alpha_1)}\boldsymbol{f}_i + \sum_{i'\in I} t_{i'i}e^{t(\alpha_i+\alpha_{i'})}\boldsymbol{f}_{i'} - \sum_{j\in I_*^c} t_{ji}e^{t(\alpha_i-\alpha_j)}\boldsymbol{e}_j, \quad i \in I,$$
$$\boldsymbol{f}_j(L_t) = \boldsymbol{f}_j + t_{1j}e^{t(\alpha_1-\alpha_j)}\boldsymbol{f}_1 + \sum_{i\in I} t_{ij}e^{t(\alpha_i-\alpha_j)}\boldsymbol{f}_i - \sum_{j'\in I_*^c} t_{j'j}e^{-t(\alpha_j+\alpha_{j'})}\boldsymbol{e}_{j'}, \quad j \in I_*^c.$$

Observe that

$$\boldsymbol{e}_1(L_t), \boldsymbol{f}_j(L_t) \perp \operatorname{span}\{\boldsymbol{e}_i; i \in I\}, \quad \forall j \in I_*^c,$$

and using the condition $L_t \to \Lambda_I \supset \operatorname{span}\{\boldsymbol{e}_i;\ i \in I\}$ we deduce

$$\operatorname{span}\big\{\boldsymbol{e}_1(L_t), \boldsymbol{f}_j(L_t);\ j \in I_*^c\big\} \to \operatorname{span}\big\{\boldsymbol{f}_j;\ j \in I^c\big\} \subset \Lambda_I.$$

On the other hand, the line spanned by $\boldsymbol{e}_1(L_t)$ converges as $t \to \infty$ to either the line spanned by $\boldsymbol{e}_1$, or to the line spanned by $\boldsymbol{f}_i$, for some $i \in I_*$. Since the line spanned by $\boldsymbol{e}_1$, and the line spanned by $\boldsymbol{f}_i$, $i \in I$, are orthogonal to Λ_I we deduce

$$\operatorname{span}\big\{\boldsymbol{e}_1(L_t)\big\} \to \operatorname{span}\{\boldsymbol{f}_1\},$$

which implies

$$t_{11} \neq 0, \qquad t_{i1} = 0, \quad \forall i \in I.$$

Hence

$$\boldsymbol{e}_i(L_t) = \boldsymbol{e}_i + \sum_{i'\in I} t_{i'i}e^{t(\alpha_i+\alpha_{i'})}\boldsymbol{f}_{i'} - \sum_{j\in I_*^c} t_{ji}e^{t(\alpha_i-\alpha_j)}\boldsymbol{e}_j, \quad \forall i \in I.$$

Now we observe that $\boldsymbol{e}_i(L_t) \perp \boldsymbol{f}_j$, $\forall j \in I^c$, and we conclude that

$$\operatorname{span}\big\{\boldsymbol{e}_i(L_t);\ i \in I\big\} \to \operatorname{span}\{\boldsymbol{e}_i;\ i \in I\}.$$

Since $(\boldsymbol{e}_i(L_t)-\boldsymbol{e}_i) \perp \boldsymbol{e}_{i'}, \forall i, i' \in I, i \neq i'$, we deduce that $\operatorname{span}\{\boldsymbol{e}_i(L_t)\} \to \operatorname{span}\{\boldsymbol{e}_i\}$. This implies that

$$t_{ii'} = 0, \quad t_{ji} = 0, \quad \forall i, i' \in I,\ j \in I^c,\ j < i.$$

This proves that $W_I^- \cap \operatorname{Lag}_h(\widehat{E})_{I_*} \subset \Omega$ and thus, also the equality (5.7). In particular, this implies that $\mathcal{Y}_I = W_I^- \cup W_{I_*}^-$ is smooth, because in the Arnold chart $\operatorname{Lag}_h(\widehat{E})_{I_*}$ which contains the stratum $W_{I_*}^-$ is described by the linear equations

$$t_{ji} = 0, \quad i \in I,\ j \leqslant i. \tag{5.8}$$

To prove that $\mathcal{Y}_I$ is orientable, we will construct an orientation $\boldsymbol{or}_{I_*}$ on $\mathcal{Y}_I \cap \operatorname{Lag}_h(\widehat{E})_{I^*}$ with the property that its restriction to

$$\mathcal{Y}_I \cap \operatorname{Lag}_h(\widehat{E})_I \cap \operatorname{Lag}_h(\widehat{E})_{I_*} \subset W_I^-$$

coincides with the canonical orientation $\boldsymbol{or}_I$ on W_I^-.

We define an orientation $\boldsymbol{or}_{I_*}$ on $\mathcal{Y}_I \cap \operatorname{Lag}_h(\widehat{E})_{I_*}$ by orienting the conormal bundle of this submanifold using the conormal volume form

$$\boldsymbol{\omega}_{I_*} = (-1)^{\boldsymbol{w}(I)} d\boldsymbol{t}_I \wedge \biggl(\bigwedge_{i \in I, j \leqslant i} \biggl(\frac{1}{2\boldsymbol{i}}\, dt_{ji} \wedge dt_{ij}\biggr)\biggr),$$

where $t_{ij}(L)$ are the Arnold coordinates in the chart $\operatorname{Lag}_h(\widehat{E})_{I_*}$. Let

$$L \in \operatorname{Lag}_h(\widehat{E})_I \cap \operatorname{Lag}_h(\widehat{E})_{I_*} \subset W_I^-.$$

Then L is spanned by the vectors

$$\boldsymbol{e}_1(L) = \boldsymbol{e}_1 + t_{11}\boldsymbol{f}_1 + \sum_{i \in I} t_{i1}\boldsymbol{f}_i - \sum_{j \in I_*^c} t_{j1}\boldsymbol{e}_j,$$

$$\boldsymbol{e}_i(L) = \boldsymbol{e}_i + t_{1i}\boldsymbol{f}_1 + \sum_{i' \in I} t_{i'i}\boldsymbol{f}_{i'} - \sum_{j \in I_*^c} t_{ji}\boldsymbol{e}_j, \quad i \in I_*,$$

$$\boldsymbol{f}_j(L) = \boldsymbol{f}_j + t_{1j}\boldsymbol{f}_1 + \sum_{i \in I} t_{ij}\boldsymbol{f}_i - \sum_{j' \in I_*^c} t_{j'j}\boldsymbol{e}_{j'}, \quad j \in I_*^c.$$

The space L belongs to $\operatorname{Lag}_h(\widehat{E})_I$ if and only if

$$L \cap \Lambda_I^{\perp} = L \cap \operatorname{span}\bigl\{\boldsymbol{e}_j, \boldsymbol{f}_i;\ i \in I,\ j \in I^c\bigr\} = 0.$$

This is possible if and only if $t_{11} \neq 0$. We set

$$\boldsymbol{f}_1'(L) := \frac{1}{t_{11}}\boldsymbol{e}_1(L) = \boldsymbol{f}_1 + \frac{1}{t_{11}}\boldsymbol{e}_1 + \sum_{i'\in I} \underbrace{\frac{t_{i'1}}{t_{11}}}_{=:x_{i'1}} \boldsymbol{f}_{i'} - \sum_{j\in I_*^c} \underbrace{\frac{t_{j1}}{t_{11}}}_{=:x_{j1}} \boldsymbol{e}_j,$$

$$\begin{aligned}\boldsymbol{e}_i'(L) &:= \boldsymbol{e}_i(L) - t_{1i}\boldsymbol{f}_1(L)\\ &= \boldsymbol{e}_i + \sum_{i'\in I} \underbrace{\left(t_{i'i} - \frac{t_{1i}t_{i'1}}{t_{11}}\right)}_{=:x_{i'i}} \boldsymbol{f}_{i'} - \underbrace{\frac{t_{1i}}{t_{11}}}_{=:x_{1i}} \boldsymbol{e}_1 - \sum_{j\in I_*^c} \underbrace{\left(t_{ji} - \frac{t_{1i}t_{j1}}{t_{11}}\right)}_{=:x_{ji}} \boldsymbol{e}_j, \quad i \in I,\end{aligned}$$

$$\begin{aligned}\boldsymbol{f}_j'(L) &:= \boldsymbol{f}_j(L) - t_{1j}\boldsymbol{f}_1(L)\\ &= \boldsymbol{f}_j + \sum_{i\in I} \underbrace{\left(t_{ij} - \frac{t_{1j}t_{i1}}{t_{11}}\right)}_{=:x_{ij}} \boldsymbol{f}_i - \underbrace{\frac{t_{1j}}{t_{11}}}_{=:x_{1j}} \boldsymbol{e}_1 - \sum_{j'\in I_*^c} \underbrace{\left(t_{j'j} - \frac{t_{1j}t_{j'1}}{t_{11}}\right)}_{=:x_{j'j}} \boldsymbol{e}_{j'}, \quad j' \in I_*^c.\end{aligned}$$

The space L is thus spanned by the vectors $\boldsymbol{e}_i'(L)$, $i \in I$ and $\boldsymbol{f}_j'(L)$, $j \in I^c$, where we recall that $1 \in I^c$. Also, since $t_{11} = \bar{t}_{11}$ we deduce that

$$x_{pq} = \bar{x}_{qp}, \quad \forall 1 \leqslant p, q \leqslant n.$$

This implies that x_{pq} must be the Arnold coordinates of L in the chart $\mathrm{Lag}_h(\widehat{E})_I$.

In these coordinates the canonical orientation $\boldsymbol{or}_I$ of W_I^- is obtained from the orientation of the conormal bundle given by the form

$$\boldsymbol{\omega}_I = (-1)^{\boldsymbol{w}(I)} d\boldsymbol{x}_I \wedge \left(\bigwedge_{j<i, i\in I} \frac{1}{2\boldsymbol{i}}\, dx_{ji} \wedge dx_{ij} \right),$$

where $d\boldsymbol{x}_I$ denotes the wedge product of the forms dx_{ii}, $i \in I$, in increasing order with respect to i. Observe that

$$\begin{aligned}&x_{ii} = t_{ii} - \frac{t_{1i}t_{i1}}{t_{11}}, \quad x_{i1} = \frac{t_{i1}}{t_{11}}, \quad \forall i \in I,\\ &x_{ij} = t_{ij} - \frac{t_{1j}t_{i1}}{t_{11}}, \quad \forall i \in I,\ j \in I^c \setminus \{1\}.\end{aligned}$$

Along W_I^- we have

$$t_{i1} = t_{ij} = 0, \quad \forall i \in I,\ j \in I^c,\ j < i.$$

We will denote by $O(1)$ any differential form on $\mathrm{Lag}_h(\widehat{E})_I \cap \mathrm{Lag}_h(\widehat{E})_{I_*}$ which is a linear combination of differential forms of the type

$$f(t_{p,q})\, dt_{p_1q_1} \wedge \cdots \wedge dt_{p_mq_m}, \quad f|_{W_I^-} = 0.$$

Then

$$dx_{ii} = dt_{ii} + O(1), \quad \forall i \in I,$$
$$dx_{ij} = dt_{ij} - \frac{t_{1j}}{t_{11}} dt_{i1} + O(1), \quad \forall i \in I,\ j \in I^c,\ j \neq 1,$$
$$dx_{i1} = \frac{1}{t_{11}} dt_{i1} + O(1).$$

We deduce that

$$\boldsymbol{\omega}_I = \frac{1}{t_{11}^{2\#I}} \boldsymbol{\omega}_{I_*} + O(1).$$

The last equality shows that the orientations $\boldsymbol{or}_I$ and $\boldsymbol{or}_{I^*}$ coincide on the overlap $W_I^- \cap \mathrm{Lag}_h(\widehat{E})_{I^*}$. This concludes the proofs of both Lemma 34 and of Proposition 33. $\square$

Remark 35. Arguing as in the first part of Lemma 34 one can prove that for every $k \in \mathbb{I}_n^+$ the smooth locus of $\mathcal{X}_\nu$ contains the strata W_m^-, $m \geqslant k$, and $W_{\{1,k\}}^-$. In particular, the singular locus of $\mathcal{X}_k$ has codimension at least 3 in $\mathcal{X}_k$.

The codimension 3 is optimal. For example, the variety $\mathcal{X}_1 \subset \mathrm{Lag}_h(2)$ is a union of three strata

$$\mathcal{X}_1 = \mathcal{W}_1^- \cup \mathcal{W}_2^- \cup \mathcal{W}_{\{1,2\}}^-.$$

The smooth locus is $\mathcal{W}_1^- \cup \mathcal{W}_2^-$. The stratum $\mathcal{W}_2^-$ is one dimensional and its closure is a smoothly embedded circle. The stratum $\mathcal{W}_{\{1,2\}}^-$ is zero dimensional. It consists of a point in $\mathcal{X}_1$ whose link is homeomorphic to a disjoint union of two S^2-s. One can prove that $\mathcal{X}_1$ is a 3-sphere with two distinct points identified.

We see that any AS cell $\mathcal{W}_I(\boldsymbol{Fl}_\bullet, \boldsymbol{g}_+, \boldsymbol{g}_-)$ defines a subanalytic cycle in $U(\widehat{E}^+)$. For fixed I, any two such cycle are homologous since any one of them is the image of $[\mathcal{W}_I^-, \boldsymbol{or}_I]$ via a real analytic map, real analytically homotopic to the identity. Thus they all determine the same homology class

$$\boldsymbol{\alpha}_I \in H_{n^2-\boldsymbol{w}(I)}\big(U\big(\widehat{E}^+\big), \mathbb{Z}\big),$$

called the *AS cycle of type $I \subset \mathbb{I}_n^+$*. By Poincaré duality we obtain cocycles

$$\boldsymbol{\alpha}_I^\dagger \in H^{\boldsymbol{w}(I)}\big(U\big(\widehat{E}^+\big), \mathbb{Z}\big).$$

We will refer to these as *AS cocycles of type I*. When $I = \{\nu\}$, $\nu \in \mathbb{I}_n^+$ we will use the simpler notations $\boldsymbol{\alpha}_\nu$ and $\boldsymbol{\alpha}_\nu^\dagger$ to denote the AS cycles and cocycles of type $\{\nu\}$. We will refer to these cycles as the *basic AS* (co)cycles.

Example 36. Observe that the AS cycle $\boldsymbol{\alpha}_\emptyset$ is the orientation cycle of $\mathrm{Lag}_h(\widehat{E})$.

The codimension 1 basic cycle $\boldsymbol{\alpha}_1$ is the so called *Maslov cycle*. It defined by the same incidence relation as the Maslov cycle defined in [1] in the case of real lagrangians.

The top codimension basic cycle $\boldsymbol{\alpha}_n$ can be identified with the integration cycle defined by the embedding $U(n-1) \ni T \mapsto T \oplus \mathbb{1} \in U(n)$.

6. A transgression formula

The basic cycles have a remarkable property. To formulate it we need to introduce some fundamental concepts. We denote by $\mathcal{E}$ the rank $n = \dim_{\mathbb{C}} \widehat{E}^+$ complex vector bundle over $S^1 \times U(\widehat{E}^+)$ obtained from the trivial vector bundle

$$\widehat{E}^+ \times \big([-1,1] \times U\big(\widehat{E}^+\big)\big) \to [-1,1] \times U\big(\widehat{E}^+\big),$$

by identifying the point $\boldsymbol{u} \in \widehat{E}^+$ in the fiber over $(-1, \boldsymbol{g}) \in [-1,1] \times U(\widehat{E}^+)$ with the point $\boldsymbol{v} = \boldsymbol{g}\boldsymbol{u} \in \widehat{E}^+$ in the fiber over $(1, \boldsymbol{g}) \in [-1,1] \times U(\widehat{E}^+)$. Equivalently, consider the $\mathbb{Z}$-equivariant bundle

$$\widetilde{\mathcal{E}} = \widehat{E}^+ \times \big(\mathbb{R} \times U\big(\widehat{E}^+\big)\big) \to \mathbb{R} \times U\big(\widehat{E}^+\big),$$

where the $\mathbb{Z}$-action is given by

$$\mathbb{Z} \times \big(\widehat{E}^+ \times \mathbb{R} \times U(\widehat{E})\big) \ni (k; \boldsymbol{u}, \theta, \boldsymbol{g}) \longmapsto \big(\boldsymbol{g}^k \boldsymbol{u}, \theta + 2k, \boldsymbol{g}\big) \in \widehat{E}^+ \times \mathbb{R} \times U\big(\widehat{E}^+\big).$$

Then $\mathcal{E}$ is the quotient vector bundle $\mathbb{Z}\backslash\widetilde{\mathcal{E}} \to \mathbb{Z}\backslash(\mathbb{R} \times U(\widehat{E}^+))$.

The sections of this bundle can be identified with maps $\boldsymbol{u} : \mathbb{R} \times U(\widehat{E}^+) \to \widehat{E}^+$ satisfying the equivariance condition

$$\boldsymbol{u}(\theta + 2, \boldsymbol{g}) = \boldsymbol{g}\boldsymbol{u}(\theta, \boldsymbol{g}), \quad \forall (t, \boldsymbol{g}) \in \mathbb{R} \times U\big(\widehat{E}^+\big).$$

Denote by

$$\boldsymbol{\pi}_! : H^\bullet\big(S^1 \times U\big(\widehat{E}^+\big), \mathbb{Z}\big) \to H^{\bullet-1}\big(U\big(\widehat{E}^+\big), \mathbb{Z}\big)$$

the Gysin map determined by the natural projection

$$\boldsymbol{\pi} : S^1 \times U\big(\widehat{E}^+\big) \to U\big(\widehat{E}^+\big).$$

For every $\nu = \{1, \dots, n\}$ we define $\boldsymbol{\gamma}_\nu \in H^{2\nu-1}(U(\widehat{E}^+), \mathbb{Z})$ by setting

$$\boldsymbol{\gamma}_\nu := \boldsymbol{\pi}_! c_\nu(\mathcal{E}),$$

where $c_\nu(\widehat{E}) \in H^{2\nu}(S^1 \times U(\widehat{E}^+), \mathbb{Z})$ denotes the ν-th Chern class of $\mathcal{E}$.

Theorem 37 *(Transgression Formula). For every $\nu = \{1, \dots, n\}$ we have the equality*

$$\boldsymbol{\alpha}_\nu^\dagger = \boldsymbol{\gamma}_\nu.$$

Proof. Here is briefly the strategy. Fix a unitary basis $\underline{\boldsymbol{e}} = \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ of $\widehat{E}^+$, and consider the AS variety

$$\mathcal{X}_\nu(-1) := \big\{\boldsymbol{g} \in U\big(\widehat{E}^+\big);\ \dim\ker(\mathbb{1} + \boldsymbol{g}) \cap \operatorname{span}_{\mathbb{C}}\{\boldsymbol{e}_\nu, \dots, \boldsymbol{e}_n\} \geqslant 1\big\}.$$

It defines a subanalytic cycle $[\mathcal{X}_\nu(-1), \boldsymbol{or}_\nu]$. We will prove that there exists a subanalytic cycle $\boldsymbol{c}$ in $S^1 \times U(\widehat{E}^+)$ such that the following happen.

