# “Motivic” Integral Geometry

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**Abstract**

I am trying to understand a beautiful work of Pierre Schapira [5].

### Notations and Conventions

• For any set $S$ we denote by $\mathbb{I}_S$ the identity map $S \rightarrow S$. For any subset $A \subset S$ we denote by $\mathbb{1}_A$ the characteristic function of $A$.
• In the sequel all manifolds will be assumed *real analytic*. A morphism of manifolds will be a real analytic map between two manifolds.

### Introduction

Suppose we are given a simple closed smooth curve $C \hookrightarrow \mathbb{R}^2$. Suppose that for every affine line $L \subset \mathbb{R}^2$ we know the number $n_C(L)$ of intersection points of $C$ with $L$. How much information about $C$ can we extract from this information?

Intuitively, this knowledge ought to differentiate between different curves. Clearly, we would like to be more specific than this, and better yet, we would like to put numbers behind such statements. We will sketch a classical approach to this problem following the beautiful presentation in [3, I.§2]. We begin by giving a more useful description of the counting function $n_C(L)$.

Denote by $\tilde{G}$ the set of affine lines in the plane. A line $L$ is determined by its *normal coordinates* $(\theta, c)$. More precisely, the line $L(\theta, c)$ is given by the equation

$$x \cos\theta + y \sin\theta = c.$$

We deduce

$$\tilde{G} \cong \Big\{ (\theta, c) \in \mathbb{R}^2 \Big\} / (\theta, c) \sim (\theta + \pi, -c).$$

*Notes for myself and whoever else is reading this footnote.

We have a natural projection $\tilde{G} \to \mathbb{RP}^1$, $(\theta, c) \mapsto \theta$ so we can identify $\tilde{G}$ with the total space of the tautological line bundle over $\mathbb{RP}^1$, i.e. a Möebius band. Note that $\tilde{G}$ is equipped with a measure

$$dL = |d\theta dc|$$

classically called the *kinematic measure*. The group of affine motions acts transitively on the space of affine lines and the kinematic measure is invariant under this group. Up to a positive multiplicative constant this is the unique invariant[1] Borel measure on $\tilde{G}$.

We have an incidence relation

$$I = \Big\{ (p, L) \in \mathbb{R}^2 \times \tilde{G};\ \ p \in L \Big\}.$$

$I$ is equipped with two natural maps

$$\begin{array}{ccc} & I & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathbb{R}^2 & & \tilde{G} \end{array}$$

The fiber of $\pi_1$ over $p$ can be identified with the pencil of lines passing through $p$ while the fiber of $\pi_2$ over $L$ can be identified with the set of all points in $L$.

Consider the pullback to $I$ of the characteristic function $\mathbb{1}_C$. The support of $\pi_1^*\mathbb{1}_C$ is the set

$$\Big\{ (p, L);\ \ p \in L \cap C \Big\}.$$

The set of lines intersecting $C$ in infinitely many points has kinematic measure zero and we will neglect these lines. This shows that for almost all lines $L$ the intersection of the fiber $\pi_2^{-1}(L)$ with the support of $\pi_1^*\mathbb{1}_C$ is finite and can be identified with the set $L \cap C$. The integral of $\pi_1^*\mathbb{1}_C$ along the fibers of $\pi_2$, denoted by $(\pi_2)_*\pi_1^*\mathbb{1}_C$ is then the cardinality $n_C(L) := \#L \cap C$. We have thus shown that

$$n_C(-) = (\pi_2)_*\pi_1^*\mathbb{1}_C.$$

The classical *Crofton fomula* states that

$$\int_{\tilde{G}} n_C(L) dL = 2 \text{ length } (C).$$

For a proof we refer to [3, I§2].

This formula shows that “averaging” the intersection count information with respect to a geometrically significant (i.e. invariant) measure on $\tilde{G}$ we can obtain nontrivial geometric information about our curve.

In this report we will describe a new kind of geometrically significant measure on the grassmannian $\tilde{G}$. It is a sort of *motivic measure*. This requires additional regularity on $C$ (subanalyticity) but a clever averaging of the count function $n_C(L)$ with respect to this measure will yield *complete* geometric information on $C$: its “shape” and “location”.

[1]The invariant measures on this affine grassmanian are classified in [2].

