On a theorem of Henri Cartan concerning the equivariant cohomology

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Abstract

We provide a new look at an old result of Henri Cartan concerning the cohomology of infinitesimally free smooth Lie group actions.[1] [2]

Introduction

The equivariant cohomology of a smooth manifold acted by a Lie group is a concept which crystallized in the works of A. Borel, H. Cartan, C. Chevalley, H. Hopf, L. Koszul and A. Weil in the late forties and early fifties.

The differential geometric approach to this subject was brilliantly described by Henri Cartan in the beautiful survey [2] which continues to be the first source for anyone interested in learning the basic facts of this theory. Recently, it has been the focus of intense research in connection with many problems in differential geometry, representation theory and quantum field theory.

A central result of [2] is Cartan's theorem which states that if a compact Lie group G acts freely on a smooth manifold M then the G-equivariant cohomology of M (as defined by Cartan) is naturally isomorphic to the DeRham cohomology of the quotient.

There are currently many proofs of this fact (e.g. [2], [4], [11]) but, in the author's view, they all suffer of the same æsthetic "deficiency". They involve a quite large amount of amazing combinatorics whose origin is somewhat obscure. Moreover, the resulting isomorphism is extremely difficult to figure out *explicitely* at *cochain level*.

The main goal of this paper is to provide a new, *direct* and more transparent proof of the following slight generalization of Cartan's theorem.

Theorem *If G is a compact Lie group acting on the smooth manifold X and N is a closed normal subgroup of G acting* freely *on X then the G-equivariant cohomology of X is* naturally *isomorphic with the G/N-equivariant cohomology of X/N.*

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We will actually establish a more general algebraic result (see Theorem 5.1). Moreover, relying on a recent result of Kalkman [6] (which provides a very explicit isomorphism between the Weil model and the Cartan model of equivariant cohomology) we will offer an *explicit* description of this isomorphism (along the lines of [8]). In the course of the proof we will provide yet another interpretation for *moment map* and the equivariant characteristic classes described for example in [1] or [3]. The very simple functorial principle behind our proof is explained in Remark 5.4. While some of the computations involved may not look too eye pleasing, they are entirely routine and more importantly, their logical succesion is very natural.

There are two surprising aspects of this proof which make it so attractive. They can best be grasped by looking at a special example. Suppose $P \to B$ is a smooth principal G-bundle. Cartan's theorem then states that $H^*_G(P)$, the G-equivariant cohomology of P, coincides with the DeRham cohomology of the base B. Naturally, one tries to construct cochain homotopy equivalences between the complexes leading to the two cohomologies. A geometer might even attempt this using purely geometric operations on the smooth manifolds involved. This approach is doomed to fail. The method we propose is to embedd the two complexes in the same larger complex consisting of "ideal" elements and then show that the two embeddings are homotopic with this larger space. The homotopies are described by the Weil transgression between a genuine G-connection on P and a certain "ideal" connection which has only a formal meaning !!! The second surprise is the amazing effectiveness of this method. Normally one expects that by "pushing" the geometric situation into an "ideal" abstract framework the resulting formulæ will be more involved. To our surprize, Kalkman's isomorphism fits perfectly in such a framework. A bonus of this proof is that the isomorphism $H^*_G(P) \cong H^*(B)$ can be *explicitely* described at the *cochain* level. More precisely, to any Cartan representative of an element in $H^*_G(P)$ we associate a closed form on B. This correspondence descends to an *isomorphism* between the two cohomologies. This map is obtained naturally, as a by-product of our computations.

The paper is divided into five section of which four are devoted to surveying the basic "players" in Cartan's approach to equivariant theory. In the first part we introduce the notion of *operation* which captures the essential features of the DeRham algebra of a smooth manifold with a Lie group action.

In the first section we introduce the main object of study, that of *operation*. It formalizes the algebra of exterior forms on a smooth manifold equipped with a Lie group action. Section 2 introduces the Cartan and Weil models of equivariant cohomology while section 3 describes the Kalkman isomorphism between them. In section 4 we review the basics of the Weil transgression trick in the framework of *operations*. In the final section we prove Cartan's theorem and discuss a few consequences.

In the sequel G will always denote a compact, connected Lie group. The prefix "s" will refer to "super" (i.e. $\mathbb{Z}_2$-graded) objects as in [1]. The bracket $[\,\cdot\,,\,\cdot\,]_s$ denotes the supercommutator in a super-algebra. Also, we will use Einstein's summation convention (unless otherwise indicated).

1 Operations

As in [3] we will consider Frechet algebras. These are associative $\mathbb{R}$-algebras such that their algebraic operations are continuous with respect to a Frechet topology. The standard example of Frechet algebra is that of the algebra of smooth functions on a smooth manifolds.

In this section we want to introduce the algebraic counterpart of the geometric notion of smooth manifold acted on by a Lie group. This object appears in literature with various names. We have chosen the terminology of [5] which stays closer to the original motivation.

Definition 1.1 An *operation* consists of the following.
(a) A $\mathbb{Z}$-graded Frechet algebra

$$\mathcal{A} = \oplus_{n\in\mathbb{Z}}\mathcal{A}^n$$

such that $a\cdot b = (-1)^{|a|\cdot|b|}b\cdot a$ for any two homogeneous elements. ($\mathcal{A}$ is naturally a s-algebra by $\mathcal{A} = \mathcal{A}^{even} \oplus \mathcal{A}^{odd}$).
(b) A *continuous odd* derivation

$$d : \mathcal{A}^* \to \mathcal{A}^{*+1}$$

such that $d^2 = 0$.
(c) A *smooth* action of the Lie group G on $\mathcal{A}$ via algebra automorphisms commuting with d. We denote by $\mathcal{L}$ the derivative of this action at $\mathbf{1} \in G$. Thus $\mathcal{L}$ defines a representation of $\mathfrak{g}$ into the Lie algebra of even derivations of $\mathcal{A}$. ($\mathcal{L}_X$ ($X \in \mathfrak{g}$) is the Lie derivative along the automorphisms $\exp(tX)$ of $\mathcal{A}$.) Note that

$$[\mathcal{L}_X, d]_s = \mathcal{L}_X d - d\mathcal{L}_X = 0.$$

(d) A continuous G-equivariant linear map $\mathcal{I}$ from $\mathfrak{g}$ (called *contraction*) to the space of *odd* derivations of $\mathcal{A}$ such that $\forall X, Y \in \mathfrak{g}$
(d1) $\mathcal{I}_X(\mathcal{A}^n) \subset \mathcal{A}^{n-1}$, $\forall n$.
(d2) $[\mathcal{I}_X, \mathcal{I}_Y]_s = \mathcal{I}_X\mathcal{I}_Y + \mathcal{I}_Y\mathcal{I}_X = 0$.
(d3) $[\mathcal{L}_X, \mathcal{I}_Y] = \mathcal{I}_{[X,Y]}$.
(d4) (Cartan formula) $[\mathcal{I}_X, d]_s = \mathcal{I}_X d + d\mathcal{I}_X = \mathcal{L}_X$.