- The (integral) homology class determined by $\boldsymbol{c}$ is Poincaré dual to $c_\nu(\mathcal{E})$.
- We have an equality of *subanalytic currents* $\boldsymbol{\pi}_*\boldsymbol{c} = [\mathcal{X}_\nu(-1), \boldsymbol{or}_\nu]$.

To construct this analytic cycle we will use the interpretation of c_ν as the Poincaré dual of a degeneracy cycle [22,29].

We set $V := \operatorname{span}_{\mathbb{C}}\{\boldsymbol{e}_\nu, \dots, \boldsymbol{e}_n\}$, and we denote by $\underline{V}$ the trivial vector bundle with fiber V over $S^1 \times U(\widehat{E})$. Denote by $\mathbb{P}(V)$ the projective space of lines in V, and by $\boldsymbol{p}$ the natural projection

$$\boldsymbol{p} : \mathbb{P}(V) \times S^1 \times U\big(\widehat{E}^+\big) \to S^1 \times U\big(\widehat{E}^+\big).$$

We have a tautological line bundle $\mathcal{L} \to \mathbb{P}(V) \times S^1 \times U(\widehat{E}^+)$ defined as the pullback to $\mathbb{P}(V) \times S^1 \times U(\widehat{E}^+)$ of the tautological line bundle over $\mathbb{P}(V)$.

To any morphism $T : \underline{V} \to \mathcal{E}$ of vector bundles over $S^1 \times U(\widehat{E}^+)$ we can associate in a canonical fashion a bundle morphism $\tilde{T} : \mathcal{L} \to \boldsymbol{p}^*\mathcal{E}$. We regard $\tilde{T}$ as a section of the bundle $\mathcal{L}^* \otimes \boldsymbol{p}^*\mathcal{E}$. If T is a C^2, subanalytic section such that the associated section $\tilde{T}$ vanishes transversally, then its zero locus $\mathcal{Z}(\tilde{T})$ is a C^1 subanalytic manifold equipped with a natural orientation and defines a subanalytic current $[\mathcal{Z}(\tilde{T})]$. Moreover, by Thom–Porteous formula (see [22, VI.1]), the subanalytic current $\boldsymbol{p}_*[\mathcal{Z}(\tilde{T})]$ is Poincaré dual to $c_\nu(\mathcal{E})$. We will produce a C^2, subanalytic bundle morphism T satisfying the above transversality condition, and satisfying the additional equality of currents

$$\boldsymbol{\pi}_*\boldsymbol{p}_*\big[\mathcal{Z}(\tilde{T})\big] = \big[\mathcal{X}_\nu(-1), \boldsymbol{or}_\nu\big].$$

To construct such a morphism T we first choose a polynomial $\eta \in \mathbb{R}[\theta]$ satisfying the following conditions

$$\eta'(\theta) \geqslant 0, \quad \forall\theta \in [-1,1],$$
$$\eta(-1) = 0, \qquad \eta(1) = 1, \qquad \eta(0) = \frac{1}{2},$$
$$\eta'(0) = \frac{1}{4}, \qquad \eta'(\pm 1) = \eta''(\pm 1) = 0.$$

Note that a bundle morphism $T : \underline{V} \to \mathcal{E}$ is uniquely determined by the sections $T\boldsymbol{e}_j$, $\nu \leqslant j \leqslant n$, of $\mathcal{E}$. Now define a vector bundle morphism

$$\mathcal{T} : V \times \big([-1,1] \times U\big(\widehat{E}^+\big)\big) \longrightarrow \widehat{E}^+ \times \big([-1,1] \times U\big(\widehat{E}^+\big)\big),$$

given by,

$$V \times \big([-1,1] \times U\big(\widehat{E}^+\big)\big) \ni (\boldsymbol{v}; \theta, \boldsymbol{g}) \mapsto \big(S(\theta)\boldsymbol{v}; \theta, \boldsymbol{g}\big) \in \widehat{E}^+ \times \big([-1,1] \times U\big(\widehat{E}^+\big)\big),$$

where

$$S(\theta) = S(\theta, \boldsymbol{g}) = \mathbb{1} + \eta(\theta)(\boldsymbol{g} - \mathbb{1}) = \big(1 - \eta(\theta)\big)\mathbb{1} + \eta(\theta)\boldsymbol{g}.$$

Observe that $S(-1)$ is the inclusion of V in $\widehat{E}$, while $S(1) = \boldsymbol{g}$. Thus, for every $\boldsymbol{v} \in V$ the map

$$\Psi_{\boldsymbol{v}} : [-1,1] \times U\big(\widehat{E}^+\big) \to \widehat{E}^+, \qquad \Psi_{\boldsymbol{v}}(\theta, \boldsymbol{g}) = S(\theta)\boldsymbol{v},$$

satisfies $\Psi_{\boldsymbol{v}}(1, \boldsymbol{g}) = \boldsymbol{g}\Psi_{\boldsymbol{v}}(-1, \boldsymbol{g})$ and defines a C^2-semi-algebraic section of $\mathcal{E}$. Hence $\mathcal{T}$ determines a C^2-semi-algebraic bundle morphism $T : \underline{V} \to \mathcal{E}$.

Let $(\ell, \theta, \boldsymbol{g}) \in \mathcal{Z}(\tilde{T}) \subset \mathbb{P}(V) \times S^1 \times U(\widehat{E}^+)$. This means that the restriction of $S(\theta)$ to the line $\ell \subset V$ is trivial, i.e.,

$$\ell \subset \ker\bigl(\bigl(1 - \eta(\theta)\bigr)\mathbb{1} + \eta(\theta)\boldsymbol{g}\bigr).$$

Clearly when $\eta(t) = 0, 1$ this is not possible. Hence $\eta(\theta) \neq 0, 1$ and thus $-\frac{1-\eta(\theta)}{\eta(\theta)}$ must be an eigenvalue of the unitary operator $\boldsymbol{g}$. Since $\eta(\theta) \in (0, 1)$, and the eigenvalues of $\boldsymbol{g}$ are complex numbers of norm 1, we deduce that $-\frac{1-\eta(\theta)}{\eta(\theta)}$ can be an eigenvalue of $\boldsymbol{g}$ if and only if $-\frac{1-\eta(\theta)}{\eta(\theta)} = -1$, so that $\eta(\theta) = \frac{1}{2}$. From the properties of η we conclude that this happens if and only if $\theta = 0$. Thus

$$\mathcal{Z}(\tilde{T}) = \bigl\{(\ell, \theta, \boldsymbol{g}) \in \mathbb{P}(V) \times S^1 \times U\bigl(\widehat{E}^+\bigr);\ \theta = 0,\ \ell \subset \ker(\mathbb{1} + \boldsymbol{g})\bigr\}.$$

Lemma 38. *The section $\tilde{T}$ constructed above vanishes transversally.*

Proof. Let $(\ell_0, 0, \boldsymbol{g}_0) \in \mathcal{Z}(\tilde{T})$. Fix $\boldsymbol{v}_0 \in V$ spanning ℓ_0. Then we can identify an open neighborhood of ℓ_0 in $\mathbb{P}(V)$ with an open neighborhood of 0 in the hyperplane $\ell_0^\perp \cap V$: to any $\boldsymbol{u} \in \ell_0^\perp \cap V$ we associate the line $\ell_{\boldsymbol{u}}$ spanned by $\boldsymbol{v}_0 + \boldsymbol{u}$. We obtain in this fashion a map

$$\bigl(\ell_0^\perp \cap V\bigr) \times (-1, 1) \times U\bigl(\widehat{E}^+\bigr) \ni (\boldsymbol{u}, \theta, \boldsymbol{g}) \stackrel{F}{\longmapsto} \bigl(\mathbb{1} + \eta(\theta)(\boldsymbol{g} - \mathbb{1})\bigr)(\boldsymbol{v}_0 + \boldsymbol{u}) \in \widehat{E}^+, \tag{6.1}$$

and we have to prove that the point $(0, 0, \boldsymbol{g}_0) \in (\ell_0^\perp \cap V) \times (-1, 1) \times U(\widehat{E}^+)$ is a regular point of this map.

Choose a smooth path $(-\varepsilon, \varepsilon) \ni t \mapsto (\boldsymbol{u}_t, \theta_t, \boldsymbol{g}_t) \in (\ell_0^\perp \cap V) \times (-1, 1) \times U(\widehat{E}^+)$ such that

$$\boldsymbol{u}_0 = 0, \qquad \theta_0 = 0, \qquad \boldsymbol{g}_{t=0} = \boldsymbol{g}_0.$$

We set

$$\dot{\boldsymbol{u}} := \frac{d}{dt}\bigg|_{t=0} \boldsymbol{u}_t, \qquad \dot{\theta} := \frac{d}{dt}\bigg|_{t=0} \theta_t, \qquad \dot{\boldsymbol{g}}_0 := \frac{d}{dt}\bigg|_{t=0} \boldsymbol{g}_t,$$

and

$$X := \boldsymbol{g}_0^{-1}\dot{\boldsymbol{g}}_0 = \boldsymbol{g}_0^*\dot{\boldsymbol{g}}_0, \quad \text{i.e.,} \quad \dot{\boldsymbol{g}}_0 = \boldsymbol{g}_0 X.$$

Observe that X is a skew-hermitian operator $\widehat{E}^+ \to \widehat{E}^+$, and we can identify the tangent space to $\ell_0^\perp \cap V \times (-1, 1) \times U(\widehat{E}^+)$ at $(0, 0, S_0)$ with the space of vectors

$$(\dot{u}, \dot{\theta}, X) \in \ell_0^\perp \cap V \times \mathbb{R} \times \underline{u}\bigl(\widehat{E}^+\bigr).$$

Then

$$\frac{d}{dt}\bigg|_{t=0} F(\boldsymbol{u}_t, \theta_t, \boldsymbol{g}_t) = \frac{1}{2}(\mathbb{1} + \boldsymbol{g})\dot{u}_0 + \eta'(0)\dot{\theta}_0(\boldsymbol{g}_0 - \mathbb{1})(\boldsymbol{v}_0) + \eta(0)\dot{\boldsymbol{g}}_0\boldsymbol{v}_0$$

$(-\mathbb{1}\boldsymbol{v}_0 = \boldsymbol{g}_0\boldsymbol{v}_0,\ \eta'(0) = \frac{1}{4})$

$$= \frac{1}{2}(\mathbb{1} + \boldsymbol{g}_0)\dot{u}_0 + \frac{1}{2}\dot{\theta}_0\boldsymbol{g}_0\boldsymbol{v}_0 + \frac{1}{2}\boldsymbol{g}_0 X\boldsymbol{v}_0 = \frac{1}{2}(\mathbb{1} + \boldsymbol{g}_0)\dot{u}_0 + \frac{1}{2}\boldsymbol{g}_0(\dot{\theta}_0\mathbb{1} + X)\boldsymbol{v}_0.$$

The surjectivity of the differential of F at $(0, 0, \boldsymbol{g}_0)$ follows from the fact that the $\mathbb{R}$-linear map

$$\mathbb{R} \times \underline{u}\big(\widehat{E}^+\big) \ni (\dot{\theta}_0, X) \longmapsto (\dot{\theta}_0 + X)\boldsymbol{v}_0 \in \widehat{E}^+$$

is surjective for any nonzero vector $\boldsymbol{v}_0 \in \widehat{E}^+$. □

The above lemma proves that $\mathcal{Z}(\tilde{T})$ is a C^1 submanifold of $\mathbb{P}(V) \times S^1 \times U(\widehat{E}^+)$. It carries a natural orientation which we will describe a bit later. It thus defines a subanalytic current $[\mathcal{Z}(\tilde{T})]$. Observe that

$$\mathcal{Z}(\tilde{T}) \subset \mathbb{P}(V) \times \{\theta = 0\} \times U\big(\widehat{E}^+\big) \subset \mathbb{P}(V) \times S^1 \times U\big(\widehat{E}^+\big).$$

The current $\boldsymbol{p}_*[\mathcal{Z}(\tilde{T})]$ is the integration current defined by $\mathcal{Z}(\tilde{T})$ regarded as submanifold of $\mathbb{P}(V) \times U(\widehat{E}^+)$. Its support has the description

$$\mathcal{Z}(\tilde{T}) = \big\{(\ell, \boldsymbol{g}) \in \mathbb{P}(V) \times U(\widehat{E});\ (\mathbb{1} + \boldsymbol{g})|_\ell = 0\big\}.$$

We set

$$\mathcal{Z}(\tilde{T})^* := \big\{(\ell, \boldsymbol{g}) \in \mathcal{Z}(\tilde{T});\ \ell = \ker(\mathbb{1} + \boldsymbol{g}),\ \boldsymbol{e}_\nu \notin \ell^\perp\big\}.$$

Note that the projection

$$\boldsymbol{\pi} := \mathbb{P}(V) \times U\big(\widehat{E}^+\big) \to U(\widehat{E}), \qquad (\ell, \boldsymbol{g}) \mapsto \boldsymbol{g},$$

maps $\mathcal{Z}(\tilde{T})$ surjectively onto $\mathcal{X}_\nu(-1)$. Moreover, $\mathcal{Z}(\tilde{T})^*$ is the preimage under $\boldsymbol{\pi}$ of the top stratum $\mathcal{W}_\nu^-(-1)$ of $\mathcal{X}_\nu(-1)$,

$$\mathcal{Z}(\tilde{T})^* = \boldsymbol{\pi}^{-1}\big(\mathcal{W}_\nu^-(-1)\big),$$

and the restriction of $\boldsymbol{\pi}$ to $\mathcal{Z}(\tilde{T}^*)$ is a bijection with inverse

$$W_\nu^-(-1) \ni \boldsymbol{g} \mapsto \big(\ker(\mathbb{1} + \boldsymbol{g}), \boldsymbol{g}\big) \in \mathcal{Z}(\tilde{T})^*.$$

Lemma 39. *The map $\boldsymbol{\pi} : \mathcal{Z}(\tilde{T})^* \to \mathcal{W}_\nu^-$ is a diffeomorphism.*

Proof. It suffices to show that the differential of $\boldsymbol{\pi}$ is everywhere injective. Let $\zeta_0 = (\ell_0, \boldsymbol{g}_0) \in \mathcal{Z}(\tilde{T})^*$. Suppose $\ell_0 = \mathrm{span}\{\boldsymbol{v}_0\}$. We have to prove that if

$$(-\varepsilon, \varepsilon) \ni (\ell_t, \boldsymbol{g}_t) \in \mathcal{Z}(\tilde{T})^*$$

is a smooth path $\mathcal{Z}(\tilde{T})^*$ passing through ζ_0 at $t = 0$ and $\dot{\boldsymbol{g}}_0 := \frac{d}{dt}|_{t=0}\boldsymbol{g}_t = 0$, then $\frac{d}{dt}|_{t=0}\ell_t = 0$.

We write $\ell_t = \mathrm{span}\{\boldsymbol{v}_0 + \boldsymbol{u}_t\}$, where $t \mapsto \boldsymbol{u}_t \in \ell_0^\perp \cap V$ is a C^1-path such that $\boldsymbol{u}_0 = 0$. Then

$$\boldsymbol{g}_t(\boldsymbol{v}_0 + \boldsymbol{u}_t) = -\boldsymbol{v}_0 - \boldsymbol{u}_t, \quad \forall t,$$

and differentiating with respect to t at $t = 0$ we get

$$-\dot{\boldsymbol{u}}_0 = \boldsymbol{g}_0 \dot{\boldsymbol{u}}_0 + \dot{\boldsymbol{g}}_0(\boldsymbol{v}_0) = \boldsymbol{g}_0 \dot{\boldsymbol{u}}_0.$$

Hence $\dot{\boldsymbol{u}}_0 \in \ker(\mathbb{1} + \boldsymbol{g}_0)$. We conclude that $\dot{\boldsymbol{u}}_0 = 0$ because $\ker(\mathbb{1} + \boldsymbol{g}_0)$ is the line spanned by $\boldsymbol{v}_0$, and $\dot{\boldsymbol{u}}_0 \perp \boldsymbol{v}_0$. $\square$

Lemma 39 implies that we have an equality of currents

$$\boldsymbol{\pi}_* \boldsymbol{p}_* \big[\mathcal{Z}(\tilde{T})\big] = \pm \big[\mathcal{X}_\nu(-1)\big].$$

To eliminate the sign ambiguity we need to understand the orientation of $\mathcal{Z}(\tilde{T})$.

We begin by describing the conormal orientation of $\mathcal{Z}(\tilde{T})$ at a special point $\xi_0 = (\ell_0, 0, \boldsymbol{g}_0)$, where

$$\ell_0 = \mathrm{span}_{\mathbb{C}}\{\boldsymbol{e}_\nu\}, \quad \text{and} \quad \boldsymbol{g}_0 \boldsymbol{e}_i = \begin{cases} -1, & i = \nu, \\ 1, & i \neq \nu. \end{cases}$$

Observe that $\boldsymbol{g}_0$ is *self-adjoint* and belongs to the top dimensional stratum $\mathcal{W}_\nu^-(-1)$ of $\mathcal{X}_\nu(-1)$. Denote by $\underline{F}$ the differential at ξ_0 of the map F described in (6.1).

The fiber at ξ_0 of the conormal bundle to $\mathcal{Z}(\tilde{T})$ is the image of the *real* adjoint of $\underline{F}$,

$$\underline{F}^\dagger : T_0^* \widehat{E}^+ \to T_{\xi_0}^*\big(\mathbb{P}(V) \times S^1 \times U\big(\widehat{E}^+\big)\big).$$

Since $\underline{F}$ is surjective, its *real* dual $\underline{F}^\dagger$ is injective. The fiber at ξ_0 of the conormal bundle is the image of $\underline{F}^\dagger$, and we have an orientation on this fiber induced via $\underline{F}^\dagger$ by the canonical orientation of $\widehat{E}^+$ as a complex vector space.