## Contents



## 1 Constructible functions

Suppose $X$ is a real analytic manifold. A subset $S \subset X$ is called *subanalytic* at $x \in X$ if there exists an open neighborhood $U$ of $x \in X$, compact manifolds $Y_i, Z_i$, $1 \leq i \leq N$ and morphisms

$$f_i : Y_i \to X, \;\; g_i : Z_i \to X$$

such that

$$S \cap U = U \cap \bigcup_{i=1}^{n} f_i(Y_i) \setminus g_i(Z_i).$$

If $S$ is analytic at each point $x \in X$, one says that $Z$ is subanalytic in $X$.

The subanalytic sets behave nicely with respect to the set theoretic operations. We list below some of the most useful properties.

• If $S \subset X$ is subanalytic then so is its closure, its interior, its complement and any of its connected components. Moreover the collection of connected components is locally finite[2].

• The union and the intersection of two subanalytic sets is subanalytic.

• Suppose $f : X \to Y$ is a morphism. If $S \subset Y$ is subanalytic then $f^{-1}(S)$ is subanalytic. If $f$ is proper and $T \subset X$ is subanalytic then so is its image $f(T) \subset Y$.

• Every close subanalytic subset $S \subset Y$ is the image of a manifold $X$ via a proper morphism $f : X \to Y$.

• (*Triangulation theorem*) If $X = \bigsqcup_{\alpha \in A} X_\alpha$ is a locally finite partition of $X$ by subanalytic subsets then there exists a simplicial complex $\mathbf{S}$ and a homeomorphism $\mathfrak{t} : |\mathbf{S}| \to X$ with the following properties.

(i) For every simplex $\sigma$ of $\mathbf{S}$ the image $\mathfrak{t}(\mathrm{int}\,|\sigma|)$ is a subanalytic submanifold of $X$.

(ii) The image of the *interior* of any simplex $|\sigma|$ via $\mathfrak{t}$ is entirely contained in a single stratum $X_\alpha$.

The pair $(\mathbf{S}, \mathfrak{t})$ is called a *subanalytic triangulation* subordinated to the subanalytic partition $\bigsqcup_{\alpha \in A} X_\alpha$. In the sequel we will omit $\mathfrak{t}$ from notations.

Suppose $X$ is a manifold. A function $f : X \to \mathbb{Z}$ is called *constructible* if it satisfies the following two conditions.

[2]A family of subsets of a topological space is called locally finite if every point of the space has a neighborhood which intersects only finitely many sets of the family.

- For every $m \in \mathbb{Z}$ the level set $f^{-1}(m)$ is subanalytic.
- The family $\left\{ f^{-1}(m);\ \ m \in \mathbb{Z} \right\}$ is locally finite.

The properties of subanalytic sets show that the sum of two constructible functions is a constructible function. For any open set $U \subset X$ we denote by $\mathcal{CF}_X(U)$ the Abelian group of constructible functions

$$f : X \to \mathbb{Z}, \quad \operatorname{supp} f \subset U.$$

We set $\mathcal{CF}(X) := \mathcal{CF}_X(X)$. The correspondence

$$U \to \mathcal{CF}_X(U), \;\; U \stackrel{open}{\hookrightarrow} X$$

defines a *soft sheaf* on $X$ which we will denote by $\mathcal{CF}_X$.

Using the triangulation theorem we deduce that for any constructible function $f : X \to \mathbb{Z}$ we can find a subanalytic triangulation $(\mathbf{S}, \mathfrak{t})$ subordinated to the level sets of $f$ and for every simplex $\sigma \in \mathbf{S}$ an integer $m_f(\sigma)$ such that

$$f = \sum_{\sigma} m_f(\sigma) \mathbb{1}_{\mathfrak{t}(|\sigma|)}.$$

More precisely we can define the integers $m_f(\sigma)$ by descending induction over dimension via the formulæ

$$m_f(\sigma) = f(\sigma) - \sum_{\tau \succ \sigma} m_f(\tau), \tag{1.1}$$

where $\sigma \prec \tau$ signifies that $\sigma$ is a face of $\tau$.

In Figure 1 we have indicated by numbers attached to the various simplices the multiplicities $m_f$, when $f$ is the characteristic function of a square, and the triangulation is as shown.

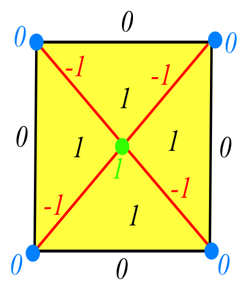



Figure 1: *A triangulation of a square.*

For any open set $U \hookrightarrow X$ we denote by $\mathcal{CF}^c_X(U)$ the subgroup of $\mathcal{CF}_X(U)$ consisting of constructible functions $F : X \to \mathbb{Z}$ with compact support contained in $U$.