Remark 1.2 The above contraction $\mathcal{I}$ extends to an algebra morphism $\mathcal{I} : \Lambda\mathfrak{g} \to \mathrm{End}\,(\mathcal{A})$ thus defining a $\Lambda\mathfrak{g}$-module structure on $\mathcal{A}$.

Example 1.3 The *right* action of a Lie group G on the smooth manifold M defines a structure of operation on $\Omega^*(M)$. For each $X \in \mathfrak{g}$ we will denote by $\mathcal{L}_X$ the Lie derivative along the flow $m \mapsto m\cdot\exp(tX)$ while $\mathcal{I}_X$ denotes the contraction along $X^\#$-the infinitesimal generator of the above flow.

Example 1.4 (The Weil algebra) Consider the Lie group G and set

$$W_G = S\mathfrak{g}^* \otimes \Lambda\mathfrak{g}^*$$

where Λ and S denote the exterior and respectively the symmetric algebra. Topologize W_G in the obvious fashion (as a space of polynomials) and equip it with the $\mathbb{Z}$-grading

$$\deg(\Lambda^p\mathfrak{g}^*) = p, \ \ \deg(S^q\mathfrak{g}^*) = 2q.$$

Denote by $\mathfrak{h}$ the obvious isomorphism

$$\mathfrak{h} : \Lambda^1\mathfrak{g}^* \to S^1\mathfrak{g}^*. \tag{1.1}$$

The usual derivations d, L_X and i_X on $\Omega^*(G)$ have an algebraic counterpart on $\Lambda\mathfrak{g}^*$ which we denote by $\mathbf{d}$, $\mathbf{L}_X$ and $\imath_X$., It is convenient to describe these operations in "local coordinates". Choose a basis (e_i) of $\mathfrak{g}$ and denote by (θ^i) the dual basis of $\mathfrak{g}^*$. Set $\Omega^i = \mathfrak{h}(\theta^i) \in S^1\mathfrak{g}^*$. Denote by $\mathbf{L}_i$ and $\imath_j$ the Lie derivative along e_i and respectively the contraction by e_j. Then

$$\mathbf{L}_i\theta^j = -C^j_{ik}\theta^k \ \text{ and } \ \imath_j\theta^k = \delta^k_j$$

where C^i_{jk} denote the structural constants of the Lie algebra $\mathfrak{g}$

$$[e_j, e_k] = C^i_{jk}e_i.$$

To describe $\mathbf{d}$ in local coordinates we use the formula

$$(\mathbf{d}\omega) = -\omega([X,Y]), \ \ \omega \in \mathfrak{g}^*, \ X, Y \in \mathfrak{g}.$$

This yields

$$\mathbf{d}\theta^i = -\frac{1}{2}C^i_{jk}\theta^j\theta^k.$$

This implies immediately Koszul's formula

$$\mathbf{d} = \frac{1}{2}\mu(\theta^k)\mathbf{L}_k = \frac{1}{2}\theta^k\mathbf{L}_k \tag{1.2}$$

where for any algebra A and any $a \in A$ we denote by $\mu(a)$ the left multiplication by a. For simplicity we will very often omit the μ symbol when there is very little room for confusion.

The action of G on W_G induced by the coadjoint action defines a Lie derivative $\mathbf{L}$ extending the Lie derivative on $\Lambda\mathfrak{g}^*$ according to the prescription

$$\mathbf{L}_X\mathfrak{h}(\omega) = \mathfrak{h}(\mathbf{L}_X\omega).$$

We can extend $\mathbf{d}$ to an odd derivation d_W of W_G uniquely determined by its action on the generators

$$d_W\theta^i = \mathbf{d}\theta^i + \mathfrak{h}\theta^i = -\frac{1}{2}C^i_{jk}\theta^j\theta^k + \Omega^i = (\frac{1}{2}\theta^j\mathbf{L}_j + \Omega^j\imath_j)\theta^i \tag{1.3}$$

$$d_W\Omega^i = -C^i_{jk}\theta^j\Omega^k = \left(\theta^j\mathbf{L}_j\right)\Omega^i. \tag{1.4}$$

Extend the contraction $\imath_X$ to W_G by imposing $\imath_X\mathfrak{h}(\omega) = 0$.

We leave the reader verify that these three derivations on W_G do indeed define a structure of operation. □

Given two operations $(\mathcal{A}_i, d_i, \mathcal{L}^i, \mathcal{I}^i)_{i=1,2}$ we can define a structure of operation on their (Grothendiek) topological tensor product $\mathcal{A}_1 \otimes \mathcal{A}_2$ (described e.g. in [10]) according to the rules (ε_1 is the grading operator of the s-algebra $\mathcal{A}_1$)

$$d = d_1 \otimes \mathbf{1} + \varepsilon_1 \otimes d_2$$

$$\mathcal{L} = \mathcal{L}^1 \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{L}^2$$

and

$$\mathcal{I} = \mathcal{I}^1 \otimes \mathbf{1} + \varepsilon_1 \otimes \mathcal{I}^2.$$

Given an operation $(\mathcal{A}, d, \mathcal{L}, \mathcal{I})$ we can define three subalgebras

$$\mathcal{A}_{inv} = \ker \mathcal{L} = \{a \in \mathcal{A} \; ; \; \mathcal{L}_X a = 0 \;\; \forall X \in \mathfrak{g}\}$$

$$\mathcal{A}_{hor} = \ker \mathcal{I} \;\text{ and }\; \mathcal{A}_{bas} = \mathcal{A}_{inv} \cap \mathcal{A}_{hor}.$$

Since $[d, \mathcal{L}]_s = 0$ we deduce $d\mathcal{A}_{inv} \subset \mathcal{A}_{inv}$. Moreover, Cartan formula implies $d\mathcal{A}_{bas} \subset \mathcal{A}_{bas}$. Thus we can define the cohomology groups

$$H^*_{inv}(\mathcal{A}) = H^*(\mathcal{A}_{inv}, d) \;\text{ and }\; H^*_{bas}(\mathcal{A}) = H^*(\mathcal{A}_{bas}, d).$$

Example 1.5 Consider a smooth principal G-bundle $G \hookrightarrow P \to B$. The right action of G on P induces a structure of operation on $\Omega^*(P)$. The basic subalgebra of this operation is then naturally isomorphic to $\Omega^*(B)$.

2 The Cartan-Weil descriptions

We will work in the more general setting of operations. We will define two notions of equivariant cohomology and in the next subsection we will show they coincide.

Consider a G-operation $(\mathcal{A}, d, \mathcal{L}, \mathcal{I})$.

The Weil description We define Weil's equivariant cohomology of $\mathcal{A}$ by

$$WH^*_G(\mathcal{A}) \stackrel{def}{=} H^*_{bas}(W_G \otimes \mathcal{A})$$

where W_G denotes the Weil algebra introduced in the previous subsection.

