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ETA INVARIANTS OF DIRAC OPERATORS ON CIRCLE BUNDLES OVER RIEMANN SURFACES AND VIRTUAL DIMENSIONS OF FINITE ENERGY SEIBERG–WITTEN MODULI SPACES

BY

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ABSTRACT

Using an adiabatic collapse trick we determine, by two different methods, the eta invariants of many Dirac type operators on circle bundles over Riemann surfaces. These results, coupled with a delicate spectral flow computation, are then used to determine the virtual dimensions of moduli spaces of finite energy Seiberg–Witten monopoles on 4-manifolds bounding such circle bundles.

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Introduction

The eta invariant was introduced in mathematics in the celebrated papers [APS1–3] as a correction term in an index formula for a non-local, elliptic boundary value problem and since then it has been subjected to a lot of scrutiny because of its appearance in many branches of mathematics.

Contrary to the index density of an elliptic operator, the eta invariant is a *non-local* object and this explains why it is so much harder to compute. Most concrete computations rely on special topologic or geometric features. For example, one can use the Atiyah–Patodi–Singer theorem to compute the eta invariant of the signature operator because in this case the eta invariant is a combination of a topological term (the signature of a $4k$-dimensional manifold with boundary) and a local contribution (the integral of the L-genus). For S^1-bundles over Riemann surfaces, this approach was successfully carried out in [Ko] (see also [O] for similar results in the more general case of Seifert manifolds).

For the Dirac operator associated to a spin structure such an approach is not possible because the index of the Atiyah–Patodi–Singer problem is notoriously dependent upon the metric. However, if all the manifolds involved have positive scalar curvature then a Lichnerowicz type argument allows the computation of the index and thus, in this case, the computation of the eta invariant is a local problem.

The first goal of this paper is to compute the eta invariant of some Dirac operators on the total space of a *nontrivial* circle bundle N over a Riemann surface Σ of genus ≥ 1. The second goal is to use the eta invariant information to determine the virtual dimensions of the moduli spaces of finite energy solutions of the Seiberg–Witten equations on a 4-manifold bounding a disjoint union of circle bundles over Riemann surfaces.

As in [N], we will work with product-like metrics on N such that the fibers are very short. Such metrics have negative scalar curvatures and thus are beyond the reach of the Lichnerowicz vanishing approach. Instead, using the results of Bismut–Cheeger [BC] and Dai [Dai] we will compute the eta invariant for the usual Dirac operator using its known adiabatic limit (i.e. its limiting value as the geometry of N changes so that the fibers become shorter and shorter). To recover the eta invariant (at least for short fibers) one can use known variational formulæ and some very precise information about the very small eigenvalues (in the sense of [Dai]) of the Dirac operators determined by metrics with very short fibers. It turns out that the variational formulæ in this case involve no spectral flow contribution.

Once this computation is performed we embark on a related problem. More precisely, we will determine the eta invariant of a very special scalar perturbation of the Dirac operator. These perturbed Dirac operators (we called them *adiabatic Dirac operators*) arose in [N] where we studied the adiabatic limits of the Seiberg–Witten equations on circle bundles (see also [MOY]). We again use a variational approach. This time, however, there is a spectral flow contribution which requires some "spectral care".

An adiabatic approach was also used in [SS] to compute the eta invariant of Dirac operators on circle bundles over Riemann surfaces of genus ≥ 2. There are two main differences. The first difference comes from the *spin* structure considered in [SS] which extends to the disk bundle bounding our circle bundle. We perform our computations on Dirac operators associated to $spin^c$ structures pulled back from the base of our fibration and these, as explained in [KS], have notable topological properties. For example, the pullback of a *spin* structure from the base does not extend to a *spin*-structure on the bounding disk bundle, though it extends as a $spin^c$-structure. This explains why the adiabatic limit in [SS] is different from ours and shows that the eta invariants can distinguish *spin* structures!!!

The second difference is in the manner in which the adiabatic limit is computed. In [SS], using the representation theory of $PSL_2(\mathbb{R})$ the authors determine explicitly the adiabatically important part of the spectrum which allows them to determine the adiabatic limit of eta itself. We achieve this in two ways. The first method uses the results of Bismut, Cheeger and Dai. In Appendix C we present a second method, which works for the adiabatic Dirac operators. Their *whole eta functions* can be computed directly and "elementarily", and can be elegantly described in terms of Riemann's zeta function and some topological invariants. This argument extends easily to the more general case of Seifert manifolds. We present this extension in a separate paper [N1] to isolate the very complex combinatorics, generated by the singular fibers, from the analytical arguments, which work without any modification in the general case.

The eta invariant is an essential ingredient in the computation of the virtual dimension of the moduli space of finite energy solutions of the Seiberg–Witten equations on a 4-manifold with boundary a disjoint union of S^1-bundles over Riemann surfaces. For closed 4-manifolds the virtual dimension of the moduli space of solutions of the Seiberg–Witten equations corresponding to the *spin*^{c} structure σ is given by

$$d(\sigma) = \frac{1}{4}(c_1(\sigma)^2 - (2e + 3\tau))$$

where $c_1(\sigma)$ denotes the Chern class of the line bundle determined by the *spin*^{c} structure, while e respectively τ are the Euler characteristic and resp. the signature of the 4-manifold.

In the non-closed case the above formula is no longer true. There is a correction term determined by the asymptotic value of a finite-energy solution.

We compute this correction term via the Atiyah–Patodi–Singer and the Seiberg–Witten analogues of the results in [MMR] describing the structure of the finite energy moduli space. There is an additional difficulty one has to overcome. The operators describing the deformation complex of this moduli space are based not just on the adiabatic operator alone. They depend on a very explicit (though complicated) perturbation of the direct sum (*Dirac operator* $\oplus$ *odd signature operator*). The final determination of the virtual dimension relies on an excision trick which requires a spectral flow computation. Some of the eigenvalues changing sign do not do this transversally and detecting them is a very delicate perturbation theoretic problem. The theoretical basis of our approach is described in [FL] and [KK] which deal with similar degeneration problems in the case of the odd signature operators twisted by flat connections.

We obtain explicit formulæ for the virtual dimensions for *any 4-manifold bounding disjoint unions of circle bundles.* We briefly describe one instance when the asymptotic limit of a finite energy solution is irreducible.

The total space N of a degree $\ell \neq 0$ S^1-bundle over a Riemann surface Σ of genus g can be equipped with a *spin* structure obtained by pullback from a fixed *spin* structure on Σ. The *spin*^{c} structures can be identified with second degree integral cohomology classes $\sigma \in H^2(N) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}_{|\ell|}$. The three dimensional Seiberg–Witten equations have solutions only if σ is a torsion class $\sigma = k \bmod |\ell|$. Set

$$R_k = \{n \in \mathbb{Z};\ 1 \leq |n| \leq g-1,\ \ n \equiv k \mod \ell\}.$$

In [N] we have shown that the space of irreducible solutions of a certain perturbation of the Seiberg–Witten equations on N is smooth, and its components are bijectively parametrized by R_k. Fix a *spin*^{c} structure $\hat{\sigma}$ on $\hat{N}$ extending the *spin*^{c} structure k on N. Suppose $\hat{N}$ is a four manifold with boundary $\partial\hat{N} = N$ and $\hat{\mathrm{C}} := (\hat{\psi}, \hat{A})$ is a finite energy solution of the Seiberg–Witten equations on $\hat{N}_\infty = \hat{N} \cup \mathbb{R}_+ \times N$. If the asymptotic limit of $\hat{\mathrm{C}}$ is an irreducible solution on N lying in the connected component labelled by $n \in R_k$, then the expected dimension of a neighborhood of $\hat{\mathrm{C}}$ in its moduli space is

$$\frac{1}{4}\left(-\frac{1}{4\pi}^2 \int_{\hat{N}_\infty} F_{\hat{A}} \wedge F_{\hat{A}} - (2e(\hat{N}) + 3\mathrm{sign}(\hat{N}))\right),$$

$$\frac{1}{2}(\mathrm{sign}(\ell) - 1) + n + \frac{1}{2}(2g-1) - \frac{1}{4}(\ell - \mathrm{sign}(\ell)).$$

We tested our results in special case of "tunnelings". These are finite energy solutions of the Seiberg–Witten equations on an infinite cylinder $\mathbb{R} \times N$. Our results are in perfect agreement with the computations in [MOY] obtained by entirely different methods.

There are similarities between our paper and [MOY], but there are also many important differences. The paper [MOY] is interested in finite energy solutions of the Seiberg–Witten equations *only on cylinders* $\mathbb{R} \times M$ where M is a Seifert fibration. The techniques used there are *algebraic-geometric* in nature and allow them to obtain detailed information about the nature of solutions, leading eventually to virtual dimension formulæ. In this paper (and its sequel [N1]) we are interested in finite energy solutions on *any 4-manifold with cylindrical ends of the form* $\mathbb{R}_+ \times M$ where M is again a Seifert manifold. This is outside the realm of algebraic geometry so we use entirely different methods, *differential-geometric* in nature. We obtain virtual dimension formulæ in this general situation and, additionally, detailed information about the eta invariants of many Dirac operators. As shown in [N1] and [N2], these eta invariants contain a remarkable amount of topological information. On the other hand, some informations about tunnelings obtained in [MOY] are not accessible by our techniques.

This paper is divided into three sections and four appendices. The first section is essentially a brief survey of known facts concerning the eta invariant: definition, the Atiyah–Patodi–Singer theorem, variational formulæ and the spectral flow. We included these facts as a service to the reader, to eliminate any ambiguity concerning the various sign conventions. There does not seem to be general agreement on these conventions and, additionally, we used some "folklore" results for which we could not indicate satisfactory references.

The second section contains the main steps in the computation of the eta invariants discussed above. We begin by describing the geometric background and the various Dirac operators. Then using variational formulæ for the eta invariant and the adiabatic results of Bismut–Cheeger–Dai we compute in the second part the eta invariant of the Dirac operator on a circle bundle with very short fibers (Theorem 2.4).

In the third part, we compute the eta invariant of the *adiabatic Dirac operator* —a perturbation of the Dirac operator which arose in [N]. This is achieved in Theorem 2.6 via a variational formula and a spectral flow computation. The computations of certain transgression terms involved in the variational formuæ are deferred to appendices. An alternative method of computation is described in Appendix C.

The last part of this section is devoted to extending the previous computations to the Dirac operators coupled with flat line bundles. We use essentially the same variational strategy. However, new phenomena arise during the computation of some spectral flow contributions.

The third section is devoted to applications to Seiberg–Witten theory. The first two subsections describe the 3- and 4-dimensional Seiberg–Witten equations and some basic facts about them established in [MOY] and [N]. The third subsection is entirely devoted to the computation of a spectral flow. This is a very delicate job since one has to worry about eigenvalues changing sign in a nontransversal manner. In the last subsection we compute virtual dimensions of finite energy Seiberg–Witten moduli spaces on 4-manifolds founding circle bundles over Riemann surfaces and we conclude by comparing our answers in the special case of tunnelings to those in [MOY].

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1. The eta invariant of a first order elliptic operator

§1.1. Definition.

The elliptic selfadjoint operators on closed compact manifolds behave in many respects as common finite dimensional symmetric matrices. The eta invariant extends the notion of signature from finite dimensional matrices to elliptic operators. We will denote the trace of an infinite dimensional operator (when it exists) by "Tr" while "tr" is reserved for finite dimensional operators. We have the following result.

PROPOSITION 1.1: (a) *Consider a closed, compact, oriented Riemann manifold* (N, g) *of dimension* d, $E \to N$ *a hermitian vector bundle and* A: $C^\infty(E) \to C^\infty(E)$ *a first-order selfadjoint elliptic operator. Then*

$$(1.1) \qquad \eta_A(s) = \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty t^{(s-1)/2} \mathrm{Tr}\,(Ae^{-tA^2})dt = \sum_{\lambda>0} \frac{\dim V_\lambda - \dim V_{-\lambda}}{\lambda^s}$$

($V_\lambda = \ker(\lambda - A)$) *is well defined for all* $\Re s \gg 0$ *and extends to a meromorphic function on* $\mathbb{C}$. *Its poles are all simple and can be located only at* $s = (d+1-n)/2$, $n = 0, 1, 2, \ldots$.

(b) *If* d *is odd, then the residue of* $\eta_A(s)$ *at* $s = 0$ *is zero so that* $s = 0$ *is a regular point.*

For a proof of this proposition we refer to [APS3]. When d is odd we define the eta invariant of A by

$$\eta(A) := \eta_A(0).$$

Remark 1.2: (a) From the definition it follows directly that $\eta(-A) = -\eta(A)$ and $\eta(\lambda A) = \eta(A)$, $\forall \lambda > 0$.

(b) In [BF] it is shown that if A is an operator of Dirac type then one can define its eta invariant directly by setting $s = 0$ in (1.1). In other words, in this case

$$\eta(A) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \mathrm{Tr}\,(Ae^{-tA^2})dt.$$

In the sequel, we will reserve the letter D to denote Dirac type operators.

§1.2. The Atiyah–Patodi–Singer theorem.

The importance of the eta invariant in mathematics is due mainly to its appearance in the formula for the index of an elliptic boundary value problem first considered by Atiyah–Patodi–Singer in [APS1].

Suppose that (M, g) is a compact, $(d+1)$-dimensional, oriented Riemann manifold with boundary $N = \partial M$. We assume d is odd and that the metric g is a product on a tubular neighborhood $(-1, 0] \times N$ of the boundary, i.e. $g = du^2 + g_0$, where g_0 is a metric on N (see Fig. 1). We orient N such that the outer normal followed by the orientation of N gives the orientation of M. (This is precisely the orientation that makes the Stokes' formula come out right.)

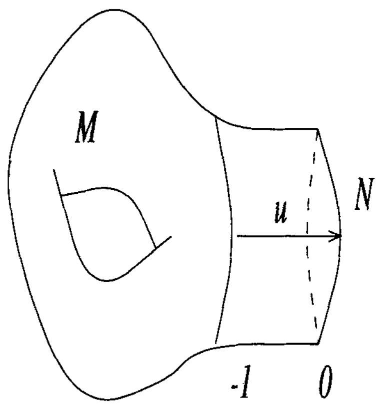

Figure 1. An oriented manifold with boundary.

Next suppose that $E_{\pm} \to M$ are two hermitian vector bundles and $L\colon C^\infty(E_+) \to C^\infty(E_-)$ is a first order elliptic operator which along the neck can be written as

$$L = G\left(\frac{\partial}{\partial u} - A\right),$$

where $G\colon E := E_+|_N \to E_-|_N$ is a bundle isomorphism and $A\colon C^\infty(E) \to C^\infty(E)$ is a selfadjoint elliptic operator. (Note that our convention differs from the one in [APS1]!) Denote by $P_{\geq}\colon L^2(E) \to L^2(E)$ the orthogonal projection onto the closed space spanned by the eigenvectors of A corresponding to eigenvalues ≥ 0. $P_<$ is defined similarly. The Atiyah–Patodi–Singer (APS) boundary value problem is

$$(APS)\colon \begin{cases} L\psi &= 0, \\ P_{\geq}\psi|_N &= 0. \end{cases}$$

Note that if ψ is a solution of (APS) then its restriction to the boundary lies in the negative eigenspace of A. Then, for all $u \geq 0$ we can define

$$\psi(u) = e^{uA}\psi|_{\partial M}\,.$$

We see that $\psi(u)$ extends ψ to an exponentially decaying solution of $L\psi = 0$ on M_∞. Here M_∞ denotes M with the half-infinite tube $[0,\infty) \times N$ attached (see Fig. 2). Thus, the solutions of (APS) can be identified with the exponentially decaying solutions of L on M_∞. The adjoint of (APS) is

$$(APS)^*\colon \begin{cases} L^*\phi &= 0, \\ P_<\phi|_N &= 0, \end{cases}$$

where L^* denotes the formal adjoint of L. (APS) is an elliptic problem which implies finite dimensional spaces of solutions for both (APS) and its adjoint.

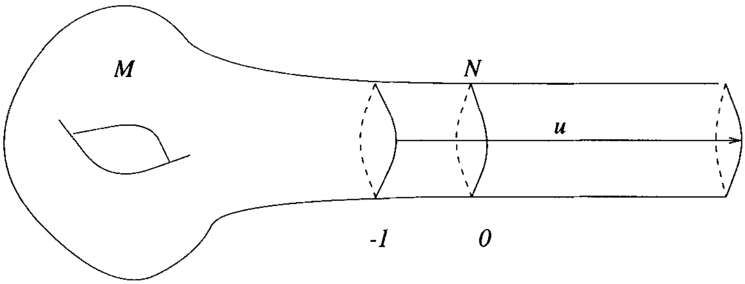

Figure 2. Attaching a half-infinite tube.

Define

$$\operatorname{ind}(L, APS) = \dim \ker(APS) - \dim \ker(APS)^*.$$

We have the following fundamental result.

THEOREM 1.3 (Atiyah–Patodi–Singer):

$$\operatorname{ind}(L, APS) = \int_M \alpha_0(x) dv_g - \frac{1}{2}(\, h(A) + \eta(A)\,)$$

where $h(A) = \dim \ker A$, $\eta(A)$ is the eta invariant of A and $\alpha_0(x)dv_g$ is the index density determined by L and is a completely local object (see [Gky], Sect. 1.8.2 for an exact definition).

Suggested by the above theorem we introduce the *ξ-invariant* (or *the reduced eta invariant*) of A by

$$\xi(A) = \frac{1}{2}(h(A) + \eta(A)).$$

Note that $\xi(-A) = (h(A) - \eta(A))/2$ so that $A \mapsto \xi(A)$ is not an odd function.

In many geometrically interesting situations the index density $\alpha_0(x)dv_g$ can be described quite explicitly. We describe below one such instance.

Suppose that M is equipped with a spin structure. Denote by $\mathbb{S} = \mathbb{S}_+ \oplus \mathbb{S}_-$ the associated superbundle of spinors. Fix a connection ∇^M on M compatible with the metric g. ∇^M *need not be the Levi-Civita connection* but we require that it looks like a product in a tubular neighborhood of the boundary. This induces in a canonical way a connection on $\mathbb{S}$ (compatible with both the metric and the splitting of $\mathbb{S}$) which we denote by $\hat{\nabla}^M$. Suppose moreover that $E \to M$ is a hermitian vector bundle equipped with a compatible connection ∇^E. We get in a standard fashion a connection on $\mathbb{S} \otimes E$ compatible with both the metric and the $\mathbb{Z}_2$-grading. Finally, this connection canonically defines a Dirac operator $\hat{\mathcal{D}} : C^\infty(\mathbb{S}_+ \otimes E) \to C^\infty(\mathbb{S}_- \otimes E)$ described by

$$\hat{\mathcal{D}}: C^\infty(\mathbb{S}_E^+) \xrightarrow{\nabla} C^\infty(T^*M \otimes \mathbb{S}_E^+) \xrightarrow{\hat{\mathbf{c}}} C^\infty(\mathbb{S}_E^-)$$

where $\hat{\mathbf{c}}$: $T^*M \to \mathrm{Hom}\,(\mathbb{S}_+ \otimes E, \mathbb{S}_- \otimes E)$ denotes the Clifford multiplication.

As required by the Atiyah–Patodi–Singer index theorem, near the boundary $\hat{\mathcal{D}}$ has the product structure

$$\hat{\mathcal{D}} = \hat{\mathbf{c}}(du)\,(\nabla_u - \mathcal{D})$$

where $\mathcal{D}$ is the Dirac operator induced by $\hat{\mathcal{D}}$ on the boundary.

The index density associated to this operator is the top degree part of the differential form $\hat{\mathbf{A}}(\nabla^M) \wedge \mathbf{ch}(\nabla^E)$ where $\hat{\mathbf{A}}$ (resp. $\mathbf{ch}$) denote the $\hat{\mathbf{A}}$-genus form (resp. the Chern character form) obtained from ∇^M (resp. ∇^E) via the Chern-Weil construction. In particular, if $\dim M = 4$ and E is the trivial line bundle equipped with the trivial connection we deduce

$$\mathrm{ind}\,(\hat{\mathcal{D}}, APS) = -\frac{1}{24}\int_M p_1(\nabla^M) - \xi(\mathcal{D}). \tag{1.2}$$

Remark 1.4: The above formula for $\alpha_0(x)dv_g$ is traditionally proved only for the special case when ∇^M is the Levi-Civita connection. However, a careful inspection of the proof in Chap. 11 of [Roe] shows it extends verbatim to the more general case when ∇^M is only metric compatible

§1.3. Variational formulæ.

While the eta invariant itself is a very complex object, its deformation theory turns out to be a lot simpler. We collect here some results we will use in our computations. More specifically, we will address the following problem.

Consider two metrics g_i $i = 0, 1$ and compatible connections ∇^i on an odd dimensional manifold N and denote by $\mathcal{D}_i$ the associated Dirac operators. Compute $\xi(\mathcal{D}_1) - \xi(\mathcal{D}_0)$.

We will soon see this problem does not have a unique answer and the reason will be very clear. Leaving this worry aside for a moment, consider a smooth path $\{(g_t, \nabla^t)\}_{t\in[0,1]}$ of metrics and compatible connections connecting (g_0, ∇^0) to (g_1, ∇^1). Denote the associated Dirac operators by $\mathcal{D}_t$ and set $\xi_t = \xi(\mathcal{D}_t)$. We want to compute $\dot{\xi}_t = d\xi_t/dt$, although at this moment we have no guarantee the map $t \mapsto \xi_t$ is differentiable.

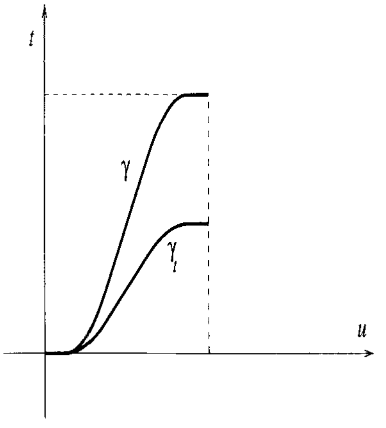

Figure 3. The smoothing function γ.

