# MICROLOCAL STUDIES OF SHAPES

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ABSTRACT. A gentle introduction to stratified Morse theory and Kashiwara's conormal cycle.

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## 1. EULER CHARACTERISTIC AND CLASSICAL MORSE THEORY

Suppose $M \hookrightarrow E$ is an embedding of a compact, connected, smooth oriented, $m$-dimensional manifold $M$ in the finite dimensional vector space $E$.

Every linear function $\xi \in E^* = \mathrm{Hom}_{\mathbb{R}}(E, \mathbb{R})$ defines by restriction a smooth function $\xi_M$ on $M$. The level sets $M_{=t} = \xi_M^{-1}(t)$ can be visualized as the intersection of $M$ with the hyperplane $\xi = t$. A point $x \in M$ is critical for $\xi_M$ if the hyperplane $\xi = \xi(x)$ is tangent to $M$ at $x$, i.e. $T_xM \subset \xi^{-1}(0)$.

For generic $\xi$ the restriction $\xi_M$ is a Morse function on $M$, i.e. all its critical points are nondegenerate. Recall that the critical point $p_0$ of a smooth function $f$ is called nondegenerate if we can find local coordinates $(x_1, \cdots, x_m)$ on $M$ near $p_0$ such that

$$x_i(p_0) = 0, \;\; f(x_1, \cdots, x_m) = f(p_0) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_m^2.$$

The integer $\lambda$ is independent of the above choices of coordinates. It is called the *Morse index of $f$ at $p_0$* and it is denoted by $\lambda(p_0) = \lambda(f, p_0)$.

Denote by $C_\xi \subset M$ the critical set of $\xi_M$. Then $C_\xi$ is finite and we denote by

$$D_\xi = \xi(C_\xi) \subset \mathbb{R}$$

the set of critical values. $D_\xi$ is a finite subset of $\mathbb{R}$ so that $\mathbb{R} \setminus D_\xi$ is a finite union of open intervals.

$$M_{<t} = \{x \in M;\;\; \xi < t\}, \quad \chi(t) := \chi(M_{<t})$$

Consider for example the situation depicted in Figure 1. The critical set $C_\xi$ and the discriminant set $D_\xi$ are marked in red.

The first theorem of classical Morse theory implies that the function $\chi(t)$ is constant on each connected component of $\mathbb{R} \setminus D_\xi$, i.e.

$$\chi(M_{<a}) = \chi(M_{<b}) \quad \text{if the interval } [a, b] \text{ does not intersect the discriminant } D_\xi.$$

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Notes for a Felix Klein Seminar, Fall 2004.

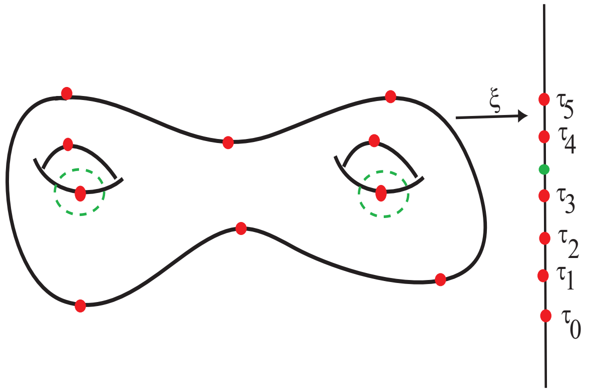


FIGURE 1. *The height function on a genus two surface.*

Thus for every $t\in\mathbb{R}$ the limits $\chi_-(t)=\lim_{s\to t^-}\chi(s)$ and $\chi_+(t)=\lim_{s\to t^+}\chi(s)$ are well defined and

$$\delta(t)=\chi_+(t)-\chi_-(t)=0,\;\;\forall t\in\mathbb{R}\setminus D_\xi.$$

We deduce that

$$\chi(M)=\chi(\infty)-\chi(-\infty)=\sum_{\tau\in D_\xi}\delta(\tau).$$

Observe that for $\tau\in D_\xi$ we have

$$\delta(\tau)=\chi(\tau+\varepsilon)-\chi(\tau-\varepsilon),\;\;\forall 0<\varepsilon\ll 1,$$

i.e.

$$\delta(\tau)=\chi(M_{\tau+\varepsilon})-\chi(M_{<\tau-\varepsilon})=\chi\big(\,H^\bullet(M_{<\tau+\varepsilon},M_{<\tau-\varepsilon})\,\big).$$

Let $C_\xi(\tau)=C_\xi\cap\{\xi_M=\tau\}$ denote the set of critical points of $\xi_M$ with critical value $\tau$. In Figure 1 We see that $C_\xi(\tau_4)$ consists of three critical points.

By choosing $\varepsilon>0$ sufficiently small we can cover $C_\xi(\tau)$ by finitely many disjoint open balls $B(x)$, $x\in C_\xi(\tau)$ such that

$$B(x)\subset\{\tau-\varepsilon<\xi_M<\xi_M+\varepsilon\},\quad B_-(x):=B(x)\cap\{\xi_M<\tau-\varepsilon/2\}\neq\emptyset,\;\;\forall x.$$

The second fundamental theorem of classical Morse theory states

$$H^\bullet(M_{<\tau+\varepsilon},M_{<\tau-\varepsilon})\cong\bigoplus_{x\in C_\xi(\tau)}H^\bullet\big(\,B(x),B_-(x)\,\big),$$

where each pair $(B(x),B_-(x)\,)$ deformation retracts to the pair $(\,D^{\lambda(x)},\partial D^{\lambda(x)}\,)$. Here $\lambda(x)$ denotes the Morse index of the critical point $x$ and $D^\lambda$ denotes the closed $\lambda$-dimensional ball.

We deduce that

$$\delta(\tau)=\sum_{x\in C_\xi(\tau)}\chi(D^{\lambda(x)},\partial D^{\lambda(x)})=\sum_{x\in C_\xi(\tau)}\chi(D^{\lambda(x)})-\chi(\partial D^{\lambda(x)}=\sum_{x\in C_\xi(\tau)}(-1)^{\lambda(x)}.$$

Hence we deduce

$$\chi(M)=\sum_{x\in C_\xi}(-1)^{\lambda(x)}.$$

Let us rephrase the above equality. Denote by

$$\langle\bullet,\bullet\rangle:E\times E^*\to\mathbb{R}$$

the natural pairing between a vector space and its dual. Consider the cotangent bundle $T^*E$ of $E$. Recall that we have a natural pairing

$$\langle \bullet, \bullet \rangle : TE \times T^*E \to \underline{\mathbb{R}}_E := \mathbb{R} \times E, \quad \big\langle\, (y,x), (\xi, x)\,\big\rangle = (\langle v, \xi\rangle, x).$$

This induces a pairing

$$TE|_M \times T^*E|_M \to \underline{\mathbb{R}}_M$$

Define the *conormal bundle* of the embedding $M \hookrightarrow X$ as the subbundle $T^*_M E$ of $T^*E|_E$ defined by the condition

$$(\xi, x) \in T^*_M E \Longleftrightarrow \langle v, \xi\rangle = 0, \ \ \forall v \in T_x M.$$

We regard $T^*_M E$ as a submanifold of $T^*E$. Observe that $\dim T^*_M E = \dim E = \frac{1}{2}\dim T^*E$. The total space of the cotangent bundle $T^*E$ caries a natural symplectic form

$$\omega_0 = d\alpha.$$

If we choose linear coordinates $(x^1, \cdots, x^N)$ on $E$ and we denote by $(\xi_1, \cdots, \xi_N)$ the dual coordinates on $E^*$ then

$$\alpha = \sum_i \xi_i dx^i, \quad \omega_0 = \sum_i d\xi_i \wedge dx^i.$$

We orient[1] the total space of $T^*E$ using the volume form

$$\Omega := d\xi_1 \wedge \cdots \wedge d\xi_N \wedge dx^1 \wedge \cdots \wedge dx^N = \frac{(-1)^{N(N-1)/2}}{N!}\omega_0^N.$$

Then $T^*_M E$ is a lagrangian submanifold of $T^*E$, i.e. $\omega_0$ restricts to the trivial form on $T^*_M E$. An orientation on $E$ induces a natural orientation on $T^*_M E$ defined as follows. Let $p \in M$ and choose local coordinates $x^1, \cdots, x^N$ on $E$ near $p$ such that

$$x^i(p) = 0, \ \ \forall i, \ \ M = \{x^1 = \cdots = x^{N-m} = 0\}$$

and the orientation of $E$ is defined by $dx^1 \wedge \cdots \wedge dx^N$. We obtain coordinates $(\xi_1, \cdots, \xi_N)$ in the fiber $T^*_p E$. Then $(x^1, \cdots, x^m, \xi_{m+1}, \cdots, \xi_N)$ define local coordinates on $T^*_M E$ and we orient this manifold using the volume form

$$d\xi_1 \wedge \cdots \wedge d\xi_{N-m} \wedge dx^{N-m+1} \wedge \cdots \wedge dx^N.$$

Suppose $\xi^0 \in E^*$. We view $\xi^0$ as a smooth function on $E$. Its differential is a section of $T^*E$ and its graph

$$\Gamma_{\xi^0} = \big\{\, (d\xi^0\,|_x, x) \in E^* \times E = T^*E\,\big\}.$$

is a Lagrangian submanifold of $T^*E$. It carries a natural orientation induced by the orientation of $E$. Observe that $p_0 \in M$ is a critical point of $\xi^0|_M$ if and only if $P_0 := (\xi^0, p_0) \in \Gamma_{\xi^0} \cap T^*_M E$.

We want to prove that if $p_0$ is nondegenerate as a critical point of index $\lambda$ then $\Gamma_{\xi^0}$ intersects $T^*_M E$ transversally[2] in $P_0$. Set for simplicity $\Lambda = T^*_M E$ and $\Gamma = \Gamma_{\xi^0}$. Since $p_0$ is a nondegenerate critical point of $\xi^0$ we can find local coordinates $(x^1, \cdots, x^N)$ in $E$ near $p_0$ such that $x^i(p_0) = 0, \ \ \forall i$, such that if we set

$$x_\perp = (x^1, \cdots, x^{N-m}), \quad x_0 = (x^{N-m+1}, \cdots, x^N)$$

then $M = \{x_\perp = 0\}$

$$\xi^0(x) = \xi^0(p_0) + \langle x^\perp, c^0\rangle + \frac{1}{2}(\epsilon_1(x^{N-m+1})^2 + \cdots + \cdots + \epsilon_m(x^N)^2)$$
$$+x^{N-m+1}\ell_1(x_\perp) + \cdots + x^N\ell_m(x_\perp) + q(x^\perp) + O(3),$$

[1] This differs from the two different orientation conventions in [5] and [10].