**Proposition 1.1.** *There exists a group morphism*

$$L : \mathcal{CF}^c(X) \to \mathbb{Z}$$

*uniquely determined by the requirement*

$$L(\mathbb{1}_\sigma) = 1,$$

*for any regular simplex $\sigma \hookrightarrow X$. We will denote this morphism by $\int_X$.*

**Proof** Any constructible compactly supported function $f$ can be written as a $\mathbb{Z}$-linear combination

$$f = \sum_{\sigma} m_f(\sigma) \mathbb{1}_\sigma,$$

where the summation is carried over a finite family of regular simplices belonging to a triangulation subordinated to the partition of $X$ given by the level sets of $f$.

$$\int_X f := \sum_{\sigma} m_f(\sigma)$$

We need to show that this integer is independent of the choice of triangulation and it suffices to prove this in the special case when $f$ is the characteristic function of a regular simplex. This follows from the following more general result.

**Lemma 1.2.** *Suppose $K \subset X$ is a compact subanalytic set and $\mathbf{S}$ is a subanalytic triangulation of $X$ subordinated to the partition $K, X \setminus K$. If*

$$\mathbb{1}_K = \sum_{\sigma \in \mathbf{S}} m_K(\sigma) \mathbb{1}_\sigma$$

*then*

$$\sum_{\sigma} m_K(\sigma) = \chi(K) = \text{the Euler characteristic of } K.$$

**Proof of the Lemma** Define

$$S_\sigma := \{\tau \in \mathbf{S};\ \ \tau \succeq \sigma\} = \text{all simplices } \tau \text{ which have } \sigma \text{ as a face.}$$

We interpret $m(\sigma)$ as a function $m_K : \mathbf{S} \to \mathbb{Z}$ and thus if we use (1.1) with $f = \mathbb{1}_K$ we deduce

$$\int_{S_\sigma} m_K = 1, \ \ \forall \sigma. \tag{1.2}$$

Denote by $V$ the set of vertices (0-simplices) of our triangulation. Observe that

$$\mathbf{S} = \bigcup_{v \in V} S_v$$

Using the inclusion exclusion principle we deduce that

$$\int_{\mathbf{S}} m_K = \sum_{k \geq 0} (-1)^k \sum_{v_0, \cdots ,, v_k \in V} \int_{S_{v_0} \cap \cdots \cap S_{v_k}} m_K.$$

Now observe that $S_{v_0} \cap \cdots \cap S_{v_k}$ is non-empty if and only if $v_0, \cdots , v_k$ span a simplex $[v_0, \cdots , v_k] \in \mathbf{S}$ in which case we have

$$S_{v_0} \cap \cdots \cap S_{v_k} = S_{[v_0, \cdots , v_k]}.$$

Hence we deduce

$$\int_{\mathbf{S}} m_K = \sum_{\sigma \in \mathbf{S}} (-1)^{\dim \sigma} \int_{S_\sigma} m_K \overset{(1.2)}{=} \sum_{\sigma \in \mathbf{S}} (-1)^{\dim \sigma} = \chi(K).$$

□

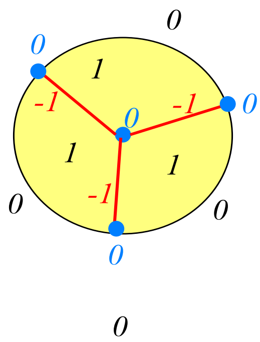



Figure 2: *A triangulation of the disk*

**Example 1.3.** (a) Denote by $D$ the unit disk

$$D = \{z \in \mathbb{C};\ \ |z| \leq 1\}.$$

and by $D^*$ the punctured disk, $D^* = D \setminus 0$. Then

$$\mathbb{1}_{D^*} = \mathbb{1}_D - \mathbb{1}_{\{0\}}$$

so that

$$\int \mathbb{1}_{D^*} = \int \mathbb{1}_D - \int \mathbb{1}_{\{0\}} = \chi(D) - \chi(\{0\}) = 0 = \chi(D^*)$$