Since the path (g_t, ∇^t) may not be independent of t near $t = 0, 1$ we need to smooth-out the corners. With this aim, consider a smooth, nondecreasing map $\gamma : [0,1] \to [0,1]$, $u \mapsto \gamma(u)$ such that $\gamma(0) = 0$, $\gamma(1)$ and $\gamma'(u) \equiv 0$ for u near 0 and 1 (see Fig. 3). Moreover, for each $0 < t \leq 1$ set $\gamma_t(u) = t\gamma(u)$ so that γ_t connects 0 to t.

Now for every $0 < t \leq 1$ form the operator L_t on $[0,1] \times N$ defined by

$$L_t = \nabla_u - \mathcal{D}_{t\gamma(u)}.$$

L_t is an elliptic operator and from the A–P–S theorem we get

$$i_t := \operatorname{ind}(L_t, APS) = \rho_t - \frac{1}{2}(h_0 + h_t) + \frac{1}{2}(\eta_0 - \eta_t)$$

where ρ_t denotes the integral of the index density of L_t, $h_t = h(\mathcal{D}_t)$, $\eta_t = \eta(\mathcal{D}_t)$. The above formula can be rewritten as

$$\xi_t - \xi_0 = \rho_t + j_t \tag{1.3}$$

where $j_t = -(h_0 + i_t)$. The term ρ_t depends smoothly on t since the coefficients of L_t do. The term j_t is $\mathbb{Z}$-valued so it cannot be smooth, unless it is constant. If $[\xi_t] = \xi_t \pmod{\mathbb{Z}}$ then the map $t \mapsto [\xi_t]$ is smooth and by (1.3)

$$\frac{d[\xi_t]}{dt} = \dot{\rho}_t. \tag{1.4}$$

We will deal with $\dot{\rho}_t$ a bit later later but first we need to better understand the special nature of the discontinuities of ξ_t.

We see from (1.1) that the discontinuities of ξ_t (and hence those of j_t) are due to jumps in h_t. We describe how the jumps in h_t affect ξ_t in a simple, yet generic situation. We assume $\mathcal{D}_t$ is a regular family i.e.

• The resonance set $\mathcal{Z} = \{t \in [0,1];\ h_t \neq 0\}$ is finite.

• For every $t_0 \in \mathcal{Z}$ there exists $\varepsilon > 0$, an open neighborhood $\mathcal{N}$ of t_0 in $[0,1]$ and smooth maps $\lambda_k : \mathcal{N} \to (-\varepsilon, \varepsilon)$, $k = 0, 1, ..., h_{t_0}$ such that for all $t \in \mathcal{N}$ the family $\{\lambda_k(t)\}_k$ describes *all* the eigenvalues of $\mathcal{D}_t$ in $(-\varepsilon, \varepsilon)$ (including multiplicities) and, moreover, $\dot{\lambda}_k(t_0) \neq 0$ for all $k = 1, 2, ..., h_{t_0}$.

Now for each $t \in \mathcal{Z}$ set

$$\sigma_{\pm}(t) = \#\{k\ ;\ \pm\dot{\lambda}_k(t) > 0\},$$

and

$$\Delta_t\sigma = \begin{cases} -\sigma_-(0), & t = 0, \\ \sigma_+(t) - \sigma_-(t), & t \in (0,1), \\ \sigma_+(1), & t = 1. \end{cases}$$

If

$$\Delta_t\xi := \lim_{\varepsilon \to 0^+} (\xi_{t+\varepsilon} - \xi_{t-\varepsilon})$$

we see that $\Delta_t\xi = 0$ if $t \notin \mathcal{Z}$ while for $t \in \mathcal{Z}$ we have

$$\Delta_t\xi = \Delta_t\sigma. \tag{1.5}$$

(To understand the above formula it is convenient to treat $\mathcal{D}_t$ as a finite dimensional symmetric matrix and then keep track of the changes in its signature as the spectrum changes in the regular way described above.) Finally, define the spectral flow of the family $\mathcal{D}_t$ by

$$\mathrm{SF}(\mathcal{D}_t) = \sum_{t \in [0,1]} \Delta_t\sigma. \tag{1.6}$$

For example, in Fig. 4 we have represented those eigenvalues λ_t of a smooth path of Dirac operators which vanish for some values of t. The ± 1's describe the jumps $\Delta_t\sigma$. Thus the spectral flow in Fig. 4 is 1.

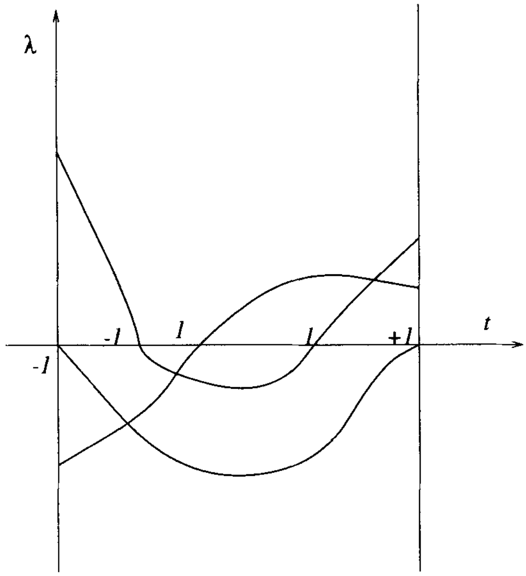

Figure 4. Spectral flow.

Using the equalities $j_1 - j_0 = \sum_t \Delta_t \xi$ and $j_0 = 0$ we deduce

$$j_1 - j_0 = -i_1 - h_0 = \sum_t \Delta_t \xi = \sum_{t \in [0,1]} \Delta_t \sigma = \mathrm{SF}(\mathcal{D}_t) \tag{1.7}$$

so that

$$i_1 = \mathrm{ind}\,(L_1, APS) = -h_0 - \mathrm{SF}(\mathcal{D}_t). \tag{1.8}$$

From the equalities (1.3) and (1.7) we now conclude

$$\xi_1 - \xi_0 = \mathrm{SF}(\mathcal{D}_t) + \int_0^1 \frac{d[\xi_t]}{dt} dt. \tag{1.9}$$

Remark 1.5: In the above two equalities we have neglected the smoothing effect of γ. However, since $\gamma(u)$ is nondecreasing, the crossing patterns of the eigenvalues of $t \mapsto \mathcal{D}_t$ and $u \mapsto \mathcal{D}_{\gamma(u)}$ are identical. This implies $\mathrm{SF}(\mathcal{D}_t) = \mathrm{SF}(\mathcal{D}_{\gamma(u)})$.

It is now the time to explain the continuous variation $\frac{d}{dt}[\xi]_t$. Formula (1.4) shows this is a locally computable quantity. In fact, one can be more accurate than this. We start with a simple situation first.

Assume (N, g) is an oriented Riemann manifold of dimension $d \equiv 3 \bmod 4$ equipped with a spin structure. Fix a smooth path $(\nabla^t)_{t\in[0,1]}$ of g-compatible connections and for each t denote by $\mathcal{D}_t$ the associated Dirac operator. Consider now the manifold $M = [0,1] \times N$ equipped with the metric $\hat{g} = du^2 + g$. The connection $\hat{\nabla} = du \wedge \partial_u + \nabla^{\gamma(u)}$ is compatible with the metric $\hat{g}$ and it determines a Dirac operator $\hat{\mathcal{D}}$ which has the form

$$\hat{\mathcal{D}} = \hat{\mathbf{c}}(du)\left(\partial_u - \mathcal{D}_{\gamma(u)}\right).$$

The A–P–S theorem then gives

$$\begin{aligned} \xi_1 - \xi_0 &= \int_M \hat{\mathbf{A}}(\hat{\nabla}) - \operatorname{ind}(\hat{\mathcal{D}}, APS) - h_0 \\ &= \mathrm{SF}(\mathcal{D}_{\gamma(u)}) + \int_M \hat{\mathbf{A}}(\hat{\nabla}). \end{aligned}$$

To further simplify this formula note firstly that

$$\mathrm{SF}(\mathcal{D}_{\gamma(u)}) = \mathrm{SF}(\mathcal{D}_t;\ 0 \le t \le 1).$$

Secondly, as in [APS2] one can show that the integral term is *independent* of the path of connections chosen to deform ∇^0 to ∇^1. Thus we can set $\nabla^t = \nabla^0 + t(\nabla^1 - \nabla^0)$. The resulting integral over M can then be rephrased as an integral over N of the transgression form from ∇^0 to ∇^1. This is defined as the degree d part of

$$T\hat{\mathbf{A}}(\nabla^1, \nabla^0) := \frac{d+1}{2} \cdot \int_0^1 \hat{\mathbf{A}}(\omega, \Omega_t)dt$$

where $\omega = \nabla^1 - \nabla^0$ and Ω_t is the curvature of $\nabla^0 + t\omega$. More explicitly,

$$\Omega_t = \Omega_0 + t d^{\nabla^0}\omega + t^2 \omega \wedge \omega$$

where d^{∇^0} denotes the exterior derivative defined by ∇^0.

In the special case when $d = 3$ the only important part of $\hat{\mathbf{A}}$ is $-\frac{1}{24}p_1$ where p_1 is the degree 2 invariant polynomial on $\underline{so}(4)$ given by

$$p_1(X, Y) = -\frac{1}{8\pi^2}\operatorname{tr}(XY).$$

(Here we use the conventions of [BGV].) In this case the transgression is a multiple of the Chern–Simons integrand, and more precisely

$$T\hat{\mathbf{A}}(\nabla^1, \nabla^0) = \frac{1}{96\pi^2}\operatorname{tr}\left(\omega \wedge \Omega^0 + \frac{1}{2}\omega \wedge d^{\nabla^0}\omega + \frac{1}{3}\omega \wedge \omega \wedge \omega\right).$$

Thus when $d = 3$ we have the following remarkable formula

$$\xi_1 - \xi_0 = \mathrm{SF}(\mathcal{D}_t) + \frac{1}{96\pi^2}\int_N \mathrm{tr}\,(\omega \wedge \Omega^0 + \frac{1}{2}\omega \wedge d^{\nabla^0}\omega + \frac{1}{3}\omega\wedge\omega\wedge\omega). \tag{1.10}$$

Now consider a more complicated problem. N is again a compact, oriented, d-dimensional manifold ($d = 3 \pmod 4$), but this time we allow the metric to vary. Thus, let $(g_t)_{t\in[0,1]}$ be a smooth path of Riemann metrics on N and, for each t, denote by ∇^t the *Levi-Civita* connection associated to the metric g_t. We obtain in this way a path of Dirac operators $(\mathcal{D}_t)_{t\in[0,1}$. We want to compute $\xi_1 - \xi_0$ assuming for simplicity that all the operators $\mathcal{D}_t$ are invertible so there is no spectral flow.

Form again the metric $\hat{g} = du^2 + g_{\gamma(u)}$ on $[0,1] \times N$ and denote by $\hat{\nabla}$ its associated Levi-Civita connection. We get a Dirac operator $\hat{\mathcal{D}}$ on M. It has the form $\hat{\mathbf{c}}(du)\left(\partial_u - \mathcal{D}_{\gamma(u)}\right)$ for u close to 0 and 1. Unfortunately, for u away from the endpoints it has the form

$$\hat{\mathbf{c}}(du)\left(\partial_u - \mathcal{D}'_{\gamma(u)}\right)$$

where $\mathcal{D}'_{\gamma(u)} = \mathcal{D}_{\gamma(u)} + T_u$ and T_u is a certain endomorphism expressible in terms of $\frac{d}{du}g_{\gamma(u)}$. If the operators $\mathcal{D}'_{\gamma(u)}$ were invertible, then their spectral flow would be zero and then $\xi_1 - \xi_0$ would be expressible as an integral of an $\hat{\mathbf{A}}$-form.

Fortunately, there is a simple way to guarantee the above invertibility, relying on the observation that the size of T_u is comparable with the size of the u-derivative of $g_{\gamma(u)}$. Consider a very large positive number L and form the tube $M_L = [0, L] \times N$ equipped with the metric

$$g_L = dv^2 + g_{\gamma(v/L)}.$$

In other words, the path $v \mapsto g_{\gamma(v/L)}$ defines a very, very slow deformation of g_0 to g_1. (A physicist would call this an adiabatic process.) In this case the v derivatives of $g_{\gamma(v/L)}$ become extremely small so that the corresponding perturbations T_v become negligible and $\mathcal{D}'_{\gamma(v/L)}$ will be invertible. If ∇^L denotes the Levi-Civita connection of g_L we get

$$\xi_1 - \xi_0 = \int_{M_L} \hat{\mathbf{A}}(\nabla^L).$$

As remarked in [APS2], the above integral does not change if we replace ∇^L by a linear connection on $[0, L] \times N$, *not necessarily compatible with* g_L, which interpolates affinely between the Levi-Civita connections of g_0 and g_1. This shows that even in this case we can express the variation of ξ as the integral of a transgression form. The only difference this time is that the transgression goes through $\mathrm{GL}(d,\mathbb{R})$-connections rather than $O(d)$-connections. This is no problem since the two groups are homotopically equivalent.

We now have (almost) all the background necessary to compute eta invariants of Dirac operators. The only missing piece of information is the Bismut–Cheeger–Dai result concerning the adiabatic limits of the eta invariants. We will state the special case we need at the opportune moment.

Remark 1.6: The above observations can be used to determine the index of an elliptic problem on a *noncompact* manifold considered in [LM].

Consider a smooth, non-decreasing function $\beta\colon \mathbb{R}\to[0,\infty)$ such that $\beta(u)\equiv 0$ for $u\leq 1/4$ and $\beta(u)\equiv u$ for $u\geq 3/4$.

Using the notations of §1.2, we define for each $\mu\in\mathbb{R}$ the weighted Sobolev spaces $L^{k,2}_\mu(E_\pm)$ as completions of $C_0^\infty(E_\pm)$ with respect to the norm

$$|\,\psi|_{\,k,2,\mu}=\left(\sum_{j=0}^{k}\int_{M_\infty}|e^{\mu\beta(u)}\nabla^j\psi|^2dV_g\right)^{1/2}.$$

Consider the bounded operator $L=\partial_u-A\colon L^{1,2}_\mu(E_+)\to L^2_\mu(E_-)$. In [LM] it was shown that L is Fredholm if and only if $A+\mu$ is invertible, i.e. $-\mu\notin\operatorname{spec}(A)$. We denote by $i_\mu(L)$ its index. For example, if A is invertible, then as pointed out in [APS1] we have

$$i_0=\operatorname{ind}(L,APS).$$

In general, to compute i_μ for an arbitrary μ note that the map

$$T_\mu : L^2_\mu\to L^2,\quad \psi\mapsto e^{\beta(u)}\psi$$

is an isometry so that $i_\mu(L)=i_0(T_\mu L T_\mu^{-1})$. A simple computation shows that

$$T_\mu L T_\mu^{-1}=L_\mu:=L-\mu\beta'(u).$$

Construct M_1 by attaching the cylinder $C_1=[0,1]\times N$ to the boundary of M. Alternatively, M_1 is the region $u\leq 1$ in M_∞. Then L_μ is well defined on M_1 and as above we conclude

$$i_\mu=\operatorname{ind}(L_\mu,APS).$$

Set $A_\mu=A+\mu$. We have

$$\operatorname{ind}_{M_1}(L_\mu,APS)-\operatorname{ind}_M(L,APS)=-(\,\xi(A_\mu)-\xi(A)\,)+\int_{C_1}\alpha_0(x)dv_g.$$

On the other hand, the above index density can be expressed as in (1.3) in terms of the APS index of the operator $L - \mu\beta'(u)$ on C_1,

$$\int_{C_1} \alpha_0(x) dv_g = \xi(A_\mu) - \xi(A) + h(A) + \operatorname{ind}_{C_1}(L - \mu\beta'(u), APS).$$

Finally, according to (1.8), the last term can be expressed as a spectral flow

$$\operatorname{ind}_{C_1}(L - \mu\beta'(u), APS) = -h(A) - \mathrm{SF}(A + t\mu,\ t \in [0,1]). \tag{1.11}$$

Putting all of the above together we obtain the following useful equality

$$i_\mu = \operatorname{ind}(L, APS) - \mathrm{SF}(A + t\mu,\ t \in [0,1]) \tag{1.12}$$

This is in perfect agreement with Theorem 1.2 in [LM]. Note also that if μ is sufficiently small and positive then there is no spectral flow correction in the above formula.

We also want to mention an immediate consequence of the above considerations. Consider two elliptic first order operators as above,

$$L_1, L_2 \colon \Gamma(E_+) \to \Gamma(E_-),$$

which have the *same principal symbol* and have the APS form $L_j = G(\partial_u - A_j)$ along the neck. Arguing as in the proof of (1.12) we deduce the *excision formula*

$$\operatorname{ind}(L_2, APS) = \operatorname{ind}(L_1, APS) - \mathrm{SF}(A_1 \to A_2) \tag{1.13}$$

where $\mathrm{SF}(A_1 \to A_2)$ denotes the spectral flow of the affine path of elliptic operators $A_t = A_1 + t(A_2 - A_1)$, $t \in [0,1]$.

2. Eta invariants of Dirac operators

§2.1. THE DIFFERENTIAL GEOMETRIC BACKGROUND.

Consider $\ell \in \mathbb{Z}$ and denote by $N = N_\ell$ the total space of a degree ℓ principal S^1 bundle over a compact oriented surface of genus g: $S^1 \hookrightarrow N_\ell \xrightarrow{\pi} \Sigma$. Denote by $\zeta \in \mathrm{Vect}(N)$ the infinitesimal generator of the S^1 action. N has a natural orientation which can be described using any splitting $TN = \langle\zeta\rangle \oplus \pi^* T\Sigma$ determined by an arbitrary connection.

Assume Σ is equipped with a *constant sectional curvature* Riemann metric h_b such that $\mathrm{vol}_{h_b}(\Sigma) = \pi$. Pick a connection form $\mathbf{i}\varphi \in \mathbf{i}\Omega^1(N)$ such that

$$-d\varphi = 2\ell dv_{h_b}.$$

This choice is possible since $\frac{-1}{2\pi}d\varphi$ represents the first Chern class of N which is ℓ. For each $0 < r \leq 1$ define a metric h_r on N by

$$h_r = \varphi_r \otimes \varphi_r \oplus \pi^* h_b, \quad \varphi_r = r\varphi.$$

Set $\zeta_r = r^{-1}\zeta$.

Using this metric we can orthogonally split $T^*N \cong \langle\varphi\rangle \oplus \pi^*T^*\Sigma$ and this defines in a natural way an orientation on N. If $*_r$ denotes the Hodge $*$ operator of the metric h_r we get

$$d\varphi_r = 2\lambda_r *_r \varphi_r \tag{2.1}$$

where $\lambda_r = -r\ell$.

Fix a local, orthonormal coframe θ^1, θ^2 on the base Σ such that

$$d\theta^1 = \kappa\theta^1 \wedge \theta^2 \tag{2.2}$$

and

$$d\theta^2 = 0 \tag{2.3}$$

where κ is a *nonnegative constant*. Such a choice is obviously possible if Σ is the flat torus. If Σ has higher genus, then any constant curvature metric on Σ admits such local coframes because it is a quotient of the hyperbolic plane and on the hyperbolic plane such choices are possible. Note that $-\kappa^2$ is actually the sectional curvature of Σ so by Gauss–Bonnet

$$\kappa^2 = 4(g-1).$$

We now get a local, oriented, orthonormal frame of T^*N $(\varphi_r, \varphi^1, \varphi^2)$, where $\varphi^i = \pi^*\theta^i$, $i = 1, 2$. Denote by $(\zeta_r, \zeta^1, \zeta^2)$ its dual frame. In [N] we showed that the 1-form associated to the Levi-Civita connection $\nabla^r = \nabla(h_r)$ by the above frame is

$$\omega_r = \begin{bmatrix} 0 & A_r & -B_r \\ -A_r & 0 & C_r \\ B_r & -C_r & 0 \end{bmatrix} \tag{2.4}$$

where

$$A_r = \lambda_r\varphi^2, \quad B_r = \lambda_r\varphi^1, \quad C_r = -\lambda_r\varphi_r - \kappa\varphi^1. \tag{2.5}$$

In our computations we will need various other connections compatible with the above metric. For $t \in (0,1]$ consider the bundle isomorphism $L_t: TN \to TN$ described locally by

$$\zeta \mapsto t\zeta, \quad \zeta_i \mapsto \zeta_i, \quad i = 1, 2.$$

Clearly L_t defines an isometry $(TN, h_{rt}) \to (TN, h_r)$, for all $r > 0$ and all $t \in (0,1]$. This implies that the connection

$$\nabla^{r,t} = L_t \nabla^{rt} L_t^{-1}$$

is compatible with the metric h_r. A simple computation shows that the 1-form associated to this connection by the frame $(\zeta_r, \zeta_1, \zeta_2)$ is

$$\omega_{r,t} = \begin{bmatrix} 0 & \lambda_r t\varphi^2 & -\lambda_r t\varphi^1 \\ -\lambda_r t\varphi^2 & 0 & -\lambda_r t^2\varphi_r - \kappa\varphi^1 \\ \lambda_r t\varphi^1 & \lambda_r t^2\varphi_r + \kappa\varphi^1 & \end{bmatrix}. \tag{2.6}$$

The connection $\nabla^{r,t}$ induces a connection $\tilde{\nabla}^{r,t}$ on $\langle\zeta_r\rangle^\perp = \pi^* T\Sigma$. Denote by ∇^Σ the Levi-Civita connection on Σ. The formulae (2.5) and (2.6) imply immediately

$$\lim_{t\to 0} \tilde{\nabla}^{r,t} = \pi^* \nabla^\Sigma, \quad \forall r > 0$$

and

$$\lim_{r\to 0} \tilde{\nabla}^{r,t} = \pi^* \nabla^\Sigma, \quad \forall t > 0.$$

Set $\nabla^\infty = \pi^* \nabla^\Sigma$.