The canonical orientation of the real cotangent space $T_0^* \widehat{E}^+$ is described by the top degree exterior monomial

$$\alpha^1 \wedge \beta^1 \wedge \cdots \wedge \alpha^n \wedge \beta^n,$$

where $\alpha^k, \beta^k \in \mathrm{Hom}_{\mathbb{R}}(\widehat{E}^+, \mathbb{R})$ are defined by

$$\alpha^k(\boldsymbol{x}) = \mathbf{Re}(\boldsymbol{x}, \boldsymbol{e}_k), \quad \beta^k(x) = \mathbf{Im}(\boldsymbol{x}, \boldsymbol{e}_k), \quad \forall \boldsymbol{x} \in \widehat{E}^+,\ k = 1, \dots, n.$$

For every $\dot{u}_0 \in V$, $\dot{u}_0 \perp \boldsymbol{e}_\nu$, $\dot{\theta}_0 \in \mathbb{R}$ and $\boldsymbol{i}Z \in \underline{u}(\widehat{E}^+)$ we have

$$\begin{aligned} \underline{F}^\dagger \alpha^k(\dot{\boldsymbol{u}}_0, \dot{\theta}_0, \boldsymbol{i}Z) &= \mathbf{Re}\big(\underline{F}(\dot{u}_0, \dot{\theta}_0, \boldsymbol{i}Z), \boldsymbol{e}_k\big) \\ &= \frac{1}{2}\mathbf{Re}\big((\mathbb{1} + \boldsymbol{g}_0)\dot{u}_0, \boldsymbol{e}_k\big) + \frac{1}{2}\mathbf{Re}\big(\boldsymbol{g}_0(\dot{\theta}_0 + \boldsymbol{i}Z)\boldsymbol{e}_\nu, \boldsymbol{e}_k\big) \\ &= \frac{1}{2}\mathbf{Re}\big(\dot{u}_0, (\mathbb{1} + \boldsymbol{g}_0)\boldsymbol{e}_k\big) + \frac{1}{2}\mathbf{Re}\big((\dot{\theta}_0 + \boldsymbol{i}Z)\boldsymbol{e}_\nu, \boldsymbol{g}_0\boldsymbol{e}_k\big), \end{aligned}$$

$$\underline{F}^{\dagger}\beta^k(\dot{\boldsymbol{u}}_0,\dot{\theta}_0,\boldsymbol{i}Z)=\frac{1}{2}\mathbf{Im}\big(\dot{u}_0,(\mathbb{1}+\boldsymbol{g}_0)\boldsymbol{e}_k\big)+\frac{1}{2}\mathbf{Im}\big((\dot{\theta}_0+\boldsymbol{i}Z)\boldsymbol{e}_\nu,\boldsymbol{g}_0\boldsymbol{e}_k\big).$$

To simplify the final result observe that the restrictions to $\ell_0^{\perp}\cap V$ of the $\mathbb{R}$-linear functions α^k, β^k, $k\geqslant\nu$ determine a basis of $\mathrm{Hom}(\ell_0^{\perp}\cap V,\mathbb{R})$ which we will continue by the same symbols. We denote by dt the tautological linear map $T_0\mathbb{R}\to\mathbb{R}$.

Recall (see Example 32) that the real dual of $\underline{u}(\widehat{E}^+)$ admits a natural basis given by the $\mathbb{R}$-linear forms

$$\theta^j(Z)=(Z\boldsymbol{e}_j,\boldsymbol{e}_j),\quad \theta^{ij}(Z)=\mathbf{Re}(Z\boldsymbol{e}_j,\boldsymbol{e}_i),\quad \varphi^{ij}=\mathbf{Im}(Z\boldsymbol{e}_j,\boldsymbol{e}_i),\quad i<j\in\mathbb{I}_n^+,\ \boldsymbol{i}Z\in\underline{u}\big(\widehat{E}^+\big).$$

Observe that

$$\theta^{ij}=\theta^{ji},\quad \varphi^{ij}=-\varphi^{ji},\quad \forall i\neq j.$$

For every $\dot{u}_0\in\ell_0^{\perp}\cap V$, $\dot{\theta}_0\in T_0\mathbb{R}$, $\boldsymbol{i}Z\in\underline{u}(\widehat{E}^+)$ we have

$$\mathbf{Re}\big(\dot{u}_0,(\mathbb{1}+\boldsymbol{g}_0)\boldsymbol{e}_k\big)=\begin{cases}2\alpha^k(\dot{u}_0), & k>\nu,\\ 0, & k\leqslant\nu,\end{cases}$$
$$\mathbf{Im}\big(\dot{u}_0,(\mathbb{1}+\boldsymbol{g}_0)\boldsymbol{e}_k\big)=\begin{cases}2\beta^k(\dot{u}_0), & k>\nu,\\ 0, & k\leqslant\nu,\end{cases}$$
$$\mathbf{Re}\big((\dot{\theta}_0+\boldsymbol{i}Z)\boldsymbol{e}_\nu,\boldsymbol{g}_0\boldsymbol{e}_k\big)=\delta_{\nu k}dt(\dot{\theta}_0)-\begin{cases}0, & k=\nu,\\ \varphi^{k\nu}(Z), & k\neq\nu,\end{cases}$$
$$\mathbf{Im}\big((\dot{\theta}_0+\boldsymbol{i}Z)\boldsymbol{e}_\nu,\boldsymbol{g}_0\boldsymbol{e}_k\big)=\begin{cases}\theta^{\nu}(Z), & k=\nu,\\ \theta^{k\nu}(Z), & k\neq\nu.\end{cases}$$

We deduce the following.

- If $k<\nu$, then

$$\underline{F}^{\dagger}\alpha^k=-\frac{1}{2}\varphi^{k\nu},\qquad \underline{F}^{\dagger}\beta^k=\frac{1}{2}\theta^{k\nu}.$$

- If $k=\nu$, then

$$\underline{F}^{\dagger}\alpha^{\nu}=\frac{1}{2}dt,\qquad \underline{F}^{\dagger}\beta^{\nu}=\frac{1}{2}d\theta^{\nu}.$$

- If $j>\nu$, then

$$\underline{F}^{\dagger}\alpha^j=\alpha^j+\frac{1}{2}\varphi^{\nu j},\qquad \underline{F}^{\dagger}\beta^k=\beta^j+\frac{1}{2}\theta^{\nu j}.$$

Thus, the conormal space of $\mathcal{Z}(\tilde{T})\hookrightarrow\mathbb{P}(V)\times S^1\times U(\widehat{E})$ at ξ_0 has an orientation given by the oriented basis

$$-\varphi^{1\nu}, \theta^{1\nu}, \dots, -\varphi^{\nu-1,\nu}, \theta^{1,\nu}, dt, d\theta^{\nu},$$
$$\alpha^{\nu+1} + \frac{1}{2}\varphi^{\nu,\nu+1}, \beta^{\nu+1} + \frac{1}{2}\theta^{\nu,\nu+1}, \dots, \alpha^{n} + \frac{1}{2}\varphi^{\nu,n}, \beta^{n} + \frac{1}{2}\theta^{\nu,n}$$

which is equivalent with the orientation given by the oriented basis

$$\theta^{1\nu}, \varphi^{1\nu}, \dots, \theta^{1,\nu}, \varphi^{\nu-1,\nu}, dt, d\theta^{\nu},$$
$$\alpha^{\nu+1} + \frac{1}{2}\varphi^{\nu,\nu+1}, \beta^{\nu+1} + \frac{1}{2}\theta^{\nu,\nu+1}, \dots, \alpha^{n} + \frac{1}{2}\varphi^{\nu,n}, \beta^{n} + \frac{1}{2}\theta^{\nu,n}.$$

We will represent this oriented basis by the exterior polynomial

$$\boldsymbol{\omega}^{\mathrm{norm}} \in \Lambda^{2n} T^*_{\xi_0}\big(\mathbb{P}(V) \times S^1 \times U\big(\widehat{E}^+\big)\big),$$
$$\boldsymbol{\omega}^{\mathrm{norm}} := \Big(\bigwedge_{k<\nu} \theta^{k\nu} \wedge \varphi^{k\nu}\Big) \wedge dt \wedge d\theta^{\nu} \wedge \Big(\bigwedge_{j=\nu+1} \Big(\alpha^j + \frac{1}{2}\varphi^{\nu j}\Big) \wedge \Big(\beta^j + \frac{1}{2}\theta^{\nu j}\Big)\Big).$$

The zero set $\mathcal{Z}(\tilde{T})$ is a smooth manifold of dimension

$$\begin{aligned}\dim_{\mathbb{R}} \mathcal{Z}(\tilde{T}) &= \dim_{\mathbb{R}}\big(\mathbb{P}(V) \times S^1 \times U\big(\widehat{E}^+\big)\big) - \dim_{\mathbb{R}} \widehat{E}^+ \\ &= (2n-2\nu) + 1 + n^2 - 2n = n^2 - (2\nu - 1) = n^2 - \boldsymbol{w}(\nu).\end{aligned}$$

The orientation of $T_{\xi_0}(\mathbb{P}(V) \times S^1 \times U(\widehat{E}^+))$ is described by the exterior monomial

$$\boldsymbol{\Omega} = dt \wedge \Big(\bigwedge_{j=\nu+1}^{n} \alpha^j \wedge \beta^j\Big) \wedge \underbrace{\Big(\bigwedge_{i=1}^{n} \theta^i\Big) \wedge \Big(\bigwedge_{j<i} \theta^{ji} \wedge \varphi^{ji}\Big)}_{\boldsymbol{\Omega}_n}.$$

The orientation of $T_{\xi_0}\mathcal{Z}(\tilde{T})$ is given by any $\boldsymbol{\omega} \in \Lambda^{n^2-\boldsymbol{w}(\nu)} T^*_{\xi_0}(\mathbb{P}(V) \times S^1 \times U(\widehat{E}^+))$ such that

$$\widehat{\boldsymbol{\Omega}} = \boldsymbol{\omega}^{\mathrm{norm}} \wedge \boldsymbol{\omega}.$$

We can take $\boldsymbol{\omega}$ to be

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{\tan} := (-1)^{\nu-1}\theta^1 \wedge \dots \wedge \theta^{\nu-1} \wedge \theta^{\nu+1} \wedge \dots \wedge \theta^n \wedge \Big(\bigwedge_{j<k,\, k\neq\nu} \theta^{jk} \wedge \varphi^{jk}\Big). \tag{6.2}$$

If we now think of $\mathcal{Z}(\tilde{T})$ as an *oriented* submanifold of $\mathbb{P}(V) \times U(\widehat{E}) \subset \mathbb{P}(V) \times S^1 \times U(\widehat{E}^+)$, we see that its conormal bundle has a natural orientation given by the exterior form

$$\boldsymbol{\omega}_0^{\mathrm{norm}} = \Big(\bigwedge_{k<\nu} \theta^{k\nu} \wedge \varphi^{k\nu}\Big) \wedge d\theta^{\nu} \wedge \Big(\bigwedge_{j=\nu+1} \Big(\alpha^j + \frac{1}{2}\varphi^{\nu j}\Big) \wedge \Big(\beta^j + \frac{1}{2}\theta^{\nu j}\Big)\Big).$$

The discussion in Example 32 shows that the tangent space $T_{S_0}\mathcal{W}_\nu^-(-1) \subset T_{S_0}U(\widehat{E}^+)$ is also oriented by $\boldsymbol{\omega}_{\tan}$. This proves that the differential $D\boldsymbol{\pi} : T_{(\ell_0,S_0)}\mathcal{Z}(\tilde{T}) \to T_{S_0}\mathcal{W}_\nu^-(-1)$ is orientation preserving. This concludes the proof of Theorem 37. □

Remark 40. The proof of Theorem 37 shows that we have a resolution $\widetilde{\mathcal{X}}_\nu \xrightarrow{\pi} \mathcal{X}_\nu$ of $\mathcal{X}_\nu$, where

$$\widetilde{\mathcal{X}}_\nu = \bigl\{(\ell, \boldsymbol{g}) \in \mathbb{P}(V_\nu) \times U\bigl(\widehat{E}^+\bigr);\ (\mathbb{1} - \boldsymbol{g})|_\ell = 0\bigr\}, \quad V_\nu := \operatorname{span}_{\mathbb{C}}\{\boldsymbol{e}_\nu, \dots, \boldsymbol{e}_n\},$$

and π is induced by the natural projection $\mathbb{P}(V_\nu) \times U(\widehat{E}) \to U(\widehat{E}^+)$. The map π is a resolution in the sense that it is semi-algebraic, proper, and it is a diffeomorphism over the top dimensional stratum $\mathcal{W}_\nu^-$ of $\mathcal{X}_\nu$.

The map π is also a Bott–Samelson cycle (see [10,39] for a definition) for the Morse function $U(\widehat{E}^+) \ni \boldsymbol{g} \mapsto -\mathbf{Re}\operatorname{tr} A\boldsymbol{g} \in \mathbb{R}$ and its critical point $\boldsymbol{g}_\nu \in U(\widehat{E}^+)$ given by

$$\boldsymbol{g}_\nu \boldsymbol{e}_k = \begin{cases} \boldsymbol{e}_\nu, & k = \nu, \\ -\boldsymbol{e}_k, & k \neq \nu. \end{cases}$$

All the AS-varieties $\mathcal{X}_I$ admit such Bott–Samelson resolutions (see [39]), $\rho_I : \tilde{X}_I \to \mathcal{X}_I$. Using these resolutions Vassiliev defined in [44] the cycles $\boldsymbol{\alpha}_I$. □

Remark 41. We want to describe explicitly an invariant differential form on $U(n)$ representing the cohomology class $\boldsymbol{\alpha}_k^\dagger \in H^{2k-1}(U(n))$ and compare it with the computations in [41, §5]. As in [41], we do this to present a consistent set of sign conventions and normalization factors.

The proof of Theorem 37 shows that the Poincaré dual of the Chern class $\boldsymbol{\pi}_! c_\nu(\mathcal{E})$ is represented by a cycle contained in the slice $\{0\} \times U(\widehat{E}^+)$. If we denote by $\boldsymbol{j}$ the inclusion $\{0\} \times U(\widehat{E}^+) \hookrightarrow S^1 \times U(\widehat{E}^+)$ we deduce

$$c_\nu(\mathcal{E}) = \boldsymbol{j}_! \alpha_\nu^\dagger = \frac{1}{2\pi} d\theta \wedge \boldsymbol{\pi}^* \alpha_\nu^\dagger, \quad \forall \nu. \tag{6.3}$$

Now denote by $\boldsymbol{ch}_k(\mathcal{E})$ the homogeneous part of degree $2k$ of the Chern character of $\mathcal{E}$. From the classical Newton identities we deduce that

$$\boldsymbol{ch}_k(\mathcal{E}) = \frac{(-1)^{k+1}}{(k-1)!} c_k(\mathcal{E}) + \frac{1}{k!} P_k\bigl(c_1(\mathcal{E}), \dots, c_{k-1}(\mathcal{E})\bigr),$$

where $P_k(c_1, \dots, c_{k-1})$ denotes a universal homogeneous polynomial of degree $2k$ with integral coefficients in the variables $c_i(\mathcal{E})$, $1 \leqslant i < k$, $\deg c_i = 2i$. Using the equality (6.3) we deduce

$$\boldsymbol{ch}_k(\mathcal{E}) = \frac{(-1)^{k+1}}{(k-1)!} c_k(\mathcal{E}),$$

so that

$$\boldsymbol{\alpha}_k^\dagger = (-1)^{k+1}(k-1)! \boldsymbol{\pi}_! \boldsymbol{ch}_k(\mathcal{E}). \tag{6.4}$$

Let $n = \dim \widehat{E}^+$ and identify $U(\widehat{E}^+)$ with $U(n)$ using a unitary basis of $\widehat{E}^+$. We can use the last equality to describe a DeRham representative for $\boldsymbol{\alpha}_k^\dagger = \boldsymbol{\alpha}_{k,n}^\dagger \in H^{2k-1}(U(n))$ using the left invariant Maurer–Cartan form $\varpi = \varpi_n = \boldsymbol{g}^{-1}d\boldsymbol{g} = -d(\boldsymbol{g})^{-1}\boldsymbol{g}$ on $U(n)$.

Consider the $\mathbb{Z}$-equivariant connection of the $\mathbb{Z}$-equivariant vector bundle $\widetilde{\mathcal{E}} \to \mathbb{R} \times U(n)$ given by

$$\nabla = \nabla^0 + f(t)\boldsymbol{g}^{-1}d\boldsymbol{g} = \nabla^0 - f(t)\big(d(\boldsymbol{g})^{-1}\big)\boldsymbol{g},$$

where ∇^0 denotes the trivial connection on the trivial rank n bundle over $\mathbb{R} \times U(n)$ and $f : \mathbb{R} \to \mathbb{R}$ is a smooth strictly decreasing function such that $f(1) = 0$, and satisfying the equivariance condition

$$f(t + 2k) = f(t) - k, \quad \forall t \in \mathbb{R},\ k \in \mathbb{Z}.$$

For example, we can take $f(t) = -\frac{t-1}{2}$. The sections of $\mathcal{E}$ can be identified with $\mathbb{Z}$-equivariant sections of $\widetilde{\mathcal{E}}$, i.e., maps $\boldsymbol{u} : \mathbb{R} \times U(n) \to \mathbb{C}^n$ satisfying the condition

$$\boldsymbol{u}(t - 2, \boldsymbol{g}) = \boldsymbol{g}^{-1}u(t, \boldsymbol{g}), \quad \forall (t, \boldsymbol{g}) \in \mathbb{R} \times U(n).$$

Observe that

$$\begin{aligned}
(\nabla\boldsymbol{u})(t-2) &= \nabla^0\boldsymbol{u}(t-2) + f(t-2)\boldsymbol{g}^{-1}(d\boldsymbol{g})\boldsymbol{u}(t-2)\\
&= \nabla^0\big(\boldsymbol{g}^{-1}\boldsymbol{u}(t)\big) + f(t)\boldsymbol{g}^{-1}(d\boldsymbol{g})\boldsymbol{g}^{-1}\boldsymbol{u}(t) + \boldsymbol{g}^{-1}(d\boldsymbol{g})\boldsymbol{g}^{-1}\boldsymbol{u}(t)\\
&= \big(d(\boldsymbol{g})^{-1}\big)\boldsymbol{u}(t) + \boldsymbol{g}^{-1}\nabla^0\boldsymbol{u}(t) - f(t)\big(d(\boldsymbol{g})^{-1}\big)\boldsymbol{u}(t) - \big(d(\boldsymbol{g})^{-1}\big)\boldsymbol{u}(t)\\
&= \boldsymbol{g}^{-1}\big(\nabla^0 - f(t)\boldsymbol{g}d(\boldsymbol{g})^{-1}\big)\boldsymbol{u}(t) = \boldsymbol{g}^{-1}\nabla\boldsymbol{u}(t).
\end{aligned}$$

Thus, ∇ induces a connection on the vector bundle $\mathcal{E}$. Using the Maurer–Cartan identity $d\varpi = -\varpi \wedge \varpi$ we deduce that its curvature is given by

$$F(\nabla) = d(f\varpi) + f^2\varpi \wedge \varpi = df \wedge \varpi + f(f-1)\varpi \wedge \varpi.$$

Thus the cohomology class $\boldsymbol{ch}_k(\mathcal{E})$ is represented by the $2k$-form

$$\frac{\boldsymbol{i}^k}{(2\pi)^k k!}\operatorname{tr}\big(F(\nabla)^k\big) = \frac{\boldsymbol{i}^k}{(2\pi)^k k!}\operatorname{tr}\Big(f^k(f-1)^k\varpi^{\wedge(2k)} + k\big(f(f-1)\big)^{k-1}df \wedge \varpi^{\wedge(2k-1)}\Big).$$

Integrating with respect to $t \in [-1, 1]$ using the (decreasing) change in variables $t \leftrightarrow f$ and the sign conventions in (B.2) we deduce that $\boldsymbol{\pi}_!\boldsymbol{ch}_k(\mathcal{E})$ is represented by the form

$$\frac{(-\boldsymbol{i})^k}{(2\pi)^k(k-1)!}\left(\int_0^1 f^{k-1}(1-f)^{k-1}\,df\right)\operatorname{tr}\varpi^{\wedge(2k-1)}.$$

In the above integral we recognize the Beta function $B(k,k) = \frac{(k-1)!(k-1)!}{(2k-1)!}$. From the equality (6.4) we deduce that $\boldsymbol{\alpha}_k^\dagger \in H^{2k-1}(U(n))$ is represented by the form

$$\boldsymbol{\Theta}_k = \boldsymbol{\Theta}_{k,n} = (-1)^{k+1}\frac{B(k,k)}{(2\pi\boldsymbol{i})^k}\operatorname{tr}\bigl(\varpi^{\wedge(2k-1)}\bigr). \tag{6.5}$$

This shows that the differential form $\frac{(-1)^{k+1}}{(k-1)!}\boldsymbol{\Theta}_k$ coincides with the differential form $\boldsymbol{ch}_{k-1/2}$ in [41, Def. 5.4].