We can confirm this using the definition of the integral. Figure 2 contains a decomposition of $\mathbb{1}_{D^*}$ as a linear combination of characteristic functions of simplices. We deduce that

$$\int \mathbb{1}_{D^*} = \sum \text{weights of vertices, edges and faces in Figure 2}$$

$$= 0 - 3 + 3 = 0$$

(b) Suppose $S$ is the surface of the tetrahedron $[P_0P_1P_2P_3]$. Then

$$\mathbb{1}_S = \sum_{i<j<k} \mathbb{1}_{[P_iP_jP_k]} - \sum_{i<j} \mathbb{1}_{[P_iP_j]} + \sum_i \mathbb{1}_{[P_i]}$$

so that

$$\int \mathbb{1}_S = \text{number of faces} - \text{number of edges} + \text{number of vertices} = \chi(S) = 2.$$

□

**Remark 1.4.** (a) The construction of the integral reflects the fact that the Euler characteristic of compact subanalytic sets satisfies the inclusion exclusion principle

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cup B).$$

This happens if the pair $(A, B)$ is *excisive*, see [6, Chap.4, Sec.6]. The special local structure of subanalytic sets implies that *any pair* of subanalytic sets is excisive.

(b) One can show that if $S$ is a precompact subanalytic set then

$$\int \mathbb{1}_S = \chi_c(S, \underline{\mathbb{R}}),$$

where $\chi_c(S, \underline{\mathbb{R}})$ denotes the Euler characteristic of the compactly supported Čech cohomology with coefficients in the constant sheaf $\underline{\mathbb{R}}$.

□

Given a morphism of manifolds $f : X \to Y$ we have a pullback map

$$f^* : \mathcal{CF}(Y) \to \mathcal{CF}(X), \;\; \mathcal{CF}(Y) \ni \varphi \mapsto f^*(\varphi) := \varphi \circ f : X \to \mathbb{Z}.$$

This induces a morphism of sheaves

$$f^\sharp : f^{-1}\,\mathcal{CF}_Y \to \mathcal{CF}_X\,.$$

**Remark 1.5.** Using the adjunction isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(f^{-1}\,\mathcal{CF}_Y, \mathcal{CF}_X) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathcal{CF}_Y, f_*\,\mathcal{CF}_X)$$

we can regard $f^\sharp$ as a morphism $\mathcal{CF}_Y \to f_*\,\mathcal{CF}_X$. Thus, if we regard the pair $(X, \mathcal{CF}_X)$ as a ringed space, we deduce that a morphism of manifolds $f : X \to Y$ induces a morphism of ringed spaces

$$(f, f^\sharp) : (X, \mathcal{CF}_X) \to (Y, \mathcal{CF}_Y).$$

□

Given a morphism $f : X \to Y$ and a constructible function $\phi : X \to \mathbb{Z}$ such that the restriction of $\phi$ to every fiber $\phi^{-1}(y)$ has compact support we can define a new function on $Y$

$$\Big(\int_f \phi\Big)(y) := \int_X \mathbb{1}_{f^{-1}(y)}\phi \text{“}='' \int_{f^{-1}(y)} \phi.$$

We have the following *nontrivial result.* For a proof we refer to [1, Chap. IX], [4], or [7] in the complex analytic case.

**Theorem 1.6.** *For every $\phi \in \mathcal{CF}(X)$ such that the restriction of $f$ to $\mathrm{supp}\,\phi$ is proper the function $\int_f \phi : Y \to \mathbb{Z}$ is constructible.*

**Remark 1.7.** Theorem 1.6 is very strong. For example, it implies that, given a compact subanalytic set $K \subset K$ then for any $n \in \mathbb{Z}$ the set

$$\Big\{ y \in Y;\;\; \chi\big(K \cap f^{-1}(y)\big) \leq n \Big\}$$

is subanalytic. Roughly speaking, the Euler characteristic of $K \cap f^{-1}(y)$ ought to depend analytically on $y$, except for a few glitches!

□

**Proposition 1.8 (Projection Formula).** *If $f : X \to Y$ is a proper morphism then for every $\phi \in \mathcal{CF}(Y)$, $\psi \in \mathcal{CF}(Y)$ we have*

$$\int_f f^*\phi \cdot \psi = \phi\Big(\int_f \psi\Big).$$

*In particular*

$$\int_f f^*\phi = \Big(\int_f \mathbb{1}_X\Big)\phi$$

**Example 1.9.** Suppose $X$ and $Y$ are two compact Riemann surfaces and $f : X \to Y$ is a non-constant holomorphic map

$$f : X \to Y$$

Let $C_f \subset X$ be the set of critical points of $f$ and $B_f \subset Y$ be the branching locus of $f$, i.e. the set of critical values. We have a constructible function $\nu = \mu_f : X \to \mathbb{Z}$ naturally associated to $f$. More precisely for every $x \in X$ we define $\mu_f(x)$ to be the *Milnor number* of $f$ at $x$. This means that we can choose local coordinates $z$ near $x$ and $w$ near $y$ such that