The bundle $\langle\varphi\rangle^\perp$ has a natural complex structure locally defined by the correspondences $\varphi_1 \mapsto -\varphi_2 \mapsto -\varphi_1$. In this way we get a complex line bundle $\mathcal{K} \to N$. It is isomorphic with the pullback of the canonical line bundle K_Σ of the base. Once we fix a *spin* structure on Σ the Levi-Civita induced connection defines a natural holomorphic structure on $K_\Sigma^{-1/2}$. In [N] we showed that this induces a *spin* structure on N with associated bundle of spinors

$$\mathbb{S} = \mathcal{K}^{-1/2} \oplus \mathcal{K}^{1/2}.$$

The Clifford multiplication is described explicitly in Appendix D. We only want to mention here that our choice is such that the Clifford multiplication by the volume form on N is equal to -1. This agrees with the conventions of [BC].

Note that the h_r-compatible connections $\nabla^{r,t}$ define a 2-parameter family of Dirac operators $\mathcal{D}_{r,t}$ on $\mathbb{S}$. To explicitly describe their form introduce as in [N] the following operators:

$$Z = Z_r = \begin{bmatrix} \mathbf{i}\partial_{\zeta_r} & 0 \\ 0 & -\mathbf{i}\partial_{\zeta_r} \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 0 & {}^\flat\bar{\partial} \\ {}^\flat\partial & 0 \end{bmatrix}$$

where (locally)

$$^\flat\bar\partial = 2^{-1/2}(\varphi^1 - \mathbf{i}\varphi^2)\otimes(\nabla^\infty_{\zeta_1} + \mathbf{i}\nabla^\infty_{\zeta_2})$$

and

$$^\flat\partial = 2^{-1/2}(\varphi^1 + \mathbf{i}\varphi^2)\otimes(\nabla^\infty_{\zeta_1} - \mathbf{i}\nabla^\infty_{\zeta_2}).$$

The computations in [N] show that

$$\mathcal{D}_{r,t} = Z_r + T + \frac{\lambda_r t^2}{2}. \tag{2.7}$$

We set

$$\mathcal{D}_r := \mathcal{D}_{r,t}\,|_{t=1}$$

and

$$D_r = \lim_{t\to 0}\mathcal{D}_{r,t} = Z_r + T = \mathcal{D}_r - \frac{\lambda_r}{2}. \tag{2.8}$$

The first goal of this paper is to compute $\eta(\mathcal{D}_r)$ and $\eta(D_r)$.

We conclude this subsection by listing some properties of the spectrum of D_r as $r \to 0$. Their proofs can be found in [N].

PROPOSITION 2.1: (a) *For all* $r \in (0, 1]$

$$\dim\ker D_r = 2h_{1/2} := 2\dim H^0(\Sigma, K^{1/2}).$$

(b) *There exists* $r_0 > 0$ *and* $z_0 > 0$ *such that for all* $r \in (0, r_0]$ *the only eigenvalue of* D_r *in* $(-z_0, z_0)$ *is 0.*

§2.2. THE ETA INVARIANT OF THE SPIN DIRAC OPERATOR.

In the sequel we assume $\ell \neq 0$, i.e. N is a *nontrivial* circle bundle.

As we mentioned in the introduction, the key step in our computation of $\eta_r := \eta(\mathcal{D}_r)$ will be the adiabatic result of Bismut–Cheeger [BC] in the more accurate form of [Dai]. Instead of formulating the most general version of their result (which would require a large preamble) we state it for the special case we have in mind. Fortunately, in this case concrete computations were performed in [Z] and [DZ]. Set $\eta_0 = \lim_{r\to 0}\eta_r$. We then have the following result.

THEOREM 2.2: *The adiabatic limit exists and moreover*

$$\eta_0 = -2\int_\Sigma \hat{\mathbf{A}}(\nabla^\Sigma)\frac{\tanh(c/2) - c/2}{c\tanh(c/2)} + \sum_\mu \operatorname{sign}\mu,$$

where $c \in H^2(\Sigma, \mathbb{R})$ is the Euler class of the S^1 bundle N and the summation is carried over all nonzero eigenvalues μ of $\mathcal{D}_r$ which are of size $O(r)$ as $r \to 0$.

Using the Taylor expansion

$$\tanh x = x - \frac{x^3}{3} + O(x^5)$$

and the fact that $\hat{\mathbf{A}}(\Sigma) = 1$, we deduce

$$\hat{\mathbf{A}}(\nabla^\Sigma)\frac{\tanh(c/2) - c/2}{c\tanh(c/2)} = -c/12.$$

Since $\mathcal{D}_r = D_r + \lambda_r/2$ we deduce from Proposition 2.1 that for $0 < r \ll r_0$ the only nonzero eigenvalue of $\mathcal{D}_r$ which is of size $O(r)$ is $\lambda_r/2 = -r\ell/2$ and it has multiplicity $\dim\ker D_r$. Putting all the above together we deduce

$$\eta_0 = \ell/6 - 2\operatorname{sign}(\ell)h_{1/2}. \tag{2.9}$$

Let r_0 be as in Proposition 2.1. For every $0 < r \ll r_0$ set $\xi_r = \xi(\mathcal{D}_r)$. Note that $\xi_r = \frac{1}{2}\eta_r$ since, by Proposition 2.1, $\ker\mathcal{D}_r = 0$ if $r \ll r_0$. Finally, denote $\xi_0 = \lim_{r\to 0}\xi_r$. From the equality (2.9) we deduce

$$\xi_0 - \xi_r = \frac{\ell}{12} - \operatorname{sign}(\ell)h_{1/2} - \xi_r, \quad \forall r \ll r_0. \tag{2.10}$$

On the other hand, as explained in subsection 1.3, for all $0 < \rho < r \ll r_0$ the difference $\xi_\rho - \xi_r$ can be expressed as the integral of an $\hat{\mathbf{A}}$-genus form. There is no spectral flow because the operators $\mathcal{D}_r$ are invertible for $0 < r \ll r_0$. Following the prescriptions at the end of subsection 1.3 one obtains the following result. (For details we refer to Appendix A.)

LEMMA 2.3 (First transgression formula): *For all $0 < r \ll r_0$ we have*

$$\lim_{\rho\to 0}(\xi_\rho - \xi_r) = -\frac{\ell}{12}(\ell^2 r^4 - \chi r^2)$$

where $\chi = \chi(\Sigma) = 2 - 2g$.

By combining all of the above we get the following result.

THEOREM 2.4: *For all $0 < r \ll r_0$ we have*

$$\frac{1}{2}\eta_r = \xi_r = \frac{\ell}{12} - \operatorname{sign}(\ell)h_{1/2} + \frac{\ell}{12}(\ell^2 r^4 - \chi r^2).$$

§2.3. THE ETA INVARIANT OF THE ADIABATIC DIRAC OPERATOR.

In this subsection we take up the computation of the eta invariant of D_r. We rely on our freshly acquired knowledge of $\xi(\mathcal{D}_r)$.

The Dirac operator $\mathcal{D}_r$ is associated to the Levi-Civita connection ∇^r while D_r is associated to the connection $\nabla^{r,0} = \lim_{t\to 0}\nabla^{r,t}$. Set $\tau = (1-t^2)$. Then, using (2.7) and (2.8) we get a path

$$\tilde{\mathcal{D}}_{r,\tau} = \mathcal{D}_r - \frac{\tau\lambda_r}{2}$$

such that $\tilde{\mathcal{D}}_{r,0} = \mathcal{D}_r$ and $\tilde{\mathcal{D}}_{r,1} = D_r$. Set $\xi_\tau = \xi(\tilde{\mathcal{D}}_{r,\tau})$. Using the variational technique described in §1.3 we deduce

$$\xi_1 = \xi_0 + \mathrm{SF}(\tilde{\mathcal{D}}_{r,\tau};\, \tau\in[0,1]) + \int_N TA(\nabla^{r,0},\nabla^{r,1}). \tag{2.11}$$

To compute the spectral flow note that according to Proposition 2.1 the operator $\tilde{\mathcal{D}}_{r,\tau}$ has a kernel only for $\tau = 1$. In this case, the kernel has dimension $2h_{1/2}$. Using (1.5) of §1.3 we deduce

$$\mathrm{SF} = \begin{cases} 2h_{1/2}, & \ell > 0,\\ 0, & \ell < 0.\end{cases} \tag{2.12}$$

As for the transgression term, it is described in the following lemma whose proof can be found in Appendix B.

LEMMA 2.5 (Second transgression formula):

$$\int_N TA(\nabla^{r,0},\nabla^{r,1}) = -\frac{\ell}{12}(\ell^2 r^4 - \chi r^2).$$

Putting together all the above we obtain from Theorem 2.4 the following result.

THEOREM 2.6: *For all $0 < r \ll 1$ we have*

$$\xi(D_r) = \frac{\ell}{12} + h_{1/2}.$$

Note that $\xi(D_r)$ is independent of r !!! In hindsight, this should not be so surprising if we think that D_r was obtained after the adiabatic deformation in (2.8). Notice that $\xi(D_r)$ still "remembers" it came from a fibration due to the term $\ell/12$. The geometry of the base is reflected in the term $h_{1/2}$. Remarkably, $\eta(D_r) = \ell/6$. Thus the base Σ is "invisible" to the eta invariant of D_r !!!

§2.4. THE ETA INVARIANT OF THE COUPLED ADIABATIC DIRAC OPERATOR. Let us begin by recalling that the Gysin exact sequence implies that

$$H^2(N;\mathbb{Z}) \cong \pi^* H^2(\Sigma;\mathbb{Z}) \oplus H^1(\Sigma,;\mathbb{Z}) \cong \mathbb{Z}_{|\ell|} \oplus \mathbb{Z}^{2g}.$$

Consider a complex line bundle $L \to N$ such that $c_1(L) = \hat{k} \in \mathbb{Z}_{|\ell|}$. Such a line bundle can be obtain as the pullback of line bundle $L_\Sigma \to \Sigma$ of degree $k \in \mathbb{Z}$. Note that k is determined only modulo ℓ. A line bundle as above admits flat connections and the holonomy of such a connection is $\exp(2\pi k\mathbf{i}/\ell)$. The collection of gauge equivalence classes of flat connections is homeomorphic to a torus T^{2g}.

These facts were proven in [N] relying on a simple observation which we repeat here, since it is relevant to our computations.

Let A be a flat connection on L and set

$$B := A + \frac{k\mathbf{i}}{\ell}\varphi. \tag{2.13}$$

Then B is a connection with trivial holonomy along fibers and it can be regarded as a pullback of a connection on a line bundle $L_\Sigma \to \Sigma$ such that $c_1(L_\Sigma) = k \in \mathbb{Z}$. Now set

$$\mathbb{S}_L = \mathbb{S} \otimes L = \mathcal{K}^{-1/2} \otimes L \oplus \mathcal{K}^{1/2} \otimes L.$$

By coupling the connection $\pi^*\nabla^\Sigma$ on $\mathbb{S}$ with the flat connection A we get a connection on $\mathbb{S}_L$ which leads to a Dirac operator $D_A = D_{A,r}$. We call this the *adiabatic operator coupled with A*.

Similarly, using the connection B on L we obtain two connections on $\mathbb{S}_L$ obtained by coupling B with the Levi-Civita connection and respectively the connection $\pi^*\nabla^\Sigma$. These lead to two Dirac operators, $\mathcal{D}_{B,r}$ and respectively $D_{B,r}$. The goal of this section is to compute the eta invariant of the operator $D_{A,r}$ which played a key role in [N] in the description of the reducible adiabatic solutions of the Seiberg–Witten equations. We will use these eta invariant computations in a forthcoming work on Seiberg–Witten equations on manifolds with cylindrical ends.

The computation of $\eta(D_{A,r})$ for $0 < r \ll 1$ is performed in three steps.

STEP 1: Compute $\xi(\mathcal{D}_{B,r})$.

STEP 2: Compute $\xi(D_{B,r})$.

STEP 3: Compute $\xi(D_{A,r})$.

While the first two steps follow closely §2.2 and respectively §2.3, interesting new phenomena arise at Step 3.

Before we carry out the computations we need to review some facts and introduce some notations.

Recall first that if $L_\Sigma \to \Sigma$ is a complex line bundle then any connection B on L_Σ introduces a holomorphic structure on L_Σ. We denote by $h_{1/2}(L_\Sigma)$ the dimension of the space of holomorphic sections of $K_\Sigma^{1/2} \otimes L_\Sigma$. Using the Riemann–Roch formula we deduce

$$\dim H^0(K_\Sigma^{1/2} \otimes L_\Sigma) - \dim H^1(K_\Sigma^{1/2} \otimes L) = \deg L_\Sigma.$$

On the other hand, Serre duality implies $\dim H^1(K_\Sigma^{1/2} \otimes L) = H^0(K_\Sigma^{1/2} \otimes L_\Sigma^*) = h_{1/2}(L_\Sigma^*)$, where L_Σ^* denotes the dual of L_Σ. Hence

$$h_{1/2}(L_\Sigma) - h_{1/2}(L^*) = \deg L_\Sigma. \tag{2.14}$$

Let A and B as in (2.13). In [N] we proved the following result.

PROPOSITION 2.7:

(a) *For all $r \in (0,1]$ the splitting $\mathbb{S}_L = \mathcal{K}^{-1/2} \otimes L \oplus \mathcal{K}^{1/2} \otimes L$ induces a splitting of $\ker D_{B,r}$ and moreover we have an isomorphism*

$$\ker D_{B,r} \cong H^0(K_\Sigma^{1/2} \otimes L_\Sigma^*) \oplus H^0(K_\Sigma^{1/2} \otimes L_\Sigma)$$

so that $\dim\ker D_{B,r} = h_{1/2}(L_\Sigma^*) + h_{1/2}(L_\Sigma)$.

(b) *There exist $r_0 > 0$ and $z_0 > 0$ such that for all $r \in (0, r_0]$ the only eigenvalue of $D_{B,r}$ in $(-z_0, z_0)$ is 0.*

With respect to the splitting $\mathbb{S}_L = \mathcal{K}^{-1/2} \otimes L \oplus \mathcal{K}^{1/2} \otimes L$ the operator $D_{B,r}$ has the block decomposition $D_{B,r} = Z_{B,r} + T_B$ where

$$Z_{B,r} = \begin{bmatrix} \mathbf{i}\nabla^B_{\zeta_r} & 0 \\ 0 & -\mathbf{i}\nabla^B_{\zeta_r} \end{bmatrix} \quad \text{and} \quad T_B = \begin{bmatrix} 0 & {}^b\bar\partial_B \\ {}^b\bar\partial_B^* & 0 \end{bmatrix}.$$

Also $\mathcal{D}_{B,r} = D_{B,r} + \frac{\lambda_r}{2}$. Another important piece of information is a supercommutator identity established in [N]. In our special case it has the form

$$\{Z_{B,r}, T_B\} := Z_{B,r}T_B + T_B Z_{B,r} = 0. \tag{2.15}$$

STEP 1: The same argument as in [Z] proves the following result.

PROPOSITION 2.8:

$$\eta_0 := \lim_{r\to 0} \eta(\mathcal{D}_{B,r}) = -2\int_\Sigma \hat{\mathbf{A}}(\nabla^\Sigma) \cdot \mathbf{ch}(B) \cdot \frac{\tanh(c/2) - c/2}{c \cdot \tanh(c/2)} + \sum_\mu \operatorname{sign}\mu$$

where $\mathbf{ch}(B)$ *denotes the Chern-Character defined in terms of the connection* B *on* $L \to \Sigma$ *and the remaining terms have the same significance as in Theorem 2.2.*

Proceeding exactly as in §2.2 we conclude (via Proposition 2.7) that

$$\eta_0 = \frac{\ell}{6} - \operatorname{sign}(\ell)\big(h_{1/2}(L^*_\Sigma) + h_{1/2}(L_\Sigma)\big).$$

If we set $\xi_r = \xi(\mathcal{D}_{B,r}) = \frac{1}{2}\eta_r$ we deduce

$$\xi_0 - \xi_r = \frac{\ell}{12} - \operatorname{sign}(\ell) \cdot \frac{h_{1/2}(L^*_\Sigma) + h_{1/2}(L_\Sigma)}{2} - \xi_r.$$

On the other hand, we have

$$\xi_0 - \xi_r = -\frac{\ell}{12}(\ell^2 r^4 - \chi r^4).$$

This follows from the first transgression formula. We can quote this formula since as $r \to 0$ the only constituent of $\mathcal{D}_{B,r}$ that changes is the Levi-Civita connection while the coupling connection is independent of r. The degree 3 part of the transgression of the index density $\hat{\mathbf{A}} \wedge \mathbf{ch}(B)$ equals precisely the transgression of the $\hat{\mathbf{A}}$-genus which was computed in the first transgression formula. We conclude

$$\text{(2.16)} \quad \frac{1}{2}\eta_r = \xi_r(\mathcal{D}_{B,r}) = \frac{\ell}{12} - \operatorname{sign}(\ell) \cdot \frac{h_{1/2}(L^*_\Sigma) + h_{1/2}(L_\Sigma)}{2} + \frac{\ell}{12}(\ell^2 r^4 - \chi r^2).$$

STEP 2: Now we "transgress" from $\mathcal{D}_{B,r}$ to $D_{B,r}$ using the same deformation $(\tilde{\mathcal{D}}_{B,r,\tau})$ as in §2.3. As in that case we have

$$\text{(2.17)} \qquad \xi(D_{B,r}) = \xi(\mathcal{D}_{B,r}) + \mathrm{SF}(\tilde{\mathcal{D}}_{B,r,\tau}; 0 \le \tau \le 1) + \int_N T\hat{\mathbf{A}}(\nabla^{r,0}, \nabla^{r,1}).$$

Again there is no transgression term coming from the coupling which does not change as τ runs from 0 to 1.

The spectral flow contribution occurs only at $\tau = 1$ and using Proposition 2.7 we determine it to be

$$\text{(2.18)} \qquad \mathrm{SF}(\tilde{\mathcal{D}}_{B,r,\tau}; 0 \le \tau \le 1) = \begin{cases} h_{1/2}(L^*_\Sigma) + h_{1/2}(L_\Sigma), & \ell > 0, \\ 0, & \ell < 0. \end{cases}$$

The transgression term is given by the second transgression formula. Putting all the above together we deduce

$$\text{(2.19)} \qquad \xi(D_{B,r}) = \frac{\ell}{12} + \frac{h_{1/2}(L^*_\Sigma) + h_{1/2}(L_\Sigma)}{2}.$$

Note that

$$\eta(D_{B,r}) = \frac{\ell}{6}. \tag{2.20}$$

Surprisingly, $\eta(D_{B,r})$ carries very little geometrical information. The extreme generality of the Bismut–Cheeger–Dai theorem may obscure some beautiful symmetries responsible for (2.20). We refer the reader to Appendix C, where we have included an elementary derivation of this equality which works in the more general context of Seifert manifolds and we believe contains several illuminating informations.

STEP 3: Finally we compute $\xi(D_{A,r})$. *In the remaining part we will assume* $k \in \mathbb{Z} \cap (0, |\ell|)$. Note that if $k = 0$ then $D_{A,r} = D_{B,r}$ and there is nothing to compute in this case. Hence we have to consider only the case $0 < k < \ell$.

The equality

$$A = B - \frac{k\mathbf{i}}{\ell}\varphi$$

suggests using the path of connections

$$B_t = B + t\dot{B}, \quad \dot{B} = -\frac{k\mathbf{i}}{\ell}\varphi_r.$$

We have (omitting the r-subscript for brevity)

$$\xi(D_A) = \xi(D_B) + \mathrm{SF}(D_{B_t}) + \int_N T(\hat{\mathbf{A}} \wedge \mathbf{ch})(B_1, B_0). \tag{2.21}$$

This time $\hat{\mathbf{A}}$ is fixed and only the coupling connection changes. We have the following result.

LEMMA 2.9 (Third transgression formula):

$$\int_N T(\hat{\mathbf{A}} \wedge \mathbf{ch})(B_1, B_0) = \frac{k^2}{2\ell}.$$

Proof of the lemma: As we mentioned before, the only part which contributes to the transgression is $\mathbf{ch}$ through its degree 4 component $c_1^2(B_t)/2$. For an arbitrary connection ∇ on L we have

$$\frac{c_1^2(\nabla)}{2} = -\frac{1}{8\pi^2} F(\nabla) \wedge F(\nabla).$$

Thus in our case the transgression is

$$T\mathbf{ch} = -\frac{1}{4\pi^2}\int_0^1 \dot{B} \wedge F_{B_t}.$$

A simple computation shows

$$F_{B_t} = F_B + td\dot{B} = F_B - \frac{tk\mathbf{i}}{\ell}d\varphi = F_B + 2tk\mathbf{i}\varphi^1 \wedge \varphi^2,$$

$$\dot{B} \wedge F_{B_t} = -\frac{k\mathbf{i}}{\ell}\varphi \wedge F_B + \frac{2tk^2}{\ell}\varphi \wedge \varphi^1 \wedge \varphi^2.$$

Hence

$$\begin{aligned} T\mathbf{ch} &= \frac{\mathbf{i}k}{4\pi^2\ell}\varphi \wedge F_B - \frac{k^2}{4\pi^2\ell}\varphi \wedge \varphi^1 \wedge \varphi^2 \\ &= \frac{k}{2\pi\ell}\varphi \wedge \frac{\mathbf{i}F_B}{2\pi} - \frac{k^2}{4\pi^2\ell}\varphi \wedge \varphi^1 \wedge \varphi^2. \end{aligned}$$

The lemma follows integrating over N and using the equalities

$$\int_{base} \frac{\mathbf{i}F_B}{2\pi} = \deg L = k, \quad \int_{fiber} \varphi = 2\pi, \quad \int_N \varphi \wedge \varphi^1 \wedge \varphi^2 = 2\pi^2. \qquad \blacksquare$$

To compute the spectral flow in (2.21) we need to go deeper inside the structure of D_{B_t}. We have

$$D_{B_t} = Z_{B_t,r} + T_{B_t}.$$

Note that $T_{B_t} = T_B$ since T involves only derivatives along horizontal directions while B_t changes only in the vertical direction. As for $Z_{B_t,r}$ we have

$$\begin{aligned} Z_{B_t,r} = Z_{B,r} &= \begin{bmatrix} \mathbf{i}\nabla^B_{\zeta_r} + \mathbf{i}t\dot{B}(\zeta_r) & 0 \\ 0 & -\mathbf{i}\nabla^B_{\zeta_r} - \mathbf{i}t\dot{B}(\zeta_r) \end{bmatrix} \\ &= Z_{B,r} + \frac{t}{r}\begin{bmatrix} k/\ell & 0 \\ 0 & -k/\ell \end{bmatrix}. \end{aligned}$$

Denote the "matrix" above by $\mathfrak{R}$ and set $Z_B := Z_{B,r=1}$. Using the equality $\zeta_r = r^{-1}\zeta$ we deduce

$$Z_{B_t,r} = \frac{1}{r}(Z_B + t\mathfrak{R}).$$

Observe now that both Z_B and $\mathfrak{R}$ anticommute with T_B so that

$$D^2_{B_t,r} = \frac{1}{r^2}(Z_B + t\mathfrak{R})^2 + T_B^2.$$

In particular, this shows

$$\ker D_{B_t,r} = \ker D^2_{B_t,r} = \ker(Z_B + r\mathfrak{R}) \cap \ker T_B.$$

Since $0 < k < |\ell|$ we see that $\ker Z_B + t\mathfrak{R} = (0)$ if $t \in (0,1]$. In other words, the only contribution to the spectral flow arises at $t = 0$. Denote by $\{\mu_i(t)\}$ the eigenvalues of $D_{B_t,r}$ such that $\mu_i(0) = 0$. There are $\dim\ker D_{B,r}$ such eigenvalues. Denote by σ_- the number of those such that $\dot\mu_i(0) < 0$. The spectral flow is then $-\sigma_-$. Determining the eigenvalues $\mu_i(t)$ may be a complicated job. We follow a different description of σ_- given in [RS].