[2] The converse is also true

where $\epsilon_j = \pm 1$,

$$\#\{j;\ \ \epsilon_j = -1\} = \lambda,$$

$c^0 \in E^* \setminus 0$ vanishes along $T_{p_0}M$, $\ell_i$ are linear functions in the variables $x_\perp$, $q(x_\perp)$ is quadratic in the same variables. Then near $P_0$ the graph of $d\xi^0$ admits the parametrization

$$\xi_k = c_k^0 + \sum_{j=1}^m x^{N-m+j}\frac{\partial \ell_j}{\partial x^k} + \frac{\partial q}{\partial x^k} + O(2), \ \ k = 1, \cdots, N-m$$

$$\xi_{N-m+j} = \epsilon_j x^{N-m+j} + \ell_j(x^\perp) + O(2), \ \ j = 1, \cdots, m$$

$$x^i = x^i, \ \ \forall i.$$

An oriented basis of $T_{P_0}\Gamma_{\xi^0}$ is given by the vectors

$$U_j = \begin{bmatrix} \frac{\partial \xi_1}{\partial x^j} \\ \vdots \\ \frac{\partial \xi_N}{\partial x^j} \\ \frac{\partial x^1}{\partial x^j} \\ \vdots \\ \frac{\partial x^n}{\partial x^j} \end{bmatrix}, \ \ j = 1, \cdots, n.$$

An oriented basis of $T_{P_0}\Lambda$ is given by the vectors

$$V_j = \begin{cases} \partial_{\xi_j} & \text{if} \quad j \leq N-m \\ \partial_{x^j} & \text{if} \quad j > N-m \end{cases}.$$

We want to compute $\Omega(U_1, \cdots, U_N, V_1, \cdots, V_N)$. Denote by $S_m$ the diagonal $m \times m$ matrix with entries $\epsilon_j$. We deduce

$$\Omega(U_1, \cdots, U_N, V_1, \cdots, V_N) = \det \begin{bmatrix} * & \vdots & * & \vdots & I_{N-m} & \vdots & 0 \\ \cdots & & \cdots & & \cdots & & \cdots \\ * & \vdots & S_m & \vdots & 0 & \vdots & 0 \\ \cdots & & \cdots & & \cdots & & \cdots \\ I_{N-m} & \vdots & 0 & \vdots & 0 & \vdots & 0 \\ \cdots & & \cdots & & \cdots & & \cdots \\ 0 & \vdots & I_m & \vdots & 0 & \vdots & I_m \end{bmatrix}$$

$$= \det \begin{bmatrix} * & \vdots & * & \vdots & I_{N-m} \\ \cdots & & \cdots & & \cdots \\ * & \vdots & S_m & \vdots & 0 \\ \cdots & & \cdots & & \cdots \\ I_{N-m} & \vdots & 0 & \vdots & 0 \end{bmatrix} = (-1)^{N(N-m)} \det \begin{bmatrix} * & \vdots & S_m \\ \cdots & & \cdots \\ I_{N-m} & \vdots & 0 \end{bmatrix}$$

$$= (-1)^{N(N-m)+m(N-m)} \det S_m = (-1)^{N-m} \det S_m = (-1)^{N-m}(-1)^\lambda.$$

Let us perform a few cosmetic changes. Observe that if $\lambda(-\xi^0, p)$ denotes the index of $p$ as a critical point of $(-\xi^0)$ then $\lambda(-\xi^0, p) = m - \lambda(\xi^0, p)$ so that

$$(-1)^{\lambda - m} = (-1)^{\lambda(-\xi^0, p)}.$$

If we consider the antipodal map ${}^a : T^*E \to T^*E$, $(\xi, x) \mapsto (-\xi, x)$ we deduce that

$$\#(\Gamma_{-\xi^0} \cap \Lambda_M, P_0^a) = (-1)^{N+\lambda(\xi^0,p)}$$

and since $\dim \Gamma = \dim \Lambda = N$ we conclude

$$\#(\Lambda_M \cap \Gamma_{-\xi^0}, P_0^a) = (-1)^{\lambda(\xi^0,p)}.$$

We obtain the following equality

$$\chi(M) = \#(\Lambda_M \cap \Gamma_{-\xi}), \quad \text{for any generic linear map } \; \xi : E \to \mathbb{R}. \tag{1.1}$$

## 2. Weyl tube formula

Suppose $M \hookrightarrow E$ is as before but we assume additionally that $M$ is equipped with an Euclidean metric $g_0$. $g_0$ induces a metric $g$ on $M$. We set $c = N - m =$ the codimension of $M$ in $E$. The normal bundle of the embedding $M \hookrightarrow E$ is the quotient bundle

$$T_M E := (TE)|_M \,/TM.$$

Since $TE$ is equipped with a metric we can identify $T_M E$ with the bundle $\mathcal{N}(M) \to M$, the orthogonal complement of $TM$ in $(TE)|_M$. The metric on $E$ defines a function

$$\rho : TE \to \mathbb{R}, \;\; \rho(Y, x) = |Y|_{g_0}.$$

We set

$$D_r(\mathcal{N}) := \big\{ p \in \mathcal{N}(M); \;\; \rho(p) \leq r \big\}, \;\; S_r(\mathcal{N}) := \partial D_r(\mathcal{N}) = \big\{ p \in \mathcal{N}(M); \;\; \rho(p) = r \big\},$$
$$S_r(TE) = \big\{ p \in TE; \;\; \rho(p) = r \big\}.$$

We have an exponential map

$$\exp : TE \to E, \;\; \exp(y, x) = x + y.$$

Define the tube of radius $r > 0$ around $M$ to be the closed set

$$\mathbb{T}_r(M) := \big\{ x \in E; \;\; \mathrm{dist}\,(x, M) \leq r \big\}.$$

For $r > 0$ sufficiently small we have a diffeomorphism

$$\exp : D_r(\mathcal{N}) \longrightarrow \mathbb{T}_r(M). \tag{2.1}$$

Let $V_M(r) = \mathrm{Vol}(\,\mathbb{T}_r(M)\,)$. We would like to understand the behavior of $V_M(r)$ as $r \searrow 0$. Denote by $dv_E$ the volume form on $E$. Using the identification (2.1) we deduce

$$V_M(r) = \int_{\mathbb{T}_r(M)} dv_E = \int_{D_r(\mathcal{N})} \exp^* dv_E.$$

In more down-to-Earth terms, we are using normal (Fermi) coordinates near $M$ to compute the volume of the tube.

Let us first understand the $N$-form

$$\Omega_E = \exp^* dV_E \in \Omega^N(TE).$$

Choose oriented, orthonormal coordinates $x = (x^1, \cdots, x^N)$ on $E$. They induce oriented orthonormal coordinates $Y = (Y^1, \cdots, Y^N)$ in each tangent space. Then

$$dv_E = dx^1 \wedge \cdots \wedge dx^N, \quad \Omega_E = \exp^* dv_E = \bigwedge_{j=1}^N (dx^j + dY^j).$$

Denote by $\partial_\rho$ the radial vector field along the fibers of $TE$

$$\partial_\rho = \nabla\rho = \frac{1}{\rho}\sum_j Y^j\partial_{Y^j}.$$

Then $\partial_\rho \lrcorner\, d\rho = 1$ and we set

$$\sigma_E := \partial_\rho \lrcorner\, \Omega_E$$

so that

$$\Omega_E = d\rho\wedge\sigma_E.$$

Then

$$V_M(r) = \int_{D_r(\mathcal{N})}\Omega_E = \int_0^r dt\int_{S_t(\mathcal{N})}\sigma_E.$$

Consider the radial projection $\nu_t : S_1(TE)\to S_t(E)$ and set $\sigma_{E,t} := \nu_t^*(\sigma_E\,|_{S_t(TE)})$ We conclude

$$V_M(r) = \int_{D_r(\mathcal{N})}\Omega_E = \int_0^r dt\int_{S_1(\mathcal{N})}\sigma_{E,t}. \tag{2.2}$$

Set

$$S := \frac{1}{\rho}Y = (s^1,\cdots,s^N).$$

Observe that $\partial_\rho \lrcorner\, dY^k = s^k$ so that

$$\sigma_E = \partial_\rho \lrcorner \bigwedge_{j=1}^N (dx^j + dY^j)$$

$$= \sum_k (-1)^{k-1}s^k\bigwedge_{j\neq k}\big(\,dx^j + dY^j\,\big) = \sum_k (-1)^{k-1}s^k\bigwedge_{j\neq k}\big(\,dx^j + \rho ds^j + s^j d\rho\,\big)$$

Hence

$$\sigma_{E,t} = \sum_k (-1)^{k-1}s^k\bigwedge_{j\neq k}\big(\,dx^j + tds^j\,\big) = \sum_{j=0}^{N-1} t^j\eta_{N-1-j},$$

where $\eta_k\in\Omega^{n-1}(S_1(TE))$ is a form independent of $t$ of degree $k$ in the variables $dx$ and of degree $N-k-1$ in the variables $ds$. We denote by $\omega_d$ the volume of the unit $d$-dimensional ball and by $\sigma_d$ the ”area” of its boundary. More explicitly

$$\omega_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)},\quad \sigma_d = d\omega_d,$$

where for every positive integer $j$ we compute $\Gamma(j/2)$ inductively using the formulæ

$$\Gamma(1) = 1,\;\; \Gamma(1/2) = \pi^{1/2},\;\; \Gamma(x+1) = x\Gamma(x).$$

We normalize

$$\hat{\eta}_k := \frac{1}{\sigma_{N-k}}\eta_k = \frac{1}{(N-k)\omega_{N-k}}\eta_k.$$

We deduce

$$V_M(r) = \sum_{j=0}^{N-1}\int_0^r\Big(\int_{S^1(\mathcal{N})}\eta_{N-1-j}\Big)t^j dt = \sum_{k=1}^N A_k(M)\omega_{N-k}t^{N-k},$$

where

$$A_k(M) = \int_{S_1(\mathcal{N})}\hat{\eta}_k$$

Observe that

$$\int_{S_1(\mathcal{N})} \eta_k = 0, \quad \text{if} \quad k > m$$

so that

$$V_M(r) = \sum_{k=0}^{m} A_k(M) t^{N-k} = \sum_{j=0}^{m} A_{m-j}(M) \omega_{c+j} t^{c+j}, \quad c = N - m = \operatorname{codim} M.$$