$$z(x) = w(f(x)) = 0, \quad f^*dw = z^\mu dz.$$

Equivalently, this means that $\mu_f(x) + 1$ is the local degree of $f$ at $x$. Observe that

$$\mu_f(x) \neq 0 \iff x \in C_f.$$

We have the *"motivic" Riemann-Hurwitz formula*

$$\forall \phi \in \mathcal{CF}(Y), \quad \int_f f^*\phi = \Big(\int_f f^*\mathbb{1}_Y\Big)\phi = (\deg f - \int_f \mu_f)\phi. \tag{1.3}$$

The first equality is the projection formula and it is tautological,

$$\Big(\int_f f^*\phi\Big)(y) = \#f^{-1}(y)\cdot\phi(y) = \Big(\int_f f^*\mathbb{1}_Y\Big)(y)\cdot\phi(y).$$

If we write $f^{-1}(y) = \{x_1, \cdots, x_k\}$ then

$$\deg f = \sum \text{local degrees} = \sum_{i=1}^k (\mu_f(x_i) + 1) = k + \sum_{i=1}^k \mu_f(x_i)$$

so that

$$\#f^{-1}(y) = k = \deg f - \sum_{i=1}^k \mu_f(x_i) = \deg f - \Big(\int_f \mu_f\Big)(y).$$

□

Given two morphisms

$$X_0 \xrightarrow{f_0} S, \quad X_1 \xrightarrow{f_1} S$$

we define

$$X_0 \times_S X_1 := \Big\{(x_0, x_1) \in X_0 \times X_1;\ \ f_0(x_0) = f_1(x_1)\Big\}$$

Observe that

$$X_0 \times_S X_1 = (f_0 \times f_1)^{-1}(\Delta_S),$$

where $\Delta_S$ denotes the diagonal of $S \times S$. $X_0 \times_S X_1$ is equipped with natural maps

$$X_0 \xleftarrow{p_0} X_0 \times_S X_1 \xrightarrow{p_1} X_1.$$

We obtain in this fashion a *Cartesian square*

$$\begin{array}{ccc} X_0 \times_S X_1 & \xrightarrow{p_1} & X_1 \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ X_0 & \xrightarrow[f_0]{} & S \end{array} \qquad (\dagger)$$

If $X_0 \times_S X_1$ is a manifold then we say that the Cartesian square is smooth. We describe below a simple way of constructing smooth Cartesian squares. First, we need to introduce some *microlocal* terminology.

The *normal bundle* of a submanifold $Y$ of a manifold $Z$ is the vector bundle $T_Y Z \to Y$ defined as the quotient

$$0 \to TY \to (TZ)\,|_Y \twoheadrightarrow T_Y Z \to 0.$$

The *conormal bundle* of a submanifold $Y$ of a manifold $Z$ is the dual of the normal bundle. It can be defined as the subbundle $T^*_Y Z$ of $(T^*Z)\,|_Y$ defined by

$$(T^*_Y Z)_y := \Big\{ \xi \in T^*_y Z;\ \ \langle \xi, v\rangle = 0,\ \ \forall v \in T_y Y \Big\}.$$

The fiber $(T^*_Y Z)_y$ is spanned by the differentials of functions vanishing on $X$.

A morphism $f : M \to N$ is called *clean* with respect to a submanifold $Y \subset N$ if

- $f^{-1}(Y)$ is a submanifold $X \subset M$.
- For every $x \in X$ and every $\xi \in (T^*_X M)_x$ there exists a smooth function $\varphi$ defined in a neighborhood of $f(x)$ such that

$$\varphi\,|_Y \equiv 0, \quad d_M(f^*\varphi)_x = \xi.$$

Equivalently this means that for every $x \in X$ the germ of $f$ at $x$ is equivalent to the germ of a linear map, i.e we can find local coordinates $\Phi_x : U_x \to T_x M$ near $x$ and local coordinates $\Psi_y : V_y \to T_y N$ near $y = f(x)$ such that $f(U_x) \subset V_y$ and the diagram below is commutative

$$\begin{array}{ccc} U_x & \xrightarrow{f} & V_y \\ {\scriptstyle \Phi_x}\downarrow & & \downarrow{\scriptstyle \Psi_y} \\ T_x M & \xrightarrow[df_x]{} & T_y Y \end{array}$$

A morphism $f : M \to N$ is called *transversal* to a submanifold $Y \subset N$ if

• $f^{-1}(Y)$ is a submanifold $X \subset M$.
• For every $x \in X$ we have

$$T_{f(x)}N = T_{f(x)}Y + df_x(T_xM).$$

Observe that $f : M \to N$ is clean with respect to $Y$ iff $f^{-1}(Y)$ is a submanifold of $M$ and the restriction of $f$ to $f^{-1}(Y)$ is a submersion onto $Y$, i.e. it is transversal to every $y \in Y$.