Set $E_0 = \ker D_{B,r}$, denote by P_0 the orthogonal projection onto E_0 and define the *resonance matrix* $R : E_0 \to E_0$ by

$$R = P_0 \dot D_{B_t,r}|_{t=0} : E_0 \to E_0.$$

Clearly R is nondegenerate and, as explained in [RS], σ_- can be identified with $\sigma_-(R)$ which is the number of negative eigenvalues of R. This number can be determined using the explicit description of $\mathfrak{R}$ and Proposition 2.7 (a). We deduce

$$\mathrm{SF}(D_{B_t,r}) = -\sigma_- = \begin{cases} -h_{1/2}(L_\Sigma), & \ell > 0, \\ -h_{1/2}(L_\Sigma^*), & \ell < 0. \end{cases}$$

Using the third transgression formula and the equalities (2.21), (2.19) we finally determine

$$\text{(2.22)} \qquad \begin{aligned} \xi(D_A) &= -\frac{\ell}{24} + \frac{k^2}{2\ell} + \mathrm{sign}\,(\ell) \cdot \frac{h_{1/2}(L_\Sigma^*) - h_{1/2}(L_\Sigma)}{2} \\ &\overset{(2.14)}{=} \frac{\ell}{12} + \frac{k^2}{2\ell} - \mathrm{sign}\,(\ell)\frac{k}{2}. \end{aligned}$$

Again $\xi(D_{A,r})$ is a topological quantity!!!

Remark 2.10: The spectral flow computation in Step 3 used in an essential way the fact that $k \in (0, \ell)$. In fact, if we started with a different $k' \equiv k \bmod \ell$ then the computations at Step 3 would be affected in both the transgression term and in the spectral flow term (which would now have several contributions). One can verify easily that these changes cancel each other so that the final result is independent of the choice of a residue of k mod ℓ.

3. Finite energy Seiberg–Witten monopoles

Throughout this section, a hat over an object will signal (unless otherwise indicated) that it is a 4-dimensional geometric object.

For example, if N is a 3-manifold then on the tube $\mathbb{R}\times N$ there exist two exterior derivatives: the 3-dimensional exterior derivative d along the slices $\{t\} \times N$ and the 4-dimensional exterior derivative $\hat d$ so that $\hat d = dt \wedge \partial_t + d$. If $A(t)$ is a family of connection on some vector bundle $E \to N$, then we get a bundle $\hat E \to \mathbb{R} \times N$ and we can think of the path $A(t)$ as a connection $\hat A$ on $\hat E$. We will denote by

$F_{A(t)}$ the curvature of $A(t)$ on the slice $\{t\} \times N$ while $\hat{F}_{\hat{A}}$ will denote the curvature of $\hat{A}$ on the tube.

§3.1. THE 4-DIMENSIONAL SEIBERG–WITTEN EQUATIONS.

Let $\hat{N}$ denote an oriented 4-manifold (*not necessarily compact*), equipped with a Riemann metric $\hat{g}$. Denote by $\hat{*}$ the Hodge star operator defined by the metric $\hat{g}$ and the orientation of $\hat{N}$. *Fix* a connection $\hat{\nabla}$ on $T\hat{N}$ compatible with $\hat{g}$. $\hat{\nabla}$ need not be the Levi-Civita connection.

Denote by $\mathrm{Spin}_c(\hat{N})$ the collection of isomorphism classes of spin^c structures on $\hat{N}$. For each $\hat{\sigma} \in \mathrm{Spin}^c(\hat{N})$ we denote by $\det \hat{\sigma}$ the associated line bundle and by $\mathbb{S}_{\hat{\sigma}} = \hat{\mathbb{S}}^+_{\hat{\sigma}} \oplus \hat{\mathbb{S}}^-_{\hat{\sigma}}$ the associated bundle of spinors. Note that $\det \hat{\sigma} \cong \det \hat{\mathbb{S}}^+_{\hat{\sigma}}$.

Denote by $\mathfrak{A}_\sigma$ the space of hermitian connections on $\mathbb{S}_\sigma$ compatible with both the $\mathbb{Z}_2$-grading and the fixed background connection $\hat{\nabla}$. More precisely, $A \in \mathfrak{A}_{\hat{\sigma}}(\hat{N})$ if for any $\alpha \in \Omega^1(N)$, any $X \in \mathrm{Vect}\,(N)$ and any $\hat{\psi} \in C^\infty(\mathbb{S}_\sigma)$ we have

$$\nabla^A_X(\hat{\mathbf{c}}(\alpha)\hat{\psi}) = \hat{\mathbf{c}}(\nabla_X \alpha)\hat{\psi} + \hat{\mathbf{c}}(\alpha)\nabla^A_X \hat{\psi},$$

where

$$\hat{\mathbf{c}}\colon T^*\hat{N} \to \mathrm{Hom}\,(\hat{\mathbb{S}}^+_{\hat{\sigma}}, \mathbb{S}^-_{\hat{\sigma}})$$

denotes the Clifford multiplication. Any connection on $\det \hat{\sigma}$ determines a connection in $\mathfrak{A}_\sigma$ and, moreover, once we fix a connection $A_0 \in \mathfrak{A}_{\hat{\sigma}}(\hat{N})$, we can identify $\mathfrak{A}_{\hat{\sigma}}(\hat{N})$ with $\mathbf{i}\Omega^1(\hat{N})$. To any connection $\hat{A} \in \mathfrak{A}_{\hat{\sigma}}(\hat{N})$ we can associate the Dirac operator

$$\hat{D}_{\hat{A}}\colon \Gamma(\hat{\mathbb{S}}^+_{\hat{\sigma}}) \to \Gamma(\hat{\mathbb{S}}^-_{\hat{\sigma}})$$

defined as the composition

$$\Gamma(\hat{\mathbb{S}}_{\hat{\sigma}}) \stackrel{\hat{\nabla}^{\hat{A}}}{\to} \Gamma(T^*\hat{N} \otimes \hat{\mathbb{S}}^+_{\hat{\sigma}}) \stackrel{\hat{\mathbf{c}}}{\to} \Gamma(\hat{\mathbb{S}}^-_{\hat{\sigma}}).$$

There is a natural quadratic map

$$q\colon \Gamma(\hat{\mathbb{S}}^+_{\hat{\sigma}}) \to \mathrm{End}\,(\hat{\mathbb{S}}^+_{\hat{\sigma}}), \quad \hat{\psi} \mapsto \tau(\hat{\psi})$$

defined by

$$q(\hat{\psi})\hat{\phi} = \langle \hat{\phi}, \hat{\psi} \rangle - \frac{1}{2}|\hat{\psi}|^2 \hat{\phi}.$$

In terms of Dirac's bra-ket notation $\tau(\hat{\psi})$ can be alternatively described as

$$q(\langle \hat{\psi}|) = |\hat{\psi}\rangle\langle\hat{\psi}| - \frac{1}{2}\langle\hat{\psi}|\hat{\psi}\rangle.$$

Note that for each $\hat{\psi}$ the endomorphism $\tau(\hat{\psi})$ is symmetric and traceless (see Appendix D).

The quantization map from the exterior algebra to the Clifford algebra extends the Clifford multiplication to a map

$$\hat{\mathbf{c}}\colon \Lambda^* T^* \hat{N} \to \operatorname{End}(\hat{\mathbb{S}}_{\hat{\sigma}}).$$

This map has the property that $\hat{\mathbf{c}}(\omega)$ is a traceless, skew-symmetric endomorphism of $\hat{\mathbb{S}}^+_{\hat{\sigma}}$ for any $\hat{g}$-self-dual real valued 2-form ω.

The Seiberg–Witten equations (associated to the spin^c structure $\hat{\sigma}$) are equations for a pair $(\hat{\psi}, \hat{A}) =$ (spinor in $\hat{\mathbb{S}}^+_{\hat{\sigma}}$, connection in $\mathfrak{A}_{\hat{\sigma}}(\hat{N})$). More precisely, they are

$$(\widehat{SW}) \quad \begin{cases} \hat{D}_{\hat{A}}\hat{\psi} & = & 0, \\ \hat{\mathbf{c}}(\hat{F}^+_{\hat{A}}) & = & \tau(\hat{\psi}). \end{cases}$$

In the remaining part of this subsection we will make further additional assumptions on the geometry and the topology of $\hat{N}$ and explain how this affects the Seiberg–Witten equations.

More precisely, assume the manifold $\hat{N}$ can be decomposed as

$$\hat{N} = \hat{N}_0 \cup [0, \infty) \times N$$

where $\hat{N}_0$ is a compact oriented 4-manifold with boundary $\partial\hat{N}_0 = N$. We will denote by t the longitudinal coordinate on the cylindrical part of N (see Fig. 5). Fix a tubular neighborhood $(-1, 0] \times N$ of N in $\hat{N}_0$, a metric g on N and a connection ∇ compatible with g, *not necessarily the Levi-Civita connection* of g. We assume that along the infinite cylinder $(-1, \infty) \times N$ the metric $\hat{g}$ is a product metric $\hat{g} = dt^2 + g$. We fix a connection $\hat{\nabla}$ compatible with $\hat{g}$ such that along the above cylindrical end it has the form

$$\hat{\nabla} = \partial_\tau \wedge dt + \nabla.$$

We denoted by ∂_τ the $\hat{g}$-gradient of τ where $\tau\colon \hat{N} \to [0, \infty)$ is a smooth function which coincides with the canonical projection $[0, \infty) \times N \to [0, \infty)$ on the infinite neck.

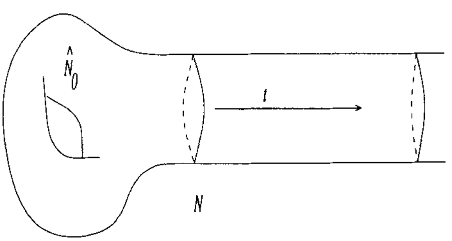

Figure 5. The background manifold $\hat{N}$.

Note that the spin^c structure $\hat{\sigma}$ induces a spin^c structure σ on $N = \partial \hat{N}_0$. Denote by $\mathbb{S}_\sigma \to N$ the associated bundle of spinors and by $\mathbf{c}\colon T^*N \to \mathrm{End}\,(N)$ the corresponding Clifford multiplication. As in the 4-dimensional case we can define $\mathfrak{A}_\sigma(N)$.

Fix a reference connection $\hat{A}_0 \in \mathfrak{A}_{\hat{\sigma}}(\hat{N})$ which, along the neck, is gauge equivalent to a product connection $dt \otimes \partial_t + A_0$, $A_0 \in \mathfrak{A}_\sigma(N)$. Now define the configuration space $\widehat{\mathfrak{C}}$ as the set of pairs $(\psi, \hat{A}_0 + \mathbf{i}\hat{a}) := (\hat{\psi}, \hat{A})$=(spinor, connection) such that $(\hat{\psi}, \mathbf{i}\hat{a}) \in L^{2,2}_{loc}(\hat{\mathbb{S}} \oplus \mathbf{i}T^*\hat{N})$ and

$$\hat{\nabla}^{\hat{A}}_{\partial_\tau}\hat{\psi} \oplus i_{\partial_\tau}\left(\mathbf{i}F_{\hat{A}}\right) \in L^2(\hat{\mathbb{S}}_{\hat{\sigma}} \oplus \mathbf{i}\Lambda^1 T^*\hat{N}).$$

We denoted by i_{∂_τ} the contraction by ∂_τ. For brevity, will denote the elements of $\widehat{\mathfrak{C}}$ by the generic symbol $\hat{\mathrm{C}}$.

Definition 3.1: (a) A finite energy solution of $(\widehat{SW}_\omega)$ is a solution $(\hat{\psi}, \hat{A})$ such that $(\hat{\psi}, \hat{A} - \hat{A}_0) \in \widehat{\mathfrak{C}}$.
(b) A Seiberg–Witten tunneling is a finite energy solution on $\hat{N} = \mathbb{R} \times N$.

There is an infinite dimensional group $\widehat{\mathfrak{G}}$ acting on the configuration space, more precisely

$$\widehat{\mathfrak{G}} = \{\gamma \in \mathrm{Map}\,(\hat{N}, S^1); \gamma \in L^{3,2}_{loc}\}.$$

The group $\widehat{\mathfrak{G}}$ acts (on the right) on $\widehat{\mathfrak{C}}$ and transforms finite energy solutions to finite energy solutions. Define

$$\widehat{\mathfrak{M}} := \{(\hat{\psi}, \hat{A}) \text{ finite energy solutions of } \widehat{SW}\}/\widehat{\mathfrak{G}}.$$

In this section we want to analyze the Fredholm properties of the deformation complex naturally associated to $\widehat{\mathfrak{M}}$ when N is a circle bundle over a Riemann surface. In particular, we will compute the virtual dimension of the space of Seiberg–Witten tunnelings.

We conclude this subsection with a simple but crucial observation which reveals the dynamical feature of the Seiberg–Witten equations on cylinders which perhaps will explain the tunneling terminology.

Note that if we set $J = \hat{\mathbf{c}}(d\tau)$ then J induces isomorphisms

$$\hat{\mathbb{S}}^+_{\hat{\sigma}}\,|_N \cong \hat{\mathbb{S}}^-_{\hat{\sigma}}\,|_N \cong \mathbb{S}_\sigma \tag{3.1}$$

and

$$\mathbf{c}(\alpha) = J\hat{\mathbf{c}}(\alpha), \quad \forall \alpha \in \Omega^1(N) \hookrightarrow \Omega^1([0,\infty) \times N). \tag{3.2}$$

A connection $\hat{A} \in \mathfrak{A}_{\hat{\sigma}}(\hat{N})$ is said to be in a **temporal gauge** if $i_{\partial_\tau}(\hat{A} - \hat{A}_0) = 0$ along the infinite neck $[0, \infty) \times N$.

Assume now that $(\hat{\psi}, \hat{A})$ is a finite energy solution of $(\widehat{SW})$ such that $\hat{A}$ is in a temporal gauge. Along the neck we can write

$$\hat{\psi} = \psi(t), \quad \hat{A} = A_0 + \mathbf{i}a(t)$$

where $A_0 = \hat{A}_0|_N$, $\psi(t) \in \Gamma(\mathbb{S}_\sigma)$, $a(t) \in \Omega^1(N)$, $\forall t \geq 0$. Then (along the neck)

$$\hat{F}^+_{\hat{A}} = \frac{1}{2}\{(F_a + *\mathbf{i}\dot{a}) + dt \wedge (\mathbf{i}\dot{a} + *F_a)\} \tag{3.3}$$

where $A_0 + a(t)$ is the connection on the line bundle $\det \sigma$ restricted to the slice $\{t\} \times N$, $F_a = F_{A_0 + \mathbf{i}a}$ denotes its curvature and $*$ denotes the Hodge star operator on N. $A_0 + \mathbf{i}a(t)$ induces a Dirac operator

$$D_a = D_{a(t)} \colon \Gamma(\mathbb{S}_\sigma) \to \Gamma(\mathbb{S}_\sigma).$$

Using (3.1) and (3.2) we deduce that along the neck

$$\hat{D}_{\hat{A}} = J\left(\partial_t - \mathcal{D}_a\right).$$

The equality (3.3) now implies

$$\hat{\mathbf{c}}(\hat{F}^+_{\hat{A}}) = \mathbf{c}(*F_a + \mathbf{i}\dot{a}).$$

Consequently, along the neck, in a temporal gauge, the Seiberg–Witten equations can be rewritten as

$$\left\{\begin{array}{rcl} \dot{\psi} & = & D_a\psi, \\ \mathbf{i}\mathbf{c}(\dot{a}) & = & q(\psi) - \mathbf{c}(*F_a). \end{array}\right. \tag{3.4}$$

The right-hand side of (3.4) arises when one considers the three dimensional counterpart of the Seiberg–Witten equations.

§3.2. The 3-dimensional Seiberg–Witten equations.

To formulate these equations we need to consider a new configuration space. Fix a connection $A_0 \in \mathfrak{A}_\sigma(N)$ and define

$$\mathfrak{C} = \{(\psi, A)\ ;\ (\psi, (A - A_0) \in L^{1,2}(\mathbb{S}_\sigma \oplus \mathbf{i}T^*N)\}.$$

For brevity, its elements will be denoted by the symbol C and we will often write $\mathrm{C} = (\psi, a)$ instead of $(\psi, A_0 + \mathbf{i}a)$ whenever no confusion is possible. There is an energy functional $\mathcal{E} \colon \mathfrak{C} \to \mathbb{R}$ defined by

$$\mathcal{E}(\psi, A) = \frac{\mathbf{i}}{2}\int_N a \wedge (F_{A_0} + F_A) + \frac{1}{2}\int_N \langle \psi, \mathcal{D}_A\psi\rangle dv_g. \tag{3.5}$$

The gauge group

$$\mathfrak{G} = \{\gamma \in \mathrm{Map}\,(N, S^1)\ ;\ \gamma \in L^{2,2}\}$$

acts on $\mathfrak{C}$ and, moreover,

$$\mathcal{E}(\gamma^{-1}\cdot(\psi, A)) - \mathcal{E}(\psi, A) = -\int_N \gamma^{-1}d\gamma \wedge F_{A_0} = 2\pi\mathbf{i}\int_N \gamma^{-1}d\gamma \wedge c_1(A_0),$$

where we denoted by $c_1(A_0)$ the 2-form representing the first Chern class of $\det\sigma$ associated to A_0 via the Chern–Weil construction. The L^2-gradient of $\mathcal{E}$ is (see [N] or [MOY])

$$\nabla\mathcal{E}(\psi, A) = \left[\begin{array}{c} \mathcal{D}_A\psi \\ q(\psi) - *F_A \end{array}\right]$$

where we tacitly identified $q(\psi)$ with a purely imaginary 1-form via the Clifford multiplication. The 3-dimensional Seiberg–Witten equations can now be described as

$$\nabla\mathcal{E}(\mathrm{C}) = 0 \Longleftrightarrow \left\{\begin{array}{rcl} \mathcal{D}_A\psi & = & 0, \\ \mathbf{c}(*F_A) & = & q(\psi). \end{array}\right.$$

We see that (3.4) can be rewritten as a gradient flow equation

$$\dot{\mathrm{C}} = \nabla\mathcal{E}(\mathrm{C}). \tag{3.6}$$

This last equation suggests that as $t \to \infty$, $\mathrm{C}(t)$ converges to a critical point of $\mathcal{E}$. Assuming the finite energy condition this can be proved for arbitrary N using the techniques of [MMR]. However, unlike the Yang–Mills situation, the nature of critical points and the manner in which they are organized are less transparent in the Seiberg–Witten case. This is the reason why we will concentrate on a special case.

In the remainder of the section N will be assumed to be a degree $\ell \neq 0$ circle bundle over a genus $g > 0$ Riemann surface Σ, $S^1 \hookrightarrow N \stackrel{\pi}{\to} \Sigma$ equipped with the metric described in §2.1 As background g-compatible connection on N we choose the adiabatic connection $\nabla^0 = \lim_{t\to 0}\nabla^{r,t}$, where r is fixed and small.

The spin^c structures on N are bijectively parameterized by the space of isomorphism classes of hermitian line bundles on N. Fix a spin structure on Σ determined by a holomorphic square root $K^{1/2}$. If $L \to N$ is such a line bundle, then the corresponding bundle of complex spinors is

$$\mathbb{S}_L = \mathcal{K}^{-1/2}\otimes L \oplus \mathcal{K}^{1/2}\otimes L.$$

Moreover, we can identify the connections in $\mathfrak{A}_\sigma(N)$ with the hermitian connections on L. The Dirac operator on $\mathbb{S}_L$ induced by ∇^0 and a connection A on L

will be denoted by D_A. If instead of ∇^0 we use the Levi-Civita connection of the metric g_r we get a different Dirac type operator that we denote by $\mathcal{D}_A$. The operator D_A can be related to $\mathcal{D}_A$ by the following simple identity (see Section §2.1):

$$D_A = \mathcal{D}_A - \frac{\lambda_r}{2}, \quad \lambda_r = -r\ell.$$

Both D_A and $\mathcal{D}_A$ have obvious extensions to $[0,\infty) \times N$ given by

$$\hat{D}_A = J(\partial_t - D_A), \quad \hat{\mathcal{D}}_A = J(\partial_t - \mathcal{D}_A).$$

Under these special geometric circumstances the Seiberg–Witten equations can be rewritten in a more useful form.