The Cartan description Consider the algebra

$$\mathcal{B} = S\mathfrak{g}^* \otimes \mathcal{A}.$$

$S\mathfrak{g}^*$ is graded as usual by

$$\deg S^p\mathfrak{g}^* = 2p.$$

G acts smoothly on $\mathcal{B}$ and so we can form the subalgebra of invariant elements

$$\mathcal{B}_{inv} = (S\mathfrak{g}^* \otimes \mathcal{A})^G.$$

Now define the operator

$$\mathfrak{d} = \mathbf{1} \otimes d - \sum_i \mu(\Omega^k) \otimes \mathcal{I}_k = \mathbf{1} \otimes d - \Omega^k \otimes \mathcal{I}_k. \tag{2.1}$$

If we regard $\omega \in S\mathfrak{g}^* \otimes \mathcal{A}$ as a polynomial map $\mathfrak{g} \to \mathcal{A}$ then $\mathfrak{d}\omega$ is the polynomial map

$$X \mapsto d(\omega(X)) - \mathcal{I}_X(\omega(X)).$$

$\mathfrak{d}$ satisfies the following conditions (see [2])

$$\mathfrak{d}^2 = -\Omega^k \otimes \mathcal{L}_k. \tag{2.2}$$

$$[\mathfrak{d}, \mathbf{L} \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{L}] = 0. \tag{2.3}$$

The equality (2.3) shows that $\mathcal{B}_{inv}$ is $\mathfrak{d}$ invariant. Moreover, on this subalgebra $\mathfrak{d}^2 = 0$. Indeed, on this subalgebra we have $\mathbf{L} \otimes \mathbf{1} = -\mathbf{1} \otimes \mathcal{L}$ so that by (2.2) we have

$$\mathfrak{d}^2 = -\Omega^i \otimes \mathcal{L}_i = \Omega^i \mathbf{L}_i \otimes \mathbf{1}.$$

Now it is not difficult to see that $\Omega^i \mathbf{L}_i \equiv 0$ on $S\mathfrak{g}^*$ due to the skew symmetry of the structural constants. Thus $(\mathcal{B}_{inv}, \mathfrak{d})$ is a cochain complex and we define the Cartan equivariant cohomology of $\mathcal{A}$ by

$$CH_G^*(\mathcal{A}) \stackrel{def}{=} H^*(\mathcal{B}_{inv}, \mathfrak{d}).$$

When $\mathcal{A}$ is the algebra of differential forms on a smooth manifold M on which G acts smoothly we will use the notations $WH_G^*(M)$ and $CH_G^*(M)$ to denote the corresponding equivariant cohomologies.

3 Weil model $\Longleftrightarrow$ Cartan model

Consider a G-operation $(\mathcal{A}, d, \mathcal{L}, \mathcal{I})$. The main result of this subsection is the following.

Theorem 3.1 There exists a *natural* isomorphism

$$WH_G^*(\mathcal{A}) \cong CH_G^*(\mathcal{A}).$$

We briefly describe the proof in [6]. For a different but related approach we refer to [9].

Consider the algebra $\mathcal{B} = W_G \otimes \mathcal{A}$. It has a tensor product structure of G-operation with structural derivations D, L and respectively I. For each $U_i^j = \theta^j \otimes e_i \in \mathfrak{g}^* \otimes \mathfrak{g}$ define the following operators (on $\mathcal{B}$)

$$\mathbb{A}_i^j = \theta^j \otimes \mathcal{I}_i$$

$$\mathbb{L}_i^j = \theta^j \otimes \mathcal{L}_i - \Omega^j \otimes \mathcal{I}_i$$

In general for any $T = t_j^i U_i^j \in \mathfrak{g}^* \otimes \mathfrak{g}$ set

$$\mathbb{A}_T = t_j^i \mathbb{A}_i^j, \;\; \mathbb{L}_T = t_j^i \mathbb{L}_i^j \;\text{ and }\; D_T = D + \mathbb{L}_T.$$

Note that $(\mathbb{A}_i^j)^2 = 0$, $\forall i, j$ and moreover $\mathbb{A}_T \mathbb{A}_S = \mathbb{A}_S \mathbb{A}_T$, $\forall S, T$. Thus $\exp(\mathbb{A}_T)$ is well defined and invertible. A simple computation shows that for all i, j the operator $\exp(\mathbb{A}_i^j) = \mathbf{1} + \mathbb{A}_i^j$ is an *algebra automorphism* of $\mathcal{B}$ so that $\exp(\mathbb{A}_T)$ is an automorphism of $\mathcal{B}$ for all $T \in \mathfrak{g}^* \otimes \mathfrak{g}$.

The key step in the proof of Theorem 3.1 is contained in the following result.

Lemma 3.2 For any $T \in \mathfrak{g}^* \otimes \mathfrak{g}$ we have

$$\exp(\mathbb{A}_T) D \exp(-\mathbb{A}_T) = D_T.$$

Proof of the lemma An elementary computation shows that for any i, j, k, ℓ we have the following "differential equations"

$$[D, \exp(\mathbb{A}^i_j)] = \mathbb{L}^i_j \exp(\mathbb{A}^i_j) \tag{3.1}$$

$$[\mathbb{L}^i_j, \exp(\mathbb{A}^k_l)] = 0. \tag{3.2}$$

The equality (3.1) can be rephrased as

$$\exp(\mathbb{A}^i_j) D \exp(-\mathbb{A}^i_j) = D_{U^i_j} := D + \mathbb{L}^i_j \tag{3.3}$$

while (3.2) is equivalent to

$$\exp(\mathbb{A}^k_\ell) \mathbb{L}^i_j \exp(-\mathbb{A}^k_\ell) = \mathbb{L}^i_j. \tag{3.4}$$

Using (3.4) in (3.3) we deduce ($\mathbb{A} = \mathbb{A}^i_j + \mathbb{A}^k_\ell$)

$$\exp(\mathbb{A}) D \exp(-\mathbb{A}) = D_{\mathbb{A}}.$$

Lemma 3.2 now follows by iterating the above procedure. □

We now have a whole family of G-structures on $\mathcal{B}$ parameterized by $\mathfrak{g}^* \otimes \mathfrak{g}$,

$$\mathcal{B}_T := (\mathcal{B}, D_T, L_T = \exp(\mathbb{A}_T) L \exp(-\mathbb{A}_T), I_T = \exp(\mathbb{A}_T) I \exp(-\mathbb{A}_T)\,), \;\; T \in \mathfrak{g}^* \otimes \mathfrak{g}.$$

Moreover an elementary computation shows that $L_T \equiv L_0$. All these structures are isomorphic with the canonical tensor product structure and in particular

$$H^*_{bas}(\mathcal{B}_T) \cong H^*_{bas}(\mathcal{B}_0) \cong WH^*(\mathcal{A}).$$

An interesting special case arises when $T = \mathbf{id} = \theta^i \otimes e_i$. In this case the derivation $D_{\mathbf{id}}$ is known as the *BRST* (= Bechi-Rouet-Stora-Tyupin) operator and it arises in the quantization of classical gauge theories.