A pair of morphisms $X_j \xrightarrow{f_j} S$, $j = 0, 1$ is called *clean* (resp. *transversal*) if the morphism $X_0 \times X_1 \xrightarrow{f_0 \times f_1} S \times S$ is clean with respect to (resp. transversal to) the diagonal $\Delta_S \subset S \times S$. We have the following implications involving Cartesian squares

$$\text{transversal} \Longrightarrow \text{clean} \Longrightarrow \text{smooth}.$$

We will take for granted the following *deep result*, [1, Chap. IX], [4] in the real case, or [7] in the complex analytic case.

**Theorem 1.10.** *(a) The operations $f^*$ and $\int_f$ are functorial in the sense that given a sequence of morphisms*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

*we have*

$$(g \circ f)^* = f^* \circ g^*, \tag{1.4a}$$

$$\int_{g \circ f} \phi = \int_g \int_f \phi, \tag{1.4b}$$

*for every constructible function $\phi : X \to \mathbb{Z}$ such that $g \circ f$ is proper on* $\operatorname{supp} \phi$.

*(b) Given a smooth Cartesian square (†) and $\psi \in \mathcal{CF}(X_0)$ such that $f_0$ restricted to* $\operatorname{supp} \psi$ *is proper we have the* **base change formula**

$$f_1^* \int_{f_0} \psi = \int_{p_0} p_1^* \psi. \tag{1.5}$$

**Example 1.11.** We include here an elementary example which in our view illustrates the strength of (1.4b). Suppose $f : X \to Y$ is a smooth map between two compact Riemann surfaces. Denote by $c_Y : Y \to \{pt\}$ the collapse map. Then

$$c_X = c_Y \circ f$$

and in particular, for every $\phi \in \mathcal{CF}(X)$ we have

$$\int_X \phi = \int_{c_X} \phi = \int_{c_Y} \Big( \int_f \phi \Big) = \int_Y \Big( \int_f \phi \Big)$$

If we take $\phi = \mathbb{1}_X = f^*(\mathbb{1}_Y)$ then we deduce

$$\chi(X) = \int_Y \Big( \int_f \phi \Big) \stackrel{(1.3)}{=} \int_Y \Big( \deg f - \int_f \mu_f \Big)$$

$$= \deg f \int_Y \mathbb{1}_Y - \int_Y \int_f \mu_f \overset{(1.4b)}{=} (\deg f)\chi(Y) - \int_X \mu_f = (\deg f)\chi(Y) - \sum_{x \in C_f} \mu_f(x).$$

We have obtained the *classical Riemann-Hurwitz formula*

$$\chi(X) = (\deg f)\chi(Y) - \sum_{x \in C_f} \mu_f(x).$$

$\square$

## 2 "Motivic" Radon Transforms

A morphism between two manifolds $f : X \to Y$ is uniquely determined by its its graph

$$\Gamma_f \subset X \times Y,\ \ \Gamma_f := \Big\{(x, y \in X \times Y;\ \ y = f(x)\Big\}$$

The graph is equipped with two natural morphisms

$$X \overset{\pi_X}{\longleftarrow} \Gamma_f \overset{\pi_Y}{\longrightarrow} Y.$$

Suppose for simplicity that $f$ is proper. Then given any constructible function $\phi \in \mathcal{CF}(X)$ we set

$$T_{\Gamma_f}\phi := \int_{\pi_Y} \pi_X^* \phi.$$

We have the following identity

$$\int_f \phi = T_{\Gamma_f}\phi. \tag{2.1}$$

To prove this notice that we have a clean Cartesian square

$$\begin{array}{ccc} \Gamma_f & \xrightarrow{\pi_Y} & Y \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \mathbb{I}_Y} \\ X & \xrightarrow[f]{} & Y \end{array}.$$