Using the decomposition $\mathbb{S}_L \cong \left(\mathcal{K}^{-1/2} \otimes L\right) \oplus \left(\mathcal{K}^{1/2} \otimes L\right)$ we can represent any section ψ of $\mathbb{S}_L$ as $\psi = \psi_- \oplus \psi_+$. Then the Seiberg–Witten equations can be rephrased as (see [N])

$$(3.7) \quad \begin{cases} \mathbf{i}\nabla^A_\zeta\psi_- & +\bar{\partial}_A\psi_+ & +\lambda\psi_- & = & 0, \\ (\bar{\partial}_A)^*\psi_- & -\mathbf{i}\nabla_\zeta\psi_+ & +\lambda\psi_+ & = & 0, \\ & & \frac{1}{2}(|\psi_-|^2 - |\psi_+|^2) & = & \mathbf{i}F_A(\zeta_1,\zeta_2), \\ & & \mathbf{i}\psi_-\bar{\psi}_+ & = & \bar{\varepsilon}\otimes F_A(\zeta_1+\mathbf{i}\zeta_2,\zeta), \end{cases}$$

where $\varepsilon = 2^{-1/2}(\varphi^1 + \mathbf{i}\varphi^2)$.

Set

$$\mathfrak{C}^* = \{(\psi, A) \in \mathfrak{C}\,;\ \psi \not\equiv 0\}.$$

The configurations in $\mathfrak{C}^*$ are called *irreducible*. As in [M] one can show that $\mathfrak{B} := \mathfrak{C}/\mathfrak{G}$ is a metric space and, moreover, $\mathfrak{B}^* = \mathfrak{C}^*/\mathfrak{G}$ is a Banach manifold. This is proved using the existence of local slices for the $\mathfrak{G}$-action exactly as in the Yang–Mills case. For every configuration $\mathrm{C} \in \mathfrak{C}$ we will denote by $[\mathrm{C}]$ its image in $\mathfrak{B}$.

The solutions of (3.7) are explicitly described in [N] and [MOY]. Here are the relevant facts.

FACT 1: If $c_1(L)$ is not torsion then (3.7) has no solutions.

Assume now that $c_1(L) \equiv \kappa \pmod{\ell}$ and define

$$R_\kappa = \{n \in \mathbb{Z}\,;\ 1 \le |n| \le g-1,\ \ n \equiv \kappa \bmod \ell\}.$$

FACT 2: Any irreducible solution (ψ, A) of (3.7) is gauge equivalent to the pullback of a pair $(\tilde{\psi}, B)$ where B is a connection in a line bundle $L_\Sigma \to \Sigma$ such that $\deg L_\Sigma \in R_\kappa$ (so that $\pi^* L_\Sigma \cong L$); $\tilde{\psi} = \tilde{\psi}_- \oplus \tilde{\psi}_-$ is a section of $C^\infty(K^{-1/2} \otimes L_\Sigma \oplus K^{1/2} \otimes L)$. The connection B defines holomorphic structures in $K^{\pm 1/2} \otimes L$. $\tilde{\psi}_-$ is an antiholomorphic section of $K^{-1/2} \otimes L$ while $\tilde{\psi}_+$ is a holomorphic section of $K^{1/2} \otimes L_\Sigma$. Moreover, one of $\tilde{\psi}_-$ or $\tilde{\psi}_+$ is zero and satisfy the identity

$$\frac{1}{4\pi} \int_\Sigma (|\tilde{\psi}_-|^2 - |\tilde{\psi}_+|^2)\, dv = \deg L_\Sigma.$$

Thus $\tilde{\psi}_+ = 0$ if $\deg L_\Sigma > 0$ and $\tilde{\psi}_- = 0$ if $\deg L_\Sigma < 0$. The irreducible part (mod $\mathfrak{G}$), denoted by $\mathfrak{M}^*$, consists of $\# R_\kappa$ components

$$\mathfrak{M}^* = \bigcup_{n \in R_\kappa} \mathfrak{M}_{\kappa,n}.$$

The component $\mathfrak{M}_n = \mathfrak{M}_{\kappa,n}$ (corresponds to the choice $\deg L_\Sigma = n$) is diffeomorphic to a symmetric product of $(g-1) - |n|$ copies of Σ and thus has real dimension $2(g-1-|n|)$. Each component is Bott nondegenerate as a critical set. (Pairs $(\tilde{\psi}_- \oplus \tilde{\psi}_+, B)$ as above are known as *vortex pairs* on Σ.)

FACT 3: The reducible solutions consist of pairs (zero spinor, flat connection). Modulo $\mathfrak{G}$ they form a space $\mathfrak{M}^0_\kappa$ homeomorphic to a $2g$-dimensional torus. Moreover, if $\kappa \not\equiv 0 \pmod{\ell}$ the reducible part is nondegenerate (in a sense described in [MOY]). If $\kappa \equiv 0$ the reducible solutions can be identified with the theta divisor W_{g-1} inside the Jacobian $J_{g-1}(\Sigma)$ (see [GH] for a definition of W_{g-1}).

Associated to each component $\mathfrak{M}$ there is a deformation theory which we now proceed to describe. We will concentrate only on the irreducible part $\mathfrak{C}^*$. Since $c_1(L)$ is torsion, the energy functional $\mathcal{E}$ is gauge invariant and thus it descends to a well defined functional

$$\underline{\mathcal{E}}: \mathfrak{B}^* \to \mathbb{R}.$$

The group $\mathfrak{G}$ is a Hilbert–Lie group and its Lie algebra can be identified with the space $\mathfrak{g} := L^{2,2}(N, \mathbf{i}\mathbb{R})$. The exponential map has the form

$$\mathfrak{g} \ni \mathbf{i}f \mapsto (\exp(\mathbf{i}f): N \to S^1).$$

The tangent space to the orbit $\mathcal{O}_{\phi,A}$ through $\mathrm{C} = (\phi, A)$ of the *right* action of $\mathfrak{G}$ is the range of the infinitesimal action operator

$$\mathcal{L} = \mathcal{L}_{\mathrm{C}}: \mathfrak{g} \to \mathcal{X} := L^{1,2}(\mathbb{S}_L) \oplus L^{1,2}(\mathbf{i}T^*N), \quad \mathbf{i}f \mapsto -\mathbf{i}f \oplus \mathbf{i}df.$$

The tangent space to $\mathfrak{B}^*$ at [C] can be identified with the orthogonal complement to the tangent space to the orbit $\mathcal{O}_C$ and ultimately with the kernel of $\mathcal{L}_C^*$, the adjoint of $\mathcal{L}_C$. An integration by parts shows

$$\mathcal{L}^*(\dot\psi \oplus \mathbf{i}\dot a) = -\mathbf{i}d^*a + \mathbf{i}\mathfrak{Im}\langle \phi, \dot\psi\rangle, \quad \forall \dot\psi \oplus \mathbf{i}\dot a \in \mathcal{X}.$$

We can use the affine structure of $\mathfrak{C}$ to linearize $\nabla\mathcal{E}$ at a given configuration $\mathrm{C} = (\phi, A)$ and we obtain the *unrestricted hessian* at C

$$\mathcal{H}_C \begin{bmatrix} \dot\psi \\ \mathbf{i}\dot a \end{bmatrix} = \frac{d}{dt}|_{t=0} \nabla\mathcal{E}(\psi + t\dot\psi, A + t\mathbf{i}\dot a) = \begin{bmatrix} D_A\dot\psi + \mathbf{c}(\mathbf{i}\dot a)\phi \\ -\mathbf{i} * d\dot a + \dot q(\phi, \dot\psi) \end{bmatrix}.$$

The term $\dot q(\phi, \dot\psi)$ is formally defined by the equality

$$\dot q(\phi, \dot\psi) := \frac{d}{dt}|_{t=0}\, q(\phi + t\dot\psi)$$

where we regard q as a quadratic map $q\colon \mathbb{S}_L \to \mathbf{i}T^*N$.

The *stabilized hessian* of $\mathcal{E}$ at $\mathrm{C} = (\phi, A)$ is the unbounded operator on $L^2(\mathbb{S}_L \oplus \mathbf{i}(\Lambda^1 \oplus \Lambda^0)T^*N)$ defined by

$$\begin{aligned}\tilde{\mathcal{H}}_C \begin{bmatrix} \dot\psi \oplus \mathbf{i}\dot a \\ \mathbf{i}f \end{bmatrix} &:= \begin{bmatrix} \mathcal{H} & \mathcal{L} \\ \mathcal{L}^* & 0 \end{bmatrix} \begin{bmatrix} \dot\psi \oplus \mathbf{i}\dot a \\ \mathbf{i}f \end{bmatrix} \\ &= \begin{bmatrix} D_A\dot\psi & & + & \mathbf{c}(\mathbf{i}\dot a)\phi - \mathbf{i}f\phi \\ & -\mathbf{i} * d\dot a + \mathbf{i}df & + & \dot q(\phi, \dot\psi) \\ & \mathbf{i}d^*\dot a & + & \mathbf{i}\mathfrak{Im}\langle\phi, \dot\psi\rangle \end{bmatrix}.\end{aligned}$$

In [N] and [MOY] it is shown that if $[\mathrm{C}] \in \mathfrak{M}_{\kappa,n}$ then the kernel of the stabilized hessian $\tilde{\mathcal{H}}_C$ is naturally isomorphic to the tangent space $T_{[\mathrm{C}]}\mathfrak{M}_{\kappa,n}$. Now define

$$\tilde{\mathcal{H}}_0 \begin{bmatrix} \dot\psi \\ \mathbf{i}\dot a \\ \mathbf{i}f \end{bmatrix} = \begin{bmatrix} D_A & 0 & 0 \\ 0 & -*d & d \\ 0 & d^* & 0 \end{bmatrix} \begin{bmatrix} \dot\psi \\ \mathbf{i}\dot a \\ \mathbf{i}f \end{bmatrix}$$

and $\mathcal{P} = \mathcal{P}_\phi$ by

$$\mathcal{P}_\phi \begin{bmatrix} \dot\psi \\ \mathbf{i}\dot a \\ \mathbf{i}f \end{bmatrix} = \begin{bmatrix} \mathbf{c}(\mathbf{i}\dot a)\phi - \mathbf{i}f\phi \\ \dot q(\phi, \dot\psi) \\ \mathbf{i}\mathfrak{Im}\langle\phi, \dot\psi\rangle \end{bmatrix}.$$

Note that $\tilde{\mathcal{H}}_0 = \tilde{\mathcal{H}}_0(\mathrm{C})$ is an elliptic selfadjoint operator for any $\mathrm{C} \in \mathfrak{C}$ and $\tilde{\mathcal{H}}_C = \tilde{\mathcal{H}}_0 + \mathcal{P}_\phi$. For every $\mathrm{C} \in \mathfrak{C}$ define $\mathrm{SF}_\pm(\mathrm{C}) \in \mathbb{Z}$ as the spectral flow of the path $\pm(\tilde{\mathcal{H}}_0(\mathrm{C}) + t\mathcal{P}_\phi)$, $t \in [0,1]$. The next subsection is devoted to the computation of $\mathrm{SF}_\pm(\mathrm{C})$ when $[\mathrm{C}] \in \mathfrak{M}_{\kappa,n}$. For the reducible component $\mathcal{P}_\phi \equiv 0$ and this problem is trivial.

Define now for later use the **resonance matrix**. This is the quadratic form $\mathcal{R}$ on $\ker \tilde{\mathcal{H}}_0$ defined by

$$\mathcal{R}\Xi = \mathcal{R}_\phi \Xi = \mathbf{Proj}\mathcal{P}_\phi \Xi, \quad \Xi = \dot{\psi} \oplus \mathbf{i}\dot{a} \oplus \mathbf{i}f \in \ker \tilde{\mathcal{H}}_0$$

where **Proj** denotes the orthogonal projection onto $\ker \tilde{\mathcal{H}}_0$. Note also that for $[\mathrm{C}] \in \mathfrak{M}_{\kappa,n}$

$$\ker \tilde{\mathcal{H}}_0([\mathrm{C}]) = H^0(K^{1/2} - L_\Sigma) \oplus H^0(K^{1/2} + L_\Sigma) \oplus H^1(\Sigma, \mathbb{R}) \oplus H^0(\Sigma, \mathbb{R})$$

where L_Σ is the holomorphic line bundle on Σ determined by [C] as in Fact 2.

§3.3. SPECTRAL FLOWS AND PERTURBATION THEORY.

Fix $[\mathrm{C}] = [\phi, A] \in \mathfrak{M}_{\kappa,n}$. Assume for simplicity that $n < 0$ so that $\phi_- = 0$. Denote by L_Σ the holomorphic line bundle on Σ ($\deg L_\Sigma = n$), by B the induced connection on L_Σ and by $\tilde{\phi}_+$ the holomorphic section of $K^{1/2} \otimes L_\Sigma$ determined by [C]. The computation of $\mathrm{SF}_+([\mathrm{C}])$ is carried out in two steps. We consider only the spectral flow SF_+. Also for simplicity we will write L instead of L_Σ, and ϕ instead of $\tilde{\phi}$.

STEP 1: Along the path $t \mapsto \tilde{\mathcal{H}}_t := \tilde{\mathcal{H}}_0 + t\mathcal{P}$, $t \in [0,1]$ there is no spectral flow contribution for $t \neq 0$.

STEP 2: Compute the spectral flow contribution at $t = 0$.

Note first that $t \mapsto \tilde{\mathcal{H}}_t$ is an analytic family of selfadjoint operators with compact resolvent and thus by known perturbation results (see [Kato], Thm. 3.9, Chap. VII) the eigenvalues and the eigenvectors of this family can be locally organized in analytic families. To complete the first step it suffices to show that $\dim \ker \tilde{\mathcal{H}}_t$ is independent of $0 < t \leq 1$.

With this aim consider as in [N], Sect. 4.2, the following elliptic complex

$$(V_{[\mathrm{C}]})\colon 0 \to \mathbf{i}\Omega^0(\Sigma) \xrightarrow{I} \Gamma(L \otimes K^{1/2}) \oplus \mathbf{i}\Omega^1(\Sigma) \xrightarrow{\Upsilon} \Gamma(L \otimes K^{-1/2}) \oplus \mathbf{i}\Omega^0(\Sigma) \to 0,$$

where

$$\Upsilon \begin{bmatrix} \dot{\beta} \\ \mathbf{i}\dot{a} \end{bmatrix} = \begin{bmatrix} \bar{\partial}\dot{\beta} + 2^{-1/2}\mathbf{i}\dot{a}^{0,1}\tilde{\phi}_+ \\ \mathbf{i} * d\dot{a} - \mathbf{i}\mathfrak{Re}\,\langle \tilde{\phi}_+, \dot{\beta}\rangle \end{bmatrix};$$

$\mathbf{i}\dot{a}^{0,1}$ component is the K^{-1}-component of $\mathbf{i}\dot{a}$ corresponding to the orthogonal decomposition $T^*\Sigma \otimes \mathbb{C} \cong K \oplus K^{-1}$. I is the infinitesimal action

$$\mathbf{i}f \overset{I}{\mapsto} (-\mathbf{i}f\tilde{\phi}_+, B + \mathbf{i}df).$$

In Sect. 4.2 of [N] it is shown that $H^0(V_{[C]}) \cong H^2(V_{[C]}) \cong 0$ and

$$\dim_{\mathbf{R}} H^1(V_{[C]}) = -\mathrm{ind}_{\mathbf{R}}(V_{[C]}) = \dim \mathfrak{M}_{\kappa,n}.$$

Arguing exactly as in Sect. 5.6 of [MOY] one can prove that

$$\ker \tilde{\mathcal{H}}_t \cong H^1(V_{[C]}), \quad \forall t \in (0,1].$$

In particular, if $\Xi = \dot{\psi} \oplus \mathbf{i}\dot{a} \oplus \mathbf{i}f \in \ker \tilde{\mathcal{H}}_t$, $t > 0$ then $f \equiv 0$. This concludes the first step in our program.

STEP 2: Before we embark on the computation of the spectral flow contribution at $t = 0$ we need to survey a few facts pertaining to perturbation theory.

As we have already mentioned, the spectral data of $\tilde{\mathcal{H}}_t$ can be organized in families depending analytically upon t. Denote by Z the set of all pairs $(\lambda(t), \Xi(t))$ where $\lambda(t)$ is an eigenvalue of $\tilde{\mathcal{H}}_t$, $\Xi(t)$ is a (length 1) eigenvector corresponding to $\lambda(t)$, $\lambda(0) = 0$ and the dependence

$$t \mapsto (\lambda(t), \Xi(t))$$

is analytic. Clearly, $\#Z = \dim \tilde{\mathcal{H}}_0$. For every $(\lambda(t), \Xi(t))$ we have Taylor expansions

$$\lambda(t) = \lambda_\nu t^\nu + \cdots, \quad \lambda_\nu \neq 0,$$

$$\Xi(t) = \Xi_0 + t\Xi_1 + \cdots, \quad \Xi_0 \in \ker \tilde{\mathcal{H}}_0, \quad |\,\Xi_0| = 1.$$

The integer ν is called **the order** of the pair $(\lambda(t), \Xi(t))$. A pair is called **degenerate** if its order is > 1 and **nondegenerate** if it has order 1. Set

$$Z^* = \{(\lambda(t), \Xi(t)) \in Z\,;\ \lambda(t) \not\equiv 0\}.$$

The complement $Z \setminus Z^*$ is determined (according to Step 1) by $\ker \tilde{\mathcal{H}}_t$ $(t > 0)$ and thus

$$\#Z^* = \dim \ker \tilde{\mathcal{H}}_0 - \dim \ker \tilde{\mathcal{H}}_1.$$

The spectral flow $\mathrm{SF}_+([C])$ is then determined by

$$\mathrm{SF}_+([C]) = -\#\{(\lambda(t), \Xi(t)) \in Z^*; \lambda_\nu < 0\}. \tag{3.8}$$

To determine this integer we will distinguish two cases.

The nondegenerate case $(\nu = 1)$: The equation $\tilde{\mathcal{H}}_t \Xi(t) = \lambda(t)\Xi(t)$ implies

$$\tilde{\mathcal{H}}_0 \Xi_0 = 0, \quad \tilde{\mathcal{H}}_0 \Xi_1 + \mathcal{P}\Xi_0 = \lambda_1 \Xi_0.$$

This shows that λ_1 is a *nonzero* eigenvalue of the resonance matrix $\mathcal{R}$ and moreover

$$\operatorname{sign}\lambda_1 = \operatorname{sign}\langle\mathcal{R}\Xi_0, \Xi_0\rangle. \tag{3.9}$$

In particular, the contribution to the spectral flow of the nondegenerate pairs is equal to the number of negative eigenvalues of the resonance matrix $\mathcal{R}$. Thus we need to better understand the structure of the resonance form

$$Q(\Xi) = \langle\mathcal{R}\Xi, \Xi\rangle, \quad \Xi \in \ker\tilde{\mathcal{H}}_0.$$

Any $\Xi \in \ker\tilde{\mathcal{H}}_0$ decomposes as

$$\Xi = \dot{\psi} \oplus \mathbf{i}\dot{a} \oplus \mathbf{i}f$$

where $\dot{\psi} = \dot{\psi}_- \oplus \dot{\psi}_+ \in \ker D_A$, $\dot{a} \in \Omega^1(N)$ is harmonic and f is constant. All these objects are pulled back from the base and moreover

- $\dot{\bar{\psi}}_- \in H^0(K^{1/2} - L)$, $\dot{\psi}_+ \in H^0(K^{1/2} + L)$.
- $\mathbf{i}\dot{a} = \frac{1}{2}(\omega - \bar{\omega})$, $\omega \in H^0(K)$.

With these observations in place we have the following result.

LEMMA 3.2:

$$Q(\Xi) = 2^{1/2} f\,\mathfrak{Im}\langle\phi_+, \dot{\psi}_+\rangle - \mathfrak{Re}(\dot{\bar{\psi}}_- \phi_+ \bar{\omega}).$$

The proof of this lemma can be found in Appendix D, equality (D.4).

We see that (up to the positive factor $2^{1/2}$) the resonance form is the direct sum of

(a) a quadratic form Q_1 on $\mathbb{R} \oplus H^0(K^{1/2} + L)$

$$Q_1(f \oplus \dot{\psi}_+) = f\,\mathfrak{Im}\langle\phi_+, \dot{\psi}_+\rangle,$$

(b) a quadratic form Q_2 on $H^0(K^{1/2} - L) \oplus H^0(K)$ defined by

$$Q_2(\dot{\bar{\psi}}_- \oplus \omega) = -\mathfrak{Re}(\dot{\bar{\psi}}_- \phi_+ \bar{\omega}).$$

If we denote by $\dim_\pm$ the dimension of the positive/negative eigenspace of a quadratic form then

$$\dim_- Q = \dim_- Q_1 + \dim_- Q_2.$$

The negative eigenspace of Q_1. Set $V = H^0(K^{1/2} + L)$ and $e_1 = \phi_+$. Then

$$\Omega(u, v) = \mathfrak{Im}\langle u, v\rangle, \quad u, v \in V$$

is a symplectic form on V. Q_1 is the quadratic form on $\mathbb{R} \oplus V$ defined by

$$Q_1(f \oplus v) = f\Omega(e_1, v).$$

To determine its negative eigenspace extend e_1 to a symplectic basis $e_1, e_2, \dots, e_{2d-1}, e_{2d}$ where $d = \dim_{\mathbf{C}} V$. If $v = \sum_j v_j e_j$ then

$$Q_1(f \oplus v) = fv_2.$$

This can be easily diagonalized and we get

$$\dim_- Q_1 = 1 = \dim_+ Q_1 \tag{3.10}$$

and

$$\dim \ker Q_1 = 2\dim_{\mathbf{C}} V - 1 = 2h_0(K^{1/2} + L) - 1 = 2h_{1/2}(L) - 1. \tag{3.11}$$

The negative eigenspace of Q_2. Consider the multiplication map

$$\mathbf{m}\colon H^0(K^{1/2} - L) \to H^0(K), \quad \dot{\bar{\psi}}_- \mapsto \dot{\bar{\psi}}_- \phi_+.$$

$\mathbf{m}$ is obviously injective. (The implied inequality $\dim H^0(K^{1/2} - L) \leq \dim H^0(K)$ is also a consequence of the classical Clifford theorem.) Set $V = H^0(K)$ and $U = \operatorname{Range} \mathbf{m}$. Q_2 can be rewritten as

$$Q_2(\dot{\bar{\psi}}, \omega) = -\mathfrak{Re}\langle \mathbf{m}\dot{\bar{\psi}}, \omega\rangle$$

and thus it can be regarded as the quadratic form on $U \oplus V$,

$$Q_2(u \oplus v) = -\mathfrak{Re}\langle u, v\rangle.$$

This can be again easily diagonalized and leads to the equalities

$$\dim_- Q_2 = \dim_{\mathbf{R}} U = 2h_{1/2}(L^*) = \dim_+ Q_2, \tag{3.12}$$

$$\dim \ker Q_2 = \dim_{\mathbf{R}} H^0(K) - \dim_{\mathbf{R}} U = 2g - 2h_{1/2}(L^*). \tag{3.13}$$

Summarizing, we deduce the following.