We leave the reader check that

$$(\mathcal{B}_{\mathbf{id}})_{hor} = \ker I_{\mathbf{id}} = S\mathfrak{g}^* \otimes \mathcal{A}.$$

Hence

$$(\mathcal{B}_{\mathbf{id}})_{bas} = (S\mathfrak{g}^* \otimes \mathcal{A})^G$$

and it is not difficult to see that on this subalgebra $D_{\mathbf{id}} = \mathfrak{d}$. Thus

$$H^*_{bas}(\mathcal{B}_{\mathbf{id}}) \cong CH^*_G(\mathcal{A}).$$

Theorem 3.1 is proved. □

Remark 3.3 Kalkman's isomorphism

$$\phi = \phi_G = \exp(\mathbb{A}_{\mathbf{id}}) : \mathcal{B}_0 \to \mathcal{B}_{\mathbf{id}} \tag{3.5}$$

has a particularly nice form when restricted to $(\mathcal{B}_0)_{bas}$. It is uniquely determined by the correspondences $\Omega^i \mapsto \Omega^i$ and $\theta^j \mapsto 0$.

4 Algebraic connections

Among the possible actions of a Lie group on a smooth manifold the free ones play a special role. Consider for example the case of a smooth principal G-bundle $G\hookrightarrow P \to B$. Such actions admit connections. Recall (see [7]) that a connection on P is an equivariant splitting

$$TP \cong \mathcal{V}P \oplus \mathcal{H}P$$

where $\mathcal{V}P$ is the bundle spanned by the infinitesimal generators of the G actions. In fact, for any $p \in P$ the correspondence

$$\mathfrak{g} \ni X \mapsto X_p^{\#}$$

identifies the fiber $\mathcal{V}_pP$ with $\mathfrak{g}$.

Alternatively, a splitting as above can be defined by a vertical projector i.e. a $\mathfrak{g}$-valued 1-form $\Theta \in \mathfrak{g} \otimes \Omega^1(P)$ which is G-invariant and satisfies

$$i_{X^{\#}}\Theta = X \quad \forall X \in \mathfrak{g}. \tag{4.1}$$

We can regard this connection as a linear map

$$\Theta : \mathfrak{g}^* \ni \theta^i \mapsto \tilde{\theta}^i \in \Omega^*(P)$$

so that

$$\Theta = e_i \otimes \tilde{\theta}^i.$$

The condition (4.1) reads

$$\mathcal{I}_j\tilde{\theta}^k = \delta_j^k \tag{4.2}$$

or equivalently,

$$\mathcal{I}_X\Theta = X, \quad \forall X \in \mathfrak{g}. \tag{4.3}$$

The invariance implies

$$\mathbf{L}_k e_i \otimes \tilde{\theta}^i + e_i \otimes \mathcal{L}_k\tilde{\theta}^i = 0$$

i.e.

$$\mathcal{L}_k\tilde{\theta}^i = -C_{kj}^i\tilde{\theta}^j. \tag{4.4}$$

or equivalently,

$$\Theta\mathbf{L}_X = \mathcal{L}_X\Theta, \quad \forall X \in \mathfrak{g}. \tag{4.5}$$

The conditions (4.2)-(4.5) are formulated using a language which involves only the structure of operation. Thus we can define an abstract notion of *algebraic connection* on any G-operation $\mathcal{A}$ as a G-equivariant linear map $\tilde{\theta} : \mathfrak{g}^* \to \mathcal{A}$ satisfying (4.2)-(4.5).

Example 4.1 The inclusion $\mathfrak{g}^*\hookrightarrow W_G$ defines an algebraic connection.

Consider now a G-operation $(\mathcal{A}, d, \mathcal{L}, \mathcal{I})$ equipped with a connection

$$\Theta : \theta^i \mapsto \tilde{\theta}^i.$$

Define

$$\tilde{\Omega}^i = d\tilde{\theta}^i - \frac{1}{2}C_{jk}^i\tilde{\theta}^j\tilde{\theta}^k.$$

The form

$$\tilde{\Omega} = e_i \otimes \tilde{\Omega}^i \in \mathfrak{g} \otimes \mathcal{A}^2$$

is independent of the basis (e_i) and it is called *the curvature of the connection.* An elementary computation shows that

$$\mathcal{I}_k \tilde{\Omega}^i = 0 \quad \forall i, k$$

i.e. $\tilde{\Omega}^i \in \mathcal{A}^2_{hor}$, $\forall i$.

The main algebraic implications of the existence of a connection derive from the following decomposition result.

Proposition 4.2 The connection induced map

$$\Lambda \mathfrak{g}^* \otimes \mathcal{A}_{hor} \to \mathcal{A}, \quad \theta^A \otimes \omega \mapsto \tilde{\theta}^A \omega$$

($\omega \in \mathcal{A}_{hor}$, A is an ordered multi-index $(a_1, a_2, \ldots)$ and $\theta^A = \theta^{a_1} \wedge \theta^{a_2} \wedge \cdots$) is an isomorphism of graded algebras.

Idea of proof The map is clearly injective. The surjectivity follows from the following simple observation

$$\forall \omega \in \mathcal{A}, \;\; \omega - \tilde{\theta}^k \mathcal{I}_k \omega \in \ker \mathcal{I}_k \;\; (\text{no summation}). \quad \Box$$

Thus we can *uniquely* represent any element $\omega \in \mathcal{A}$ as a polynomial

$$\omega = \tilde{\theta}^A \omega_A$$

where in the above sum A runs through all ordered multi-indices. ω is said to be *horizontally homogeneous* if all the coefficients $\omega_A \in \mathcal{A}_{hor}$ have the same degree called the *horizontal degree* and denoted by $\deg_h$.

The component $\omega_\emptyset \in \mathcal{A}_{hor}$ of $\omega \in \mathcal{A}$ is called the *horizontal component*, the map $\omega \mapsto \omega_\emptyset$ will be denoted by h and will be named the *horizontal projection.*

Remark 4.3 It is not difficult to see that the horizontal projection can be explicitly described by

$$h = \prod_k \left(\mathbf{1} - \tilde{\theta}^k \otimes \mathcal{I}_k \right) = \exp(-\tilde{\theta}^k \otimes \mathcal{I}_k).$$

We can now define the covariant derivative of the connection Θ as the composition

$$\nabla = h \circ d.$$

A simple computation shows that

$$\nabla \tilde{\theta}^i = \tilde{\Omega}^i \quad (\text{Maurer} - \text{Cartan}) \tag{4.6}$$

and

$$\nabla \tilde{\Omega}^i = 0 \quad (\text{Bianchi}) \tag{4.7}$$

Set

$$\mathfrak{cw} : W_G \to \mathcal{A}, \quad \theta^i \mapsto \tilde{\theta}^i, \;\; \Omega^i \mapsto \tilde{\Omega}^i.$$

This map is independent of the basis (e_i) and it is called *the Chern-Weil correspondence.* The following result explains the universal role played by the "exotic" structure of W_G.