**Example 2.1.** (a) Suppose $E = \mathbb{R}^2$ with Euclidean coordinates $(x, y)$. In each fiber of $TE$ we choose polar coordinates $(r, \theta)$ so that

$$\exp(r, \theta; x, y) = (x + r\cos\theta, y + r\sin\theta), \quad \exp^* dv_E = d(x + r\cos\theta) \wedge d(y + r\sin\theta)$$

$$\partial_r \lrcorner \exp^* dv_E = (\cos\theta dy - \sin\theta dx) + \rho d\theta,$$

so that

$$\hat{\eta}_1 = \frac{1}{2}(\cos\theta dy - \sin\theta dx), \quad \hat{\eta}_0 = \frac{1}{2\pi} d\theta.$$

□

The integrals of the forms $\eta_k$ over $S_1(\mathcal{N})$ can be expressed in terms of the second fundamental form of $M \hookrightarrow E$. This is also known as the *shape operator* and it is a bilinear map

$$S : TM \times TM \to \mathcal{N}$$

defined as follows. Given vector fields $X, Y$ tangent to $M$ we denote by $\nabla^E_X Y$ the Euclidean covariant derivative of $Y$ along $X$

$$\nabla^E_{X^i \partial_i} Y^j \partial_j = \big( X^i \partial_i Y^j \big) \partial_j.$$

We have an orthogonal decomposition of $\nabla^E_X Y$ into a tangential and a normal part

$$\nabla^E_X Y = (\nabla^E_X Y)^\tau + (\nabla^E_X Y)^\nu.$$

Then

$$S(X, Y) = (\nabla^E_X Y)^\nu.$$

The shape operator enjoys several nice properties (see [9, §4.2.4]).

**Proposition 2.2.** *(a) $S$ is symmetric in its arguments, i.e.*

$$S(X, Y) = S(Y, X), \quad \forall X, Y \in \mathbf{Vect}(M).$$

*(b) For all $N \in C^\infty(\mathcal{N}_M)$ and $X, Y \in \mathbf{Vect}(M)$ we have*

$$g_0(S(X, Y), N) = g_0(\nabla^E_X N, Y).$$

The shape operator is related to the Gauss map $\Gamma_M : M \to \mathrm{Gr}_m(E)$ = the Grassmanian of $m$-dimensional subspaces in $E$

$$M \ni p \mapsto T_p M \in \mathrm{Gr}_m(E).$$

For a $m$-dimensional vector space $V \subset E$ the tangent space of the Grassmanian at $V$ is described by

$$T_V \mathrm{Gr}_m(E) = \mathrm{Hom}(V, V^\perp).$$

The differential at $p \in M$ of the Gauss map can therefore be viewed as a map

$$D\Gamma : T_p M \to T_{\Gamma(p)} \mathrm{Gr}_m(E) = \mathrm{Hom}(T_p M, \mathcal{N}_p).$$

One can show that for every $X, Y \in T_p M$ the linear map $D\Gamma_p(X) \in \mathrm{Hom}(T_p M, \mathcal{N}_p)$ is given by

$$Y \longmapsto S_p(X, Y).$$

**Theorema Egregium** shows that the shape operator determines the Riemann tensor of $(M,g)$ via the formula

$$R_{ijk\ell} = g\big( S(\partial_i,\partial_k), S(\partial_j,\partial_\ell)\big) - g\big( S(\partial_i,\partial_\ell), S(\partial_j,\partial_k)\big)$$

For any local coordinate system $(x^i)$ on $M$.

The forms $\eta_k$ can be explicitly expressed in terms of the shape operator. More precisely, for every unit normal vector $\vec{\nu}\in\mathcal{N}_p$ we obtain a symmetric bilinear form on $T_pM$

$$S_{\vec{\nu}}(X,Y) = g_0( S(X,Y),\vec{\nu}).$$

Using an orthonormal basis of $T_pM$ we can identify it with a symmetric matrix. We denote by $P_\nu(t)=\det( \mathbb{1}_{T_pM}+tS_{\vec{\nu}} )$ its characteristic polynomial. Then

$$\Omega_E = \exp^* dv_E\,|_{\mathcal{N}} = P_\nu(\rho)\rho^{c-1}d\rho d\vec{\nu} dV_M,$$

where $d\vec{\nu}$ denotes the volume form on the unit sphere $S_1(\mathcal{N}_p)$. We obtain (see the beautiful original source [12] for details)

$$V_M(r) = \int_{D_r(\mathcal{N})}\Omega_E = \sum_{k\geq 0}\omega_{c+2k}r^{c+2k}\underbrace{\int_M \mathcal{P}_k(R)dV_M}_{:=\lambda_{2k}(M)},$$

where $\mathcal{P}_k(R)$ is a *universal* degree $k$-polynomial in the curvature tensor $(R_{ijk\ell})$. Hence $\mu_k(M)$ is an *intrinsic invariant of the Riemann manifold* $(M,g)$. We have an equality

$$\lambda_k(M,g) = \frac{1}{\sigma_{c+2k}}\int_{S_1(\mathcal{N})}\eta_{m-2k}.$$

Note that the quantity $\lambda_k(M)$ is measured in $meters^{m-2k}$. For this reason we introduce the notation

$$\mu_{m-2k}(M,g)=\lambda_{2k}(M,g).$$

We can then rewrite

$$V_M(r) = \sum_{k\geq 0}\mu_{m-2k}(M,g)\cdot \operatorname{vol}\big( B^{c+2k}(r)\big),$$

where $B^d(r)$ denotes the $d$-dimensional Euclidean ball of radius $r$.

There are some old acquaintances amongst the quantities $\mu_j(M,g)$. For example

$$\mu_m(M,g)=\operatorname{vol}(M,g).$$

If $m$ is even, $m=2m_0$ then $\mathcal{P}_{m_0}dV_M\in\Omega^m(M)$ is the Euler form determined by the metric and the Gauss-Bonnet theorem implies

$$\mu_0(M,g)=\chi(M).$$

In general we have

$$\mu_{m-2}(M,g) = \frac{1}{4\pi}\int_M s dV_M,$$

where $s: M\to\mathbb{R}$ denotes the scalar curvature of $(M,g)$.

The quantities $\mu_k$ are related by the so called *reproducing formulæ*. Denote by $\operatorname{Graff}^c(E)$ the Grassmanian of *affine* subspaces in $E$ of codimension $c$. More precisely we have the following result (see [3])

$$\mu_k(M) := A(N,m,c,k)\cdot\int_{\operatorname{Graff}^c(E)}\mu_{k-c}(M\cap P)|dP|,$$

where $|dP|$ is a $O(E)$- invariant measure on $\operatorname{Graff}^c(E)$. If we set $c=k$ we deduce

$$\mu_k(M) := A(N,m,k)\cdot\int_{\operatorname{Graff}^k(E)}\chi(M\cap P)|dP|.$$

We can interpret $\mu_k(M)$ as an average of the Euler characteristics of the intersections of $M$ with codimension $k$ affine planes. If we take $k = \dim M$ we deduce

$$\mathrm{vol}\,(M, g) = A(N, m) \int_{\mathrm{Graff}^m(E)} \chi(M \cap P) |dP|.$$

The intersection of $M$ with a generic codimension $m$ affine subspace $P$ is a finite set so that

$$\chi(M \cap P) = |M \cap P|.$$

The last formula can be rewritten as

$$\mathrm{vol}\,(M, g) = A(N, m) \int_{\mathrm{Graff}^m(E)} |M \cap P|\, |dP|.$$

This generalizes the classical Crofton formula for curves in $\mathbb{R}^2$.

As explained in [7], we can normalize the invariant measures in $\mathrm{Graff}^m(E)$ in a very clever way so that $A(N, m) = 1$.

## 3. Singular Morse theory

To understand how to extend the previous facts to more singular situations we need to produce more flexible definitions of the notions of critical points and critical values.

We will begin by defining the notion of regular value. This will require the notion of *local cohomology*

Suppose $X$ is a locally compact metric space, and $S$ is a closed subset. To eliminate many pathological phenomena *we will assume that $X$ and $S$ are locally contractible*, i.e. every point admits a basis of contractible neighborhoods. This condition implies for example that $X$ and $S$ are $ENR$'s (Euclidean Neighborhood Retract). We denote by $i : S \hookrightarrow X$ and $j : X \setminus S \hookrightarrow X$ the natural inclusions We define the local cohomology of $X$ along $S$ (with real coefficients) to be

$$H^\bullet_S(X) := H^\bullet(X, X \setminus S; \mathbb{R}).$$

For every topological space $Y$ we denote by $H^\bullet(Y)$ its (Čech) cohomology with real coefficients. A cohomology class $c \in H^\bullet(X \setminus S)$ is said to *propagate across* $S$ if it belongs to the image of the morphism

$$j^* : H^\bullet(X) \to H^\bullet(X \setminus S).$$

Observe that we have a long exact sequence (called the *adjunction sequence*)

$$\cdots \to H^k(X) \xrightarrow{j^*} H^k(X \setminus S) \xrightarrow{\delta} H^{k+1}_S(X) \to \cdots \tag{3.1}$$

We see that a cohomology class $c \in H^\bullet(X \setminus S)$ propagates across $S$ if and only if $\delta(c) = 0 \in H^{\bullet+1}_S(X)$. We can the regard the local cohomology of $X$ along $S$ as collecting the obstructions to the propagation across $S$ of the cohomology classes in the complement of $S$. If the inclusion $j$ induces an isomorphism in cohomology then $H^\bullet_S(X) = 0$. This is the case if for example $X \setminus S$ is a deformation retract of $X$.

Observe that if $V$ is an open neighborhood of $S$ in $X$ then $X \setminus V$ is a closed subset in $X \setminus S$ and we obtain an excision isomorphism

$$H^\bullet_S(X) = H^\bullet(X, X \setminus S) \cong H^\bullet\big(\, X \setminus (X \setminus V), (X \setminus S) \setminus (X \setminus V)\,\big) = H^\bullet(V, V \setminus S) = H^\bullet_S(V).$$

This shows that the local cohomology reflects the local behavior of $X$ near $S$ and it is blind to what is happening further away from $S$.