Using (1.5) we deduce

$$\int_f \phi = \mathbb{I}_Y^* \int_f \phi = \int_{\pi_Y} \pi_X^* \phi$$

which is exactly (2.1). Notice in passing that for every $x \in X$ we have

$$T_{\Gamma_f} \mathbb{1}_{\{x\}} = \mathbb{1}_{\{f(x)\}}$$

so that the transformation

$$T_{\Gamma_f} : \mathcal{CF}(X) \to \mathcal{CF}(Y)$$

completely determines $f$. The above construction works for any locally closed subanalytic set $S \subset X \times Y$ and properly supported function $\phi$. We assume

$$\pi_X \text{ is proper on the closure of } S \text{ in } X \times Y. \tag{$*$}$$

We define

$$T_S : \mathcal{CF}_c(X) \to \mathcal{CF}(Y),\ \ \phi \mapsto \int_{\pi_Y} \pi_X^*\phi = \int_{\pi_Y} \mathbb{1}_S \pi_X^*\phi.$$

We would like to describe a situation when $T_S$ is bijective and hopefully describe a computable way of finding its inverse.

Given a locally closed subanalytic set

$$R \subset Y \times X$$

we form the Cartesian square

$$\begin{array}{ccc} S \times_Y R & \xrightarrow{\ \pi_R\ } & R \\ \downarrow \pi_S & \searrow^{p} \ X \times X & \downarrow \pi_Y \\ S & \xrightarrow[\ \pi_Y\ ]{} & Y \end{array}\ ,\ \ p(s,r) = (\pi_X(s), \pi_X(r)). \tag{2.2}$$

Assume $R$ satisfies the following conditions

$$\text{the Cartesian square (2.2) is smooth.} \tag{2.3a}$$

$$\pi_X \text{ is proper on the closure of } R \text{ in } Y \times X. \tag{2.3b}$$

$$\exists \mu \neq \nu \in \mathbb{Z} :\ \ \chi(p^{-1}(x,x')) = \begin{cases} \mu & \text{if}\quad x \neq x' \\ \nu & \text{if}\quad x = x' \end{cases}. \tag{2.3c}$$

Note that condition (2.3c) is equivalent to

$$\int_p \mathbb{1}_{S\times_Y R} = \nu \mathbb{1}_{\Delta_X} + \mu \mathbb{1}_{X\times X\setminus\Delta_X} = (\nu - \mu)\mathbb{1}_{\Delta_X} + \mu \mathbb{1}_{X\times X}. \tag{2.4}$$

The fiber of $p$ over $(x, x')$ is

$$p^{-1}(x,x') = \Big\{ [(x,y),(y,x')] \in S \times R \Big\}.$$

To get a feeling on the meaning of this space, we need to elaborate significance of the fiber product $S \times_Y R$.

It is convenient to think of $S$ as the graph of a multivalued map $F : X \dashrightarrow Y$,

$$(x,y) \in S \Longleftrightarrow x \xrightarrow{F} y.$$

Similarly ,we can think of $R$ as the graph of a multivalued map $G : Y \dashrightarrow X$. Then the graph of $G \circ F : X \to X$ is the image of $S \times_Y R$ via the map $p$. In other words

$$x \xrightarrow{G\circ F} x' \Longleftrightarrow p^{-1}(x,x') \neq \emptyset.$$

Also we can identify the fiber $p^{-1}(x,x')$ with the set of paths

$$x \xrightarrow{F} y \xrightarrow{G} x'$$

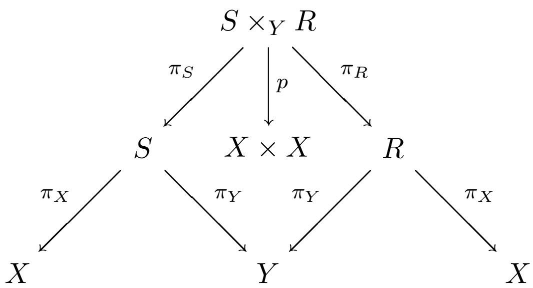
We can loosely reformulate (2.4) as follows.

- The "number" of paths $x \xrightarrow{F} y \xrightarrow{G} x$ is independent of $x$ and it is the integer $\nu$.
- The "number" of paths $x \xrightarrow{F} y \xrightarrow{G} x'$ is independent of $x \neq x'$ and it is the integer $\mu$.