Proposition 4.4 The Chern-Weil correspondence induced by a connection is a morphism of G-operations. Moreover, given two connections $\tilde{\theta}_i$, $i = 0, 1$ on $\mathcal{A}$ the corresponding Chern-Weil maps $\mathfrak{cw}_i$ are homotopic as morphisms of cochain complexes.

Proof The first part is left to the reader. To prove the second part we use a familiar trick from the theory of characteristic classes.

Form the algebra $\hat{\mathcal{A}} = \Omega^*(\mathbb{R})\otimes\mathcal{A}$. (If $\mathcal{A}$ were the algebra of differential forms $\Omega^*(M)$ on a smooth manifold M then $\hat{\mathcal{A}} \cong \Omega^*(\mathbb{R}\times M)$.) Clearly $\hat{\mathcal{A}}$ is a G-operation and $\hat{\theta} = (1-t)\tilde{\theta}_0 + t\tilde{\theta}_1$ defines a connection on $\hat{\mathcal{A}}$. Denote by Ψ_i $(i = 0, 1)$ the the maps $\Psi_i : \hat{\mathcal{A}} \to \mathcal{A}$ defined by the localizations at $t = 1$

$$\Omega^*(\mathbb{R}) \to \mathbb{R}, \;\; f(t) \mapsto f(i), \;\; dt \mapsto 0.$$

We have a fiberwise integration morphism

$$\int_I : \hat{\mathcal{A}} \to \mathcal{A}$$

defined by

$$\int_I f(t)\otimes\omega = \begin{cases} 0 & \text{if} \;\; f \in \Omega^0(\mathbb{R}) \\ \left(\int_0^1 f(t)\right)\omega & \text{if} \;\; f \in \Omega^1(\mathbb{R}) \end{cases}$$

The fundamental theorem of calculus implies immediately the following homotopy formula

$$\forall \hat{\omega} \in \hat{\mathcal{A}} : \;\; \Psi_1\hat{\omega} - \Psi_0\hat{\omega} = d\int_I \hat{\omega} + \int_I \hat{d}\hat{\omega}$$

where $\hat{d}$ is the exterior derivative in $\hat{\mathcal{A}}$ defined by

$$\hat{d} = dt\frac{\partial}{\partial t}\otimes\mathbf{1} + \epsilon\otimes d.$$

d is the exterior derivative in $\mathcal{A}$ while ϵ is the s-grading operator in $\Omega^*(\mathbb{R})$. From the equalities $\mathfrak{cw}_i = \Psi_i \circ \hat{\mathfrak{cw}}$ we deduce *Weil's transgression formula*

$$\mathfrak{cw}_1 - \mathfrak{cw}_0 = d\int_I c\hat{w} + \int_I \hat{d}\hat{\mathfrak{cw}} = d\int_I \hat{\mathfrak{cw}} + \int_I \hat{\mathfrak{cw}} d_W. \tag{4.8}$$

Thus the map

$$K = K(\tilde{\theta}_1, \tilde{\theta}_0) = \int_I \hat{\mathfrak{cw}} : W_G^* \to \mathcal{A}^{*-1}$$

is a cochain homotopy connecting $\mathfrak{cw}_0$ to $\mathfrak{cw}_1$. □

Remark 4.5 (a) It is instructive to compute $K(\theta^i)$ and $K(\Omega^j)$. We have

$$K(\theta^i) = \int_I \hat{\theta}^i = 0.$$

To compute $K(\Omega^j)$ we need to compute the curvature $\hat{\Omega}$ of $\hat{\theta}$. Set $\dot{\tilde{\theta}} = \tilde{\theta}_1 - \tilde{\theta}_0$. We have

$$\hat{\Omega} = \hat{d}\hat{\theta} + \frac{1}{2}[\hat{\theta}, \hat{\theta}]$$

$$= dt \otimes \dot{\tilde{\theta}} + d\tilde{\theta}_0 + \frac{1}{2}[\tilde{\theta}_0, \tilde{\theta}_0] + td\dot{\tilde{\theta}} + t[\tilde{\theta}_0, \dot{\tilde{\theta}}] + \frac{t^2}{2}[\dot{\tilde{\theta}}, \dot{\tilde{\theta}}]$$

$$= dt \otimes \dot{\tilde{\theta}} + \tilde{\Omega}_0 + t\left(d\tilde{\theta}_1 + [\tilde{\theta}_1, \tilde{\theta}_0] - \tilde{\Omega}_0 - \frac{1}{2}[\tilde{\theta}_0, \tilde{\theta}_0]\right) + \frac{t^2}{2}[\dot{\tilde{\theta}}, \dot{\tilde{\theta}}]$$

$$= dt \otimes \dot{\tilde{\theta}} + (1-t)\tilde{\Omega}_0 + t\left(d\tilde{\theta}_1 + [\tilde{\theta}_1, \tilde{\theta}_0] - \frac{1}{2}[\tilde{\theta}_0, \tilde{\theta}_0]\right) + \frac{t^2}{2}[\dot{\tilde{\theta}}, \dot{\tilde{\theta}}].$$

where $\tilde{\Omega}_0$ is the curvature of $\tilde{\theta}_0$. Set

$$X(t) = X(t, \tilde{\theta}_1, \tilde{\theta}_0) = (1-t)\tilde{\Omega}_0 + t\left(d\tilde{\theta}_1 + [\tilde{\theta}_1, \tilde{\theta}_0] - \frac{1}{2}[\tilde{\theta}_0, \tilde{\theta}_0]\right) + \frac{t^2}{2}[\dot{\tilde{\theta}}, \dot{\tilde{\theta}}].$$

Thus

$$K(\Omega) = \int_I \left(X(t) + dt \otimes \dot{\tilde{\theta}}\right) = \tilde{\theta}_1 - \tilde{\theta}_0.$$

More generally if $P \in S\mathfrak{g}^*$ and $Q \in \Lambda\mathfrak{g}^*$ we have

$$K(P \otimes Q) = \int_I P(\hat{\Omega}) \otimes Q(\hat{\theta})$$

$$= \int_I \left\{P(X(t) + dt \otimes \dot{\tilde{\theta}}) \otimes Q(\hat{\theta})\right\}.$$

Using the Taylor expansion of P at $X(t)$ we get

$$= \int_0^1 \left(\sum_i (\tilde{\theta}_1^i - \tilde{\theta}_0^i)\frac{\partial P}{\partial \Omega^i}(X(t)) \otimes Q(\hat{\theta})\right) dt. \tag{4.9}$$

In particular, this shows that (i) K commutes with the G action and (ii) $K(W_{bas}) \subset \mathcal{A}_{bas}$. We call such a homotopy a *basic homotopy*. We will use the symbol "$\simeq_b$" to denote the basic homotopy equivalence relation. In particular, we say that two G-operations $\mathcal{A}, \mathcal{B}$ are b-homotopic if there exist morphisms $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{A}$ such that $f \circ g \simeq_b \mathbf{id}$ and $g \circ f \simeq_b \mathbf{id}$. We write this as $\mathcal{A} \simeq_b \mathcal{B}$.