We can now define the *local cohomology sheaves* $\mathcal{H}^\bullet_S$ to be the sheaves associated to the presheaves

$$U \longmapsto H^\bullet_{S \cap U}(U).$$

If $x \in X$ and $U_n(x)$ denotes the open ball of radius $1/n$ centered at $x$ then for every $m \leq n$ we have morphisms

$$H^\bullet_{S\cap U_m}(U_m) \to H^\bullet_{S\cap U_n}(U_n)$$

and then the stalk of $\mathcal{H}^p_S$ at $x$ is the inductive limit

$$\mathcal{H}^\bullet_S(x) := \lim_{n\to\infty} H^\bullet_{S\cap U_n}(U_n)$$

Observe that since $X$ is locally contractible we have

$$\mathcal{H}^\bullet_S(x) = 0 \text{ for every } \; x \in (X \setminus S) \cup \operatorname{int}(S). \tag{3.2}$$

We set

$$\chi_S(X) = \sum_k (-1)^k \dim H^k_S(X), \;\; \chi_S(x) := \sum_{k\geq 0} (-1)^k \dim \mathcal{H}^k_S(x).$$

**Example 3.1.** Assume $X$ is the planar three arm star depicted in Figure 2 and $P_0$ is the center of the star. Assume $S = \{P_0\}$. In this case we have

$$\mathcal{H}^\bullet_S(P_0) = H^\bullet_{\{P_0\}}(X) \cong H_\bullet(X, X \setminus P_0)^*$$

and we deduce

$$\mathcal{H}^0_S(P_0) = 0, \;\; \mathcal{H}^1_S(P_0) \cong \mathbb{R}^2, \;\; \chi_S(P_0) = \chi(X, X \setminus P_0) = \chi(X) - \chi(X \setminus P_0) = -2.$$

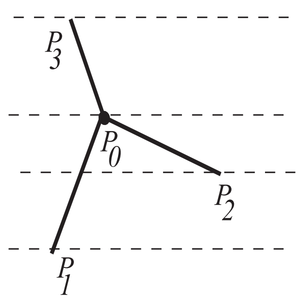


FIGURE 2. *A planar star.*

□

An iterated application of the Mayer-Vietoris sequence shows that the local cohomology sheaves determine the local cohomology along $S$. More precisely we have a Grothendieck spectral sequence converging to $H^\bullet_S(X)$ whose $E_2$ term is

$$E_2^{p,q} = H^p(X, \mathcal{H}^q_S).$$

If it happens that the local cohomology sheaves are supported by finite sets then

$$H^{p,q}(X, \mathcal{H}^q_S) = 0, \;\; \forall p > 0,$$

so that the spectral sequence degenerates at the $E_2$-terms. In this case we have

$$H^q_S(X) \cong H^0(X, \mathcal{H}^q_S) \cong \bigoplus_{x\in X} \mathcal{H}^q_S(x). \tag{3.3}$$

In particular

$$\chi_S(X) = \sum_{x\in X} \chi_S(x). \tag{3.4}$$

Observe that if the local cohomology sheaves are trivial then so is the local cohomology. The converse need not be true.[3]

Before we proceed with our search for a new definition for a regular value let us mention that if $V = \oplus_{n\in\mathbb{Z}} V_n$ is a graded vector space we denote by $V[\mu]$ the shift by $\mu$

$$V[\mu]_n = V_{n+\mu}$$

We will identify $\mathbb{R}$ with the graded vector space $V$ defined by

$$V_n = 0, \;\; \forall n \neq 0, \;\; V_0 = \mathbb{R}.$$

Then

$$\mathbb{R}[-\mu]_n = 0, \;\; \forall n \neq \mu, \;\; \mathbb{R}[-\mu]_\mu = \mathbb{R}.$$

Suppose $M$ is a smooth manifold and $f : M \to \mathbb{R}$ is a smooth function. For every $c \in \mathbb{R}$ we set $M_{\geq c} = \{f \geq c\}$, $M^{<c} := \{f < c\}$ etc. If $c$ is a regular value of $f$ then the level set $\{f = c\}$ is a smooth hypersurface. Moreover, every point $x$ on this level surface admits a fundamental system of neighborhoods $U_n(x)$ such that the set $U_n \cap \{M^{<c}\} = U_n \setminus M_{\geq c}$ is a deformation retract of $U_n$. This implies

$$H^\bullet_{U_n \cap M_{\geq c}}(U_n) = 0$$

These non-obstructions to local propagation are patched together in the next result.

**Theorem 3.2** (Kashiwara's Lemma)**.** *Suppose $K \xrightarrow{f} \mathbb{R}$ is a continuous function on the compact space $K$. If for every $p \in K$ we have*

$$\mathcal{H}^\bullet_{K_{\geq f(p)}}(p) = 0$$

*then for every $t \in \mathbb{R}$ the inclusion induced morphism $H^\bullet(K) \to H^\bullet(K^{<t})$ is an isomorphism.*

For a proof of this result we refer to [6, §2.7]. The above result shows that if the interval $[a, b]$ contains no critical value of $f$ the for every $a \leq s < t \leq b$ the inclusion induced morphism

$$H^\bullet(M^{<t}) \to H^\bullet(M^{<s})$$

is an isomorphism. Thus we obtain a fact we knew already that when going through regular values the sublevel sets do not undergo changes detectable homologically.

Suppose now that the level set contains a critical point $p$ of index $\lambda$. Denote by $W_p^-$ the unstable manifold of $p$. for a small coordinate ball $U$ around $b$ we have $U \cap W_p^- \cong D^\lambda =$ open $\lambda$-dimensional disk centered at $p$ and we have an isomorphism

$$H^\bullet_{U_{\geq c}}(U) = H^\bullet(U, U^{<c}) \cong H^\bullet(D^\lambda, D^\lambda \setminus p) = H^\bullet_{\{p\}}(D^\lambda) \cong \mathbb{R}[-\lambda].$$

The critical point $p$ distinguishes itself from other points on the level set $\{f = f(p)\}$ by the condition

$$\mathcal{H}^\bullet_{M_{\geq f(p)}}(p) \neq 0.$$

We will use this as our criticality test.

**Definition 3.3.** Suppose $M$ is a compact connected (subanalytic) subset in an Euclidean space and $f : M \to \mathbb{R}$ is smooth function. A point $p \in M$ is said to be *critical* for $f$ if

$$\mathcal{H}^\bullet_{M_{\geq f(p)}}(p) \neq 0.$$

We set

$$\delta(f, p) := \chi(\mathcal{H}^\bullet_{M_{\geq f(p)}}(p)).$$

[3] Can you find and example?

Observe that any relative minimum of $f$ is necessarily a critical point. Suppose $M$ is as above and $f : M \to \mathbb{R}$ is a smooth function with finitely many critical points $p_1, \cdots, p_\nu$ with critical values $c_1 \leq \cdots \leq c_\nu$. For the simplicity of the exposition we assume that the critical values are distinct, i.e. there is at most one critical point on each level set.

Observe that any relative minimum of $f$ is necessarily a critical point. $c_1$ must be the absolute minimum of $f$ so that $M^{<c_1} = \emptyset$. From Kashiwara's lemma we deduce

$$\chi(M) = \chi(M^{<c_\nu+\varepsilon})$$

and we deduce

$$\chi(M) = \underbrace{\chi(M^{<c_1})}_{=0} + \chi(M^{<c_2}) - \chi(M^{c_1}) + \cdots + \chi(M^{<c_\nu}) - \chi(M^{<c_{\nu-1}}) + \chi(M^{<c_\nu+\varepsilon}) - \chi(M^{<c_\nu})$$

$$= \chi(\, H^\bullet(M^{<c_2}, M^{<c_1})\,) + \cdots + \chi(\, H^\bullet(M^{<c_\nu}, M^{<c_{\nu-1}})\,) + \chi(\, M^{c_\nu+\varepsilon}, M^{<c_\nu})\,)$$

$$= \sum_{k=1}^{\nu} \chi\big(\, M^{<c_k+\varepsilon}, M^{<c_k}\,\big).$$

Due to Kashiwara's Lemma we have

$$\chi\big(\, M^{<c_k+\varepsilon}, M^{<c_k}\,\big) = \lim_{\varepsilon \searrow 0} \chi\big(H^\bullet_{M_{\geq c_k}}(M^{<c_k+\varepsilon})\,\big).$$

Since

$$\bigcap_{\varepsilon>0} M^{<c_k+\varepsilon} = M^{\leq c_k}$$

we deduce

$$H^\bullet_{M_{\geq c_k}}(M^{\leq c_k}) = \varinjlim_n H^\bullet_{M_{\geq c_k}}(M^{<c_k+1/n}) = H^\bullet_{M_{\geq c_k}}(M^{<c_k+\varepsilon}), \;\; \forall 0 < \varepsilon < c_{k+1} - c_k.$$

Now observe that the restriction of $\mathcal{H}^\bullet_{M_{\geq c_k}}$ on $M^{\leq c_k}$ is supported exactly at the point $p_k$ so that

$$H^\bullet_{M_{\geq c_k}}(M^{\leq c_k}) \stackrel{(3.3)}{=} \mathcal{H}^\bullet_{M_{\geq c_k}}(p_k).$$

Hence

$$\chi\big(\, M^{<c_k+\varepsilon}, M^{<c_k}\,\big) = \chi_{M_{\geq c_k}}(p_k)$$

and

$$\chi(M) = \sum_{k=1}^{\nu} \chi_{M_{\geq c_k}}(p_k) = \sum_{k=1}^{\nu} \delta(f, p).$$

If $M$ is a compact smooth manifold and $M$ is a Morse function then

$$\chi_{f \geq c_i}(p_i) = (-1)^{\lambda_i},$$

where $\lambda_i$ denotes the Morse index of $p_i$.

**Example 3.4.** Suppose $C \subset E$ is a simplicial complex linearly embedded in the Euclidean space $E$. We denote by $V(C)$ the set of vertices of $C$. Suppose $\xi : E \to \mathbb{R}$ is a linear function in general position with respect to $C$, i.e. its restriction to the set of vertices is one-to-one. Then the set of critical points of $\xi$ is contained in the set of vertices, and in fact there is at most one critical point in each level set.