Observe that when $k=1$ and $n=1$, so that $U(1)=S^1$, then $\boldsymbol{\Theta}_{1,1} = \frac{1}{2\pi}d\theta$ is the angular form on S^1 with integral periods. Note that for every $n>0$ we have a determinant map $\det : U(n) \to S^1 = U(1)$ and $\boldsymbol{\Theta}_{1,n} = \det^* \boldsymbol{\Theta}_{1,1}$.

7. The Morse–Floer complex and intersection theory

It is well known that the integral cohomology ring of $U(n)$ is an exterior algebra freely generated by elements $x_i \in H^{2i-1}(U(n),\mathbb{Z})$, $i=1,\dots,n$. The transgression formula implies that as generators x_i of this ring we can take the AS cocycles $\boldsymbol{\alpha}_i^\dagger$. In this section we would like to prove this by direct geometric considerations, and then investigate the cup product of two arbitrary AS cocycles.

Proposition 42. *The AS-cycles $\boldsymbol{\alpha}_I$, $I \subset \mathbb{I}_n^+$, form a $\mathbb{Z}$-basis of $\boldsymbol{H}_\bullet(\mathrm{Lag}_h(\widehat{E}),\mathbb{Z})$.*

Proof. We will use a Morse theoretic approach. Consider again the Morse flow $\Psi_A^t = e^{t\widehat{A}}$ on $\mathrm{Lag}_h(\widehat{E})$.

Lemma 43. *The flow Ψ^t is a Morse–Stokes flow, i.e., the following hold.*

(a) *The flow Ψ^t is a finite volume flow, i.e., the (n^2+1)-dimensional manifold*

$$\bigl\{\bigl(t,\Psi^{1/t}(L),L\bigr);\ t\in(0,1],\ L\in\mathrm{Lag}_h(\widehat{E})\bigr\} \subset (0,1]\times\mathrm{Lag}_h(\widehat{E})\times\mathrm{Lag}_h(\widehat{E}),$$

has finite volume.

(b) *The stable and unstable manifold $W_I^\pm$ have finite volume.*

(c) *If there exists a tunnelling from Λ_I to Λ_J then* $\dim W_J^- < \dim W_I^-$.

Proof. From Theorem 64 in Appendix A we deduce that Ψ^t is a tame flow. Proposition 63 now implies that the flow satisfies (a) and (b). Property (c) follows from Proposition 24. □

As in Harvey and Lawson [21], we consider the subcomplex $\boldsymbol{C}_\bullet(\Psi^t)$ of the complex $\mathcal{C}_\bullet(\mathrm{Lag}_h(\widehat{E}))$ of subanalytic chains generated by the analytic chains $[W_I^-, \boldsymbol{or}_I]$, and their boundaries. According to [21, Thm. 4.1], the inclusion $\boldsymbol{C}_\bullet(\Psi^t) \hookrightarrow \mathcal{C}_\bullet$ induces an isomorphism in homology.

Proposition 33 implies that the complex $\boldsymbol{C}_\bullet(\Psi^t)$ is perfect. Hence the AS cycles, which form an integral basis of the complex $\boldsymbol{C}_\bullet(\Psi^t)$, also form an integral basis of the integral homology of $\mathrm{Lag}_h(\widehat{E})$. □

Remark 44. The complex $\boldsymbol{C}_\bullet(\Psi^t)$ is isomorphic to the Morse–Floer complex of the flow Ψ^t [35, §2.5].

The results of Harvey and Lawson [21] have the following consequence.

Corollary 45. *Consider the invariant forms $\boldsymbol{\Theta}_k \in H^{2k-1}(U(n))$ defined in the previous section. For every $t \in \mathbb{R}$ define*

$$\boldsymbol{\Theta}_k(t) := \left(\Psi^{-t}\right)^* \boldsymbol{\Theta}_k.$$

Then, as $t \to \infty$, the form $\boldsymbol{\Theta}_k(t)$ converges in the sense of currents to the integration current defined by $\boldsymbol{\alpha}_k$.

Using the Poincaré duality on $U(\widehat{E}^+)$ we obtain intersection products

$$\bullet : H_{n^2-p}\big(U\big(\widehat{E}^+\big), \mathbb{Z}\big) \times H_{n^2-q}\big(U\big(\widehat{E}^+\big), \mathbb{Z}\big) \to H_{n^2-p-q}\big(U\big(\widehat{E}^+\big), \mathbb{Z}\big).$$

For every pair of nonempty, disjoint subsets $I, J \subset \mathbb{I}_n^+$ such that

$$I = \{i_1 < \cdots < i_p\}, \qquad J = \{j_1 < \cdots < j_q\},$$

we define $\epsilon(I, J) = \pm 1$ to be the signature of the permutation $(i_1, \ldots, i_p; j_1, \ldots, j_q)$ of $I \cup J$.

Proposition 46. *Let $I, J \subset \mathbb{I}_n^+$ such that $\boldsymbol{w}(I) + \boldsymbol{w}(J) = \boldsymbol{w}(\mathbb{I}_n^+) = n^2$. Then*

$$\alpha_I \bullet \alpha_J = \begin{cases} 0, & I \cap J \neq \emptyset, \\ \epsilon(I, J), & I = J^c. \end{cases}$$

Proof. Fix a unitary basis $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ of $\widehat{E}^+$, and consider the symmetric operator

$$A_0 : \widehat{E}^+ \to \widehat{E}^+, \qquad A_0 \boldsymbol{e}_i = \frac{2i-1}{2} \boldsymbol{e}_i.$$

We form as usual the associated symmetric operator $\widehat{A}_0 : \widehat{E} \to \widehat{E}$, and the positive gradient flow $e^{t\widehat{A}_0}$ on $\mathrm{Lag}_h(\widehat{E})$ associated to the Morse function

$$\varphi_0 : \mathrm{Lag}_h(\widehat{E}) \to \mathbb{R}, \qquad L \mapsto \mathbf{Re}\,\mathrm{tr}(\widehat{A}_0 P_L).$$

For every critical point Λ_K of φ_0 we have

$$\dim W_K^- = \boldsymbol{w}(K) = \varphi_0(\Lambda_K) + \frac{n^2}{2}.$$

For every $M \subset \mathbb{I}_n^+$ we denote by W_M^+ the stable manifold at Λ_M^+.

Let $\boldsymbol{w}(I) + \boldsymbol{w}(J) = \boldsymbol{w}(\mathbb{I}_n^+)$. Using the equality

$$W_{J^c}^+ = \boldsymbol{J} W_J^-$$

we deduce that $W_{J^c}^+$ is also an AS cell of type J, so that we can represent the homology class $\boldsymbol{\alpha}_J$ by the subanalytic cycle given as the integration over the stable manifold $W_{J^c}^+$ equipped with the orientation induced by the diffeomorphism $\boldsymbol{J} : W_J^- \to W_{J^c}^+$. We denote by $\mathfrak{X}_{J^c}^+$ its closure.

We have $\varphi_0(\Lambda_{J^c}) = -\varphi_0(\Lambda_J)$, and the equality $\boldsymbol{w}(I) + \boldsymbol{w}(J) = n^2$ translates into the equality

$$\varphi_0(\Lambda_{J^c}) = \varphi_0(\Lambda_I) =: \kappa.$$

Observe that,

$$\mathcal{X}^+_{J^c} \setminus \{\Lambda_{J^c}\} \subset \{\varphi_0 > \kappa\}, \qquad \mathcal{X}^-_I \setminus \{\Lambda_I\} \subset \{\varphi_0 < \kappa\}.$$

This shows that if $I^c \neq J$ and $\boldsymbol{w}(I^c) = \boldsymbol{w}(J)$ the supports of the subanalytic currents $[\mathcal{X}^+_{J^c}]$ and $[\mathcal{X}^-_I]$ are disjoint, so that, in this case, $\alpha_I \bullet \alpha_J = 0$.

When $J = I^c$ we see that the supports of the above subanalytic cycles intersect only at Λ_I. In fact, only the top dimensional strata of their supports intersect, and they do so transversally. Hence the intersection of the analytic cycles $[\mathcal{X}^+_{J^c}]$ and $[\mathcal{X}^-_I]$ is well defined, and from Proposition 69 in Appendix B we deduce

$$\left[\mathcal{X}^+_{J^c}\right] \bullet \left[\mathcal{X}^-_I\right] = \pm[\Lambda_I],$$

where $[\Lambda_I]$ denotes the Dirac 0-dimensional current supported at Λ_I. The fact that the correct choice of signs is $\epsilon(I, I^c)$ follows from our orientation conventions. □

From the above result we deduce that for every cycle $c \in H_k(U(n), \mathbb{Z})$ we have a decomposition

$$c = \sum_{\boldsymbol{w}(I)=k} \epsilon\big(I, I^c\big)(c \bullet \boldsymbol{\alpha}_{I^c})\boldsymbol{\alpha}_I. \tag{7.1}$$

Theorem 47 *(Odd Schubert calculus). If $I = \{i_k < \cdots < i_1\} \subset \mathbb{I}^+_n$ then*

$$\boldsymbol{\alpha}_I = \boldsymbol{\alpha}_{i_k} \bullet \cdots \bullet \boldsymbol{\alpha}_{i_1}, \tag{7.2}$$

or equivalently,

$$\boldsymbol{\alpha}^\dagger_I = \boldsymbol{\alpha}^\dagger_{i_k} \wedge \cdots \wedge \boldsymbol{\alpha}^\dagger_{i_1}. \tag{7.3}$$

Proof. Let us first describe our strategy. Fix a unitary basis $\underline{\boldsymbol{e}}$ of $\widehat{E}^+$, an injection $\rho : I \to S^1 \setminus \{\pm 1\}$, $i \mapsto \rho_i$, and consider the AS varieties $\mathcal{X}_i(\rho_i) = \mathcal{X}_i(\rho_i, \underline{\boldsymbol{e}})$, $i \in I$, defined in (5.1). We denote by $[\mathcal{X}_i(\rho_i)]$ the associated subanalytic cycles. We will prove the following facts.

A. The varieties $\mathcal{X}_{i_\ell}$ intersect quasi-transversally, i.e., for any subset $J \subset I$ we have

$$\operatorname{codim} \bigcap_{j \in J} \mathcal{X}_j(\rho_j) \geqslant \boldsymbol{w}(J).$$

B. There exists a continuous semi-algebraic map $\Xi : U(\widehat{E}^+) \to U(\widehat{E}^+)$, semi-algebraically homotopic to the identity such that

$$\Xi\left(\bigcap_{i \in I} \mathcal{X}_i(\rho_i)\right) \subset \mathcal{X}_I = \mathcal{X}_I(1).$$

C. The intersection current $[\mathcal{X}_{i_k}(\rho_{i_k})] \bullet \cdots \bullet [\mathcal{X}_{i_1}(\rho_{i_1})] \bullet [\mathcal{X}_{I^c}(1)]$ is a well defined zero dimensional subanalytic current consisting of a single point with multiplicity $\epsilon(I, I^c)$.

We claim that the above facts imply (7.2). To see this, note first that **A** implies that, according to [18] (see also Appendix B), we can form the intersection current

$$\eta = \big[\mathcal{X}_{i_k}(\rho_{i_k})\big] \bullet \cdots \bullet \big[\mathcal{X}_{i_1}(\rho_{i_1})\big].$$

The current η is a subanalytic current whose homology class is $\boldsymbol{\alpha}_{i_k} \bullet \cdots \bullet \boldsymbol{\alpha}_{i_1}$, and its support is

$$\operatorname{supp}(\eta) = \left(\bigcap_{i \in I} \mathcal{X}_i(\rho_i)\right).$$

The push-forward $\Xi_*(\eta)$ is also a subanalytic current and it represents the same homology class since Ξ is homotopic to the identity. Moreover, property **B** shows that

$$\operatorname{supp} \Xi_*(\eta) \subset \mathcal{X}_I(1).$$

Consider again the dual AS varieties $\mathcal{X}_J^+$, $\boldsymbol{w}(J) = \boldsymbol{w}(I)$. In the proof of Proposition 46 we have seen that

$$\mathcal{X}_I \cap \mathcal{X}_J^+ = \emptyset, \quad \text{if } J \neq I.$$

Hence, the equality (7.1) implies that there exists an integer $k = k_I$ such that

$$\boldsymbol{\alpha}_{i_k} \bullet \cdots \bullet \boldsymbol{\alpha}_{i_1} = k_I \boldsymbol{\alpha}_I,$$

where

$$k_I = \epsilon\big(I, I^c\big)(\boldsymbol{\alpha}_{i_k} \bullet \cdots \bullet \boldsymbol{\alpha}_{i_1}) \bullet \boldsymbol{\alpha}_{I^c}.$$

The equality $k_I = 1$ now follows from **C**.

Proof of A. Since the set of unitary operators with simple eigenvalues is open and dense, we deduce that the set

$$\mathcal{Y}_J := \bigcap_{j \in J} \mathcal{X}_j(\rho_j)$$

contains a dense open subset $\mathcal{O}_J$ consisting of operators $\boldsymbol{g}$ such that

$$\dim_{\mathbb{C}} \ker(\rho_j - \boldsymbol{g}) = 1, \quad \forall j \in J.$$

For $\nu \in \mathbb{I}_n^+$ we set $\boldsymbol{F}_\nu := \operatorname{span}\{\boldsymbol{e}_i, i \leqslant \nu\}$.

Observe that if $\boldsymbol{g} \in \mathcal{O}_J$, then for every $j \in J$ we have $\ker(\rho_j - \boldsymbol{g}) \subset \boldsymbol{F}_{j-1}^\perp$. Suppose that $J = \{j_m < \cdots < j_1\}$, and define

$$\Phi : \mathcal{O}_J \to \mathbb{P}\big(\boldsymbol{F}_{j_m}^\perp\big) \times \cdots \times \mathbb{P}\big(\boldsymbol{F}_{j_1 - 1}^\perp\big), \qquad \boldsymbol{g} \mapsto \big(\ker(\rho_{j_m} - \boldsymbol{g}), \ldots, \ker(\rho_{j_1} - \boldsymbol{g})\big).$$

The image of Φ is

$$\Phi(\mathcal{O}_J) = \{(\ell_m, \dots, \ell_1) \in \mathbb{P}(\boldsymbol{F}^{\perp}_{j_m}) \times \cdots \times \mathbb{P}(\boldsymbol{F}^{\perp}_{j_1-1}); \ell_i \perp \ell_{i'}, \forall i \neq i'\}.$$

The resulting map $\mathcal{O}_J \to \Phi(\mathcal{O}_J)$ is a fibration with fiber over $(\ell_m, \dots, \ell_1)$ diffeomorphic to the manifold $\mathcal{F}$ consisting of the unitary operators on the subspace $(\ell_m \oplus \cdots \oplus \ell_1)^{\perp} \subset \widehat{E}^+$ which do not have the numbers ρ_j, $j \in J$, in their spectra. The manifold $\mathcal{F}$ is open in this group of unitary operators. Now observe that

$$\begin{aligned}\dim_{\mathbb{R}} \Phi(\mathcal{O}_J) &= 2(n - j_m) + 2(n - 1 - j_{m-1}) + \cdots + 2(n - (m-1) - j_1) \\ &= 2nm - m(m-1) - 2\sum_{j \in J} j.\end{aligned}$$

The fiber $\mathcal{F}$ has dimension $(n-m)^2$ so that

$$\dim_{\mathbb{R}} \mathcal{O}_J = (n-m)^2 + 2nm - m(m-1) - 2\sum_{j\in J} j = n^2 - \sum_{j\in J}(2j-1).$$

Hence

$$\operatorname{codim} \mathcal{Y}_J = \operatorname{codim} \mathcal{O}_J = \boldsymbol{w}(J).$$

Proof of B. In the proof we will need the following technical result.

Lemma 48. *For any finite subset $R \subset S^1 \setminus \{-1\}$ there exists a C^2-semi-algebraic map $\xi : S^1 \to S^1$ which is semi-algebraically homotopic to the identity and satisfies the condition*

$$\xi^{-1}(1) = R.$$

Proof. For every $\rho \in R$, we define $t(\rho)$ to be the unique real number $t \in (-1, 1)$ such $\rho = e^{\boldsymbol{i}t}$, $t_i \in (-\pi, \pi)$, and we consider a C^2-semi-algebraic map

$$f : [-\pi, \pi] \to [-\pi, \pi]$$

satisfying the following conditions (see Fig. 1, where $k = \#R$)

- $f(\pm\pi) = \pm\pi$.
- $f^{-1}(0) = \{t(\rho); \rho \in R\}$.

Now define $\xi : S^1 \to S^1$ by setting

$$\xi(e^{\boldsymbol{i}t}) = e^{\boldsymbol{i}f(t)}, \quad t \in [-\pi, \pi].$$

We have a C^2 semi-algebraic homotopy between ξ and the identity map given by

$$\xi_s(e^{\boldsymbol{i}t}) = e^{\boldsymbol{i}((1-s)f(t)+st)}, \quad s \in [0,1],\ t \in [-\pi, \pi]. \qquad \square$$

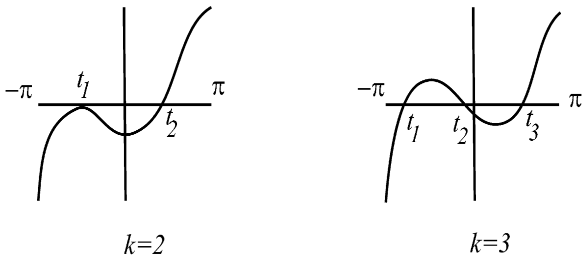

Fig. 1. Constructing degree 1 self maps of the circle.