(b) The above proposition shows that we could define the notion of connection as a morphism of G-operations $W_G \to \mathcal{A}$. We see that W_G is extremely rigid since for any G-operation the collection $[W_G, \mathcal{A}]_b$ of classes of morphisms of G-operations modulo basic homotopies is very small. It consists of at most one element. A G-operation $\mathcal{W}$ equipped with a G-connection satisfying the above rigidity condition (i.e. $[\mathcal{W}, \mathcal{A}]_b$ consists of at most one element for any G-operation $\mathcal{A}$) will be called an *universal G-operation*. Note also the similarity between this result and the topological one: two continuous maps $f_1, f_2 : B \to BG$ which induce isomorphic principal G-bundles are homotopic.

(c) The proof of the above proposition continues to hold in the following more general form: any two equivariant morphisms $\phi_i : W_G \to \mathcal{A}$ of graded differential algebras are homotopic as cochain maps. In particular, this shows that (W_G, d_W) is acyclic i.e. $H^k(W_G) = 0$ for $k > 0$.

5 The basic cohomology of a G-operation with connection

As we explained in the previous subsection, the G-operations with connections represent the algebraic counterpart of a smooth manifold M on which G acts freely. In such a case, the (Borel) equivariant cohomology of M is naturally isomorphic with the ordinary cohomology of the quotient

$$H_G^*(M) \cong H^*(M/G).$$

In the subsection we will establish the algebraic counterpart of this result. In fact, we will deal with a more general situation.

Assume we are given the following collection of data.

• A Lie group G and a closed normal subgroup $N \subset G$. Set $Q = G/N$. Since N is invariant under the adjoint of G there is an induced action on $\mathfrak{n}$-the Lie algebra of N and in particular, $\mathfrak{n}$ is a Lie algebra ideal of $\mathfrak{g}$.

• A G-operation $(\mathcal{A}, D, \mathcal{L}, \mathcal{I})$ equipped with a G-invariant N-*connection* i.e. a G-equivariant morphism of N-*operations*

$$\tilde{\theta} : W_N \to \mathcal{A}.$$

By regarding $\mathcal{A}$ as an N-operation we can form the subalgebra

$$\mathcal{B} = \mathcal{A}_{N,bas} = \{\omega \in \mathcal{A}\ ;\ \omega \text{ is } N \text{ invariant}, \ \mathcal{I}_X\omega = 0,\ \forall X \in \mathfrak{n}\}.$$

The G-operation structure on $\mathcal{A}$ induces a residual G/N-operation structure on $\mathcal{B}$. Note that we have an inclusion

$$j : W_Q \otimes \mathcal{B} \to W_G \otimes \mathcal{A}$$

such that

$$j(W_Q \otimes \mathcal{B})_{Q,bas} \subset (W_G \otimes \mathcal{A})_{G,bas}.$$

The geometric intuition behind this algebraic situation is that of a smooth G-space E such that the action of N is free. In Borel cohomology we have an isomorphism

$$H_G^*(E) \cong H_Q^*(E/N).$$

We will establish the algebraic analogue of this result.

Theorem 5.1 (Cartan) The inclusion j induces an isomorphism

$$WH_Q^*(\mathcal{B}) \cong WH_G^*(\mathcal{A}).$$

Proof Our proof will be a simple application of Weil's transgression trick. For different approaches we refer to [5], Chap. VIII, [4], [3] or [8].

We will construct a G-connection on $W_G \otimes \mathcal{A}$ starting from the G-equivariant N-connection $\tilde{\theta} \in \mathcal{A}^1 \otimes \mathfrak{n}$.

Define the linear map

$$\mu : \mathfrak{g} \to \mathcal{A}^0 \otimes \mathfrak{n},\ \ \mu(X) = -\mathcal{I}_X\tilde{\theta}.$$

μ is called the *moment map* of the G-equivariant connection $\tilde{\theta}$. We can also regard it as an element of $\mathfrak{g}^* \otimes \mathcal{A}^0 \otimes \mathfrak{n}$.

Lemma 5.2 The moment map $\mu : \mathfrak{g} \to \mathcal{A}^0 \otimes \mathfrak{n}$ is G-equivariant.

Proof Regard $\tilde{\theta}$ as a G-equivariant map

$$\mathfrak{g}^* \to \mathfrak{n}^* \to \mathcal{A}^1.$$

For each $X \in \mathfrak{g}$ regard $\mu(X)$ as a map

$$\mathfrak{g}^* \to \mathfrak{n}^* \to \mathcal{A}^0.$$

The equivariance of μ is equivalent to

$$\mu(Ad_g X) = g\mu(X)Ad^*_{g^{-1}}$$

where Ad^* denotes the coadjoint action of G. We have

$$\mu(Ad_g X) = -\mathcal{I}_{Ad_g X}\tilde{\theta} = -g\mathcal{I}_X g^{-1}\tilde{\theta} = -g\mathcal{I}_X\tilde{\theta}Ad^*_{g^{-1}} = g\mu(X)Ad^*_{g^{-1}}.$$

(The second equality is the G-equivariance of $\mathcal{I}$ while the third equality is the G-equivariance of $\tilde{\theta}$.) □

Define $\mathbf{q} : \mathfrak{g} \to \mathcal{A}^0 \otimes \mathfrak{n}$ as

$$\mathbf{q}(X) = X + \mu(X).$$

Note that $\mathbf{q}(X) = 0$ for $X \in \mathfrak{n}$ so that $\mathbf{q}$ descends to a map

$$\mathbf{q} = \mathfrak{q} = \mathfrak{g}/\mathfrak{n} \to \mathcal{A}^0 \otimes \mathfrak{n}.$$

Set $\Xi = \mathbf{q} + \tilde{\theta}$. Note that

$$\mathbf{q} \in \mathfrak{g}^* \otimes \mathcal{A}^0 \otimes \mathfrak{n} \subset W^1_G \otimes \mathcal{A}^0 \otimes \mathfrak{g}$$

and

$$\tilde{\theta} \in \mathcal{A}^1 \otimes \mathfrak{n} \subset \mathcal{A}^1 \otimes \mathfrak{g}.$$

Thus $\Xi \in (W_G \otimes \mathcal{A})^1$.