A. The spectral flow contribution of the nondegenerate pairs in Z^* is

$$-1 - 2h_{1/2}(-L).$$

B. The number of degenerate pairs $(\lambda(t), \Xi(t)) \in Z$ is equal to

$$\dim_{\mathbf{R}} \ker \mathcal{R} = 2h_{1/2}(L) + 2g - 2h_{1/2}(-L) - 1 = 2(g - |n|) - 1.$$

Recall that $\dim_{\mathbf{R}} \ker \tilde{\mathcal{H}}_t = 2(g-1-|n|)$ (if $t > 0$) and the pairs $(\lambda(t), \Xi(t))$ spanning $\ker \tilde{\mathcal{H}}_t$ do not contribute to the spectral flow. Hence there can be at most $\dim_{\mathbf{R}} \ker \mathcal{R} - \dim \ker_{\mathbf{R}} \tilde{\mathcal{H}}_t = 1$ degenerate pairs contributing to the spectral flow.

The degenerate case ($\nu > 1$): Set $d = \dim \ker \tilde{\mathcal{H}}_1$. We have $d+1$ degenerate pairs

$$\{(\lambda^k(t), \Xi^k(t))\ ;\ k = 0, \dots, d\}$$

where the labeling is such that $\ker \tilde{\mathcal{H}}_t = \operatorname{span}_{k \geq 1}(\Xi^k(t))$. Thus we need to determine the contribution of the pair $(\lambda^0(t), \Xi^0(t))$ to the spectral flow. First we claim that this pair has order two. To achieve this we argue by contradiction.

Set

$$S = \{\Xi_0 \oplus \Xi_1\ ;\ \Xi_0 \in \ker \mathcal{R},\ \ \tilde{\mathcal{H}}_0 \Xi_1 + \mathcal{P}\Xi_0 = 0\}.$$

Using the perturbation series

$$\lambda^0(t) = \lambda^0_\nu t^\nu + \cdots, \quad \nu \geq 3,$$

$$\lambda^k(t) \equiv 0, \ \ \forall k = 1, \dots, d,$$

$$\Xi^k(t) = \Xi^k_0 + \Xi^k_1 t + \Xi^k_2 t^2 + \cdots, \quad k = 0, \dots, d,$$

we deduce that for all $k = 0, \dots, d$

$$(3.14) \qquad \begin{cases} \tilde{\mathcal{H}}_0 \Xi^k_0 = 0, \\ \tilde{\mathcal{H}}_0 \Xi^k_1 + \mathcal{P}\Xi^k_0 = 0, \\ \tilde{\mathcal{H}}_0 \Xi^k_2 + \mathcal{P}\Xi^k_1 = 0. \end{cases}$$

Thus $\Xi^k_0 \oplus \Xi^k_1 \in S\ \forall k$. Taking the inner product with Ξ^j_0 in the last inequality we get

$$(3.15) \qquad \langle \mathcal{P}\Xi^k_1, \Xi^j_0 \rangle = 0, \ \ \forall j, k = 0, \dots, d.$$

Now observe the following elementary fact. Given

$$(\Xi_0, \Xi_1),\ (\Xi_0, \Xi'_1),\ (U_0, U_1) \in S$$

i.e. $\Xi_0 \in \ker \mathcal{P}$ and $\tilde{\mathcal{H}}_0 \Xi_1 = \tilde{\mathcal{H}}_0 \Xi'_1 = -\mathcal{P}\Xi_0$, then

$$\langle \mathcal{P}\Xi_1, U_0 \rangle = \langle \mathcal{P}\Xi'_1, U_0 \rangle.$$

Indeed,

$$\langle \mathcal{P}\Xi_1 - \mathcal{P}\Xi'_1, U_0 \rangle = \langle \Xi_1 - \Xi'_1, \mathcal{P}U_0 \rangle = 0$$

since $\Xi_1 - \Xi_1' \in \ker \tilde{\mathcal{H}}_0$ and $\mathcal{P}U_0 \perp \ker \tilde{\mathcal{H}}_0$. In other words, the quantity

$$\mathcal{B}(\Xi_0 \oplus, \Xi_1, U_0 \oplus U_1) = \langle \mathcal{P}\Xi_1, U_0 \rangle$$

depends bilinearly only upon Ξ_0 *and* U_0. Thus, it defines a bilinear form on $\ker \mathcal{R}$ and one can check it is also symmetric. The equality (3.15) implies

$$\mathcal{B}(\Xi_0^j, \Xi_0^k) = 0 \quad \forall j, k = 0, \ldots, d,$$

i.e. $\mathcal{B}$ is trivial on $\ker \mathcal{R}$. We will show that this is not the case, thus establishing that the order of $\lambda^0(t)$ must be 2.

Let

$$\Xi_0 = \begin{bmatrix} 0 \oplus \phi_+ \\ 0 \\ 0 \end{bmatrix} \in \ker \tilde{\mathcal{H}}_0.$$

Using the identity (D.3) in Appendix D we get

$$\mathcal{P}\Xi_0 = \begin{bmatrix} 0 \\ \mathbf{i}|\phi_+|^2\varphi \\ 0 \end{bmatrix} \in \left(\ker \tilde{\mathcal{H}}_0\right)^{\perp},$$

where φ is the global angular form on N. Hence $\Xi_0 \in \ker \mathcal{R}$. We have to solve

$$\tilde{\mathcal{H}}_0 \Xi_1 + \mathcal{P}\Xi_0 = 0.$$

If we write $\Xi_1 = \dot{\psi} \oplus \mathbf{i}\dot{a} \oplus \mathbf{i}f$ then the above equation can be rewritten as

$$\text{(3.16)} \qquad \begin{cases} D_A\dot{\psi} &= 0, \\ -*d\dot{a} + df + |\phi_+|^2\varphi &= 0, \\ d^*\dot{a} &= 0. \end{cases}$$

One can say quite a lot about Ξ_1. First note that since $|\phi_+|^2\varphi$ is co-closed (Appendix D) and $f \perp \{\text{constants}\}$ we conclude that $f \equiv 0$. We deduce

$$\text{(3.17)} \qquad \begin{cases} *d\dot{a} &= |\phi_+|^2\varphi, \\ d^*\dot{a} &= 0. \end{cases}$$

The above equation has a unique solution $\dot{a}$ orthogonal to the space of harmonic 1-forms. It is given explicitly by

$$\text{(3.18)} \qquad \dot{a} = -\frac{1}{2\ell}|\phi_+|^2\varphi.$$

Taking the inner product with $\dot{a}$ of the second equation of (3.16) we deduce that

$$\mathcal{B}(\Xi_0) = \langle \Xi_1, \mathcal{P}\Xi_0 \rangle = -\langle \tilde{\mathcal{H}}_0\Xi_1, \Xi_1 \rangle$$

$$= \int_N \langle *d\dot{a}, \dot{a}\rangle dv_N = \int_N |\phi_+|^2 \langle \varphi, \dot{a}\rangle dv_N = -\frac{1}{2\ell}\int_N |\phi_+|^2 dv_N.$$

We conclude from (3.18) that

$$\mathcal{B}(\Xi_0) = -\operatorname{sign}\ell. \tag{3.19}$$

This shows that $\mathcal{B}$ is nontrivial. Since $\mathcal{B}$ can have at most one nonzero eigenvalue the above equality shows that this eigenvalue has the same sign as $-\ell$. If we now use the perturbation equations for $\lambda^0(t)$ we obtain

$$\begin{cases} \tilde{\mathcal{H}}_0\Xi_0^0 = 0, \\ \tilde{\mathcal{H}}_0\Xi_1^0 + \mathcal{P}\Xi_0^0 = 0, \\ \tilde{\mathcal{H}}_0\Xi_2^0 + \mathcal{P}\Xi_1^0 = \lambda_2^0\Xi_0^0. \end{cases}$$

We deduce

$$\operatorname{sign}\lambda_2^0 = \operatorname{sign}\langle \mathcal{P}\Xi_1^0, \Xi_0^0\rangle = \operatorname{sign}\mathcal{B}(\Xi_0^0) = -\operatorname{sign}\ell.$$

Thus the degenerate part contributes to the spectral flow only when $\ell > 0$.

We can now assemble all the information we have collected so far in the following result.

THEOREM 3.3:

$$\mathrm{SF}_+([\mathrm{C}]) = \begin{cases} -2 - 2h_{1/2}(L^*), & \text{if } \ell > 0, \\ -1 - 2h_{1/2}(L^*), & \text{if } \ell < 0. \end{cases}$$

$$\mathrm{SF}_-([\mathrm{C}]) = \begin{cases} -2 - 2h_{1/2}(L^*), & \text{if } \ell < 0, \\ -1 - 2h_{1/2}(L^*), & \text{if } \ell > 0. \end{cases}$$

Remark 3.4: We have considered only the case $[\mathrm{C}] \in \mathfrak{M}_{\kappa,n}$ with $n < 0$. The case $n > 0$ can be approached by entirely similar methods and can be safely left to the reader. The corresponding formulæ can be obtained from the above by making the Serre duality substitution $L \longleftrightarrow L^*$.

§3.4. VIRTUAL DIMENSIONS.

In this final subsection we will show how one can use Theorem 3.3 to compute virtual dimensions of finite energy moduli spaces. We will rely heavily on the techniques of [MMR].

Consider a 4-manifold $\hat{N}$ with a cylindrical end isometric to $[0,\infty) \times N$, where N is the disjoint union of nontrivial circle bundles $\{N_j\ ;\ j = 1,\ldots,m\}$ of degrees ℓ_j over Riemann surfaces Σ_j of genera $g_j \geq 1$ (see Fig. 6). Fix spin structures on each of the Riemann surfaces Σ_j which induce by pullback spin structures on

N_j. Next fix a spin^c structure $\hat{\sigma}$ on $\hat{N}$ which induces spin^c structures σ_j on N_j. Set

$$\sigma = \coprod \sigma_j.$$

The metric and compatible connections on the end of $\hat{N}$ are prescribed as indicated in §3.2. This means that we use as background connection on N the adiabatic connection ∇^0. Consider a finite energy solution $\hat{\mathrm{C}} = (\hat{\phi}, \hat{A})$ of the Seiberg–Witten equations associated to the structure $\hat{\sigma}$. We assume that along the neck it is in temporal gauge

$$\hat{\mathrm{C}} = \{t \mapsto \mathrm{C}(t) = (\phi(t), A(t))\}.$$

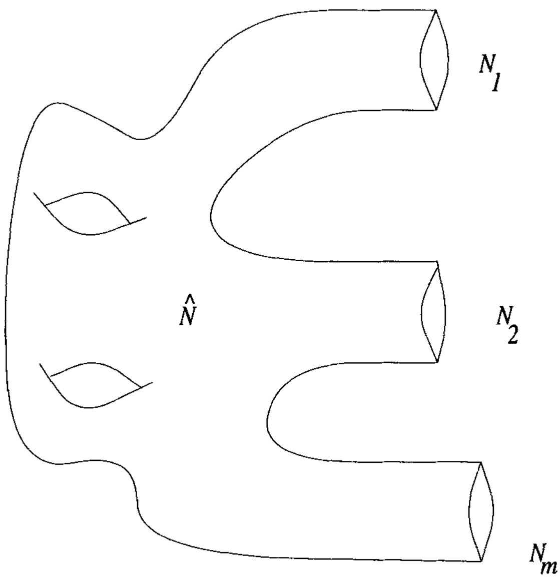

Figure 6. Multiple cylindrical ends.

The techniques of [MMR] work with no essential changes in the Seiberg–Witten context and show that $[\mathrm{C}(t)]$ converges to $[\mathrm{C}_\infty] \in \mathfrak{M}_\sigma$, where by $\mathfrak{M}_\sigma$ we denoted the Seiberg–Witten moduli space determined by the spin^c structure σ on N. The first conclusion we draw from this fact is that σ must be a pulled back spin^c structure, since otherwise $\mathfrak{M}_\sigma = \emptyset$. Suppose this is indeed the case.

The moduli space $\mathfrak{M}_\sigma$ is a disjoint union

$$\mathfrak{M}_\sigma = \coprod \mathfrak{M}_{\sigma_j}$$

and thus the asymptotic limit is a collection

$$[\mathrm{C}_\infty] = ([\mathrm{C}_1], \dots, [\mathrm{C}_m]).$$

Assume first that all the configurations C_j are irreducible,

$$\mathrm{C}_j \in \mathfrak{M}_{\kappa_j, n_j}(\sigma_j).$$

Again, to reduce the accounting job we consider that $n_j < 0\,\forall j$. The irreducibility condition implies that $[\mathrm{C}(t)]$ converges *exponentially* to its asymptotic limit.

We are interested in describing a neighborhood of $\hat{\mathrm{C}}$ in the moduli space of finite energy solutions and we will begin as in [MMR] by studying a simpler problem.

Define $\widehat{\mathfrak{M}}([\mathrm{C}_\infty])$ the moduli space of finite energy solutions with asymptotic limit $[\mathrm{C}_\infty]$. We want to understand the structure of a small neighborhood of $\hat{\mathrm{C}} \in \widehat{\mathfrak{M}}([\mathrm{C}_\infty])$. More precisely, we would like to compute the virtual dimension of such a neighborhood. This is achieved using Kuranishi's deformation picture of the moduli space which requires a suitable functional framework. Since the convergence to the asymptotic limit is exponential, one can use the very convenient weighted Sobolev spaces $L^{k,p}_w$ where w is a very small positive number. The resulting deformation complex can be described as in Chap. 8 of [MMR] and is

$$0 \to X_0 \xrightarrow{\hat{\mathcal{L}}_{\hat{\mathrm{C}}}} X_1 \xrightarrow{\underline{sw}} X_2 \to 0, \tag{3.20}$$

where X_0 is the Lie algebra of the group of gauge transformations on $\hat{N}$ exponentially converging to 1 along the neck

$$X_0 = L^{3,2}_w(\mathbf{i}\Lambda^0 T^*\hat{N}),$$

X_1 is the tangent space to the space of configurations of the 4-dimensional equations

$$X_1 = L^{2,2}_w(\hat{\mathbb{S}}^+_{\hat{\sigma}} \oplus \mathbf{i}\Lambda^1 T^*\hat{N}),$$

X_2 is the space of "obstructions"

$$X_2 = L^{1,2}_w(\hat{\mathbb{S}}^-_{\hat{\sigma}} \oplus \mathbf{i}\Lambda^2_+ T^*\hat{N}),$$

$\hat{\mathcal{L}} = \hat{\mathcal{L}}_{\hat{\mathrm{C}}}$ is the infinitesimal gauge group action at $\hat{\mathrm{C}}$ and $\underline{sw}$ is the linearization at $\hat{\mathrm{C}}$ of the SW-equations on $\hat{N}$.

We can now form the operator

$$\hat{\mathcal{O}}_w : X_1 \to X_2 \oplus X_0, \quad \hat{\mathcal{O}}_w = \underline{sw} \oplus \hat{\mathcal{L}}^{*w}$$

where $*_w$ denotes the L^2_w-adjoint of $\hat{\mathcal{L}}$. This is an elliptic operator and a computation *à la* [MMR] (Chap. 8) shows that along the neck it has the *APS* form

$$\hat{\mathcal{O}}_w = \text{something} \times (\nabla_t - \mathcal{O}_w)$$

where

$$\mathcal{O}_w \begin{bmatrix} \dot\psi \oplus \mathbf{i}\dot a \\ \mathbf{i}f \end{bmatrix} = \begin{bmatrix} D_A\dot\psi & & + & \mathbf{c}(\mathbf{i}\dot a)\phi - \mathbf{i}f\phi \\ & -\mathbf{i} * d\dot a + \mathbf{i}df & + & \dot q(\phi, \dot\psi) \\ & \mathbf{i}d^*\dot a - 2w\mathbf{i}f & + & \mathbf{i}\mathfrak{Im}\langle \phi, \dot\psi\rangle \end{bmatrix}$$

and $[\phi, A] = [\mathrm{C}_\infty]$. Note that $\tilde{\mathcal{H}}_1(\mathrm{C}_\infty) = \mathcal{O}_w\,|_{w=0}$. $\hat{\mathcal{O}}_w$ is a Fredholm operator and its index (over $\mathbb{R}$) is equal to the virtual dimension of a small neighborhood of $[\hat{\mathrm{C}}]$ in $\widehat{\mathfrak{M}}([\mathrm{C}_\infty])$. Remark 1.6 at the end of §1.3 shows that the index of $\hat{\mathcal{O}}_w$ is equal to the *APS* index of $\hat{\mathcal{O}}_w$.

Denote by $\mathcal{A}$ the anti-selfduality operator on $\hat N$

$$\mathcal{A} = d_+ \oplus d^*\colon \mathbf{i}\Omega^1(\hat N) \to \mathbf{i}\Omega^2(\hat N) \oplus \mathbf{i}\Omega^0(\hat N).$$

Using the connection $\hat A$ and the spin^c structure $\hat\sigma$ we can form the Dirac operator

$$\hat D_{\hat A}\colon \Gamma(\hat{\mathbb{S}}^+_{\hat\sigma}) \to \Gamma(\hat{\mathbb{S}}^-_{\hat\sigma}).$$

Along the neck the direct sum $\mathcal{N}_{\hat{\mathrm{C}}} = \hat D_{\hat A} \oplus \mathcal{A}$ has the *APS* form

$$\mathcal{N} = \text{something} \times \left(\nabla_t - \tilde{\mathcal{H}}_0(\mathrm{C}_\infty)\right).$$

Using the excision formula (1.13) in Remark 1.6 we deduce

$$\text{(3.21)} \qquad \mathrm{ind}_{APS}(\hat{\mathcal{O}}_w) = \mathrm{ind}_{APS}(\mathcal{N}) - \mathrm{SF}(\tilde{\mathcal{H}}_0 \to \tilde{\mathcal{H}}_1) - \mathrm{SF}(\tilde{\mathcal{H}}_1 \to \mathcal{O}_w)$$

where $\mathrm{SF}(A \to B)$ denotes $\mathrm{SF}(A + t(B - A),\ t \in [0,1])$. All the indices and the spectral flows above are *real* quantities.

We now proceed to determine the three terms in the right-hand side of the above formula.

Corollary D.3 shows that the third term above vanishes. The second spectral term can be rewritten as

$$\text{(3.22)} \qquad \mathrm{SF}(\tilde{\mathcal{H}}_0 \to \tilde{\mathcal{H}}_1) = \sum_j \mathrm{SF}_+([\mathrm{C}_j]).$$

We denote by ρ_{asd} (resp. ρ_{dir}) the index densities of $\mathcal{A}$ (resp. $\hat D_{\hat A}$),

$$\rho_{asd} = -\frac{1}{2}\left(\mathbf{e}(\hat N) + \mathbf{L}(\hat N)\right),$$

where $\mathbf{e}(\hat{N})$ and $\mathbf{L}(\hat{N})$ denote respectively the Euler and the L-genus forms on $\hat{N}$ constructed using the Levi-Civita connection. Also

$$\rho_{dir} = 2\hat{\mathbf{A}}(\hat{\nabla}^0) \wedge \exp(\frac{1}{2}c_1(\det\hat{\sigma})),$$

where on $\det\hat{\sigma}$ we used the connection induced by $\hat{A}$. The factor 2 appears since we are interested in the *real* index of $\hat{D}$. The $\hat{\mathbf{A}}$-genus form is computed using the metric compatible connection $\hat{\nabla}^0$ which, along the neck, has the product form $dt \otimes \partial_t + \nabla^0$. For simplicity, set $c(\hat{A}) = c_1(\det\hat{\sigma})$.