Using the map ξ in Lemma 48 where $R = \{\rho_i; i \in I\}$ we define

$$\Xi : U(\widehat{E}^+) \to U(\widehat{E}^+), \qquad \boldsymbol{g} \mapsto \Xi(\boldsymbol{g}).$$

The map Ξ is semi-algebraic because its graph $\Gamma_\Xi \subset U(\widehat{E}^+) \times U(\widehat{E}^+)$ can be given the description

$$\Gamma_\Xi = \{(\boldsymbol{g}, \boldsymbol{g}'); \ \exists A \in U(\widehat{E}^+), \ A\boldsymbol{g}A^* = \mathrm{Diag}(\lambda_1, \dots, \lambda_n), \ A\boldsymbol{g}'A^* = \mathrm{Diag}(\xi(\lambda_1), \dots, \xi(\lambda_n))\}.$$

The continuity of Ξ is classical; see e.g. [11, Theorem X.7.2].

If we consider the set $\mathcal{O}_I$ defined in the proof of **A** then we notice that $\Xi(\mathcal{O}_I) \subset \mathcal{W}_I^-(\underline{\boldsymbol{e}}, 1)$ and thus

$$\Xi\left(\bigcap_{i \in I} \mathcal{X}_i(\rho_i)\right) = \Xi(\boldsymbol{cl}(\mathcal{O}_I)) \subset \boldsymbol{cl}(\mathcal{W}_I^-(1)) = \mathcal{X}_I(1).$$

This proves **B**.

Proof of C. For $\nu \in \mathbb{I}_n^+$ and $\rho \neq -1$ we set

$${}^*\mathcal{W}_\nu^-(\rho) := \{S \in \mathcal{W}_\nu^-; \ \dim_{\mathbb{C}} \ker(\rho - S) = 1, \ \ker(1 + S) = 0\}.$$

Note that ${}^*\mathcal{W}_\nu^-(\rho)$ is an open and dense subset of $\mathcal{W}_\nu^-(\rho)$. We first want to produce a natural trivializing frame of the conormal bundle of ${}^*\mathcal{W}_\nu^-(\rho)$. Set $\lambda := -\boldsymbol{i}\frac{1-\rho}{1+\rho}$.

The Cayley transform

$$\boldsymbol{g} \mapsto -\boldsymbol{i}(\mathbb{1} - \boldsymbol{g})(\mathbb{1} + \boldsymbol{g})^{-1}$$

maps ${}^*\mathcal{W}_\nu^-(\rho)$ onto the subset $\mathcal{R}_\nu^*$ of the space of hermitian operators $A : \widehat{E}^+ \to \widehat{E}^+$ such that

- $\dim \ker(\lambda - A) = 1$.
- $\boldsymbol{e}_j \perp \ker(\lambda - A)$, $\forall j < \nu$.
- $(\boldsymbol{e}_\nu, \boldsymbol{u}) \neq 0$, $\forall u \in \ker(\lambda - A)$, $\boldsymbol{u} \neq 0$.

Note that for any $A \in \mathcal{R}_\nu^*$ there exists a unique vector $\boldsymbol{u} = \boldsymbol{u}_A \in \ker(\lambda - A)$ such that $(\boldsymbol{u}, \boldsymbol{e}_\nu) = 1$. For $A \in \mathcal{R}_\nu^*$ we denote by $(\lambda - A)^{[-1]}$ the unique hermitian operator $\widehat{E}^+ \to \widehat{E}^+$ such that

$$(\lambda - A)^{[-1]}\boldsymbol{u}_A = 0, \quad (\lambda - A)^{[-1]}(\lambda - A)\boldsymbol{v} = \boldsymbol{v}, \quad \forall \boldsymbol{v} \perp \boldsymbol{u}_A.$$

If $(-\varepsilon, \varepsilon) \ni t \mapsto A_t \in \mathcal{R}_\nu^*$ is a smooth path, and $\boldsymbol{u}_t := \boldsymbol{u}_{A_t}$, then differentiating the equality $A_t\boldsymbol{u}_t = \lambda\boldsymbol{u}_t$ at $t = 0$ we deduce

$$\dot{A}_0\boldsymbol{u}_0 = (\lambda - A_0)\dot{\boldsymbol{u}}_0.$$

Taking the inner product with $\boldsymbol{u}_0$ we deduce

$$(\dot{A}_0\boldsymbol{u}_0, \boldsymbol{u}_0) = 0.$$

We write $\dot{\boldsymbol{u}}_0 = c\boldsymbol{u}_0 + \dot{\boldsymbol{v}}_0$, where $(\boldsymbol{v}_0, \boldsymbol{u}_0) = (\boldsymbol{v}_0, \boldsymbol{e}_j) = 0, \forall j < \nu$. We deduce

$$\boldsymbol{v}_0 = (\lambda - A_0)^{[-1]}\dot{A}_0\boldsymbol{u}_0,$$

so that

$$\big(\dot{A}_0\boldsymbol{u}_0, (\lambda - A_0)^{[-1]}\boldsymbol{e}_j\big) = (\boldsymbol{v}_0, \boldsymbol{e}_j) = 0, \quad \forall j < \nu.$$

This shows that the fiber at A_0 of the conormal bundle of $\mathcal{R}_\nu^*$ contains the $\mathbb{R}$-linear forms

$$\begin{aligned}
\dot{A}_0 &\mapsto u^\nu(\dot{A}_0) = u^\nu_{A_0}(\dot{A}_0) = (\dot{A}_0\boldsymbol{u}_0, \boldsymbol{u}_0),\\
\dot{A}_0 &\mapsto u^{j\nu}(\dot{A}_0) = u^{j\nu}_{A_0}(\dot{A}_0) = \mathbf{Re}\big(\dot{A}_0\boldsymbol{u}_0, (\lambda - A_0)^{[-1]}\boldsymbol{e}_j\big),\\
\dot{A}_0 &\mapsto v^{j\nu}(\dot{A}_0) = v^{j\nu}_{A_0}(\dot{A}_0) = \mathbf{Im}\big(\dot{A}_0\boldsymbol{u}_0, (\lambda - A_0)^{[-1]}\boldsymbol{e}_j\big).
\end{aligned}$$

Since the vectors $\boldsymbol{e}_j$, $j < \nu$ lie in the orthogonal complement of $\ker(\lambda - A_0)$ we deduce that the vectors $(\lambda - A_0)^{[-1]}\boldsymbol{e}_j$, $j < \nu$ are linearly independent over $\mathbb{C}$. A dimension count now implies that the above linear forms form a basis of the fiber at A_0 of the conormal bundle of $\mathcal{R}_\nu^*$. Since the forms u^ν_A, $u^{j\nu}_A$, $v^{j\nu}_A$ depend smoothly on $A \in \mathcal{R}_\nu^*$, we deduce that they define a smooth frame of the conormal bundle. Moreover, the canonical orientation of $\mathcal{R}_\nu^*$ is given by

$$(-1)^{\boldsymbol{w}(\nu)}u^\nu \wedge \bigwedge_{j<\nu} u^{j\nu} \wedge v^{j\nu}.$$

In particular, we deduce that if $\boldsymbol{g} \in {}^*\mathcal{W}_\nu^-(\rho)$ is a unitary operator such that the vectors $\boldsymbol{e}_i$ are eigenvectors of $\boldsymbol{g}$, then the canonical orientation of the fiber of S of the conormal bundle of ${}^*\mathcal{W}_\nu^-(\rho)$ is given by

$$\theta^\nu \wedge \bigwedge_{j<\nu} \theta^{j\nu} \wedge \varphi^{j\nu},$$

where for any $\dot{\boldsymbol{g}} \in T_{\boldsymbol{g}}U(\widehat{E}^+)$ we have

$$\theta^{\nu}(\dot{\boldsymbol{g}}) = \big(-\boldsymbol{i}\boldsymbol{g}^{-1}\dot{\boldsymbol{g}}\boldsymbol{e}_{\nu}, \boldsymbol{e}_{\nu}\big), \qquad \theta^{j\nu}(\dot{\boldsymbol{g}}) = \mathbf{Re}\big(-\boldsymbol{i}S^{-1}\dot{\boldsymbol{g}}\boldsymbol{e}_{\nu}, \boldsymbol{e}_{j}\big),$$
$$\varphi^{j\nu}(\dot{\boldsymbol{g}}) = \mathbf{Im}\big(-\boldsymbol{i}\boldsymbol{g}^{-1}\dot{\boldsymbol{g}}\boldsymbol{e}_{\nu}, \boldsymbol{e}_{j}\big).$$

We deduce that if $\rho : I \to S^1 \setminus \{\pm 1\}$ is an injective map, then the manifolds ${}^*\mathcal{W}_i^-(\rho_i)$, $i \in I$, intersect transversally.

Now observe that the manifolds ${}^*\mathcal{W}_i^-(\rho_i)$, $i \in I$, and $\mathcal{W}_{I^c}^-(1)$ intersect at a unique point $\boldsymbol{g}_0 \in U(\widehat{E}^+)$, where

$$\boldsymbol{g}_0\boldsymbol{e}_j = \begin{cases} \rho_j\boldsymbol{e}_j, & j \in I, \\ \boldsymbol{e}_j, & j \in I^c. \end{cases}$$

The computations in Example 32 show that this intersection is transversal, and moreover, at S_0 we have

$$\boldsymbol{or}_{i_k}^{\perp} \wedge \cdots \wedge \boldsymbol{or}_{i_1}^{\perp} \wedge \boldsymbol{or}_{I^c}^{\perp} = \epsilon\big(I, I^c\big)\boldsymbol{or}\big(T_{\boldsymbol{g}_0}^* U\big(\widehat{E}^+\big)\big).$$

The equality $k_I = 1$ now follows by invoking Proposition 69 in Appendix B applied to the elementary cycles $[{}^*W_i^-(\rho_i), \boldsymbol{or}_i^{\perp}] = [\mathcal{X}_i, \boldsymbol{or}_i^{\perp}]$, $i \in I$, and $[W_{I^c}^-, \boldsymbol{or}_{I^c}^{\perp}]$ which intersect conveniently. □

We conclude this section with a simple application of the above Schubert calculus motivated by the index theoretic results in [7].

Proposition 49. *For $1 < k \leqslant n$ we denote by $\Sigma_{k,n}$ the locus of unitary $n \times n$ matrices that have an eigenvalue of multiplicity $\geqslant k$. Then $\Sigma_{k,n}$ is a semi-algebraic set, and the integration current defined by $\Sigma_{k,n}$ equipped with a suitable orientation is a cycle Poincaré dual to the cohomology class $n\boldsymbol{\alpha}_2^{\dagger} \cup \cdots \cup \boldsymbol{\alpha}_k^{\dagger} \in H^{\bullet}(U(n))$.*

Proof. The set $\{\boldsymbol{g} \in U(n);\ \dim\ker(\mathbb{1} - \boldsymbol{g}) \geqslant k\}$ is the closure $\mathcal{X}_{1,2,\dots,k}$ of the AS-cell $\mathcal{W}_{1,2,\dots,k}$. This determines the AS cycle $\boldsymbol{\alpha}_{1,\dots,k}$. Consider now the double fibration

$$\begin{tikzcd} & S^1 \times U(n) \arrow[dl, "\boldsymbol{\pi}"'] \arrow[dr, "\boldsymbol{\mu}"] & \\ U(n) & & U(n) \end{tikzcd} \tag{7.4}$$

where $\boldsymbol{\pi}$ is the natural projection, and $\boldsymbol{\mu}$ is the multiplication map

$$S^1 \times U(n) \ni (z, \boldsymbol{g}) \mapsto z^{-1}\boldsymbol{g} \in U(n).$$

Both maps π and $\boldsymbol{\mu}$ define trivial principal S^1-bundles over $U(n)$. Moreover, we have the set theoretic equality

$$\Sigma_{k,n} = \boldsymbol{\pi}\big(\boldsymbol{\mu}^{-1}(\mathcal{X}_{1,\dots,k})\big). \tag{7.5}$$

Set $\mathcal{Z}_k := \boldsymbol{\mu}^{-1}(\mathcal{X}_{1,\dots,k})$, and denote by $\mathcal{Z}_k^*$ the subset of $\mathcal{Z}_k$ consisting of pairs $(z, \boldsymbol{g}) \in S^1 \times U(n)$ such that z is an eigenvalue of $\boldsymbol{g}$ of multiplicity precisely k, while all the other eigenvalues of $\boldsymbol{g}$ are simple. Note that $\mathcal{Z}_k$ is a semi-algebraic variety of dimension

$$\dim \mathcal{Z}_k = 1 + \dim \mathcal{X}_{1,2,\dots,k},$$

while $\mathcal{Z}_k^*$ is an open and dense semi-algebraic subset of $\mathcal{Z}_k$. Hence $\dim \mathcal{Z}_k^* = \dim \mathcal{Z}_k$.

The induced map $\mathcal{Z}_k^* \xrightarrow{\boldsymbol{\pi}} \Sigma_{k,n}$ is semi-algebraic. Since $k > 1$, this map is also injective, and its image $\Sigma_{k,n}^*$ is open and dense in $\Sigma_{k,n}$. The scissor principle (see Appendix A) implies that $\Sigma_{n,k}^*$ is a semi-algebraic set of dimension

$$\dim \Sigma_{k,n} = \dim \Sigma_{k,n}^* = \dim \mathcal{Z}_k^* = 1 + \dim \mathcal{X}_{1,2,\dots,k} = \dim \mathcal{X}_{2,\dots,k}.$$

We can use the diagram (7.4) to define the (closed) current $\boldsymbol{\pi}_* \mu^{-1}[\mathcal{X}_{1,2,\dots,k}]$. The above discussion shows that it can be identified with the current of integration along $\Sigma_{k,n}^*$ equipped with an appropriate orientation. We will denote this current by $[\Sigma_{k,n}]$.

The Poincaré dual of $[\Sigma_{k,n}]$ is given by the cohomology class

$$\boldsymbol{\sigma}_{k,n}^\dagger := \boldsymbol{\pi}_! \boldsymbol{\mu}^* \boldsymbol{\alpha}_{1,2,\dots,k}^\dagger,$$

where $\pi_! : H^\bullet(S^1 \times U(n), \mathbb{Z}) \to H^{\bullet-1}(U(n), \mathbb{Z})$ denotes the Gysin morphism induced by $\boldsymbol{\pi}$.

We can write $\boldsymbol{\mu}$ as the composition

$$\boldsymbol{\mu} = \boldsymbol{\pi} \circ \Phi, \quad \Phi : S^1 \times U(n) \to S^1 \times U(n), \quad (z, \boldsymbol{g}) \mapsto \bigl(z, z^{-1}\boldsymbol{g}\bigr).$$

Hence

$$\boldsymbol{\sigma}_{k,n}^\dagger = \boldsymbol{\pi}_! \Phi^* \boldsymbol{\pi}^* \boldsymbol{\alpha}_{1,2,\dots,k}^\dagger.$$

Set $\Xi_k := \Phi^* \boldsymbol{\pi}^* \boldsymbol{\alpha}_k^\dagger$. From the Schubert calculus we deduce that

$$\boldsymbol{\sigma}_{k,n} = \boldsymbol{\pi}_!(\Xi_1 \wedge \cdots \wedge \Xi_k).$$

Denote by θ the angular coordinate on S^1. Recall that $\boldsymbol{\alpha}_k^\dagger$ is represented by the form $\boldsymbol{\Theta}_k$ defined in (6.5)

$$\boldsymbol{\Theta}_k = r_k \operatorname{tr}\bigl(\varpi^{2k-1}\bigr), \qquad r_k = (-1)^{k+1} \frac{B(k,k)}{(2\pi \boldsymbol{i})^k}, \qquad \varpi = \boldsymbol{g}^{-1} d\boldsymbol{g}.$$

We have the following result whose proof will be presented later.

Lemma 50.

(a) $\Phi^* \boldsymbol{\pi}^* \Theta_1 = -\frac{n}{2\pi} d\theta + \boldsymbol{\pi}^* \Theta_1$.
(b) *For any $j \geqslant 2$ the form $\Phi^* \boldsymbol{\pi}^* \Theta_j - \boldsymbol{\pi}^* \Theta_j$ is exact.*

From the above lemma we deduce that

$$\Xi_1 = -n[d\theta/2\pi] + \boldsymbol{\pi}^*\boldsymbol{\alpha}_1^\dagger, \qquad \Xi_k = \boldsymbol{\pi}^*\boldsymbol{\alpha}_j^\dagger,$$

so that

$$\Phi^*\boldsymbol{\pi}^*\boldsymbol{\alpha}_{1,\dots,k}^\dagger = -n[d\theta/2\pi] \wedge \boldsymbol{\pi}^*\boldsymbol{\alpha}_{2,\dots,k}^\dagger + \pi^*\boldsymbol{\alpha}_{1,\dots,k}.$$

Given that $\boldsymbol{\pi}_![d\theta/2\pi] = 1$ we deduce from the projection formula that

$$\boldsymbol{\sigma}_{k,n}^\dagger = -n\boldsymbol{\alpha}_{2,\dots,k}^\dagger.$$

Proof of Lemma 50. For simplicity we write

$$\boldsymbol{\pi}^* \operatorname{tr}\big(\varpi^{2j-1}\big) = \operatorname{tr}\big(\boldsymbol{g}^{-1} d\boldsymbol{g}\big)^{\wedge(2j-1)} = \operatorname{tr}(\varpi)^{2j-1}.$$

Then

$$\begin{aligned}
\Phi^* \operatorname{tr}\big(\boldsymbol{g}^{-1}\, d\boldsymbol{g}\big)^{\wedge(2j-1)} &= \operatorname{tr}\big(z\boldsymbol{g}^{-1}\, d\big(z^{-1}\boldsymbol{g}\big)^{\wedge(2j-1)}\big) \\
&= \operatorname{tr}\big(\big(-z^{-1}\, dz\, \mathbb{1}_{\mathbb{C}^n} + \boldsymbol{g}^{-1}\, d\boldsymbol{g}\big)^{\wedge(2j-1)}\big) \\
&= \operatorname{tr}\big(\boldsymbol{g}^{-1} d\boldsymbol{g}\big)^{\wedge(2j-1)} \\
&\quad - \operatorname{tr}\left(\sum_{i=0}^{2j-2} \big(\boldsymbol{g}^{-1}\, d\boldsymbol{g}\big)^{\wedge i} \wedge \big(z^{-1}\, dz\big) \wedge \big(\boldsymbol{g}^{-1}\, d\boldsymbol{g}\big)^{\wedge(2j-2-i)}\right) \\
&= \operatorname{tr}\big(\boldsymbol{g}^{-1}\, d\boldsymbol{g}\big)^{\wedge(2j-1)} - \operatorname{tr}\left(\sum_{i=0}^{2j-2} (-1)^i \big(z^{-1}\, dz\big) \wedge \big(\boldsymbol{g}^{-1} d\boldsymbol{g}\big)^{\wedge(2j-2)}\right) \\
&= \operatorname{tr}(\varpi)^{\wedge(2j-1)} - \operatorname{tr}\big(\big(z^{-1}\, dz\big) \wedge \varpi^{\wedge(2j)}\big).
\end{aligned}$$

From the Maurer–Cartan equality $d\varpi = -\varpi \wedge \varpi$ we deduce

$$\Phi^* \operatorname{tr}\big(\varpi^{2j-1}\big) = \operatorname{tr}(\varpi)^{\wedge(2j-1)} + (-1)^j \operatorname{tr}\big(z^{-1}\, dz \wedge (d\varpi)^{\wedge(j-1)}\big).$$

If $j = 1$, then

$$\Phi^* \operatorname{tr}\varpi = \operatorname{tr}\varpi - \operatorname{tr}\big(z^{-1}\, dz\, \mathbb{1}_{\mathbb{C}^n}\big) = \operatorname{tr}\varpi - \boldsymbol{i}\, d\theta,$$

so that

$$r_1\Phi^* \operatorname{tr}\varpi = r_1 \operatorname{tr}\varpi - \frac{n}{2\pi}\, d\theta.$$

If $j > 1$, then we observe that

$$\operatorname{tr}\big(z^{-1}\, dz \wedge (d\varpi)^{\wedge(j-1)}\big) = -d \operatorname{tr}\big(z^{-1}\, dz \wedge \varpi \wedge (d\varpi)^{\wedge(j-2)}\big). \qquad \square$$

Remark 51. The factor n in the equality $\boldsymbol{\sigma}_{k,n} = \pm n\boldsymbol{\alpha}_2^\dagger \cup \cdots \cup \boldsymbol{\alpha}_k^\dagger$ can be an annoyance if the goal is to investigate the case $n = \infty$. Here is one possible strategy of getting rid of it.

