Let X be a closed manifold having integral homology of $S^1 \times S^3$. Define $\rho(X)$ to be the Rohlin invariant of any spin 3–manifold in X carrying $H_3(X)$. A well known question is whether $\pi_1(X) = \mathbb{Z}$ implies that $\rho(X) = 0$. Motivated by Casson's approach to proving that the Rohlin invariant of a homotopy 3-sphere vanishes, we study an analogue of Casson's invariant in dimension four, obtained by "creatively" counting irreducible flat connections over X. This invariant was introduced by Furuta and Ohta, who also conjectured that its modulo 2 reduction equals $\rho(X)$.

We derive explicit formulas for the Furuta-Ohta invariant for mapping tori of finite order diffeomorphisms and for certain circle bundles. In both instances, it is expressed in terms of classical invariants, and reduces modulo 2 to the Rohlin invariant. As a part of our broader surgery program aimed at proving the Furuta-Ohta conjecture, we investigate an analogue of the Furuta-Ohta invariant for homology 4-tori T and prove that its mod 2 reduction equals the quadruple cup product of vectors forming a basis in $H^1(T)$.

Finally, we relate the Furuta-Ohta invariant to the Floer Lefschetz number and show how a positive solution of the Furuta-Ohta conjecture would imply existence of non–triangulable topological manifolds of dimension five and higher.

This is a joint work with Daniel Ruberman.