For each $p \in E$ we denote by $H^<_{\xi,p}$ the half space

$$H^<_{\xi,p} = \{ v \in E : \;\; \langle v, \xi \rangle < \langle p, \xi \rangle \} = \{ \xi < \xi(p) \}.$$

Denote by $B_\varepsilon(p)$ the open ball of radius $\varepsilon$ centered at $p$. Then for every vertex $p$ of $C$ we have

$$\mathcal{H}_{\xi\geq\xi(p)}(p) = H^\bullet\big(B_\varepsilon(p)\cap C,\ B_\varepsilon(p)\cap C\cap H^<_{\xi,p}\big),\quad \forall 0<\varepsilon\ll 1.$$

Denote by $St(p)$, the star at $p$ which is the union of all simplices in $C$ which have $p$ as a vertex. The set $B_\varepsilon(p)\cap C = B_\varepsilon(p)\cap St(p)$ deformation retracts to $p$ and we deduce

$$\chi_{\xi\geq\xi(p)} = \chi(B_\varepsilon(p)\cap C) - \chi(B_\varepsilon(p)\cap C\cap H^<_{\xi,p}) = 1-\chi(\,St(p)\cap H^<_{\xi,p}\,).$$

For every simplex $\sigma$ in $St(p)$ we denote by $V_-(\sigma)$ the collection of vertices $v$ of $\sigma$ such that $\xi(v)<\xi(p)$. We denote by $V_+(\sigma)$ the collection of vertices $v\neq p$ of $\sigma$ such that $\xi(v)\geq\xi(p)$. Projecting from the face $[V_+(\sigma)]$ of $\sigma$ spanned by $V_+(\sigma)$ onto the face spanned by $p$ and $V_-(\sigma)$ we obtain a deformation retraction (see Figure 3)

$$D_\sigma:\sigma\to[p,V_-(\sigma)] = \text{the face of } \sigma \text{ spanned by } \{p\}\cup V_-(\sigma)$$

This induces a linear deformation retraction

$$D_\sigma:\sigma\cap H^<_{\xi,p}\to[p,V^-_\sigma]\cap H^<_{\xi,p}.$$

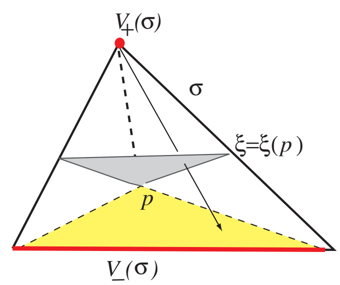


FIGURE 3. *The local homotopic structure of critical sublevel sets.*

If we denote by $St^-(p)$ the union of all simplices of $C$ contained in $H^{\leq}_{\xi,p}$ which have $p$ as a vertex. If $\sigma$ is a simplex in $St^-(p)$ and $v\neq p$ is a vertex then $\xi(v)<\xi(p)$ since $\xi$ was chosen in general position.

Hence we obtain a deformation retract of $St(p)\cap H^<_{\xi,p}$ onto $St^-(p)\setminus p$. Denote by $Lk^-(p) = Lk^-_\xi(p)$ the *descending link of* $p$ defined as the simplicial subcomplex of $St^-(p)$ spanned by the vertices $v\neq p$. Then $St^-(p)-p$ deformation retracts to $Lk^-(p)$ and we deduce

$$\chi(St(p)\cap H^<_{\xi,p}) = \chi(Lk^-(p)).$$

Observe that $Lk^-(p)$ consists of the simplices $[V_-(\sigma)]$, where $\sigma$ is a simplex in $St^-(p)$, other than $p$. Hence

$$\chi(Lk^-(p)) = \sum_{\sigma\in St^-(p)\setminus[p]}(-1)^{\dim[V_-(\sigma)]} = -\sum_{\sigma\in St^-(p)\setminus[p]}(-1)^{\dim\sigma}$$

so that

$$\chi_{\xi\geq\xi(p)} = 1+\sum_{\sigma\in St^-(p)\setminus[p]}(-1)^{\dim\sigma} = \sum_{\sigma\in St^-(p)}(-1)^{\dim\sigma} =: a(\xi,p).$$

We deduce

$$\chi(C) := \sum_{p\in V(C)} a(\xi,p) = \sum_{p\in V(C)}\big(1-\chi(Lk^-_\xi(p))\big).$$

The first equality was proved by T. Banchoff in [1] using a direct elementary method.

For example, consider the simplicial complex depicted in Figure 2 where the horizontal dotted lines depict the level sets containing the vertices. The Euler characteristic of the star is $4 - 3 = 1$. Upon inspecting the figure we deduce

$$Lk^-(P_1) = Lk^-(P_2) = \emptyset,\;\; Lk^-(P_0) = \{P_1, P_2\},\;\; Lk^-(P_3) = \{P_0\}$$

so that

$$a(P_0) = a(P_1) = 1,\;\; a(P_0) = -1,\;\; a(P_3) = 0$$

so that

$$a(P_0) + a(P_1) + a(P_2) + a(P_3) = 1 = \chi(C).$$

Observe that $P_3$ is an absolute maximum of the height function yet it is not a critical point in our sense. In fact if $C$ is a convex simplex then a generic linear function $\xi$ will have exactly one critical point on $C$, the absolute minimum. The hyperplane $\xi = \xi(p)$ passing through the absolute minimum $p$ will be a supporting hyperplane of $C$. In particular, a point could be critical for $f$ but it may not be critical for $-f$. □

## 4. THE CHARACTERISTIC VARIETY AND THE CONORMAL CYCLE OF A SIMPLICIAL COMPLEX

Suppose $X$ is a compact simplicial complex inside the Euclidean vector space $E$.

The characteristic variety of $X$ is the closed subset of the cotangent bundle $T^*E = R^* \times E$ of $E$ which is the closure of the set

$$\big\{\, (\xi, p) \in E^* \times E;\quad p \text{ is a critical point of } (-\xi)\,|_X \,\big\}.$$

The last condition signifies that $p$ admits a fundamental system of neighborhoods $U_n$ in $X$ such that

$$H^\bullet(U_n, U_n \cap \{\xi > \xi(p)\} \neq 0, \quad \forall n.$$

Loosely speaking this means that the region $U_n \cap \{\xi > \xi(p)\}$ is structurally different from $U_n$. We set

$$Ch_p(X) := Ch(X) \cap T^*_p E.$$

**Example 4.1.** Suppose $E = \mathbb{R}^2$ equipped with the standard Euclidean metric so we will identify $E^* = E$. Assume $X$ is a horizontal line segment. Denote by $\mathbb{T}_r(X)$ the tube of radius $r$ around $X$

$$\mathbb{T}_r(X) := \big\{ x \in E;\;\; \mathrm{dist}\,(x, X) \leq r \,\big\}.$$

For each $q \in \partial \mathbb{T}_r(c)$ there exists a unique point $\pi(q) \in X$ such that

$$\mathrm{dist}\,(q, \pi(p)\,) = r.$$

Denote by $R_q$ the ray which starts at $\pi(q)$ and goes through $q$ (see Figure 4). We can regard it as a ray in $T_{\pi(q)}\mathbb{R}^2 \cong T^*_{\pi(q)}\mathbb{R}^2$. Then

$$Ch(X) = \bigcup_{q \in \partial \mathbb{T}_r(X)} R_q.$$

We see that $Ch(X)$ is homeomorphic to the "aura" $\mathrm{Int}\,(\mathbb{T}_r)$ □

Motivated by this example we introduce the subbundle $D_r(TE^*) \to E$ of $TE^*$ of radius $r$ closed disks and we set

$$Ch_r(X) = Ch(X) \cap D_r(TE^*).$$

If $X$ is as in the above example then $Ch_r(X) \cong \mathbb{T}_r(X)$.

We have the following elementary facts.

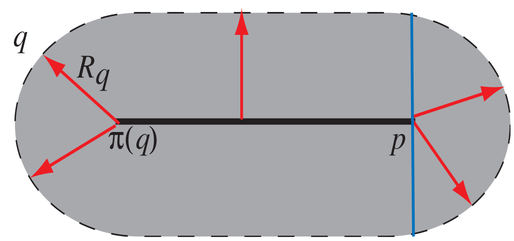


FIGURE 4. *The "aura" of a straight line segment in the plane*

**Proposition 4.2.** *(a)* $(0, p) \in Ch(X)$, $\forall p \in X$.
*(b) If* $(\xi, p) \in Ch(X)$ *then* $(t\xi, p) \in Ch(X)$, $\forall t \geq 0$. *(We say that* $Ch(X)$ *is a conic subset of the cotangent bundle.)*
*(c) If* $\sigma$ *is a simplex of* $X$, $p$ *is an interior point of* $\sigma$ *and* $(\xi, p) \in Ch(X)$ *then the simplex* $\sigma$ *is contained in the hyperplane* $\xi = \xi(p)$. *Equivalently this means that* $(\xi, p)$ *belongs to the conormal bundle* $T^*_{\mathrm{Int}\,\sigma}E$.

□

Given a point $p \in X$ there exists a unique simplex $\sigma$ such that $p \in \mathrm{Int}\,\sigma$. Suppose the simplex $\sigma$ is a face of the simplex $\tau$ (written $\sigma \preceq \tau$). We set

$$\Lambda_{\sigma,\tau}(p) := \left\{\, \xi \in E^*;\ \text{ the hyperplane } \xi = \xi(p) \text{ contains } \tau \,\right\}$$

$$\cong \left\{\text{the set of lines through } p \text{ perpendicular to } \tau\,\right\},$$

$$\Lambda(p) = \Lambda_\sigma(p) := \Lambda_{\sigma,\sigma}(p),$$

$$Ch_p(X, \tau) := Ch_p(X) \cap \Lambda_{\sigma,\tau}(p).$$

Observe that $\Lambda_\sigma(p)$ can be identified with the fiber at $p$ of the conormal bundle $T^*_{\mathrm{Int}\,\sigma}E$, or equivalently with the set of lines trough $p$ perpendicular to $\sigma$. In Figure 4 if we take $\sigma = p$ and $\tau =$ the segment $X$ then $Ch_p(X, \tau)$ is the vertical line through $p$ since any line through $p$ and perpendicular to that line will contain the segment $X$. Observe that

$$\mathrm{codim}\big(\, Ch_p(X, \tau) \hookrightarrow Ch_p(X)\,\big) = \mathrm{codim}(\sigma \hookrightarrow \tau) = \dim \tau - \dim \sigma.$$

Note that

$$\sigma \preceq \tau_1 \preceq \tau_2 \Longrightarrow \Lambda_{\sigma,\tau_1} \supseteq \Lambda_{\sigma,\tau_2}.$$

The star of $\sigma$ in $X$, denoted by $St(\sigma)$, is the subcomplex determined by all the simplices $\tau$ which admit $\sigma$ as a face

$$St(\sigma) := \bigcup_{\tau \succeq \sigma} \tau.$$

We get a collection (arrangement) of subspaces in $\Lambda_\sigma(p)$

$$\mathcal{A}_\sigma(p) = \Big\{\Lambda_{\sigma,\tau}(p);\ \ \tau \in St(\sigma)\Big\}.$$

We denote by $\Lambda^0_\sigma(p)$ the complement of this arrangement of planes. Its connected components are open polyhedral cones. We will refer to them as *chambers*. We denote by $\mathcal{C}_{\sigma,p}$ the collection of chambers of $\Lambda_\sigma(p)$. The covectors in $\Lambda^0_\sigma$ are called *nondegenerate covectors* (for $X$ at $p$). We set

$$Ch_p(X)^0 = Ch_p(X) \cap \Lambda^0_\sigma(p).$$

The covectors in $Ch_p(X)^0$ are called *nondegenerate characteristic vectors* (for $X$ at $p$). Observe that if $p, q \in \operatorname{Int}\sigma$ then

$$\Lambda^0_\sigma(p) = \Lambda^0_\sigma(q), \;\; \mathcal{C}_{\sigma,p} = \mathcal{C}_{\sigma,q},$$

so $\Lambda^0_\sigma(p)$ is really an invariant of the embedding $\operatorname{Int}\sigma \hookrightarrow X$. Since every point belongs to the interior of a single simplex so we can safely drop $p$ or $\sigma$ from the notations $\Lambda_\sigma(p)$, $\mathcal{C}_{\sigma,p}$.