Suppose X is compact, oriented real analytic manifold and $g : X \to U(n)$, $x \mapsto g_x$ is smooth tame (e.g. subanalytic) map. We define the spectral flow of the map g to be the cohomology class $SF(g) \in H^1(X, \mathbb{Z})$

$$SF(g) = g^*\boldsymbol{\alpha}_1^\dagger = (\det g)^*\left(\frac{1}{2\pi}\,d\theta\right),$$

where $\frac{1}{2\pi}\,d\theta$ denotes the canonical generator of $H^1(S^1, \mathbb{Z})$. Note that we have a branched cover $\pi : \tilde{X} \to X$, where

$$\tilde{X} := \bigl\{(z, x) \in S^1 \times X;\ \det(z - g_x) = 0\bigr\},$$

and π is induced by the canonical projection $S^1 \times X \to X$. Under certain transversality assumptions one can prove that, if the spectral flow of the family g is trivial, then we can find continuous tame maps

$$u_1, \ldots, u_n : X \to \mathbb{C}^n, \qquad \lambda_1, \ldots, \lambda_n : X \to \mathbb{R}$$

such that for every $x \in X$ the following hold.

- The collection $(u_1(x), \ldots, u_n(x))$ is an orthonormal frame of $\mathbb{C}^n$.
- $\lambda_1(x) \leqslant \cdots \leqslant \lambda_n(x)$.
- $g_x u_k(x) = e^{i\lambda_k(x)} u_k(x)$, $\forall k = 1, \ldots, n$.

For every $1 \leqslant i \leqslant n$ we consider the locus

$$\Sigma_{i,k,n} := \bigl\{x \in X;\ \lambda_i(x) = \cdots = \lambda_{i+k-1}(x)\bigr\},$$

where we for any $i \in \mathbb{Z}$ we set $\lambda_{i+n} = \lambda_i$. The loci $\Sigma_{i,k,n}$ define closed subanalytic currents satisfying

$$n[\Sigma_{i,k,n}] = [\Sigma_{k,n}], \quad \forall 1 \leqslant i \leqslant n.$$

We will carry out the details elsewhere. □

8. Symplectic reductions

The stratifications we have constructed behave well with respect to a standard operation in symplectic geometry, namely the operation of symplectic reduction. We briefly recall a special case.

Suppose $\mathcal{I} \subset \widehat{E}^+$ is an *isotropic* subspace of $\widehat{E}$, so that $\mathcal{I} \subset \boldsymbol{J}\mathcal{I}^\perp$. The quotient space

$$\widehat{E}_{\mathcal{I}} := \bigl(J\mathcal{I}^\perp\bigr)/\mathcal{I}$$

is a hermitian symplectic space called the *symplectic reduction* of $\widehat{E}$ mod $\mathcal{I}$. If we denote by $\mathcal{I}_{\mathrm{hor}}^{\perp}$ the orthogonal complement of $\mathcal{I}$ in $\widehat{E}^{+}$, then we can identify $\widehat{E}_{\mathcal{I}}$ with the subspace $\mathcal{I}_{\mathrm{hor}}^{\perp} \oplus \boldsymbol{J}\mathcal{I}_{\mathrm{hor}}^{\perp} \subset \widehat{E}$. We denote by $\mathcal{P}_{\mathcal{I}}$ the orthogonal projection onto this subspace.

We say that a *lagrangian* $L \in \mathrm{Lag}_h(\widehat{E})$ is *clean mod* $\mathcal{I}$ if $L \cap \mathcal{I} = 0$, and we denote by $\mathrm{Lag}_h(\widehat{E}, \mathcal{I})$ the open and dense subset consisting of such lagrangians. The *symplectic reduction mod* $\mathcal{I}$ is the map

$$\mathcal{R}_{\mathcal{I}} : \mathrm{Lag}_h(\widehat{E}) \to \mathrm{Lag}_h(\widehat{E}_{\mathcal{I}}), \qquad \mathrm{Lag}_h(\widehat{E}) \ni L \longmapsto L_{\mathcal{I}} := \mathcal{P}_{\mathcal{I}}\big(L \cap \big(\widehat{E}^{+} + \boldsymbol{J}\mathcal{I}_{\mathrm{hor}}^{\perp}\big)\big) \in \mathrm{Lag}_h(\widehat{E}_{\mathcal{I}}).$$

The restriction of $\mathcal{R}_I$ to $\mathrm{Lag}_h(\widehat{E}, \mathcal{I})$ is continuous. We want to analyze in greater detail a special case of this construction since it is relevant to our main problem.

For every positive integer N we denote by $\widehat{E}_N$ the hermitian symplectic space

$$\widehat{E}_N := \mathbb{C}^N \oplus \mathbb{C}^N$$

equipped with the natural symplectic structure, and we set $\mathrm{Lag}_h(N) := \mathrm{Lag}_h(\widehat{E}_N)$. We denote by $(\boldsymbol{e}_i)$ the natural unitary basis of $\mathbb{C}^N \oplus 0$ and by $(\boldsymbol{f}_j)$ the natural unitary basis of $0 \oplus \mathbb{C}^N$. We set $\mathcal{I}^N = 0$, and for $n < N$ we denote by $\mathcal{I}^n$ the isotropic subspace of $\widehat{E}_N$ defined by

$$\mathcal{I}^n = \mathrm{span}_{\mathbb{C}}\{\boldsymbol{e}_i;\ i > n\}.$$

We have a natural isomorphism $\widehat{E}_n \cong (\widehat{E}_N)_{\mathcal{I}^n}$.

For $I \subset \{1, 2, \dots, N\}$ we denote by $\mathcal{X}_I^N$ the Arnold–Schubert variety determined by I and the decreasing isotropic flag $(\mathcal{I}^{\bullet})$. In particular, when $I = \{k\}$ we have

$$\mathcal{X}_{k+1}^N = \big\{L \in \mathrm{Lag}_h(N);\ L \cap \mathcal{I}^k \neq 0\big\}.$$

We set $\mathrm{Lag}_h(N; n) := \mathrm{Lag}_h(\widehat{E}_N, \mathcal{I}^n)$. Observe that

$$\mathrm{Lag}_h(N; n) = \mathrm{Lag}_h(N) \setminus \mathcal{X}_{n+1}^N.$$

We obtain an increasing filtration by open subsets

$$\mathrm{Lag}_h(N, 1) \subset \mathrm{Lag}_h(N, 2) \subset \cdots \subset \mathrm{Lag}_h(N, N) = \mathrm{Lag}_h(N). \tag{8.1}$$

For every $I \subset \mathbb{I}_n^{+} = \{1, \dots, n\}$ we set

$$\Lambda_I^N = \mathrm{span}_{\mathbb{C}}\{\boldsymbol{e}_i;\ i \in I\} \oplus \mathrm{span}_{\mathbb{C}}\big\{\boldsymbol{f}_j;\ j \in \mathbb{I}_N^{+} \setminus I\big\} \in \mathrm{Lag}_h(N),$$

and

$$\mathrm{Lag}_h(N)_I = \big\{L \in \mathrm{Lag}_h(N);\ L \cap \boldsymbol{J}_N \Lambda_I^N = 0\big\}.$$

Observe that $\mathcal{I}^n \subset \boldsymbol{J}_N \Lambda_I^N$, $\forall I \subset \mathbb{I}_n^{+}$, so that

$$\bigcup_{I \subset \mathbb{I}_n^{+}} \mathrm{Lag}_h(N)_I \subset \mathrm{Lag}_h(N, n).$$

In fact, we can be more precise.

Proposition 52. *Let $\mathcal{R}_{n,N}$ denote the symplectic reduction map* $\mathrm{Lag}_h(N,n) \to \mathrm{Lag}_h(n)$. *Then for every $I \subset \mathbb{I}_n^+$ we have*

$$\mathrm{Lag}_h(N)_I = \mathcal{R}_{n,N}^{-1}\big(\mathrm{Lag}_h(n)_I\big), \qquad \mathcal{R}_{n,N}\big(\mathrm{Lag}_h(N)_I\big) = \mathrm{Lag}_h(n)_I.$$

In particular,

$$\bigcup_{I \subset \mathbb{I}_n^+} \mathrm{Lag}_h(N)_I = \mathrm{Lag}_h(N;n).$$

Proof. Let $L \in \mathrm{Lag}_h(N,n)$, and denote by L' the symplectic reduction of L mod $\mathcal{I}^n$. We have to prove two things.

A. If $L' \in \mathrm{Lag}_h(n)_I$ for some $I \subset \mathbb{I}_n^+$, then $L \in \mathrm{Lag}_h(N)_I$.
B. If $L \in \mathrm{Lag}_h(N)_I$ for some $I \subset \mathbb{I}_n^+$, then $L' \in \mathrm{Lag}_h(n)_I$.

Proof of A. We know that $L' \cap \boldsymbol{J}_n \Lambda_I^n = 0$ and have to prove that $L \cap \boldsymbol{J}_N \Lambda_I^N = 0$. We argue by contradiction. Suppose that $\exists \boldsymbol{u} \in L \cap \boldsymbol{J}_N \Lambda_I^N$, $\boldsymbol{u} \neq 0$. Observe that

$$\Lambda_I^N = \Lambda_I^n \oplus \mathrm{span}_{\mathbb{C}}\{\boldsymbol{f}_j;\ j > n\} = \Lambda_I^n \oplus \boldsymbol{J}_N \mathcal{I}_n.$$

Note that $u \in L \cap (\boldsymbol{J}\Lambda_I^n \oplus \mathcal{I}_n)$ so we can write

$$\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}, \quad \boldsymbol{v} \in \boldsymbol{J}\Lambda_I^n,\ \boldsymbol{w} \in \mathcal{I}_n.$$

Since $L \cap \mathcal{I}_n = 0$ we deduce $\boldsymbol{v} \neq 0$. Observe that $\widehat{E}_n = \mathcal{I}_n^\perp \oplus \boldsymbol{J}\mathcal{I}_n^\perp = \Lambda_I^n \oplus \boldsymbol{J}\Lambda_I^n$, where $\mathcal{I}_n^\perp$ denotes the orthogonal complement of $\mathcal{I}_n$ in $\mathbb{C}^N \oplus 0$. If $\mathcal{P}_n : \widehat{E}_N \to \widehat{E}_n$ denotes the orthogonal projection, then we have

$$L' = \mathcal{R}_{n,N} L = \mathcal{P}_n\big(L \cap (\widehat{E}_n \oplus \mathcal{I}_n)\big).$$

Note that $\boldsymbol{u} \in L \cap (\widehat{E}_n \oplus \mathcal{I}_n)$. We conclude that $\boldsymbol{v} \in L' \cap \boldsymbol{J}\Lambda_I^n = 0$. This contradiction proves **A.**

Proof of B. Since $L \in \mathrm{Lag}_h(\widehat{E}_N)_I$ we can find a hermitian matrix $(x_{ij})_{1 \leqslant i,j \leqslant N}$ such that L is spanned by the vectors $\boldsymbol{e}_i(L)$ $i \in I$, and $\boldsymbol{f}_j(L)$, $j \in I_N^c = \mathbb{I}_N^+ \setminus I$, where,

$$\boldsymbol{e}_i(L) = \boldsymbol{e}_i + \sum_{i' \in I} x_{i'i} \boldsymbol{f}_{i'} - \sum_{j \in I_N^c} x_{ji} \boldsymbol{e}_j, \quad i \in I, \tag{8.2a}$$

$$\boldsymbol{f}_j(L) = \boldsymbol{f}_j + \sum_{i \in I} x_{ij} \boldsymbol{f}_i - \sum_{j' \in I_N^c} x_{j'j} \boldsymbol{e}_{j'}, \quad j \in I_N^c. \tag{8.2b}$$

From the above we deduce that the collection of vectors

$$\big\{\boldsymbol{e}_i(L), \boldsymbol{f}_k(L);\ i \in I,\ k \in I_n^c := \mathbb{I}_n^+ \setminus I\big\}$$

is a basis of $L \cap (\widehat{E}_n \oplus \mathcal{J}_n)$. Hence $L \cap (\widehat{E}_n \oplus \mathcal{J}_n)$ has complex dimension n, and we deduce that the vectors

$$\boldsymbol{e}_i(L') = \mathcal{P}_n \boldsymbol{e}_i(L) = \boldsymbol{e}_i + \sum_{i' \in I} x_{i'i} \boldsymbol{f}_{i'} - \sum_{k \in I_n^c} x_{ki} \boldsymbol{e}_k, \quad i \in I, \tag{8.3a}$$

$$\boldsymbol{f}_k(L') = \mathcal{P}_n \boldsymbol{f}_k(L) = \boldsymbol{f}_k + \sum_{i \in I} x_{ik} \boldsymbol{f}_i - \sum_{k' \in I_n^c} x_{k'jk} \boldsymbol{e}_{k'}, \quad k \in I_n^c \tag{8.3b}$$

form a basis of L'. This shows that $L' \in \mathrm{Lag}_h(n)_I$, and in fact, that the Arnold coordinates of L' in the chart $\mathrm{Lag}_h(n)_I$ are described by the hermitian matrix $(t_{ij})_{1 \leqslant i,j \leqslant n}$. □

Denote by $\{W^-_{I,N};\, I \subset \mathbb{I}_N^+\}$ the Arnold–Schubert stratification of $\mathrm{Lag}_h(N)$ determined by the isotropic flag $\mathcal{J}^0 \supset \mathcal{J}^1 \supset \cdots$. Observe that if $I \subset \mathbb{I}_n^+$, then

$$\mathrm{Lag}_h(N,n) = \bigcup_{I \subset \mathbb{I}_n^+} W^-_{I,N}. \tag{8.4}$$

From the proof of Proposition 52 we deduce the following consequences.

Corollary 53. *The symplectic reduction map $\mathcal{R}_{n,N} : \mathrm{Lag}_h(N,n) \to \mathrm{Lag}_h(n)$ is a surjective submersion with fibers diffeomorphic to $\mathbb{R}^{N^2-n^2}$. Moreover, for every $I \subset \mathbb{I}_n^+$ we have*

$$W^-_{I,n} = \mathcal{R}_{n,N}\big(W^-_{I,N}\big), \qquad W^-_{I,N} = \mathcal{R}^{-1}_{n,N}\big(W^-_{I,n}\big).$$

Moreover, if $N > n > m$, then

$$\mathcal{R}_{m,N} = \mathcal{R}_{m,n} \circ \mathcal{R}_{n,N}.$$

The fibration $\mathcal{R}_{n,N} : \mathrm{Lag}_h(N,n) \to \mathrm{Lag}_h(n)$ admits a natural section (*extension map*)

$$\mathrm{Lag}_h(n) \ni L' \longmapsto \mathcal{E}_{N,n}\big(L'\big) = L' \oplus \mathrm{span}_{\mathbb{C}}\{\boldsymbol{f}_k;\ k > n\} \in \mathrm{Lag}_h(N,n).$$

Using the extension map $\mathcal{E}_{N,n}$ we can give a Morse theoretic interpretation of the symplectic reduction map.

Consider the hermitian matrix $\widehat{B} = \widehat{B}_{N,n} : \widehat{E}_N \to \widehat{E}_N$ given such that

$$-\widehat{B}\boldsymbol{f}_k = \widehat{B}\boldsymbol{e}_k = \begin{cases} 0, & i \leqslant n, \\ k, & n < k \leqslant N. \end{cases} \tag{8.5}$$

We obtain as before a Morse–Bott function $f_n : \mathrm{Lag}_h(N) \to \mathbb{R}$, $f_n(L) = \mathbf{Re}\,\mathrm{tr}(\widehat{B}P_L)$. The (positive) gradient flow of this function is given by

$$\Phi_n^t(L) = e^{t\widehat{B}}L, \quad \forall L \in \mathrm{Lag}_h(N).$$

Using (8.2a)–(8.2b) we deduce that if $L \in \mathrm{Lag}_h(N)_I$, $I \subset \mathbb{I}_n^+$, and $(x_{ij})_{1 \leqslant i,j \leqslant N}$ are its Arnold coordinates then (8.3a)–(8.3b) and (8.5) we deduce the following result.

Proposition 54.