Lemma 5.3 Ξ defines a G-connection on $W_G \otimes \mathcal{A}$

Proof For $X \in \mathfrak{g}$ denote by I_X the contraction by X in $W_G \otimes \mathcal{A}$. Then

$$I_X\Xi = \mathbf{q}(X) + \mathcal{I}_X\tilde{\theta} = X.$$

The G-invariance of Ξ now follows from the G-invariance of $\mathbf{q}$ and $\tilde{\theta}$. □

The G-operation $W_G \otimes \mathcal{A}$ admits the tautological connection

$$\mathbf{1} : W_G \mapsto W_G \otimes \mathcal{A}, \;\; w \mapsto w \otimes 1.$$

Denote by $K = K(\mathbf{1}, \Xi)$ the Weil transgression

$$K : W_G \to W_G \otimes \mathcal{A}$$

so that for all $w \in W_G$

$$w - \Xi w = \delta K w + K d_W w$$

where δ denotes the exterior derivation in $W_G \otimes \mathcal{A}$. Now define

$$T_0 : W_G \otimes \mathcal{A} \to W_G \otimes \mathcal{A}, \;\; w \otimes a = (\Xi w) \cdot a \tag{5.1}$$

$$T_1 = \mathbf{id} : W_G \otimes \mathcal{A} \to W_G \otimes \mathcal{A}$$

and

$$\mathcal{K} : W_G \otimes \mathcal{A} \to W_G \otimes \mathcal{A}, \;\; w \otimes a \mapsto Kw \cdot a.$$

Then for all $x \in W_G \otimes \mathcal{A}$

$$x - T_0 x = T_1 x - T_0 x = \delta \mathcal{K} x + \mathcal{K} \delta x.$$

Both T_0 and T_1 are morphisms of G-operations and $\mathcal{K}$ is a *basic homotopy.* Also note that

$$T_0(W_G \otimes \mathcal{A}) \subset W_Q \otimes \mathcal{A}$$

and

$$T_0(W_G \otimes \mathcal{A})_{G,bas} \subset (W_Q \otimes \mathcal{B})_{Q,bas}.$$

Moreover, along the basic subalgebras $T_0 \circ j = \mathbf{id}$. In (the basic) cohomology T_0 is bijective since it is homotopic to the identity. This completes the proof of Theorem 5.1. □

Remark 5.4 The reduction theorem we have just proved generalizes as follows. Consider a G-operation $\mathcal{W}$. Then the transgression trick in the proof of Theorem 5.1 can be used to show the statements below are equivalent.
(i) $\mathcal{W}$ is universal (in the sense defined in Remark 4.5 (b)).
(ii) Any morphism of G-operations $\varphi : \mathcal{W} \to \mathcal{A}$ induces a b-homotopy equivalence

$$\mathcal{W} \otimes \mathcal{A} \simeq_b \mathcal{A}.$$

Note in particular that if $\mathcal{W}_0$, $\mathcal{W}_1$ are two universal G-operations and $\alpha : \mathcal{W}_0 \to \mathcal{W}_1$ is a morphism then any morphism $\tau : \mathcal{W}_1 \to \mathcal{W}_0 \otimes \mathcal{B}$ induces a b-homotopy equivalence

$$\mathcal{W}_0 \otimes \mathcal{B} \simeq_b \mathcal{W}_1 \otimes \mathcal{B}. \tag{5.2}$$

Indeed, by (ii) τ induces a b-homotopy equivalence

$$\mathcal{W}_0 \otimes \mathcal{W}_1 \otimes \mathcal{B} \simeq_b \mathcal{W}_1 \otimes (\mathcal{W}_0 \otimes \mathcal{B}) \simeq_b \mathcal{W}_0 \otimes \mathcal{B}$$

(since $\mathcal{W}_1$ is universal) and on the other hand $\alpha : \mathcal{W}_0 \to \mathcal{W}_1 \otimes \mathcal{B}$ induces the b-homotopy equivalence

$$\mathcal{W}_0 \otimes \mathcal{W}_1 \otimes \mathcal{B} \simeq_b \mathcal{W}_1 \otimes \mathcal{B}$$

since $\mathcal{W}_0$ is universal. If we take $\mathcal{W}_0 = W_{G/N}$ and $\mathcal{W}_1 = W_G$ (note that these Weil algebras are clearly universal G-operations) then the equivalence (5.2) is precisely the content of Theorem 5.1.

Corollary 5.5 Let E be a smooth G-space. If N acts freely on E then

$$WH_G^*(E) \cong H_Q^*(E/N).$$

It is instructive to describe the reduction isomorphism

$$WH_G^*(E) \stackrel{\cong}{\to} WH_Q^*(E/N)$$

using the Cartan model. Denote by $W_{\tilde{\theta}}$ the Weil model description of the above reduction isomorphism (defined in (5.1)). We denote by $C_{\tilde{\theta}}$ its correspondent in the Cartan model. Denote by ϕ_G (resp. ϕ_Q) the Kalkman isomorphism (cf. (3.5))

$$WH_G^* \to CH_G^* \quad (\text{resp.} WH_Q^* \to CH_Q^*).$$

We then have

$$C_{\tilde{\theta}} = \phi_Q \circ W_{\tilde{\theta}} \circ \phi_G^{-1}.$$

To get a better feeling on the structure of $C_{\tilde{\theta}}$ we will work in local coordinates. Choose a basis (e_i) of $\mathfrak{n}$ and then extend it to a basis $\{e_i\,;\,f_a\}$ of $\mathfrak{g}$. Via the natural projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{n}$ the collection (f_a) induces a basis of $\mathfrak{g}/\mathfrak{n}$ which we continue to denote by the same symbols. Denote the dual basis of $\{e_i\,;\,f_a\}$ by $\{\theta^i\,;\,\varphi^a\}$. We can regard (θ^i) as a basis of $\mathfrak{n}^*$ and (φ^a) as a basis of $\mathfrak{q}^*$. We denote the image of θ^i in $\mathfrak{n}^*$ by Ω^i and the image of φ^a in $S\mathfrak{q}^*$ by Ψ^a. Set $\theta = \theta^i \otimes e_i$, $\varphi = \varphi^a \otimes f_a$, $\Omega = \Omega^i \otimes e_i$ and $\Psi = \Psi^a \otimes f_a$.Then any element in $(S\mathfrak{g}^* \otimes \Omega^*(E))^G$ is a G-equivariant polynomial map $P : \mathfrak{g} \to \Omega^*(E)$ which we will schematically describe it as $P = P(\Omega \oplus \Psi)$. Then

$$\phi_G^{-1} P(\Omega \oplus \Psi) = \exp(-\varphi^a \otimes \mathcal{I}_a) \exp(-\theta^i \otimes \mathcal{I}_i) P(\Omega \oplus \Psi).$$

The map $\Xi : W_G \to W_G \otimes \Omega^*(E)$ is determined from the assignments

$$\theta^i \mapsto \tilde{\theta}^i - \tilde{\theta}^i(f_a)\varphi^a, \quad \varphi^a \mapsto \varphi^a.$$