For every $(\xi, p) \in T^*E$ we set

$$m(\xi, p, X) := \chi\big(\mathcal{H}^\bullet_{X_{\leq \xi(p)}}(p)\big) = \lim_{r\searrow 0} \chi\big(B_r(p) \cap X\big) - \chi\big(B_r(p) \cap X_{>\xi(p)}\big).$$

We will refer to $m(\xi, p, X)$ as the multiplicity of the generic covector $(x, p)$. Note that if $p \in X \setminus E$ $m(x, p, X) = 0$ for any $\xi \in \Lambda_p = T^*_p E$. On the other hand

$$m(\xi, p, X) = m(\xi, q, X), \;\; \forall p, q \in \operatorname{Int}\sigma,$$

so we can use the notation $m_\sigma(\xi, X)$ for $m(\xi, p, X)$, $p \in \operatorname{Int}\sigma$. To provide a combinatorial description of these integers we need to introduce some terminology.

Given two simplices $\sigma \prec \tau$ we denote by $Lk(\sigma, \tau)$ the maximal face of $\tau$ which is disjoint from $\sigma$. In other words $Lk(\sigma, \tau)$ is the face of $\tau$ "opposite" to $\sigma$. Observe that

$$\dim\sigma + \dim Lk(\sigma, \tau) = \dim\tau + 1.$$

Given a simplicial complex $K$ and $\sigma$ a simplex in define the *link* of $\sigma$ in $K$ to be the subcomplex

$$Lk(\sigma, K) := \bigcup_{\sigma \precneqq \tau} Lk(\sigma, \tau). \tag{4.1}$$

Fix a point $p \in X$ and denote by $\sigma$ the unique simplex $\sigma$ in $X$ such that $p \in \operatorname{Int}(\sigma)$. For $\xi \in \Lambda_\sigma$ we define

$$St^+_\xi(p) = St^+_\xi(\sigma) = \big\{\tau \in St(\sigma);\;\; \xi(x) \geq \xi(p), \;\; \forall x \in \tau\big\} = \big\{\tau \in St(\sigma);\;\; \tau \subset \{\xi \geq \xi(p)\}\big\},$$
$$Lk^+_\xi(p) = Lk^+_\xi(\sigma) = Lk^+_\xi(p, X) = Lk(\sigma, St^+_\xi(\sigma))$$

**Proposition 4.3.** *Suppose $p \in \operatorname{Int}\sigma$ and $\xi \in \Lambda^0_\sigma$ is a nondegenerate vector. Then*

$$m(\xi, p) = 1 - \chi\big(Lk^+_\xi(p)\big) = (-1)^{\dim\sigma} \sum_{\tau \succ_\xi \sigma} (-1)^{\dim\tau},$$

*where $\tau \succ_\xi \sigma$ signifies that $\tau \succ \sigma$ and $\tau \subset \{\xi \geq \xi(p)\}$*

**Proof** For $r > 0$ sufficiently small $B_r(p) \cap X$ is a deformation retract of $St(\sigma)$ so that

$$\chi(B_r(p) \cap X) = \chi(St(\sigma)) = 1.$$

Arguing exactly as in Example 3.4 one proves that $B_r(p) \cap X_{\{\xi > \xi(p)\}}$ is a deformation retract of $St(\sigma)_{\{\xi > \xi(p)\}}$ and then that $St(\sigma)_{\{\xi > \xi(p)\}}$ deformation retracts onto the complement of $\sigma$ in $St^+_\xi(\sigma)$. Finally this complement deformation retracts onto $Lk^+_\xi(\sigma)$. Hence

$$\chi\big(B_r(p) \cap X_{\{\xi > \xi(p)\}}\big) = \chi(Lk^+_\xi(p)).$$

Next, observe that

$$\chi(Lk^+_\xi(p)) = \sum_{\tau \succ_\xi \sigma} (-1)^{\dim Lk(\sigma,\tau)} = \sum_{\tau \succ_\xi \sigma} (-1)^{\dim\tau + 1 - \dim\sigma} = -(-1)^{\dim\sigma} \sum_{\tau \succ_\xi \sigma} (-1)^{\dim\tau}.$$

Hence

$$1 - \chi(Lk^+_\xi(p)) = 1 + (-1)^{\dim\sigma} \sum_{\tau \succ_\xi \sigma} (-1)^{\dim\tau} = (-1)^{\dim\sigma} \sum_{\tau \succ_\xi \sigma} (-1)^{\dim\tau}$$

□

We see that the multiplicity of a generic covector as defined above coincides with the multiplicity defined in [2]. The above result has an important consequence.

**Corollary 4.4.** *Suppose the generic covectors* $(\xi_i, p)$, $i = 0, 1$ *belong to the same chamber* $C \in \mathcal{C}_p$*. Then*
$$m(\xi_0, p, X) = m(\xi_1, p, X).$$

**Proof** Since $x_0$ and $\xi_1$ belong to the same chamber we deduce
$$Lk^+_{\xi_0}(\sigma) = Lk^+_{\xi_1}(\sigma)$$
whence the equality of the two multiplicities.

□

The multiplicity function we have just constructed associates to each chamber at $p \in X$ an integer and thus can be viewed as a function $m_p : \mathcal{C}_{\sigma,p} \to \mathbb{Z}$. Again $m_p = m_q$ for all $p, q \in \operatorname{Int} \sigma$.

**Example 4.5.** Consider again the planar star in Figure 2. We denote it by $X$ and we denote by $P_0$ its center. In Figure 5 this simplicial complex is described with dotted lines. We would like to describe the chamber structure at $P_0$. Assume for simplicity that $P_0$ is the origin. We identify $T^*\mathbb{R}^2$ with $T\mathbb{R}^2$. The linear functionals containing an arm of the start in a level set can be identified with the line orthogonal to that arm at $P_0$. We get three such lines are depicted as continuous lines in Figure 5.

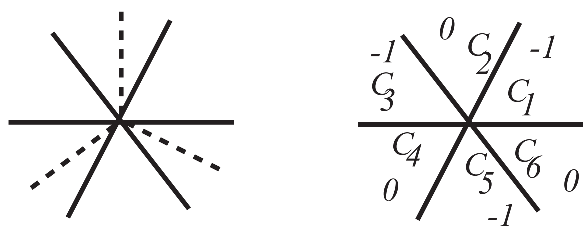


FIGURE 5. *The chambers at the vertex of a three-armed star*

They divide the plane into six cones denoted by $C_1, \cdots, C_6$. The multiplicities of the corresponding chamber are indicated in the right-hand-side of Figure 5. More precisely
$$m(P_0, C_k) = \frac{-1 + (-1)^k}{2} = \begin{cases} 0 & \text{if} \quad k \text{ is even} \\ -1 & \text{if} \quad k \text{ is odd} \end{cases}.$$

□

**Proposition 4.6.** *Suppose* $X_1, X_2 \subset$ *are two simplicial complexes such that* $X_1 \cap X_1$ *is a subcomplex of both. Then for every* $(x, p) \in T^*E$ *we have*
$$m(\xi, p, X_1 \cup X_2) = m(x, p, X_1) + m(\xi, p, X_2) - m(\xi, p, X_1 \cap X_2).$$

**Proof** For $r > 0$ sufficiently small and $Y = X_1 \cup X_2, X_1, X_2$ or $X_1 \cap X_2$ we have the equality
$$m(\xi, p, Y) = \chi(B_r \cap Y) - \chi(B_r \cap Y_{\{\xi > \xi(p)\}}).$$
The proposition now follows from the inclusion-exclusion property of the Euler characteristic.

□

To define the characteristic cycle we need a brief detour in the theory of currents. For more details we refer to [4].

Suppose $V$ is a connected, oriented smooth manifold of dimension $n$. We denote by $\Omega^k(V)$ the vector space of smooth $k$-dimensional forms and by $\Omega^k_{cpt}(V)$ the space of smooth, compactly supported $k$-dimensional forms. They have natural structure of locally convex topological vector spaces with the topology given by the uniform convergence on compacts of the forms and their partial derivatives.

For every $k \geq 0$ we denote by $\Omega_k(V)$ the topological dual of $\Omega^k_{cpt}(V)$, i.e. the space of continuous linear functionals $\Omega^k_{cpt}(V) \to \mathbb{R}$. Similarly we define $\Omega^{cpt}_k(V)$ to be the topological dual of $\Omega^k(V)$. For $C \in \Omega_k(V)$ we denote its action on $\eta \in \Omega^k_{cpt}(V)$ by $\langle C, \eta \rangle$.

Observe that we have an embedding

$$D : \Omega^{n-k}(V) \hookrightarrow \Omega_k(V), \;\; \omega \longmapsto D_\omega : \Omega^k_{cpt}(V) \to \mathbb{R}, \;\; \langle D_\omega, \eta \rangle = \int_M \omega \wedge \eta, \;\; \forall \eta \in \Omega^k_{cpt}(M).$$

We will refer to $D$ as the *Poincaré duality map*. We have a boundary operator

$$\partial \Omega_k(V) \to \Omega_{k-1}(V), \;\; \langle \partial C, \eta \rangle = \langle C, d\eta \rangle, \;\; \forall \eta \in \Omega^{k-1}_{cpt}(V).$$

We obtain in this fashion of chain complex $(\Omega_\bullet(V), \partial)$. Its homology is called the *Borel-Moore homology* of $V$, or the homology of $V$ with closed supports. It will be denoted by $H^{cl}_\bullet(V)$. The Poincaré duality map induces an isomorphism

$$H^\bullet(V) \to H^{cl}_{n-\bullet}(V).$$

**Example 4.7.** Suppose $V$ is an oriented real vector space and $P$ is a polyhedral region, i.e. a finite intersection of half-spaces (closed or open). Let $p = \dim P$. In other words $p$ is the dimension of the affine subspace $\mathrm{span}\,(P)$ spanned by $P$.