(a) *The submanifold* $\mathcal{E}_{N,n}(\mathrm{Lag}_h(n)) \subset \mathrm{Lag}_h(N)$ *is the critical submanifold of* f_n *consisting of the absolute minima. The region* $\mathrm{Lag}_h(N,n)$ *is the repelling region (unstable variety) of this critical set.*
(b) *For every* $L \in \mathrm{Lag}_h(N,n)$ *we have*

$$\lim_{t\to\infty} e^{-t\widehat{B}}(L) = \mathcal{E}_{N,n}\big(\mathcal{R}_{n,N}(L)\big),$$

where $\widehat{B}$ *is described by* (8.5).
(c) *The flow* $L \mapsto e^{-t\widehat{B}}L$ *extends to a smooth map*

$$\widehat{\Psi} : [0,\infty] \times \mathrm{Lag}_h(N,n) \to \mathrm{Lag}_h(N,n), \qquad (t,L) \mapsto \lim_{s\nearrow t} e^{-t\widehat{B}}L.$$

Proof. Suppose that $L \in \mathrm{Lag}_h(N)_I$, $I \subset \mathbb{I}_n^+$, and $(x_{ij})_{1\leqslant i\leqslant j\leqslant N}$ are its Arnold coordinates. Using (8.2a)–(8.2b) we deduce that the Arnold coordinates of $L_t = e^{t\widehat{B}}$ are

$$x_{ij}(t) = \begin{cases} x_{ij} & \text{if } i \leqslant j \leqslant n, \\ e^{jt}x_{ij} & \text{if } i \leqslant n < j, \\ e^{(i+j)t}x_{ij} & \text{if } n < i \leqslant j. \end{cases}$$

The last equality proves (a) and (b). The smoothness of Ψ_n away from $t = \infty$ is obvious. Near $t = \infty$ we use the coordinate s defined by $e^{-t} = \frac{1}{s}$, i.e., $t = \frac{1}{\log s}$. Then

$$x_{ij}(-t) = \begin{cases} x_{ij} & \text{if } i \leqslant j \leqslant n, \\ s^j x_{ij} & \text{if } i \leqslant n < j, \\ s^{(i+j)}x_{ij} & \text{if } n < i \leqslant j, \end{cases}$$

which proves the smoothness of $\widehat{\Psi}$ near $t = \infty$. □

Let us observe that the extension map $\mathcal{E}_{N,n} : \mathrm{Lag}_h(n) \to \mathrm{Lag}_h(N)$ corresponds via the Cayley diffeomorphisms $\mathrm{Lag}_h(m) \to U(m)$ to the map

$$\boldsymbol{E}_{N,n} : U(n) \to U(N), \qquad U(n) \ni \boldsymbol{g} \mapsto \boldsymbol{g} \oplus (-\mathbb{1}_{\mathbb{C}^{N-n}}) \in U(N).$$

This map has the property that

$$\boldsymbol{E}_{N,n}^* \boldsymbol{\Theta}_{k,N} = \boldsymbol{\Theta}_{k,n}, \quad \forall k \leqslant n, \tag{8.6}$$

where the closed differential forms $\boldsymbol{\Theta}_{k,m} \in \Omega^{2k-1}(U(m))$ are defined by (6.5). We denote by $\widehat{\boldsymbol{\Theta}}_{k,m} \in \Omega^{2k-1}(\mathrm{Lag}_h(m))$ the pullbacks of $\boldsymbol{\Theta}_{k,m}$ via the Cayley transform. Using (8.6) and Proposition 54 we obtain the following result.

Corollary 55. *For every integers $1 \leqslant k \leqslant n < N$ there exists a canonical form $\widehat{\boldsymbol{\theta}}_{k,n,N} \in \Omega^{2k-2}(\mathrm{Lag}_h(N,n))$ such that*

$$\widehat{\boldsymbol{\Theta}}_{k,N}|_{\mathrm{Lag}_h(N,n)} - \mathcal{R}^*_{N,n}\widehat{\boldsymbol{\Theta}}_{k,n} = d\widehat{\boldsymbol{\theta}}_{k,n,N}.$$

Proof. The form $\widehat{\boldsymbol{\theta}}_{k,n,N}$ is obtained by integrating along the fibers of the natural projection

$$[0,\infty] \times \mathrm{Lag}_h(N,n) \xrightarrow{\pi} \mathrm{Lag}_h(N,n)$$

the form $\widehat{\Psi}^*\widehat{\boldsymbol{\Theta}}_{k,N} \in \Omega^{2k-1}([0,\infty] \times \mathrm{Lag}_h(N,n))$, i.e., $\widehat{\boldsymbol{\theta}}_{k,n,N} = \boldsymbol{\pi}_!\widehat{\Psi}^*\widehat{\boldsymbol{\Theta}}_{k,N}$. □

Remark 56. In the most trivial case, $n = 0$, we have $B = \mathbb{1}$ and the flow $(t, L) \to e^{-t\hat{B}}L$ is precisely the deformation used by Quillen in [41, §5.B].

The above result has a homological counterpart.

Proposition 57. *Suppose X is a compact, oriented real analytic manifold and $F : X \to \mathrm{Lag}_h(N)$ is a real analytic map.*

(a) *There exists a decreasing isotropic flag $(\mathcal{I}^m)_{0 \leqslant m \leqslant N}$ of $\mathbb{C}^N \oplus \mathbb{C}^N$ such that F is transversal to the Arnold–Schubert strata $(W^-_{I,N})_{I \subset \mathbb{I}^+_N}$ defined by $\mathcal{I}^\bullet$. We set*
(b) *Fix a transversal isotropic flag $(\mathcal{I}^\bullet)$ as above. Then $F(X) \subset \mathrm{Lag}_h(N,n)$ if $\dim X < (2n+1)$.*
(c) *Fix n such that $(2n+1) > \dim X$, denote by $\mathcal{R}_{n,N} : \mathrm{Lag}_h(N,n) \to \mathrm{Lag}_h(n)$ the symplectic reduction map, and by F_n the composition*

$$F_n = \mathcal{R}_{n,N} \circ F.$$

Then for every $1 \leqslant k \leqslant n$ we have $F^{-1}(\mathcal{X}^N_k) = F_n^{-1}(\mathcal{X}^n_k)$.
(d) *The current of integration over the smooth locus of $F^{-1}(\mathcal{X}^N_k)$ is a subanalytic cycle Poincaré dual to $F^*(\boldsymbol{\alpha}^\dagger_{k,N})$.*

Proof. Part (a) follows from Remark 30. Observe that the complement $\mathcal{X}^N_{n+1} = \mathrm{Lag}_h(N) \setminus \mathrm{Lag}_h(N,n)$ has codimension $(2n+1)$. Part (b) now follows from the transversality of F. Part (c) follows from Proposition 52.

Via the map F the Whitney stratification of $\mathrm{Lag}_h(N)$ by Arnold–Schubert strata pulls back to a Whitney regular stratification of X by real analytic and globally subanalytic strata. The smooth locus of $F^{-1}(\mathcal{X}^N_k)$ is naturally co-oriented, it has finite volume, and its complement has codimension $\geqslant 2$. Denote by $[F^{-1}(\mathcal{X}^N_k)]$ the current of integration over the smooth locus of $F^{-1}(\mathcal{X}^N_k)$. This current is a subanalytic cycle.

In $\mathrm{Lag}_h(N)$, the subanalytic current defined by $\mathcal{X}^N_k$ is Poincaré dual to $\boldsymbol{\alpha}^\dagger_{k,N}$. The fact that $[F^{-1}(\mathcal{X}^N_k)]$ is the Poincaré dual of $F^*(\boldsymbol{\alpha}^\dagger_{k,N})$ follows from the (proof of the) slicing theorem (Theorem 66) coupled with the transversality and the tameness of F. □

Remark 58. The real analyticity of the map F seems unavoidable if we are to stay in the realm of currents. We can dispense with this assumption, and allow F to be only C^∞ if we define the cocycles $\alpha_I^\dagger$ using Čech cohomology. This approach is explained in great detail in Cibotaru's dissertation [5].

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Appendix A. Tame geometry

Since the subject of tame geometry is not very familiar to many geometers we devote this section to a brief introduction to this topic. Unavoidably, we will have to omit many interesting details and contributions, but we refer to [6,8,9] for more systematic presentations. For every set X we will denote by $\mathcal{P}(X)$ the collection of all subsets of X.

An $\mathbb{R}$-*structure*[4] is a collection $\mathcal{S} = \{\mathcal{S}^n\}_{n\geqslant 1}$, $\mathcal{S}^n \subset \mathcal{P}(\mathbb{R}^n)$, with the following properties.

E$_1$. $\mathcal{S}^n$ contains all the real algebraic subvarieties of $\mathbb{R}^n$, i.e., the zero sets of finite collections of polynomial in n real variables.
E$_2$. For every linear map $L : \mathbb{R}^n \to \mathbb{R}$, the half-plane $\{\vec{x} \in \mathbb{R}^n; L(x) \geqslant 0\}$ belongs to $\mathcal{S}^n$.
P$_1$. For every $n \geqslant 1$, the family $\mathcal{S}^n$ is closed under boolean operations, $\cup$, $\cap$ and complement.
P$_2$. If $A \in \mathcal{S}^m$, and $B \in \mathcal{S}^n$, then $A \times B \in \mathcal{S}^{m+n}$.
P$_3$. If $A \in \mathcal{S}^m$, and $T : \mathbb{R}^m \to \mathbb{R}^n$ is an affine map, then $T(A) \in \mathcal{S}^n$.

Example 59 *(Semi-algebraic sets).* Denote by $\mathcal{S}_{alg}$ the collection of real semi-algebraic sets. Thus, $A \in \mathcal{S}^n_{alg}$ if and only if A is a finite union of sets, each of which is described by finitely many polynomial equalities and inequalities. The celebrated Tarski–Seidenberg theorem states that $\mathcal{S}_{alg}$ is a structure.

Let $\mathcal{S}$ be an $\mathbb{R}$-structure. Then a set that belongs to one of the $\mathcal{S}^n$s is called $\mathcal{S}$-*definable*. If A, B are $\mathcal{S}$-definable, then a function $f : A \to B$ is called $\mathcal{S}$-*definable* if its graph $\Gamma_f := \{(a, b) \in A \times B; b = f(a)\}$ is $\mathcal{S}$-definable.

Given a collection $\mathcal{A} = (\mathcal{A}_n)_{n\geqslant 1}$, $\mathcal{A}_n \subset \mathcal{P}(\mathbb{R}^n)$, we can form a new structure $\mathcal{S}(\mathcal{A})$, which is the smallest structure containing $\mathcal{S}$ and the sets in $\mathcal{A}_n$. We say that $\mathcal{S}(\mathcal{A})$ is obtained from $\mathcal{S}$ by *adjoining the collection* $\mathcal{A}$.

Definition 60. An $\mathbb{R}$-structure is called *o-minimal* (order minimal) or *tame* if it satisfies the property

T. Any set $A \in \mathcal{S}^1$ is a *finite* union of open intervals (a, b), $-\infty \leqslant a < b \leqslant \infty$, and singletons $\{r\}$.

[4] This is a highly condensed and special version of the traditional definition of structure. The model theoretic definition allows for ordered fields, other than $\mathbb{R}$, such as extensions of $\mathbb{R}$ by "infinitesimals". This can come in handy even if one is interested only in the field $\mathbb{R}$.

Example 61.

(a) (Tarski–Seidenberg) The collection $\mathcal{S}_{alg}$ of real semi-algebraic sets is a tame structure.
(b) (A. Gabrielov, R. Hardt, H. Hironaka [16,20,23]) A *restricted* real analytic function is a function $f : \mathbb{R}^n \to \mathbb{R}$ with the property that there exists a real analytic function $\tilde{f}$ defined in an open neighborhood U of the cube $C_n := [-1, 1]^n$ such that

$$f(x) = \begin{cases} \tilde{f}(x), & x \in C_n, \\ 0, & x \in \mathbb{R}^n \setminus C_n, \end{cases}$$

we denote by $\mathcal{S}_{an}$ the structure obtained from $\mathcal{S}_{alg}$ by adjoining the graphs of all the restricted real analytic functions. Then $\mathcal{S}_{an}$ is a tame structure, and the $\mathcal{S}_{an}$-definable sets are called *globally subanalytic sets*.
(c) (A. Khovanskii, P. Speissegger, A. Wilkie [25,43,47]) There exists a tame structure $\widehat{\mathcal{S}}_{an}$ with the following properties
(c_1) $\mathcal{S}_{an} \subset \widehat{\mathcal{S}}_{an}$.
(c_2) If $U \subset \mathbb{R}^n$ is open, connected and $\widehat{\mathcal{S}}_{an}$-definable, $F_1, \dots, F_n : U \times \mathbb{R} \to \mathbb{R}$ are $\widehat{\mathcal{S}}_{an}$-definable and C^1, and $f : U \to \mathbb{R}$ is a C^1 function satisfying

$$\frac{\partial f}{\partial x_i} = F_i\big(x, f(x)\big), \quad \forall x \in \mathbb{R},\ i = 1, \dots, n, \tag{A.1}$$

then f is $\widehat{\mathcal{S}}_{an}$-definable.
(c_3) The structure $\widehat{\mathcal{S}}_{an}$ is the minimal structure satisfying (d_1) and (d_2).
The structure $\widehat{\mathcal{S}}_{an}$ is called the *pfaffian closure*[5] of $\mathcal{S}_{an}$.
Note that $\log x$ and e^x are $\widehat{\mathcal{S}}_{an}$-definable. Moreover if $f : (a, b) \to \mathbb{R}$ is C^1, $\widehat{\mathcal{S}}_{an}$-definable, and $x_0 \in (a, b)$ then the antiderivative $F(x) = \int_{x_0}^{x} f(t)\,dt$, $x \in (a, b)$, is also $\widehat{\mathcal{S}}_{an}$-definable.

The definable sets and function of a tame structure have rather remarkable *tame* behavior which prohibits many pathologies. It is perhaps instructive to give an example of function which is not definable in any tame structure. For example, the function $x \mapsto \sin x$ is not definable in a tame structure because the intersection of its graph with the horizontal axis is the countable set $\pi\mathbb{Z}$ which violates the tameness condition **T**.

We will list below some of the nice properties of the sets and function definable in a tame structure $\mathcal{S}$. Their proofs can be found in [6,8].

- (*Piecewise smoothness of tame functions*) Suppose A is an $\mathcal{S}$-definable set, p is a positive integer, and $f : A \to \mathbb{R}$ is a definable function. Then A can be partitioned into finitely many $\mathcal{S}$ definable sets $S_1, \dots, S_k$, such that each S_i is a C^p-manifold, and each of the restrictions $f|_{S_i}$ is a C^p-function.
- (*Triangulability*) For every compact definable set A, and any finite collection of definable subsets $\{S_1, \dots, S_k\}$, there exists a compact simplicial complex K, and a definable homeomorphism $\Phi : |K| \to A$ such that all the sets $\Phi^{-1}(S_i)$ are unions of relative interiors of faces of K.

[5] Our definition of pfaffian closure is more restrictive than the original one in [25,43], but it suffices for the geometrical applications we have in mind.

- (*Dimension*) The dimension of an $\mathcal{S}$-definable set $A \subset \mathbb{R}^n$ is the supremum over all the non-negative integers d such that there exists a C^1 submanifold of $\mathbb{R}^n$ of dimension d contained in A. Then $\dim A < \infty$, and $\dim(\boldsymbol{cl}(A) \setminus A) < \dim A$.
- (*The scissor principle*) Suppose A and B are two tame sets. Then the following are equivalent
  - The sets A and B have the same Euler characteristic and dimension.
  - There exists a tame bijection $f : A \to B$. (The map f *need not be continuous*.)
- (*Crofton formula* [4], [15, Thm. 2.10.15, 3.2.26]) Suppose E is an Euclidean space, and denote by $\mathbf{Graff}^k(E)$ the Grassmannian of affine subspaces of codimension k in E. Fix an invariant measure μ on $\mathbf{Graff}^k(E)$.[6] Denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure. Then there exists a constant $C > 0$, depending only on μ, such that for every compact, k-dimensional tame subset $S \subset E$ we have

$$\mathcal{H}^k(S) = C \int_{\mathbf{Graff}^k(E)} \chi(L \cap S)\, d\mu(L).$$

- (*Finite volume.*) Any compact k-dimensional tame set has finite k-dimensional Hausdorff measure.

In the remainder of this section, by a tame set (or map) we will understand a $\widehat{\mathcal{S}}_{an}$-definable set (or map).

Definition 62. A tame flow on a tame set X is a topological flow

$$\Phi : \mathbb{R} \times X \to X, \quad (t, x) \mapsto \Phi_t(x),$$

such that the map Φ is tame.

We list below a few properties of tame flows. For proofs we refer to [37].

Proposition 63. *Suppose Φ is a tame flow on a compact tame set X. Then the following hold.*

(a) *The flow Φ is a finite volume flow in the sense of* [21].
(b) *For every $x \in X$ the limits $\lim_{t \to \pm\infty} \Phi_t(x)$ exist and are stationary points of Φ. We denote them by $\Phi_{\pm\infty}(x)$.*
(c) *The maps $x \mapsto \Phi_{\pm\infty}(x)$ are definable.*
(d) *For any stationary point y of Φ, the unstable variety $W_y^- = \Phi_{-\infty}^{-1}(y)$ is a definable subset of X. In particular, if $k = \dim W_y^-$, then W_y^- has finite k-th dimensional Hausdorff measure.*

Theorem 64. *(See Theorem 4.3, [37].) Suppose M is a compact, connected, real analytic, m-dimensional manifold, $f : M \to \mathbb{R}$ is a real analytic Morse function, and g is a real analytic*

[6] The measure μ is unique up to a multiplicative constant.

metric on M such that in the neighborhood of each critical point p there exists real analytic coordinates $(x^i)_{1\leqslant i\leqslant m}$ and nonzero real numbers $(\lambda_i)_{1\leqslant i\leqslant m}$ such that,

$$\nabla^g f=\sum_{i=1}^m \lambda_i \partial_{x^i}, \quad \textit{near } p.$$

Then the flow generated by the gradient $\nabla^g f$ is a tame flow.

Appendix B. Subanalytic currents

In this appendix we gather without proofs a few facts about the subanalytic currents introduced by R. Hardt in [20]. Our terminology concerning currents closely follows that of Federer [15] (see also the more accessible [31]). However, we changed some notations to better resemble notations used in algebraic topology.