If we define $\Xi(e_i) = e_i$ and $\Xi(f_a) = f_a$ then we can rewrite

$$\Xi(\theta \oplus \varphi) = \tilde{\theta} + \varphi - \tilde{\theta}(f_a)\varphi^a = \tilde{\theta} + \varphi - \tilde{\theta} \circ \varphi. \tag{5.3}$$

Moreover

$$\Xi(\Omega \oplus \Psi) = \delta \Xi(\theta \oplus \varphi) + \frac{1}{2}\left[\Xi(\theta \oplus \varphi), \Xi(\theta \oplus \varphi)\right]$$

$$= d\tilde{\theta} + d_W \varphi + \frac{1}{2}\left\{[\tilde{\theta} + \varphi, \tilde{\theta} + \varphi] + [\tilde{\theta} \circ \varphi, \tilde{\theta} \circ \varphi]\right\} - [\tilde{\theta} + \varphi, \tilde{\theta} \circ \varphi] - \delta(\tilde{\theta} \circ \varphi)$$

$$= d\tilde{\theta} + \frac{1}{2}[\tilde{\theta}, \tilde{\theta}] + d_W \varphi + \frac{1}{2}[\varphi, \varphi] + [\tilde{\theta}, \varphi] + \frac{1}{2}[\tilde{\theta} \circ \varphi, \tilde{\theta} \circ \varphi] - [\tilde{\theta} + \varphi, \tilde{\theta} \circ \varphi] - \delta(\tilde{\theta} \circ \varphi)$$

$$= \tilde{\Omega} + \Psi + [\tilde{\theta}, \varphi] + \frac{1}{2}[\tilde{\theta} \circ \varphi, \tilde{\theta} \circ \varphi] - [\tilde{\theta} + \varphi, \tilde{\theta} \circ \varphi] - \delta(\tilde{\theta} \circ \varphi). \tag{5.4}$$

On the other hand

$$W_{\tilde{\theta}} \circ \phi_G^{-1} P(\Omega \oplus \Psi) = \exp(-\varphi^a \otimes \mathcal{I}_a) \exp(-\Xi(\theta^j) \otimes \mathcal{I}_j) P(\Xi(\Omega \oplus \Psi)). \tag{5.5}$$

Since $W_{\tilde\theta}\circ\phi_G^{-1}P$ is a Q-basic element of $W_Q\otimes\Omega^*(E/N)$ the action of ϕ_Q on this element is determined according to Remark 3.3 by setting $\varphi=0$ in (5.5). Using the equalities (5.3) and (5.4) we deduce

$$C_{\tilde\theta}P=\phi_Q\circ W_{\tilde\theta}\circ\phi_G^{-1}P=\exp(-\tilde\theta^j\otimes\mathcal{I}_j)P(\tilde\Omega+\Psi-\delta(\tilde\theta\circ\varphi)\,|_{\varphi=0}\,). \tag{5.6}$$

We need to understand the term $\delta(\tilde\theta\circ\varphi)\,|_{\varphi=0}$. We have

$$\begin{aligned}\delta(\tilde\theta^i(f_a)\varphi^a)&=d(\tilde\theta^i(f_a))\varphi^a+\tilde\theta^i d_W\varphi^a\\&=d(\tilde\theta^i(f_a))\varphi^a+\tilde\theta^i(f_a)\{\Psi^a-\mathcal{Q}(\varphi)\}\end{aligned}$$

where $\mathcal{Q}$ denotes a quadratic term in the φ's. Thus when setting $\varphi=0$ we get

$$\delta(\tilde\theta^i(f_a)\varphi^a\circ\varphi)\,|_{\varphi=0}=\tilde\theta^i(f_a)\Psi^a.$$

Symbolically

$$\delta(\tilde\theta\circ\varphi)\,|_{\varphi=0}=\tilde\theta(\Psi)=-\mu(\Psi)\in(S^1\mathfrak{q}^*\otimes\Omega^0(E))\otimes\mathfrak{n}.$$

(Recall that μ denotes the moment map of the connection.) Note that the differential form components in $\mu(\Psi)$ are N-basic so they can be regarded as forms on the basis E/N. Substituting this back in (5.6) we get

$$C_{\tilde\theta}P=\exp(-\tilde\theta\otimes\mathcal{I}_j)P(\Psi+\mu(\Psi)+\tilde\Omega).$$

The exponential factor is precisely the horizontal projection $h_{\tilde\theta}:\Omega^*(E)\to\Omega^*(E/N)$ defined by the N-connection $\tilde\theta$. On the other hand the term $\tilde\Omega+\mu(\Psi)\in(\Omega^2(E/N)\oplus(S^1\mathfrak{q}^*\otimes\Omega^0(E/N))\otimes\mathfrak{n}$ is already Q-basic. It is called the *equivariant curvature* of the connection $\tilde\theta$ and will be denoted by $\tilde\Omega_Q$. Note that $\tilde\Omega_Q$ is an element of degree 2 in $\tilde\Omega_Q\in(S\mathfrak{q}^*\otimes\Omega^*(E/N))^Q\otimes\mathfrak{n}$. Thus

$$(C_{\tilde\theta}P)(\Psi)=h_{\tilde\theta}P(\Psi+\tilde\Omega_Q). \tag{5.7}$$

We still need to give an accurate definition of the right-hand-side term above. For any $X\in\mathfrak{g}$ define $P(X+\tilde\Omega_Q)$ imitating the Taylor expansion at X

$$P(X+\tilde\Omega_Q)=\exp(\tilde\Omega_Q^i\partial_i)P(X)$$

where $\tilde\Omega_Q=\tilde\Omega_Q^i\otimes e_i\in(S\mathfrak{q}^*\otimes\Omega^*(E))^2\otimes\mathfrak{n}$ while ∂_i denotes the partial derivative along the direction $e_i\in\mathfrak{n}\subset\mathfrak{g}$. Note that $P(X+\tilde\Omega_Q)=P(X+\mu(X)+\tilde\Omega)=P(\tilde\Omega)$ for all $X\in\mathfrak{n}$ so that $P(X+\tilde\Omega_Q)$ descends to a well defined map $\mathfrak{q}\to\Omega^*(E/N)$. Thus the polynomial in the right-hand-side of (5.7) should be rather viewed as an $\Omega^*(E/N)$-valued polynomial on $\mathfrak{g}$ which descends to a polynomial on $\mathfrak{q}=\mathfrak{g}/\mathfrak{n}$.

In particular, to any G-invariant polynomial $P\in S\mathfrak{g}^*$ one can associate an equivariantly closed element

$$\Psi\mapsto P(\Psi+\tilde\Omega_Q)\in(S\mathfrak{q}^*\otimes\Omega^*(E/N))^Q.$$

This element clearly depends on the connection but its equivariant cohomology class does not. It will be denoted by $P(E)\in CH_Q^*((E/N)$ and will be called the *equivariant characteristic class* of $E\to E/N$ corresponding to P. Note that when $G=N$ this correspondence is none other than the traditional Chern-Weil construction of the characteristic classes of a principal G-bundle.

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