Any orientation $or$ on $\mathrm{span}\,(P)$ determines a $p$-current $[P] = [P, or]$ defined by

$$\langle [P], \eta \rangle = \int_{P, or} \eta, \;\; \forall \eta \in \Omega^p_{cpt}(V).$$

We will say that $[P, or]$ is the *integration current* defined by $P$ and the orientation $or$.

Denote by $\mathcal{F}(P)$ the collection of $(p-1)$-dimensional faces of $P$. For every face $F \in \mathcal{F}(P)$ the orientation $or$ on $P$ induces an orientation$or_F$ on $F$ determined by the *outer-normal-first* convention. For example, in Figure 6 where we depicted a 2-dimensional polyhedron in $\mathbb{R}^2$ equipped with the orientation induced from the canonical orientation of $\mathbb{R}^2$. The classical Stokes formula implies

$$\int_{[P,or]} d\eta = \sum_{F \in \mathcal{F}(P)} \int_{[F, or_F]} \eta, \;\; \forall \eta \in \Omega^{p-1}_{cpt}(V).$$

Hence

$$\partial [P, or] = \sum_{F \in \mathcal{F}(P)} [F, or_F].$$

Note that if we remove from $P$ a finite collection of polyhedral regions of dimensions $< p$ and we integrate on the remaining region, the integration current thus obtained is equal to $P$.

□

For each simplex $\sigma \in X$ and each chamber $C \in \mathcal{C}_\sigma$ we consider the open polyhedral subset $\mathbf{Ch}(\sigma, C)^0 := C \times \mathrm{Int}\,\sigma$ of the conormal bundle of $\mathrm{Int}\,\sigma$. The *characteristic variety* of $X$ is the closed set

$$\mathbf{Ch}(X) = \bigcup_{\sigma \in X, \; C \in \mathcal{C}_\sigma} \overline{\mathbf{Ch}(\sigma, C)^0}.$$

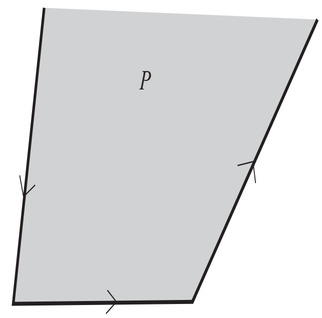


FIGURE 6. *A polyhedron in $\mathbb{R}^2$ and its boundary.*

The smooth part of the characteristic variety, denoted by $\mathbf{Ch}(X)^0$, is filled-up by the nondegenerate characteristic vectors

$$\mathbf{Ch}(X)^0 = \bigcup_{\sigma\in X,\ C\in\mathcal{C}_\sigma} \mathbf{Ch}(\sigma, C)^0.$$

It is a finite disjoint union of oriented polyhedral regions $\mathbf{Ch}(\sigma, C)^0$ of dimension $N$. Each defines a $N$-dimensional current $\mathbf{CC}(\sigma, C)^0$ and we define

$$\mathbf{CC}(X) = \sum_{\sigma\in X}\Big(\sum_{C\in\mathcal{C}_\sigma} m_\sigma(C)\,\mathbf{CC}(\sigma, C)\Big) \in \Omega_N(T^*E).$$

We say that $\mathbf{CC}(X)$ is the *conormal chain* of $X$.

For any two sets $A, B \subset T^*E$ we use the notation $A \approx B$ to signify that

$$A \cup S = B \cup R,$$

where $S, R$ are unions of polyhedral sets of dimension $< N$. This is an equivalence relation and we denote by $[A]$ the equivalence class of $[A]$. Note that if $A$, $B$ are two oriented $N$-dimensional polyhedral sets and $A \approx B$ then $A$ and $B$ define the same integration current which we denote by $[A]$. We regard the multiplicity function $m_X$ as a function defined on a set $\approx T^*E$. Its level sets carry a natural orientation and for every $k \in \mathbb{Z}$ we denote by $[m_X = k]$ the current defined by the $\approx$ class of the level set $m_X^{-1}(k)$. We see that we can define the conormal cycle by the formula

$$\mathbf{CC}(X) = \sum_{k\in\mathbb{Z}} k[m_X = k].$$

**Proposition 4.8.** *Suppose $X_1, X_2$ are finite simplicial complexes in the oriented vector space $E$ such that $X_1 \cap X_2$ is a subcomplex of both. Then we have the following equality in $\Omega_N(T^*E)$.*

$$\mathbf{CC}(X_1 \cup X_2) = \mathbf{CC}(X_1) + \mathbf{CC}(X_2) - \mathbf{CC}(X_1 \cap X_2).$$

**Proof** Using Proposition 4.6 we deduce

$$\{m_{X_1\cup X_2} = \ell\} \approx \bigsqcup_{i+j+k=\ell} \{m_{X_1} = i\} \cap \{m_{X_2} = j\} \cap \{m_{X_1\cap X_2} = -k\}.$$

We deduce the following equality of currents.

$$\mathbf{CC}(X_1 \cup X_2) = \sum_\ell \ell[m_{X_1\cup X_2} = \ell] = \sum_{i,j,k} (i + j + k)[m_{X_1} = i, m_{X_2} = j, m_{X_1\cap X_2} = -k]$$

$$= \sum_{i,j,k} i[m_{X_1} = i, m_{X_2} = j, m_{X_1\cap X_2} = -k] + \sum_{i,j,k} j[m_{X_1} = i, m_{X_2} = j, m_{X_1\cap X_2} = -k]$$

$$+\sum_{i,j,k} k[m_{X_1}=i, m_{X_2}=j, m_{X_1\cap X_2}=-k]$$
$$=\sum_i i[m_{X_1}=i]+\sum_j [m_{X_2}=j]-\sum_k k[m_{X_1\cap X_2}=k]$$
$$=\mathbf{CC}(X_1)+\mathbf{CC}(X_2)-\mathbf{CC}(X_1\cap X_2).$$

□

For any compact simplicial complex $X\subset E$ we denote by $\mathbb{1}_X$ its characteristic function. If $X_1, X_2$ are simplicial complexes then we can subdivide each of them so that $X_1\cap X_2$ is a subcomplex of both. Moreover

$$\mathbb{1}_{X_1\cup X_2}=\mathbb{1}_{X_1}+\mathbb{1}_{X_2}-\mathbb{1}_{X_1\cap X_1}=\mathbb{1}_{X_1}+\mathbb{1}_{X_2}-\mathbb{1}_{X_1}\mathbb{1}_{X_2}. \tag{4.2}$$

We can rewrite the last equality as

$$1-\mathbb{1}_{X_1\cup X_2}=(1-\mathbb{1}_{X_1})(1-\mathbb{1}_{X_2}).$$

For every simplicial complex $X$ we denote by $\mathcal{F}(X)$ the Abelian subgroup of the group of $\mathbb{Z}$-valued functions on $E$ spanned by the characteristic functions of subcomplexes of $X$. If $G$ is an Abelian group, then a *$G$-valued measure on $X$* is a function which associates to each subcomplex $K$ an element $m(K)\in G$ such that the inclusion-exclusion principle is satisfied

$$m(K_1\cup K_2)=m(K_1)+m(K_2)-m(K_1\cap K_2).$$

A *$G$-valuation* on $X$ is a morphism of Abelian groups $\mathcal{F}(X)\to G$.

*Remark* 4.9. The equality (4.2) shows that every $G$-valuation $\mu$ on $X$ defines a $G$-valued measure via the equality

$$m(K)=\mu(\mathbb{1}_K).$$

We obtain a map

$$\Psi_{X,G}:\mathrm{Hom}(\mathcal{F}(X),G)\to \mathrm{Meas}_G(X):=G\text{-valued measures on }X$$

Observe also that the correspondence $K\longmapsto \mathbb{1}_K$ is a $\mathcal{F}(X)$-valued measure on $X$. We want to prove that $\Psi_{X,G}$ is a bijection.

□

**Proposition 4.10.** *$\mathcal{F}(X)$ is a free Abelian group generated by the characteristic functions of the (closed) simplices in $X$.*

**Proof** We first prove that the family of functions $\mathbb{1}_\sigma$ is $\mathbb{Z}$-linearly independent. Suppose we have an equality of the form

$$a:=\sum_{\sigma\in X} a_\sigma \mathbb{1}_\sigma=\sum_{\sigma\in X} b_\sigma\mathbb{1}_\sigma=:b, \tag{4.3}$$

where $a_\sigma, b_\sigma\in\mathbb{Z}_{\geq 0}$. Let

$$A=\{\sigma\in X;\;\; a_\sigma\neq 0\},\;\; B=\{\sigma\in X;\;\; b_\sigma\neq 0\}.$$

We have to prove that

$$A=B,\;\; a_\sigma=b_\sigma,\;\; \forall\sigma\in A. \tag{4.4}$$

Let $\alpha$ be an element in $A$ maximal with respect to the order relation "$\prec$". Let $p$ be a point in $\mathrm{Int}\,|\alpha|$. Then

$$0<a_\alpha=a(p)=b(p)$$

and from the equality (4.3) we deduce that the set

$$B_\alpha := \{\sigma \in B;\ \ \alpha \prec \sigma\}$$

is nonempty. Let $\beta$ be a maximal element in $B_\alpha$. We claim that $\beta = \alpha$ If $q \in \operatorname{Int}|\beta|$ then

$$0 < b_\beta = b(q) = a(q).$$

Hence there must exists an element in $\gamma \in A$ such that $\gamma \succ \beta \succ \alpha$. Since $\alpha$ is maximal we deduce

$$\alpha = \beta = \gamma,\ \ a_\alpha = b_\beta.$$

We deduce that $\max A$, the set of maximal elements in $A$, is contained in $B$ and

$$a_\sigma = b_\sigma,\ \ \forall \sigma \in \max A.$$

If we set $A_1 = A \setminus \max A$, $B_1 = B \setminus \max A$ we deduce an equality

$$\sum_{\sigma \in A_1} a_\sigma \mathbb{1}_\sigma = \sum_{\sigma \in B_1} b_\sigma \mathbb{1}_\sigma,\ \ |A_1| < |A|,\ \ |B_1| < |B|$$

Iterating this procedure we deduce (4.4).