On a 4-manifold the above equality has a simpler form

$$\rho_{dir} = \frac{1}{4}(c(\hat{A})^2 - \mathbf{L}(\hat{\nabla}^0)).$$

The ξ invariant of $\tilde{\mathcal{H}}_0$ is the sum $\xi(\mathcal{A}\,|_N) + 2\xi(D_A)$ (the factor 2 is present for reality reasons),

$$\xi(\tilde{\mathcal{H}}_0) = \frac{1}{2}(\dim_{\mathbf{R}}\ker\tilde{\mathcal{H}}_0 - \eta_{\mathrm{sign}} + 2\eta(D_A)),$$

where η_{sign} denotes the eta invariant of the odd signature operator. We deduce

$$\begin{aligned}
\mathrm{ind}_{APS}(\mathcal{N}) &= \int_N (\rho_{\mathrm{asd}} + \rho_{\mathrm{dir}}) - \xi(\tilde{\mathcal{H}}_0) \\
&= -\frac{1}{2}\int_{\hat{N}} \mathbf{e} - \frac{1}{2}\left(\int_{\hat{N}} \mathbf{L} - \eta_{\mathrm{sign}}\right) + \frac{1}{4}\int_{\hat{N}} (c^2(\hat{A}) - \mathbf{L}(\hat{\nabla}^0)) \\
&\quad - \frac{1}{2}\dim_{\mathbf{R}}\ker\tilde{\mathcal{H}}_0 - \eta(D_A) \\
&= -(\chi(\hat{N}) + \mathrm{sign}\,(\hat{N}))/2 + \frac{1}{4}\int_{\hat{N}} (c^2(\hat{A}) - \mathbf{L}(\hat{\nabla}^0)) \\
&\quad - \sum_j \dim_{\mathbf{C}}\ker D_{A_j} - \frac{1}{2}\sum_j (2g_j + 1) - \frac{1}{6}\sum_j \ell_j.
\end{aligned}$$

Using (3.21) and (3.22) we deduce

$$\begin{aligned}
\mathrm{ind}\,(\hat{\mathcal{O}}_w) = &- \frac{1}{2}(\chi(\hat{N}) + \mathrm{sign}\,(\hat{N})) + \frac{1}{4}\int_{\hat{N}} (c^2(\hat{A}) - \mathbf{L}(\hat{\nabla}^0)) \\
&- \sum_j \left(\dim_{\mathbf{C}}\ker D_{A_j} + \underset{+}{\mathrm{SF}}([\mathrm{C}_j])\right) - \frac{1}{2}\sum_j (1 + 2g_j) - \sum_j \frac{\ell_j}{6}.
\end{aligned} \tag{3.23}$$

This formula can be further simplified. We can replace the integral of $\mathbf{L}(\hat{\nabla}^0)$ with the integral of $\mathbf{L}(\hat{N})$ plus a correction term given by the second transgression formula. Assume for simplicity that all components of N have fibers of the same radius r and the bases have common area π. We get

$$\int_{\hat{N}} \mathbf{L}(\hat{\nabla}^0) - \int_{\hat{N}} \mathbf{L}(\hat{N}) = \sum_j \frac{2\ell_j}{3}(\ell_j^2 r^4 - \chi_j r^2) \tag{3.24}$$

where $\chi_j = 2 - 2g_j$. Denote by η_j the signature eta invariant of N_j. This was computed in [Ko] and [O] and we have

$$\eta_j = -\mathrm{sign}\,(\ell_j) - \frac{2\ell_j}{3}(\ell_j^2 r^4 - \chi_j r^2) + \frac{\ell_j}{3}.$$

If we set $\eta = \sum \eta_j$ we deduce from (3.24)

$$\int_{\hat{N}} \mathbf{L}(\hat{\nabla}^0) + \eta - \int_{\hat{N}} \mathbf{L}(\hat{N}) = \sum_j (\ell_j/3 - \mathrm{sign}\,(\ell_j)).$$

The term

$$\eta - \int_{\hat{N}} \mathbf{L}(\hat{N})$$

is equal to $-\mathrm{sign}\,(\hat{N})$, so that we deduce

$$\int_{\hat{N}} \mathbf{L}(\hat{\nabla}^0) = \mathrm{sign}\,(\hat{N}) + \sum_j \Big(\frac{\ell_j}{3} - \mathrm{sign}\,(\ell_j)\Big).$$

If we use this equality in (3.23) we deduce

$$\mathrm{ind}\,(\hat{O}_w) = \frac{1}{4}\int_{\hat{N}} c^2(\hat{A}) - \frac{1}{4}(2\chi(\hat{N}) + 3\mathrm{sign}\,(\hat{N}))$$

$$-\sum_j \big(\dim_{\mathbf{C}} \ker D_{A_j} + \mathrm{SF}_+([\mathrm{C}_j])\big) - \frac{1}{2}\sum(2g_j+1) - \frac{1}{4}\sum_j(\ell_j - \mathrm{sign}\,(\ell_j)). \tag{3.25}$$

To find the virtual dimension $\dim_v(\hat{\mathrm{C}})$ of a neighborhood of $\hat{\mathrm{C}}$ in the *entire* moduli space $\widehat{\mathfrak{M}}$ we only have to add the dimensions of the asymptotic limit sets $\dim \mathfrak{M}_{\kappa_j, n_j} = 2(g_j - 1 + n_j)$ (recall that we have assumed $n_j < 0$):

$$\begin{aligned}\dim_v(\hat{\mathrm{C}}) =& \frac{1}{4}\int_{\hat{N}} c^2(\hat{A}) - \frac{1}{4}(2\chi(\hat{N}) + 3\mathrm{sign}\,(\hat{N})) \\ &- \sum_j \big(\dim_{\mathbf{C}} \ker D_{A_j} + \mathrm{SF}_+([\mathrm{C}_j])\big) \\ &+ \frac{1}{2}(2g_j - 1) + 2\sum_j (n_j - 1) - \frac{1}{4}\sum_j(\ell_j - \mathrm{sign}\,(\ell_j)).\end{aligned} \tag{3.26}$$

We can now use Theorem 3.3 in the form

$$\mathrm{SF}_+(\mathrm{C}_j) = -1 - 2h_{1/2}(L_j^*) - \varepsilon_j$$

where

$$\varepsilon_j = \frac{1}{2}(1 + \mathrm{sign}\,(\ell_j)).$$

Since $\dim_{\mathbf{C}} \ker D_{A_j} = h_{1/2}(L_j) + h_{1/2}(L_j^*)$ we deduce

$$\dim_{\mathbf{C}} \ker D_{A_j} + \mathrm{SF}_+([\mathrm{C}_j]) = h_{1/2}(L_j) - h_{1/2}(L_j^*) - 1 - \varepsilon_j = n_j - 1 - \varepsilon_j.$$

Using this in (3.26) we get

$$\dim_v(\hat{\mathrm{C}}) = \frac{1}{4}\int_{\hat{N}} c^2(\hat{A}) - \frac{1}{4}(2\chi(\hat{N}) + 3\mathrm{sign}\,(\hat{N}))$$

$$(3.27) \qquad + \sum_j (\varepsilon_j + n_j - 1) + \frac{1}{2}\sum_j (2g_j - 1) - \frac{1}{4}\sum_j (\ell_j - \mathrm{sign}\,(\ell_j)).$$

If we define the boundary contribution of the asymptotic limit $[\mathrm{C}_j]$ by

$$\beta([\mathrm{C}_j]) = (\varepsilon_j + n_j - 1) + \frac{1}{2}(2g_j - 1) - \frac{1}{4}(\ell_j - \mathrm{sign}\,(\ell_j)),$$

then we can rewrite formula (3.27) as

$$(3.28) \qquad \dim_v(\hat{\mathrm{C}}) = \frac{1}{4}\int_{\hat{N}} c^2(\hat{A}) - \frac{1}{4}(2\chi(\hat{N}) + 3\mathrm{sign}\,(\hat{N})) + \sum_j \beta([\mathrm{C}_j]).$$

The first two terms in (3.28) represent formally the expression which computes the virtual dimension of the Seiberg–Witten moduli spaces on a closed compact 4-manifold.

We can now apply this formula to the special situation of tunnelings. In this case $\hat{N} = \mathbb{R} \times N$ and thus $m = 2$, $g_1 = g_2 = g$, $\ell_1 + \ell_2 = 0$. Moreover, $\mathrm{sign}\,(\hat{N}) = \chi(\hat{N}) = 0$. The equality (3.27) gives the virtual dimension of the space of tunnelings between $\mathfrak{M}_{\kappa,n_1}$ (at $-\infty$) and $\mathfrak{M}_{\kappa,n_2}$ (at $+\infty$). Denote this dimension by $\tau(\kappa; n_1, n_2)$. We have

$$\tau(\kappa; n_1, n_2) = \int_{\hat{N}} c(\hat{\sigma})^2 + n_1 + n_2 + 2g - 2.$$

The integral term can be computed via transgression exactly as in the third transgression formula. It suffices to pick arbitrary $[\mathrm{C}_j] = [\phi_j, A_j] \in \mathfrak{M}_{\kappa,n_j}$,

$j = 1, 2$, and an arbitrary path of connections $A(t)$ such that $A(0) = A_1$ and $A(1) = A_2$. We choose A_j such that

$$\frac{\mathbf{i}}{2\pi}F_{A_j} = \frac{n_j}{\pi}dv_\Sigma \quad \text{and} \quad A_2 - A_1 = \mathbf{i}c\varphi,$$

where $c = (n_2 - n_1)/\ell$. $A(t)$ will be the affine path $A_1 + t\mathbf{i}c\varphi$. (The connection on $\det\hat{\sigma}$ will be $\hat{A}^{\otimes 2}$ since $\det \mathbb{S}_L = L^2$.) We get a connection $\hat{A}$ on $[0,1] \times N$. A computation entirely similar to the one in the third transgression formula leads to the equality

$$\frac{1}{2}\int_{\hat{N}} c(\hat{\sigma})^2 = -\frac{1}{4\pi^2}\int_{[0,1]\times N} F_{\hat{A}} = \frac{n_1^2 - n_2^2}{\ell}.$$

Thus we get

$$\tau(\kappa; n_1, n_2) = \frac{n_1^2 - n_2^2}{\ell} + n_1 + n_2 + 2g - 2.$$

This agrees with Corollary 1.0.5 of [MOY]. (In the notation of [MOY], $e_j = n_j + g - 1$.)

We have omitted from our discussion the case when one (or several) asymptotic limits $[\mathsf{C}_j]$ is reducible. The degenerate case, $g_j - 1 \equiv 0 \pmod{\ell_j}$, requires special care and will not be discussed here. In the remaining cases the problem is actually simpler than the case dealing with irreducible limits.

First of all the convergence to such a nondegenerate reducible continues to be exponential and thus we can use the same functional framework as above. Assume for simplicity the boundary has only one component. We have to compute the APS index of a new operator $\hat{\mathcal{O}}_w$ which, along the neck, has the form

$$\hat{\mathcal{O}}_w = \text{something} \times (\nabla_t - \mathcal{O}_w),$$

where this time

$$\mathcal{O}_w \begin{bmatrix} \dot{\psi} \oplus \mathbf{i}\dot{a} \\ \mathbf{i}f \end{bmatrix} = \begin{bmatrix} D_A\dot{\psi} & \\ & -\mathbf{i} * d\dot{a} + \mathbf{i}df \\ & \mathbf{i}d^*\dot{a} - 2w\mathbf{i}f \end{bmatrix}.$$

(The spinor part ϕ of the asymptotic limit $[\mathsf{C}] = [\phi, A]$ is zero and thus $\mathcal{P}_\phi \equiv 0$.) Thus

$$\mathrm{ind}_{\mathrm{aps}}(\hat{\mathcal{O}}_w) = \mathrm{ind}_{\mathrm{aps}}(\mathcal{N}) - \mathrm{SF}(\mathcal{O}_0 \to \mathcal{O}_w).$$

The spectral flow contribution is easy to determine. The only eigenvalue of $\mathcal{O}_{tw}$ contributing to the spectral flow is $-2wt\,|_{t=0}$ with a single eigenfunction $\dot{\psi} \oplus \mathbf{i}\dot{a} \oplus \mathbf{i}f$, where $\dot{\psi} = 0$, $\mathbf{i}\dot{a} = 0$ and $f \equiv 1$. Hence

$$\mathrm{ind}_{\mathrm{aps}}(\hat{\mathcal{O}}_w) = \mathrm{ind}_{\mathrm{aps}}(\mathcal{N}) + 1.$$

The index of $\mathcal{N}$ can be determined as above using instead the eta invariant of the adiabatic operator coupled with the flat connection A determined in §2.4. The nondegeneracy condition also implies $\ker D_A = 0$. Hence this eta invariant is twice the ξ-invariant in (2.22) so that

$$\eta(D_A) = \frac{\ell}{6} + \frac{\kappa^2}{\ell} - \kappa \cdot \operatorname{sign}(\ell).$$

We deduce (using $\dim_{\mathbf{R}} \ker \tilde{\mathcal{H}}_0 = b_0(N) + b_1(N) = 2g+1$)

$$\operatorname{ind}_{APS}(\mathcal{O})_w = 1 - (\chi(\hat{N}) + \operatorname{sign}(\hat{N}))/2 + \frac{1}{4}\int_{\hat{N}} (c^2(\hat{A}) - \mathbf{L}(\hat{\nabla}^0))$$
$$-\frac{1}{2}(1+2g) - \frac{\ell}{6} - \frac{\kappa^2}{\ell} + \kappa \cdot \operatorname{sign}(\ell).$$

Arguing as in the irreducible case we deduce

$$\begin{aligned}\operatorname{ind}(\hat{\mathcal{O}}_w) =& \frac{1}{4}\int_{\hat{N}} c^2(\hat{A}) - \frac{1}{4}(2\chi(\hat{N}) + 3\cdot\operatorname{sign}(\hat{N}))\\ &- \frac{1}{2}(2g-1) - \frac{1}{4}(\ell - \operatorname{sign}(\ell)) - \frac{\kappa^2}{\ell} + \kappa\cdot\operatorname{sign}(\ell).\end{aligned}$$

To find the virtual dimension of a neighborhood in the entire moduli space $\widehat{\mathfrak{M}}$ we proceed as in Sect. 8.5 of [MMR]. We need to add the dimension of the reducible limit set (which is a $2g$-torus) and subtract the dimension of the stabilizer of the asymptotic limit (which is S^1). We deduce

$$\begin{aligned}\dim_v(\hat{\mathrm{C}}) =& \frac{1}{4}\int_{\hat{N}} c^2(\hat{A}) - \frac{1}{4}(2\chi(\hat{N}) + 3\cdot\operatorname{sign}(\hat{N}))\\ &+ \frac{1}{2}(2g-1) - \frac{1}{4}(\ell - \operatorname{sign}(\ell)) - \frac{\kappa^2}{\ell} + \kappa\cdot\operatorname{sign}(\ell).\end{aligned} \tag{3.29}$$

We see that the boundary contribution of a reducible limit is

$$\beta([\mathrm{C}]) = \frac{2g-1}{2} - \frac{\ell - \operatorname{sign}(\ell)}{4} - \frac{\kappa^2}{\ell} + \kappa\cdot\operatorname{sign}(\ell). \tag{3.30}$$

We can now easily write the virtual dimension $\tau(\kappa : 0, n)$of the space of tunnelings from a reducible solution $[\mathrm{C}_1]$ to an irreducible one $[\mathrm{C}_2] \in \mathfrak{M}\kappa, n$. It is

$$\tau(\kappa; 0, n) = \int_N \mathbf{T}c_1^2(A_2, A_1) + \beta([\mathrm{C}_1]) + \beta([\mathrm{C}_2]),$$

where $\mathbf{T}$ stands for the transgression form. We denote by ℓ the degree of the boundary at $+\infty$. Then as in the third transgression formula we get

$$\int_N \mathbf{T}c_1^2(A_2, A_1) = -\frac{n^2}{\ell}.$$

Also

$$\beta([C_2]) = \Big(\frac{1+\operatorname{sign}(\ell)}{2} + n - 1\Big) + \frac{2g-1}{2} - \frac{\ell - \operatorname{sign}(\ell)}{4}$$

and (using the opposite orientation at $-\infty$)

$$\beta([C_1]) = \frac{2g-1}{2} + \frac{\ell-\operatorname{sign}(\ell)}{4} + \frac{\kappa^2}{\ell} - \kappa\operatorname{sign}(\ell).$$

We get

$$\tau(\kappa;0,n) = \frac{\kappa^2-n^2}{\ell} + 2g - 2 + n + \frac{1+\operatorname{sign}(\ell)}{2}.$$

Appendix A. Proof of the first transgression formula

Let $0 < \rho \ll r \ll 1$. *The parameter r will stay fixed throughout this section. ρ will eventually go to* 0. We have a *fixed* local frame $(\zeta, \zeta_1, \zeta_2)$. This is *not an orthonormal frame* for either of the metric h_r or h_ρ but it is an orthogonal frame.

We first want to compute the 1-forms associated by the above frame to the connections ∇^r and ∇^ρ. We denote them by Γ_r and resp. Γ_ρ. Using the equalities (2.4) and (2.5) we get after some simple manipulations that

$$\Gamma_r = \begin{bmatrix} 0 & \lambda\varphi^2 & -\lambda\varphi^1 \\ -\lambda\varphi^2 & 0 & -r^2\lambda\varphi - \kappa\varphi^1 \\ r^2\lambda\varphi^1 & r^2\lambda\varphi + \kappa\varphi^1 & 0 \end{bmatrix}.$$

We get a similar result for Γ_ρ. Set $\Xi_\rho = \Gamma_\rho - \Gamma_r$ and $\Xi_0 = \lim_{\rho\to 0}\Xi_\rho$. Note that

$$\Xi_0 = \begin{bmatrix} 0 & 0 & 0 \\ r^2\lambda\varphi^2 & 0 & r^2\lambda\varphi \\ -r^2\lambda\varphi^1 & -r^2\lambda\varphi & 0 \end{bmatrix}.$$

We have to compute $\lim_{\rho\to 0} T\hat{\mathbf{A}}(\nabla^\rho, \nabla^r)$. Note that

$$T\hat{\mathbf{A}}(\nabla^\rho,\nabla^r) = \frac{1}{96\pi^2}\operatorname{tr}\Big\{\ \Xi_\rho \wedge \Big(\ \Omega_r + \frac{1}{2}(d\Xi_\rho + \Gamma_r\wedge\Xi_\rho + \Xi_\rho\wedge\Gamma_r) + \frac{1}{3}\Xi_\rho\wedge\Xi_\rho\wedge\Xi_\rho\ \Big)\ \Big\}.$$

Above, Ω_r is the curvature 2-form of the connection ∇^r. By letting $\rho\to 0$ in the above equality we deduce

$$\lim_{\rho\to 0} T\hat{\mathbf{A}}(\nabla^\rho,\nabla^r)$$

(A.1)

$$= \frac{1}{96\pi^2}\operatorname{tr}\left\{\Xi_0\wedge\left(\Omega_r + \frac{1}{2}(d\Xi_0 + \Gamma_r\wedge\Xi_0 + \Xi_0\wedge\Gamma_r) + \frac{1}{3}\Xi_0\wedge\Xi_0\wedge\Xi_0\right)\right\}.$$

We now proceed to describe each of the above constituents, one by one. Since $\Omega_r = d\Gamma_r + \Gamma_r \wedge \Gamma_r$ we deduce using (2.1), (2.2) and (2.3) that

$$\Omega_r = \begin{bmatrix} 0 & r^2\lambda^2\varphi\wedge\varphi^1 & -\lambda^2r^2\varphi^2\wedge\varphi \\ -\lambda^2r^4\varphi\wedge\varphi^1 & 0 & -(\kappa^2+3\lambda^2r^2)\varphi^1\wedge\varphi^2 \\ \lambda^2r^4\varphi^2\wedge\varphi & (\kappa^2+3\lambda^2r^2)\varphi^1\wedge\varphi^2 & 0 \end{bmatrix}.$$

Then

$$\text{(A.2)} \qquad \Xi_0 \wedge \Omega_r = \begin{bmatrix} 0 & * & * \\ * & (4\lambda^3r^4+\lambda\kappa^2r^2) & * \\ * & * & (4\lambda^3r^4+\lambda\kappa^2r^2) \end{bmatrix} \varphi\wedge\varphi^1\wedge\varphi^2,$$

$$d\Xi_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2\lambda^2r^2 \\ -\lambda\kappa r^2 & -2\lambda^2r^2 & 0 \end{bmatrix} \varphi^1\wedge\varphi^2.$$

Then

$$\text{(A.3)} \qquad \Xi_0\wedge d\Xi_0 = \begin{bmatrix} 0 & * & * \\ * & -2\lambda^3r^4 & * \\ * & * & -2\lambda^3r^4 \end{bmatrix} \varphi\wedge\varphi^1\wedge\varphi^2.$$

Simple manipulations yield

$$\text{(A.4)} \quad \Xi_0\wedge\Xi_0\wedge\Gamma_r + \Xi_0\wedge\Gamma_r\wedge\Xi_0 = \begin{bmatrix} 0 & * & * \\ * & -2\lambda^3r^4 & * \\ * & * & -2\lambda^3r^4 \end{bmatrix} \varphi\wedge\varphi^1\wedge\varphi^2.$$

An immediate computation (eased by the large number of vanishing entries in Ξ_0) shows that $\operatorname{tr}(\Xi_0\wedge\Xi_0\wedge\Xi_0) = 0$. By combining (A.1) with (A.2)–(A.4) we get

$$\lim_{\rho\to 0} T\hat{\mathbf{A}}(\nabla^\rho, \nabla^r) = \frac{1}{96\pi^2}(4\lambda^3r^4 + 2\lambda\kappa^2r^2)\varphi\wedge\varphi^1\wedge\varphi^2.$$

The first transgression formula now follows by integrating over N and using the equalities $\lambda = -\ell$, $\kappa^2 = 4(g-1)$ and

$$\int_N \varphi\wedge\varphi^1\wedge\varphi^2 = 2\pi^2.$$

Appendix B. Proof of the second transgression formula

We have to compute $\lim_{t\to 0} T\hat{\mathbf{A}}(\nabla^{r,t},\nabla^r)$. As in Appendix A we get

$$\lim_{t\to 0} T\hat{\mathbf{A}}(\nabla^{r,t},\nabla^r)$$

(B.1)
$$= \frac{1}{96\pi^2}\mathrm{tr}\left\{\Xi_0\wedge\left(\Omega_r+\frac{1}{2}(d\Xi_0+\omega_r\wedge\Xi_0+\Xi_0\wedge\omega_r)+\frac{1}{3}\Xi_0\wedge\Xi_0\wedge\Xi_0\right)\right\},$$

where $\Omega_r = d\omega_r + \omega_r\wedge\omega_r$ and (using (2.5) and (2.6))

$$\Xi_0 = \omega_{r,0} - \omega_r = -\lambda_r\begin{bmatrix} 0 & \varphi^2 & -\varphi^1 \\ -\varphi^2 & 0 & -\varphi_r \\ \varphi^1 & \varphi_r & 0 \end{bmatrix}.$$

Using (2.1), (2.2) and (2.3) we get after some simple manipulations

$$\Omega_r = \begin{bmatrix} 0 & \lambda_r^2\varphi_r\wedge\varphi^1 & -\lambda_r^2\varphi^2\wedge\varphi_r \\ -\lambda_r^2\varphi_r\wedge\varphi & 0 & -(3\lambda_r^2+\kappa^2)\varphi^1\wedge\varphi^2 \\ \lambda_r^2\varphi^2\wedge\varphi_r & (3\lambda_r^2+\kappa^2)\varphi^1\wedge\varphi^2 & 0 \end{bmatrix}.$$

Then

$$\Xi_0\wedge\Omega_r = -\lambda_r\begin{bmatrix} -2\lambda_r^2 & * & * \\ * & -(4\lambda_r^2+\kappa^2) & * \\ * & * & -(4\lambda_r^2+\kappa^2) \end{bmatrix}\varphi_r\wedge\varphi^1\wedge\varphi^2, \tag{B.2}$$

$$d\Xi_0 = -\lambda_r\begin{bmatrix} 0 & 0 & -\kappa \\ 0 & 0 & -2\lambda_r \\ \kappa & 2\lambda_r & 0 \end{bmatrix}\varphi^1\wedge\varphi^2,$$

$$\Xi_0\wedge d\Xi_0 = \lambda_r^2\begin{bmatrix} 0 & * & * \\ * & -2\lambda_r & * \\ * & * & -2\lambda_r \end{bmatrix}\varphi_r\wedge\varphi^1\wedge\varphi^2, \tag{B.3}$$

$$\Xi_0\wedge\Xi_0\wedge\Xi_0 = 2\lambda_r^3\begin{bmatrix} 1 & * & * \\ * & 1 & * \\ * & * & 1 \end{bmatrix}\varphi_r\wedge\varphi^1\wedge\varphi^2, \tag{B.4}$$

$$\Xi_0\wedge\omega_r\wedge\Xi_0+\Xi_0\wedge\Xi_0\wedge\omega_r = -4\lambda_r^3\begin{bmatrix} 1 & * & * \\ * & 1 & * \\ * & * & 1 \end{bmatrix}\varphi_r\wedge\varphi^1\wedge\varphi^2. \tag{B.5}$$

Putting together (B.2)–(B.5) we deduce

$$\lim_{t\to 0} T\hat{\mathbf{A}}(\nabla^{r,t}, \nabla^r) = \frac{1}{96\pi^2}(4\lambda_r^3 + 2\kappa^2\lambda_r)\varphi_r \wedge \varphi^1 \wedge \varphi^2$$
$$= \frac{1}{96\pi^2}(4\lambda^3 r^4 + 2\lambda\kappa^2 r^2)\varphi \wedge \varphi^1 \wedge \varphi^2.$$

We now conclude exactly as in Appendix A. ∎

Appendix C. Elementary computation of the eta invariants

We include here an elementary derivation of the equality (2.20). For brevity we present a proof only for circle bundles over smooth Riemann surfaces but the arguments extend to the more general case of Seifert manifolds, i.e. smooth circle bundles over Riemann V-surfaces (2-orbifolds). The changes from the smooth to the orbifold case are only cosmetic ("orbify" everything, i.e. add the prefix V to all the intervening geometric objects and use known V-theorems: V-Riemann-Roch, V-Serre duality etc.).