Suppose X is a C^2, oriented Riemann manifold of dimension n. We denote by $\Omega_k(X)$ the space of k-dimensional currents in X, i.e., the topological dual space of the space $\Omega^k_{cpt}(X)$ of smooth, compactly supported k-forms on X. We will denote by

$$\langle\bullet,\bullet\rangle:\Omega^k_{cpt}(X)\times\Omega_k(X)\to\mathbb{R}$$

the natural pairing. The boundary of a current $T\in\Omega_k(X)$ is the $(k-1)$-current defined via the Stokes formula

$$\langle\alpha,\partial T\rangle:=\langle d\alpha,T\rangle,\quad \forall\alpha\in\Omega^{k-1}_{cpt}(X).$$

For every $\alpha\in\Omega^k(X)$, $T\in\Omega_m(X)$, $k\leqslant m$ define $\alpha\cap T\in\Omega_{m-k}(X)$ by

$$\langle\beta,\alpha\cap T\rangle=\langle\alpha\wedge\beta,T\rangle,\quad \forall\beta\in\Omega^{n-m+k}_{cpt}(X).$$

We have

$$\begin{aligned}\big\langle\beta,\partial(\alpha\cap T)\big\rangle&=\big\langle d\beta,(\alpha\cap T)\big\rangle=\langle\alpha\wedge d\beta,T\rangle\\&=(-1)^k\big\langle d(\alpha\wedge\beta)-d\alpha\wedge\beta,T\big\rangle=(-1)^k\langle\beta,\alpha\cap\partial T\rangle+(-1)^{k+1}\langle\beta,d\alpha\cap T\rangle\end{aligned}$$

which yields the *homotopy formula*

$$\partial(\alpha\cap T)=(-1)^{\deg\alpha}\big(\alpha\cap\partial T-(d\alpha)\cap T\big). \tag{B.1}$$

The manifold X together with its orientation defines a current $[X]\in\Omega_n(X)$. The induced map

$$\cap[X]:\Omega^k(X)\to\Omega_{n-k}(X)$$

is the Poincaré duality map and the morphism of complexes

$$\big(\Omega^\bullet(X),d\big)\xrightarrow{\cap[X]}\big(\Omega_{n-\bullet}(X),\partial\big)$$

induces an isomorphism in cohomology. We denote this isomorphism by PD_X. The homology of the complex of currents $(\Omega_\bullet(X), \partial)$ is the Borel–Moore homology of X with real coefficients (if X is not too wild at ∞). We denote it by $H_\bullet^{BM}(X)$ so that the Poincaré duality is a map $PD_X : H^\bullet(X) \to H_{n-\bullet}^{BM}(X)$.

If X and Y are compact oriented real analytic manifolds of dimensions n and respectively m, and $f : X \to Y$ is a smooth map, then we have a pull-back map $f^* : H^\bullet(Y) \to H^\bullet(X)$, a push-forward morphism $f_* : H_\bullet^{BM}(X) \to H_\bullet^{BM}(Y)$ and a Gysin map

$$f_! : H^\bullet(X) \to H^{\bullet-(n-m)}(Y)$$

defined by

$$f_! = PD_Y^{-1} f_* PD_X.$$

Let us say a few words about the various sign conventions hidden in the above definition.

If $\pi : S^1 \times M \to M$ is the canonical projection, where M is a compact oriented manifold, and we orient $S^1 \times M$ by using the orientation of S^1 followed by the orientation of M, then for every $\alpha \in H^k(M)$ we have

$$\pi_!\big(d\theta \wedge \pi^*\alpha\big) = 2\pi\alpha. \tag{B.2}$$

In other words, $\pi_!$ coincides with the integration-along-the-fibers map, where we use the *fiber-first orientation convention* to fix an orientation on the total space of the bundle defined by π.

We say that a set $S \subset \mathbb{R}^n$ is *locally subanalytic* if for any $p \in \mathbb{R}^n$ we can find an open ball B centered at p such that $B \cap S$ is globally subanalytic.

Remark 65. There is a rather subtle distinction between globally subanalytic and locally subanalytic sets. For example, the graph of the function $y = \sin(x)$ is a locally subanalytic subset of $\mathbb{R}^2$, but it is not a globally subanalytic set. Note that a compact, locally subanalytic set is globally subanalytic. □

If $S \subset \mathbb{R}^n$ is an orientable, locally subanalytic, C^1 submanifold of $\mathbb{R}^n$ of dimension k, then any orientation $\boldsymbol{or}_S$ on S determines a k-dimensional current $[S, \boldsymbol{or}_S]$ via the equality

$$\big\langle \alpha, [S, \boldsymbol{or}_S] \big\rangle := \int_S \alpha, \quad \forall \alpha \in \Omega_{cpt}^k\big(\mathbb{R}^n\big).$$

The integral in the right-hand side is well defined because any bounded, k-dimensional globally subanalytic set has finite k-dimensional Hausdorff measure. For any open, locally subanalytic subset $U \subset \mathbb{R}^n$ we denote by $[S, \boldsymbol{or}_S] \cap U$ the current $[S \cap U, \boldsymbol{or}_S]$.

For any locally subanalytic subset $X \subset \mathbb{R}^n$ we denote by $\mathcal{C}_k(X)$ the Abelian subgroup of $\Omega_k(\mathbb{R}^n)$ generated by currents of the form $[S, \boldsymbol{or}_S]$, as above, where $\boldsymbol{cl}(S) \subset X$. The above operation $[S, \boldsymbol{or}_S] \cap U$, U open subanalytic extends to a morphism of Abelian groups

$$\mathcal{C}_k(X) \ni T \mapsto T \cap U \in \mathcal{C}_k(X \cap U).$$

We will refer to the elements of $\mathcal{C}_k(X)$ as *subanalytic (integral) k-chains* in X.

Given compact subanalytic sets $A \subset X \subset \mathbb{R}^n$ we set

$$\mathcal{Z}_k(X, A) = \{T \in \mathcal{C}_k(\mathbb{R}^n);\ \operatorname{supp} T \subset X,\ \operatorname{supp} \partial T \subset A\},$$

and

$$\mathcal{B}_k(X, A) = \{\partial T + S;\ T \in \mathcal{Z}_{k+1}(X, A),\ S \in \mathcal{Z}_k(A)\}.$$

We set

$$\mathcal{H}_k(X, A) := \mathcal{Z}_k(X, A)/\mathcal{B}_k(X, A).$$

R. Hardt has proved in [19,20] that the assignment

$$(X, A) \longmapsto \mathcal{H}_\bullet(X, A)$$

satisfies the Eilenberg–Steenrod homology axioms with $\mathbb{Z}$-coefficients. This implies that $\mathcal{H}_\bullet(X, A)$ is naturally isomorphic with the integral homology of the pair. In fact, we can be much more precise.

If X is a compact subanalytic we can form the chain complex

$$\cdots \xrightarrow{\partial} \mathcal{C}_k(X) \xrightarrow{\partial} \mathcal{C}_{k-1}(X) \xrightarrow{\partial} \cdots$$

whose homology is $\mathcal{H}_\bullet(X)$.

If we choose a subanalytic triangulation $\Phi : |K| \to X$, and we linearly orient the vertex set $V = V(K)$, then for any k-simplex $\sigma \subset K$ we get a subanalytic map from the standard affine k-simplex Δ_k to X

$$\Phi^\sigma : \Delta_k \to X.$$

This defines a current $[\sigma] = \Phi^\sigma_*([\Delta_k]) \in \mathcal{C}_k(X)$. By linearity we obtain a morphism from the group of simplicial chains $C_\bullet(K)$ to $\mathcal{C}_\bullet(X)$ which commutes with the respective boundary operators. In other words, we obtain a morphism of chain complexes

$$C_\bullet(K) \to \mathcal{C}_\bullet(\Phi|K|).$$

The arguments in [13, Chap. III] imply that this induces an isomorphism in homology.

To describe the intersection theory of subanalytic chains we need to recall a fundamental result of R. Hardt, [18, Theorem 4.3]. Suppose E_0, E_1 are two oriented real Euclidean spaces of dimensions n_0 and respectively n_1, $f : E_0 \to E_1$ is a real analytic map, and $T \in \mathcal{C}_{n_0-c}(E_0)$ a subanalytic current of codimension c. If y is a regular value of f, then the fiber $f^{-1}(y)$ is a submanifold equipped with a natural co-orientation and thus defines a subanalytic current $[f^{-1}(y)]$ in E_0 of codimension n_1, i.e., $[f^{-1}(y)] \in \mathcal{C}_{d_0-d_1}(E_0)$. We would like to define the intersection of T and $[f^{-1}(y)]$ as a subanalytic current $T \bullet [f^{-1}(y)] \in \mathcal{C}_{n_0-c-n_1}(E_0)$. It turns out that this is possibly quite often, even in cases when y is not a regular value.

Theorem 66 *(Slicing Theorem). Let E_0, E_1, T and f be as above, denote by dV_{E_1} the Euclidean volume form on E_1, by $\boldsymbol{\omega}_{n_1}$ the volume of the unit ball in E_1, and set*

$$\mathcal{R}_f(T) := \big\{ y \in E_1;\ \operatorname{codim}(\operatorname{supp} T) \cap f^{-1}(y) \geqslant c + n_1,$$
$$\operatorname{codim}(\operatorname{supp} \partial T) \cap f^{-1}(y) \geqslant c + n_1 + 1 \big\}.$$

For every $\varepsilon > 0$ and $y \in E_1$ we define $T \bullet_\varepsilon f^{-1}(y) \in \Omega_{n_0-c-n_1}(E_0)$ by

$$\big\langle \alpha, T \bullet_\varepsilon f^{-1}(y) \big\rangle := \frac{1}{\boldsymbol{\omega}_{n_1} \varepsilon^{n_1}} \big\langle \big(f^* dV_{E_1} \big) \wedge \alpha, T \cap \big(f^{-1}\big(B_\varepsilon(y)\big)\big)\big\rangle, \quad \forall \alpha \in \Omega_{cpt}^{n_0-c-n_1}(E_0).$$

*Then for every $y \in \mathcal{R}_f(T)$, the currents $T \bullet_\varepsilon f^{-1}(y)$ converge weakly as $\varepsilon > 0$ to a subanalytic current $T \bullet f^{-1}(y) \in \mathcal{C}_{n_0-c-n_1}(E_0)$ called the f-*slice *of T over y, i.e.,*

$$\big\langle \alpha, T \bullet f^{-1}(y) \big\rangle = \lim_{\varepsilon \searrow 0} \frac{1}{\boldsymbol{\omega}_{n_1} \varepsilon^{n_1}} \big\langle \big(f^* dV_{E_1} \big) \wedge \alpha, T \cap \big(f^{-1}\big(B_\varepsilon(y)\big)\big)\big\rangle, \quad \forall \alpha \in \Omega_{cpt}^{n_0-c-n_1}(E_0).$$

Moreover, the map

$$\mathcal{R}_f \ni y \mapsto T \bullet f^{-1}(y) \in \mathcal{C}_{d_0-c-d_1}\big(\mathbb{R}^n\big)$$

is continuous in the locally flat topology.

We will refer to the points $y \in \mathcal{R}_f(T)$ as the *quasi-regular values of f relative to T*.

Consider an oriented real analytic manifold M of dimension m, and $T_i \in \mathcal{C}_{m-c_i}(M)$, $i = 0, 1$. We would like to define an intersection current $T_0 \bullet T_1 \in \mathcal{C}_{m-c_0-c_1}(M)$. This will require some very mild transversality conditions.

The slicing theorem describes this intersection current when T_1 is the integration current defined by the fiber of a real analytic map. We want to reduce the general situation to this case. We will achieve this in two steps.

- Reduction to the diagonal.
- Localization.

To understand the reduction to the diagonal let us observe that if T_0, T_1 were homology classes, then their intersection $T_0 \bullet T_1$ satisfies the identity

$$j_*(T_0 \bullet T_1) = (-1)^{c_0(m-c_1)} (T_0 \times T_1) \bullet \Delta_M,$$

where Δ_M denotes the diagonal class in $M \times M$, and $j : M \to M \times M$ denotes the diagonal embedding; see [34, Sec. 7.3.2].

We use this fact to define the intersection current in the special case when M is an open subset of $\mathbb{R}^m$. In this case the diagonal Δ_M is the fiber over 0 of the difference map

$$\delta : M \times M \to \mathbb{R}^m, \quad \delta(m_0, m_1) = m_0 - m_1.$$

If the currents T_0, T_1 are *quasi-transversal*, i.e.,

$$\operatorname{codim}(\operatorname{supp} T_0) \cap (\operatorname{supp} T_1) \geqslant c_0 + c_1, \tag{B.3a}$$

$$\operatorname{codim}\big((\operatorname{supp} T_0 \cap \operatorname{supp} \partial T_1) \cup (\operatorname{supp} \partial T_0 \cap \operatorname{supp} T_1)\big) \geqslant c_0 + c_1 + 1, \tag{B.3b}$$

then $0 \in \mathbb{R}^m$ is a $T_0 \times T_1$-quasi-regular value of δ so that the intersection

$$(T_0 \times T_1) \bullet \delta^{-1}(0) = (T_0 \times T_1) \bullet \Delta_M$$

is well defined.

The intersection current $T_0 \bullet T_1$ is then the unique current in M such that

$$j_*(T_0 \bullet T_1) = (-1)^{c_0(m-c_1)}(T_0 \times T_1) \bullet \delta^{-1}(0).$$

If M is an arbitrary real analytic manifold and the subanalytic currents are quasi-transversal, then we define $T_0 \bullet T_1$ to be the unique subanalytic current such that for any open subset U of M real analytically diffeomorphic to an open ball in $\mathbb{R}^m$ we have

$$(T_0 \bullet T_1) \cap U = (T_0 \cap U) \bullet (T_1 \cap U).$$

One can prove that

$$\partial(T_0 \bullet T_1) = (-1)^{c_0+c_1}(\partial T_0) \bullet T_1 + T_0 \bullet (\partial T_1), \tag{B.4}$$

whenever the various pairs of chains in the above formula are quasi-transversal.

One of the key results in [19,20] states that this intersection of quasi-transversal chains induces a well defined intersection pairing

$$\bullet : \mathcal{H}_{m-c_0}(M) \times \mathcal{H}_{m-c_1}(M) \to \mathcal{H}_{m-c_0-c_1}(M).$$

These intersections pairings coincide with the intersection pairings defined via Poincaré duality. This follows by combining two facts.

- The subanalytic homology groups can be computed via a triangulation, as explained above.
- The classical proof of the Poincaré duality via triangulations (see [30, Chap. 5]).

For a submanifold $S \subset M$ of dimension k we define the conormal bundle T_S^*M to be the kernel of the natural bundle morphism

$$i^* : T^*M|_S \to T^*S,$$

where $i : S \hookrightarrow M$ is the inclusion map. A co-orientation of S is then an orientation of the conormal bundle. This induces an orientation on the cotangent bundle of S as follows.

- Fix $s_0 \in S$, and a positively basis $\underline{b}_0 = \{e^1, \dots, e^k\}$ of the fiber of T_S^*M over s_0.
- Extent the basis $\underline{b}_0$ to a positively oriented basis $\underline{b} = \{e^1, \dots, e^n\}$ of $T_{s_0}^*M$.
- Orient $T_{s_0}^*S$ using the ordered basis $\{i^*(e^{k+1}), \dots, i^*(e^m)\}$.

We see that a pair $(S, \boldsymbol{or}^{\perp})$ consisting of a C^1, locally subanalytic submanifold $S \hookrightarrow M$, and a co-orientation $\boldsymbol{or}^{\perp}$ defines a subanalytic chain $[S, \boldsymbol{or}^{\perp}] \in \mathcal{C}_k(M)$. Observe that

$$\operatorname{supp} \partial\big[S, \boldsymbol{or}^{\perp}\big] \subset \boldsymbol{cl}(S) \setminus S.$$

Thus, if $\dim(\boldsymbol{cl}(S) \setminus S) < \dim S - 1$, then $\partial[S, \boldsymbol{or}^{\perp}] = 0$.

Definition 67. An *elementary cycle* of M is a co-oriented locally subanalytic submanifold $(S, \boldsymbol{or}^{\perp})$ such that $\partial[S, \boldsymbol{or}^{\perp}] = 0$.

We say that two elementary cycles $(S_i, \boldsymbol{or}_i^{\perp})$, $i = 0, 1$, intersect *conveniently* if the following hold.

- The submanifolds S_0, S_1 intersect transversally.
- $\boldsymbol{cl}(S_0) \cap \boldsymbol{cl}(S_1) = \boldsymbol{cl}(S_0 \cap S_1)$.

Remark 68. Let us point out some simple properties of elementary cycles intersecting conveniently. The last condition in the above definition implies that

$$\begin{aligned} \operatorname{codim}\big(\boldsymbol{cl}(S_0) \cap \boldsymbol{cl}(S_1) \setminus (S_0 \cap S_1)\big) &= \operatorname{codim}\big(\boldsymbol{cl}(S_0 \cap S_1) \setminus (S_0 \cap S_1)\big) \\ &> \operatorname{codim}(S_0 \cap S_1) = \operatorname{codim} S_0 + \operatorname{codim} S_1. \end{aligned} \tag{B.5}$$

The intersection $S_0 \cap S_1$ is transversal and the conormal bundle of $S_0 \cap S_1$ is the direct sum of the restrictions of the conormal bundles of S_0 and S_1,

$$T^*_{S_0 \cap S_1} M = \big(T^*_{S_0} M\big)\big|_{S_0 \cap S_1} \oplus \big(T^*_{S_1} M\big)\big|_{S_0 \cap S_1}.$$

There is natural induced co-orientation $\boldsymbol{or}_0^{\perp} \wedge \boldsymbol{or}_1^{\perp}$ on $S_0 \cap S_1$ given by the above *ordered* direct sum.

Proposition 69. *Suppose $(S_i, \boldsymbol{or}_i^{\perp})$, $i = 0, 1$, are elementary cycles intersecting conveniently. Then*

$$\big[S_0, \boldsymbol{or}_0^{\perp}\big] \bullet \big[S_1, \boldsymbol{or}_1^{\perp}\big] = \big[S_0 \cap S_1, \boldsymbol{or}_0^{\perp} \wedge \boldsymbol{or}_1^{\perp}\big].$$

Proof. From (B.4) we deduce that

$$\partial\big(\big[S_0, \boldsymbol{or}_0^{\perp}\big] \bullet \big[S_1, \boldsymbol{or}_1^{\perp}\big]\big) = 0.$$

On the other hand,

$$\operatorname{supp}\big(\big[S_0, \boldsymbol{or}_0^{\perp}\big] \bullet \big[S_1, \boldsymbol{or}_1^{\perp}\big]\big) \subset \boldsymbol{cl}(S_0) \cap \boldsymbol{cl}(S_1) = \boldsymbol{cl}(S_0 \cap S_1).$$

Using (B.5) we deduce that in order to find the intersection current $[S_0, \boldsymbol{or}_0^{\perp}] \bullet [S_1, \boldsymbol{or}_1^{\perp}]$ it suffices to test it with differential forms $\alpha \in \Omega^{c_0 + c_1}(M)$ such that

$$\operatorname{supp} \alpha \cap \boldsymbol{cl}(S_0) \cap \boldsymbol{cl}(S_1) \subset S_0 \cap S_1.$$

Via local coordinates this reduces the problem to the special case when S_0, S_1 are co-oriented subspaces of $\mathbb{R}^n$ intersecting transversally in which case the result follows by direct computation from the definition. We leave the details to the reader. □

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