Now let us prove that the family $\{\mathbb{1}_\sigma;\ \ \sigma \in X\}$ spans $\mathcal{F}(X)$. Let $K$ be a subcomplex of $X$. We want to prove that we can write

$$\mathbb{1}_K = \sum_{\sigma \in K} \nu_K(\sigma) \mathbb{1}_\sigma,\ \ \nu_K(\sigma) \in \mathbb{Z}.$$

We define $\nu_K(\sigma)$ by descending induction

$$\nu_K(\sigma) = 1 - \sum_{\tau \succ \sigma} \nu_K(\tau) \tag{4.5}$$

□

From Proposition 4.10 we deduce that a valuation $\mu$ is uniquely determined by the quantities

$$\mu(\sigma) := \mu(\mathbb{1}_\sigma).$$

*Remark* 4.11. Suppose $K$ is a finite simplicial complex and that $R$ is a commutative ring with 1. We consider the space $I(K)$ of $K \times K$-matrices $A$ with entries in $R$ such that

$$A(\sigma, \tau) \neq 0 \Longrightarrow \sigma \preceq \tau.$$

The $\zeta$-*function* of $K$ is the incidence matrix of the face relation

$$\zeta_K(\sigma, \tau) = \begin{cases} 1 & \text{if}\ \ \sigma \preceq \tau \\ 0 & \text{if}\ \ \sigma \npreceq \tau \end{cases}.$$

Observe that $I(K)$ is a $R$-algebra with respect to the addition and the usual multiplication if matrices. Note that $\zeta_K \in I(K)$. $\zeta_K$ is an invertible element of $I(K)$ and, following the terminology of [11, Chap.3], we denote by $\mu_K$ its inverse. It is known as the *Möbius function of* $K$. The matrices in $I(K)$ act in the usual on $R^K$. We denote by $\mathbb{X}_K \in R^K$ the vector

$$\mathbb{X}_K(\sigma) = 1,\ \ \forall \sigma.$$

We regard the correspondence $\sigma \mapsto \nu_K$ in the proof of Proposition 4.10 as a vector in $R^K$. The equality (4.5) can be rewritten as

$$\zeta_K \cdot \nu_K = \mathbb{X}_K \Longrightarrow \nu_K = \mu_K \cdot \mathbb{X}_K. \tag{4.6}$$

We can be more specific about $\mu$. Observe that $\zeta_K$ is an upper triangular matrix with 1's along the diagonal. In particular we deduce that $\zeta_K - 1$ is a nilpotent matrix so that

$$\mu_K = \zeta_K^{-1} = \Big(1 + (\zeta_K - 1)\Big)^{-1} = \sum_{n \geq 0} (-1)^n (\zeta_K - 1)^n.$$

Now observe that

$$(\zeta_K - 1)^n(\sigma, \tau) = \sum_{\sigma \prec \sigma_1 \prec \cdots \prec \sigma_n = \tau} 1$$

$$= \text{the number of increasing chains of length } n \text{ from } \sigma \text{ to } \tau : \quad \sigma \prec \sigma_1 \prec \cdots \prec \sigma_n = \tau.$$

We denote this number by $c_n(\sigma, \tau) = c_n(\sigma, \tau; K)$. Hence

$$\mu_K(\sigma, \tau) = \sum_{n \geq 0} (-1)^n c_n(\sigma, \tau).$$

This alternating sum can be computed directly[4] or we can invoke [11, Ex. 3.8.3] to conclude that

$$\mu_K(\sigma) = (-1)^{\dim \tau - \dim \sigma}.$$

Hence

$$\nu_K(\sigma) = \sum_{\tau \succeq \sigma} (-1)^{\dim \tau - \dim \sigma} = 1 - \sum_{\tau \succ \sigma} (-1)^{\dim \tau - \dim \sigma - 1}.$$

The last sum is precisely the Euler characteristic of the link of $\sigma$ in $K$ as defined in (4.1). Hence

$$\nu_K(\sigma) = 1 - \chi\big(\operatorname{Lk}(\sigma, K)\big) \tag{4.7}$$

If we denote by $H^\bullet_\sigma(K)$ the local cohomology of $K$ along $\sigma$ then we have

$$\nu_K(\sigma) = \chi_\sigma(K) := \chi\big(H^\bullet_\sigma(K)\big). \tag{4.8}$$

The number $\chi_\sigma(K)$ can be computed as follows. Consider the plane $P$ of codimension $= \dim \sigma$ perpendicular to $\sigma$ at its barycenter $b_\sigma$. Consider a sphere $S(\varepsilon, \sigma)$ of radius $\varepsilon$ in $P$ centered at $b_\sigma$. Then

$$\chi_\sigma(K) = 1 - \lim_{\varepsilon \searrow 0} \chi(S(\varepsilon, \sigma) \cap K). \tag{4.9}$$

We deduce that if $\mu$ is a valuation on $X$ we have

$$\mu(\mathbb{1}_X) = \sum_{\sigma \in X} \chi_\sigma(X)\mu(\mathbb{1}_\sigma). \tag{4.10}$$

□

**Proposition 4.12.** *Suppose $m$ is a $G$-valued measure on $X$. Denote by $\lambda$ the valuation determined by*

$$\lambda(\mathbb{1}_\sigma) = m(\sigma).$$

*Then for every subcomplex $K$ of $X$ we have*

$$m(K) = \lambda(\mathbb{1}_K).$$

[4]Let $S_{n,k}$ denote the number of $k$-chains $\emptyset \subsetneq T_1 \subsetneq \cdots \subsetneq T_k = \{1, \cdots, n\}$. Then $S_{n,k} = \sum_{j>0} \binom{n}{j} S_{n-j,k-1}$. If we set $c_n = \sum_k (-1)^k S_{n,k}$ we deduce that $c_n = -\sum_{j>0} \binom{n}{j} c_{n-j}$. The last equality implies inductively that $c_n = (-1)^n$.

**Proof** Consider again the quantity $\nu_K$ defined inductively in (4.5). The Möbius inversion formula [11, Prop. 3.7.1] implies

$$m(K) = \sum_{\sigma\in K} \nu_K(\sigma)m(\sigma) = \lambda\Big(\sum_{\sigma\in K}\nu_K(\sigma)\mathbb{1}_\sigma\Big) = \lambda(\mathbb{1}_K).$$

□

*Remark* 4.13. Observe that if $m$ is a $G$ valued measure and $\varphi : G \to H$ is a morphism of groups we obtain a new measure

$$\varphi_* m(K) := \varphi(m(K)).$$

We denote by $\mathbb{1}$ the $\mathcal{F}(X)$-valued measure $K \mapsto \mathbb{1}_K$. The results established so far show that for any $G$-valued measure $m$ on $X$ there exists a unique morphism $\varphi : \mathcal{F}(X) \to G$ such that

$$m = \varphi_* \mathbb{1}.$$

□

**Example 4.14.** Suppose $\Delta$ is a $n$-dimensional simplex and $m$ is a measure on $\Delta$. We want to compute $m(\partial\Delta)$. Using (4.10) we deduce

$$m(\partial\Delta) = \sum_{\dim\sigma<n} \chi_\sigma(\partial\Delta)m(\sigma).$$

$\partial\Delta$ is a topological manifold and using (4.9) we deduce

$$\chi_\sigma(\partial\Delta) = (-1)^{n-1-\dim\sigma}$$

Hence

$$m(\partial\Delta) = (-1)^{\dim\partial\Delta}\sum_{\sigma\in\partial\Delta}(-1)^{\dim\sigma}m(\sigma).$$

□

Given a simplicial complex $X$ in the $N$-dimensional oriented real vector space $E$ and $\sigma \in X$ we denote by $\mathbf{CC}(\sigma)$ the conormal chain of $\sigma$. The correspondence $\sigma \longmapsto \mathbf{CC}(\sigma)$ is a $\Omega_N(T^*E)$-valued measure and as such it extends to a valuation

$$\mathbf{CC} : \mathcal{F}(X) \to \Omega_N(T^*E).$$

Now observe that for every simplex $\sigma$ we have $\partial\,\mathbf{CC}(\sigma) = 0$ so that $\mathbf{CC}(\sigma)$ is a Lagrangian cycle. We deduce that $\mathbf{CC}(X) = \mathbf{CC}(\mathbb{1}_X)$ is a cycle as well. We denote by $\mathcal{Z}_N(T^*E) \subset \Omega_N(T^*E)$ the subgroup of $N$-cycles. Note that we have an equality

$$\mathbf{CC}(X) = \sum_{\sigma\in X}\chi_\sigma(X)\,\mathbf{CC}(\sigma). \tag{4.11}$$

Suppose $X$ is a finite simplicial complex in the oriented $N$-dimensional vector space $E$ and $X'$ is a simplicial subdivision of $X$. We write this $X < X'$. The subcomplexes of $X$ are subcomplexes of $X'$ and thus we have a natural map

$$I_{X'X} : \mathcal{F}(X) \longrightarrow \mathcal{F}(X').$$

The conormal cycle construction defines group morphisms

$$\mathbf{CC}_X : \mathcal{F}(X) \to \mathcal{Z}_N(T^*E), \quad \mathbf{CC}_{X'} : \mathcal{F}(X) \to \mathcal{Z}_N(T^*E)$$

such that the diagram below is commutative.

$$\begin{tikzcd}
\mathcal{F}(X') \arrow[dr, "\mathbf{CC}_{X'}"] & \\
& \mathcal{Z}_N(T^*E) \\
\mathcal{F}(X) \arrow[uu, "I_{X'X}"] \arrow[ur, "\mathbf{CC}_X"'] &
\end{tikzcd} \;. \tag{4.12}$$

We denote by $|X|$ the topological space subjacent to the complex $X$ and we set

$$\mathcal{F}(|X|) = \varinjlim_{X'>X} \mathcal{F}(X').$$

The group $\mathcal{F}(|X|)$ is the subgroup of $\mathbb{Z}$-valued functions on $|X|$ corresponding to (linearly) triangulable subsets. We obtain in this fashion a morphism

$$\mathbf{CC}_{|X|} : \mathcal{F}(|X|) \to \mathcal{Z}_N(T^*E).$$

Finally, we denote by $\mathcal{T}(E)$ the collection of compact, triangulable subsets of $E$. For any $A, B \in \mathcal{T}(E)$ we have a morphism

$$\mathcal{F}(A) \to \mathcal{F}(B)$$

and a commutative diagram similar to (4.12). We set

$$\mathcal{F}(E) = \varinjlim_{A} \mathcal{F}(A)$$

and we deduce in a similar fashion the existence of a group morphism

$$\mathbf{CC}_E : \mathcal{F}(E) \to \mathcal{Z}_N(T^*E).$$

$\mathbf{CC}_E$ associates to each triangulable compact set $A$ its conormal cycle in $E$, $\mathbf{CC}_E(A)$. When $E$ is obvious from the context we will drop it from the notation.

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