Our circle bundle N equipped with the metric described in §2.1 determines a hermitian metric and compatible connection on a degree ℓ hermitian line bundle over Σ. This connection determines a holomorphic structure and we denote by L_0 the holomorphic line bundle thus obtained. Consider another line bundle $L \to \Sigma$ of degree k equipped with a hermitian metric and compatible connection B. The connection B determines a holomorphic structure on L and we will denote by $h(L)$ the dimension of the space of holomorphic sections.

Consider now the *spin*^{c} structure on N whose associated spinor bundle is

$$\text{(C.1)} \qquad \mathbb{S}_k = \mathcal{K}^{-1} \otimes \pi^* L \oplus \pi^* L.$$

In terms of the notations in §2.4 we have $\mathbb{S}_k = \mathbb{S} \otimes \mathcal{K}^{-1/2} \otimes L$. Note that $\mathbb{S}_k$ makes no reference to a choice of *spin* structure on the base Σ and thus a similar object can also be defined when N is a Seifert fibration over a not necessarily *spin*-orbifold. This is very similar to the case of *spin*^{c} structures over (even dimensional) almost complex manifolds.

Using the pullback of B on $\pi^* L$, the pullback of the Levi-Civita connection on K_Σ to $\mathcal{K}$ and the adiabatic Levi-Civita connection $\lim_{t\to 0} \nabla^{r,t}$ we obtain as in §2.4 the adiabatic Dirac operator $D = D_B$. For each $\mu \in \mathbb{R}$ define

$$V_\mu = \ker(\mu - D), \quad v_\mu = \dim V_\mu.$$

We want to compute the eta function of D

$$\text{(C.2)} \qquad \eta(s) = \eta_D(s) = \sum_{\mu > 0} \frac{v_\mu - v_{-\mu}}{\mu^s}.$$

Recall (§2.4) that D has a decomposition $D = Z + T$. Now define

$$E_\mu = \{v \in V_\mu; ZTv = 0\}, \quad e_\mu = \dim E_\mu.$$

Denote by $E_\mu^\perp$ the orthogonal complement of E_μ in V_μ. Since $\{Z,T\} = 0$ we also have $\{D, ZT\} = 0$ and it is easy to check that $ZT(E_\mu^\perp) \subset E_{-\mu}^\perp$. The definition of E_μ implies that the induced map $ZT\colon E_\mu^\perp \to E_{-\mu}^\perp$ is injective. Thus $\dim E_\mu^\perp \leq \dim E_{-\mu}^\perp$ and by symmetry $\dim E_\mu^\perp = \dim E_{-\mu}^\perp$. Using this in (C.2) we deduce

$$\eta(s) = \sum_{\mu>0} \frac{e_\mu - e_{-\mu}}{\mu^s}. \tag{C.3}$$

After some elementary manipulation which can be safely left to the reader, we deduce that E_μ is both Z and T invariant. Since $\{Z,T\} = ZT = 0$ we deduce that Z, T *commute* as operators on E_μ and, moreover, $Z + T = D \equiv \mu$ on E_μ. Standard spectral theory for commuting symmetric operators implies that E_μ admits an orthogonal decomposition $E_\mu = F_\mu \oplus B_\mu$ with respect to which Z and T have the block decompositions

$$Z = \begin{bmatrix} \mu & 0 \\ 0 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & 0 \\ 0 & \mu \end{bmatrix}.$$

Set $f_\mu = \dim F_\mu$ and $b_\mu = \dim B_\mu$. We claim that

$$b_\mu = b_{-\mu}, \quad \forall \mu > 0. \tag{C.4}$$

Indeed, if $\psi \in B_\mu \setminus \{0\}$ we deduce $Z\psi = 0$ and $T\psi = \mu\psi$. The first equality implies that ψ is covariant constant along the fibers of N and thus ψ is the pullback of some section $\hat{\psi}$ on $K^{-1} \otimes L \oplus L \to \Sigma$. The second equality implies that $\hat{\psi}$ is a μ-eigenvector of the $\mathbb{Z}_2$-graded, L-twisted, Hodge–Dolbeault operator $\bar{\partial} + \bar{\partial}^*$ on Σ. The equality (C.4) is now obvious. Hence

$$\eta(s) = \sum_{\mu>0} \frac{f_\mu - f_{-\mu}}{\mu^s}. \tag{C.5}$$

At this point, the dimensions f_μ can be described quite explicitly. More precisely we have

$$f_\mu \neq 0 \Rightarrow \mu \in \mathbb{Z} \quad \text{and} \quad f_\mu = h(K - L - \mu L_0) + h(L - \mu L_0). \tag{C.6}$$

Before we prove the above equality we want to show its impact on the computation of $\eta(s)$. Using (C.5) we deduce

$$\eta(s) = \sum_{\mu>0} \frac{1}{\mu^s}\Big(\big(h(K-L-\mu L_0) - h(L+\mu L_0)\big) + \big(h(L-\mu L_0) - h(K-L+\mu L_0)\big)\Big),$$

(Riemann-Roch)

$$= \sum_{\mu>0} \frac{-(\deg L + \mu \deg L_0 + 1 - g) + (\deg L - \mu \deg L_0 + 1 - g)}{\mu^s}$$

$$= \sum_{\mu>0} \frac{-2\mu}{\mu^s} = -2\ell\zeta(s-1)$$

where $\zeta(s)$ is Riemann's zeta function. In particular,

$$\eta(0) = -2\ell\zeta(-1)$$

while by [WW], $\zeta(-1) = -\frac{1}{12}$. This agrees with (2.20).

Proof of (C.6): Let $\psi \in F_\mu$. We decompose $\psi = \alpha \oplus \beta$ using the splitting (C.1). Then

$$\mathbf{i}\nabla^B_\zeta \alpha = \mu\alpha, \quad \bar\partial^*_B \alpha = 0, \tag{C.7}$$

$$-\mathbf{i}\nabla^B_\zeta \beta = \mu\beta, \;\; \bar\partial_B \beta = 0. \tag{C.8}$$

Denote by f^-_μ (resp. by f^+_μ) the dimension of the space of solutions of (C.7) (resp. (C.8)). (The connections intervening in the above equations are connections on $\mathcal{K}^{-1} \otimes \pi^* L$ and $\pi^* L$ obtained by pullback from connections on line bundles over the base.) We will only show how to determine f^+_μ since the determination of f^-_μ is entirely similar.

Set for simplicity $\hat B = \pi^* B$, $\hat B^\pm_\mu = \hat B \mp \mathbf{i}\mu\varphi$. Note first that $\bar\partial_{\hat B}\beta = \bar\partial_{\hat B^\pm_\mu}$ since the transition $\hat B \to \hat B^\pm_\mu$ does not alter the derivatives along horizontal directions. On the other hand, the equation $-\mathbf{i}\nabla^{\hat B}_\zeta \beta = \mu\beta$ can be rewritten as

$$-\mathbf{i}\nabla^{\hat B^+_\mu}_\zeta \beta = 0.$$

Thus the equations (C.8) are equivalent to

$$-\mathbf{i}\nabla^{\hat B^+_\mu}_\zeta \beta = 0, \quad \bar\partial_{B^+_\mu}\beta = 0. \tag{C.9}$$

If (C.9) admits a nontrivial solution β then β must be B_μ^+-covariant constant along the fibers. This implies that the pair (π^*L, B_μ^+) is the pullback of a pair (line bundle L'+connection B' on L') on the base Σ. The curvature of the connection B' can be determined from

$$F_{B'} = F_B + 2\ell\mu\mathbf{i}d \cdot \mathrm{vol}_\Sigma,$$

so that

$$-\mu\ell + \deg L = \frac{\mathbf{i}}{2\pi}\int_\Sigma F_{B'} \in \mathbb{Z}.$$

On the other hand, since $\pi^*L' \cong \pi^*L$ we deduce that $\mu\ell \cong 0 \pmod{\ell}$ so that $\mu \in \mathbb{Z}$. In fact one can see that we have an isomorphism of *holomorphic line bundles*

$$(L, B') \cong L \otimes L_0^{-\mu}.$$

The second equation of (C.9) implies that β is a holomorphic section of $L \otimes L_0^{-\mu}$. Hence $f_\mu^+ = h(L - \mu L_0)$. Similarly $f_\mu^- = h(K - L - \mu L_0)$, which concludes the proof of (C.6).

Remark C.1: A similar argument allows one to compute the entire eta function of an adiabatic Dirac coupled with a flat connection of the type discussed in §2.4. In the notation of that section we have

$$\text{(C.10)} \qquad \eta(s) = -\ell\left(\zeta(s-1, k/|\ell|) + \zeta(1-s, 1-k/|\ell|)\right)$$

where, for any $a \in (0,1]$, we denoted by $\zeta(s,a)$ the Riemann–Hurwitz function

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}.$$

In [WW] it is shown that $\zeta(-1,a) = -B_3'(a)/6$, where B_3' denotes the derivative of the Bernoulli polynomial $B_3(z) = z^3 - \frac{3}{2}z^2 + \frac{1}{2}z$. Substituting this in (C.10) we re-obtain the main result of §2.4.

Appendix D. Technical identities

We gather here some technical results used at various places in the main body of the paper.

Consider a local orthonormal coframe $\{\varphi, \varphi_1, \varphi_2\}$ of T^*N as in §2.1. Set

$$\varepsilon = \frac{1}{\sqrt{2}}(\varphi_1 + \mathbf{i}\varphi_2), \quad \bar{\varepsilon} = \frac{1}{\sqrt{2}}(\varphi_1 - \mathbf{i}\varphi_2).$$

Note that ε is a local section of $\mathcal{K}$. With respect to the splitting

$$\mathbb{S}_L = \mathcal{K}^{-1/2} \otimes L \oplus \mathcal{K}^{1/2} \otimes L$$

the Clifford multiplication has the block decomposition (see [N])

$$\mathbf{c}(a\varphi + b\varphi_1 + c\varphi_2) = \begin{bmatrix} \mathbf{i}a & (b+\mathbf{i}c)\bar{\varepsilon} \\ -(b-\mathbf{i}c)\varepsilon & -\mathbf{i}a \end{bmatrix}.$$

In particular, $\mathbf{c}(\varphi)\mathbf{c}(\varphi_1)\mathbf{c}(\varphi_2) = -1$ which agrees with the conventions described in Lemma 1.22 of [BC].

We compute easily

$$\mathbf{c}(\varepsilon) = \begin{bmatrix} 0 & \\ -\sqrt{2}\varepsilon & 0 \end{bmatrix}, \quad \mathbf{c}(\bar{\varepsilon}) = \begin{bmatrix} 0 & \sqrt{2}\varepsilon \\ 0 & 0 \end{bmatrix}.$$

If $\mathbf{i}\dot{a} \in \mathbf{i}\Omega^1(N)$ has the orthogonal decomposition

$$\mathbf{i}\dot{a} = \mathbf{i}\dot{a}_0\varphi + \frac{1}{2}(\omega - \bar{\omega}), \quad \omega \in C^\infty(\mathcal{K}), \quad a_0 \in C^\infty(N),$$

then

$$\text{(D.1)} \qquad \mathbf{c}(\mathbf{i}\dot{a}) = \begin{bmatrix} -\dot{a}_0 & -2^{-1/2}\bar{\omega} \\ -2^{-1/2}\omega & \dot{a}_0 \end{bmatrix}.$$

The quadratic map $q(\phi)$, viewed as an endomorphism of $\mathbb{S}_L$, has the block decomposition

$$q(\phi) = \begin{bmatrix} \frac{1}{2}(|\phi_-|^2 - |\phi_+|^2) & \phi_-\bar{\phi}_+ \\ \bar{\phi}_-\phi_+ & -\frac{1}{2}(|\phi_-|^2 - |\phi_+|^2) \end{bmatrix}.$$

If ϕ is such that $\phi_- = 0$, then

$$\dot{q}(\phi, \dot{\psi}) \stackrel{def}{=} \frac{d}{dt}|_{t=0}\, q(\phi + t\dot{\psi}) = \begin{bmatrix} -\mathfrak{Re}\langle \phi_+, \dot{\psi}_+ \rangle & \dot{\psi}_-\bar{\phi}_+ \\ \bar{\dot{\psi}}_-\phi_+ & \mathfrak{Re}\langle \phi_+, \dot{\psi}_+ \rangle \end{bmatrix}.$$

Using (D.1) we obtain a description of $\dot{q}(\phi, \dot{\psi})$ as a *purely imaginary 1-form*

$$\text{(D.2)} \qquad \dot{q}(\phi, \dot{\psi}) = \mathbf{i}\mathfrak{Re}\langle \phi_+, \dot{\psi}_+ \rangle\varphi - 2^{-1/2}(\dot{\psi}_-\bar{\phi}_+ - \bar{\dot{\psi}}_-\phi_+).$$

Let ϕ be as above. Given $\Xi = \dot{\psi} \oplus \mathbf{i}\dot{a} \oplus \mathbf{i}f$ where $\mathbf{i}\dot{a} = \frac{1}{2}(\omega - \bar{\omega})$, $\omega \in C^\infty(\mathcal{K})$, then using (D.1) and (D.2) we deduce

$$\text{(D.3)} \qquad \mathcal{P}_\phi\Xi = \begin{bmatrix} \mathbf{c}(\mathbf{i}\dot{a})\phi - \mathbf{i}f\phi \\ \dot{q}(\phi, \dot{\psi}) \\ \mathbf{i}\mathfrak{Im}\langle \phi, \dot{\psi} \rangle \end{bmatrix}$$

$$= \begin{bmatrix} (-\bar{\omega}\phi_+) \oplus (-\mathbf{i}f\phi_+) \\ = \mathbf{i}\mathfrak{Re}\langle \phi_+, \dot{\psi}_+ \rangle\varphi + 2^{-1/2}(\dot{\psi}_-\bar{\phi}_+ - \bar{\dot{\psi}}_-\phi_+) \\ \mathbf{i}\mathfrak{Im}\langle \phi_+, \dot{\psi}_+ \rangle \end{bmatrix}.$$

We now deduce easily that

$$\langle \mathcal{P}\Xi, \Xi\rangle = 2^{1/2} f \mathfrak{Im}\langle \phi_+, \dot\psi_+\rangle - \mathfrak{Re}(\dot{\bar\psi}_- \phi_+ \bar\omega). \tag{D.4}$$

The term $\dot q(\phi, \dot\psi)$ has nice divergence properties. More precisely we have the following result.

LEMMA D.1: *Consider a spin^c structure σ on an oriented, Riemannian 3-manifold (M, g). Fix a connection A on* $\det\sigma$ *and denote by $\mathcal{D}_A$ the Dirac operator on $\mathbb{S}_\sigma$ induced by the* Levi-Civita connection *coupled with A. Then for every $\psi \in C^\infty(\mathbb{S}_\sigma)$ we have*

$$d^* q(\psi) = -\mathbf{i}\mathfrak{Im}\langle \psi, \mathcal{D}_A \psi\rangle.$$

Proof: Fix an arbitrary point $p_0 \in M$, choose normal coordinates (x^1, x^2, x^3) near p_0 and set $e^i = dx^i$. Note that *at* p_0 we have $d^* e^i = 0$ for all i. In [N] we showed that, viewed as a 1-form, $q(\psi)$ has the local description

$$q(\psi) = \frac{1}{2}\sum_i \langle \psi, \mathbf{c}(e^i)\psi\rangle e^i.$$

At p_0 we have

$$\begin{aligned} 2d^* q(\psi) &= -\sum_i \partial_i(\langle \psi, \mathbf{c}(e^i)\psi\rangle) e^i \\ &= -\sum_i \langle \nabla_i^A \psi, \psi\rangle - \sum_i \langle \psi, \mathbf{c}(e^i)\nabla_i^A\psi\rangle \quad (\text{since } \nabla_i e^i = 0 \text{ at } p_0) \\ &= \sum_i \overline{\langle \psi, \mathbf{c}(e^i)\nabla_i^A \psi\rangle} - \sum_i \langle \psi, \mathbf{c}(e^i)\nabla_i^A \psi\rangle = -2\mathbf{i}\mathfrak{Im}\langle \psi, \mathcal{D}_A\psi\rangle. \end{aligned}$$

Since p_0 is arbitrary this proves the lemma. ∎

On our circle bundle N we have $\mathcal{D}_A = D_A + \lambda_r/2$, so that

$$d^* q(\phi) = -\mathbf{i}\mathfrak{Im}\langle \phi, (D_A + \lambda_r/2)\phi\rangle = -\mathbf{i}\mathfrak{Im}\langle \phi, D_A\phi\rangle. \tag{D.5}$$

Suppose now $\phi \in \ker D_A$. We derivate (D.5) along $\dot\psi$ and we get

$$d^* \dot q(\phi, \dot\psi) = -\mathfrak{Im}\langle \phi, D_A \dot\psi\rangle. \tag{D.6}$$

This identity plays an important role in the proof of the following result.

LEMMA D.2: *Consider an irreducible solution* $\mathrm{C} = (\phi, A)$ *of the Seiberg–Witten equation on N. We assume for simplicity $\phi_- = 0$. For each $w \geq 0$ we have an* *operator $\mathcal{O}_w = \mathcal{O}_w(\mathrm{C})$ as in §3.4. (Recall that $\mathcal{O}_0 = \tilde{\mathcal{H}}_1$.) Then for all $w \geq 0$ we have*

$$\ker \tilde{\mathcal{H}}_1 = \ker \mathcal{O}_w.$$

Proof: Note that if $\Xi = \dot{\psi} \oplus \mathbf{i}\dot{a} \oplus \mathbf{i}f \in \ker \mathcal{O}_w$ is such that $f \equiv 0$, then the definition of $\mathcal{O}_w$ implies immediately that $\Xi \in \ker \tilde{\mathcal{H}}_1$. Conversely, any $\Xi \in \tilde{\mathcal{H}}_0$ has vanishing third coordinate. Hence it suffices to show that the third component of any $\Xi \in \ker \mathcal{O}_w$ vanishes.

Let $\Xi = \dot{\psi} \oplus \mathbf{i}\dot{a} \oplus \mathbf{i}f \in \ker \mathcal{O}_w$. This means

$$\text{(D.7)} \qquad \begin{cases} D_A\dot{\psi} & + \quad \mathbf{c}(\mathbf{i}\dot{a})\phi - \mathbf{i}f\phi \quad = \quad 0, \\ -\mathbf{i} * d\dot{a} + \mathbf{i}df & + \quad \dot{q}(\phi, \dot{\psi}) \quad = \quad 0, \\ \mathbf{i}d^*\dot{a} - 2w\mathbf{i}f & + \quad \mathbf{i}\mathfrak{Im}\langle \phi, \dot{\psi} \rangle \quad = \quad 0. \end{cases}$$

Take the inner product of the second equation with $\mathbf{i}df$. After an integration by parts we get

$$\int_N |df|^2 dv_N - \int_N f \cdot d^*\mathbf{i}\dot{q}(\phi, \dot{\psi} dv_N = 0.$$

Using (D.5) and the first equation in (D.7) we get

$$\int_N (|df|^2 + |f|^2 \cdot |\phi|^2) dv_N = 0.$$

This shows $f \equiv 0$ and completes the proof of the lemma. ∎

The above lemma has the following important consequence.

COROLLARY D.3:

$$\mathrm{SF}(\tilde{\mathcal{H}}_1 \to \mathcal{O}_w) = 0.$$

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