**Theorem 2.1 (The Inversion Formula).** *Suppose $S$ and $R$ satisfy the conditions $(*)$, (2.3a), (2.3b) and (2.3c). Then for every compactly supported $\varphi \in \mathcal{CF}(X)$ we have*

$$T_R \circ T_S(\varphi) = (\nu - \mu)\varphi + \mu\Big(\int_X \varphi\Big)\mathbb{1}_X.$$

**Proof** Consider the following diagram

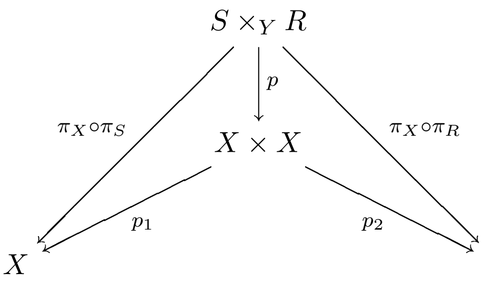



Since the square (2.2) is smooth Cartesian we deduce from (1.5) that

$$T_R \circ T_S(\varphi) = \int_{\pi_X} \circ \underbrace{\pi_Y^* \circ \int_{\pi_Y}} \circ \pi_X^*(\varphi) = \int_{\pi_X} \circ \underbrace{\int_{\pi_R} \circ \pi_S^*} \circ \pi_X^*(\varphi)$$

$$\stackrel{(1.4b)}{=} \int_{\pi_X \circ \pi_R} \circ (\pi_X \circ \pi_S)^*(\varphi)$$

Now look at the commutative diagram




$$\Longleftrightarrow \pi_X \circ \pi_S = p_1 \circ p, \quad \pi_X \circ \pi_R = p_2 \circ p.$$

We deduce

$$T_R \circ T_S(\varphi) = \int_{p_2 \circ p} \circ (p_1 \circ p)^*(\varphi) = \int_{p_2} \circ \int_p p^*(p_1^*(\varphi)).$$

At this point observe that for every $\psi \in \mathcal{CF}(\mathcal{X} \times X)$ we have according to the projection formula,

$$\int_p p^*\psi = \underbrace{\Big(\int_p p^*\mathbb{1}_{X \times X}\Big)}_{:=K(x,x')}\psi.$$

Hence

$$T_R \circ T_S(\varphi)(x') = \int_{p_2} K(x, x')\varphi(x)$$

More precisely, this means that

$$T_R \circ T_S(\varphi)(x') = \int_X K(-, x')\varphi(-).$$

Using the condition (2.4) we deduce

$$K(x, x') = (\nu - \mu)\mathbb{1}_{\Delta_X} + \mu\mathbb{1}_{X\times X}.$$

Hence

$$\int_X K(-, x')\varphi(-) = (\nu - \mu)\phi(x') + \mu\Big(\int_X \varphi\Big)\mathbb{1}_X.$$

This concludes the proof of the Inversion Formula

□

Here is a beautiful application of the Inversion Formula. Let $X$ be a $n$-dimensional real vector space, $Y =$ the Grassmanian of *affine* hyperplanes in $X$. We can describe $Y$ as the total space of the tautological line bundle over $\mathbb{P}(X^*)$ via the projection $\vec{n} : Y \to \mathbb{P}(X^*)$ which associates to each affine hyperplane the hyperplane through the origin parallel to it. For each hyperplane $H$ we can think of $\vec{n}(H)$ as a unit normal to $H$ and we can rewrite the condition $\vec{v} \in \vec{n}(H)$ as $\vec{v} \perp \vec{n}(H)$.

Consider the incidence relation

$$I = \Big\{(x, H) \in X \times Y;\ \ x \in H\Big\}$$

and its dual

$$I^* = \Big\{(H, x) \in Y \times X;\ \ H \ni x\Big\}.$$

They satisfy conditions $(*)$ and (2.3b). The transformation

$$T_I : \mathcal{CF}^c(X) \to \mathcal{CF}(Y)$$

is called the *"motivic" Radon transform*. In this case we have

$$I \times_Y I^* = \Big\{(x, H, x');\ \ x, x' \in H \in Y\Big\}.$$

Observe that if $\vec{n} = \vec{n}(H)$ denotes a unit normal vector to a hyperplane $H$ then

$$(x, H, x') \in I \times_Y I^* \Longleftrightarrow (x' - x) \perp \vec{n}.$$

This shows that we have a locally trivial fibration

$$I \times_Y I^* \stackrel{\pi}{\twoheadrightarrow} X \times \mathbb{P}(X^*) \cong \mathbb{R}^n \times \mathbb{RP}^{n-1},$$

$$\pi(x, H, x') = (x, \vec{n}(H)),\ \ \pi^{-1}\big(x, \vec{n}(H)\ \big) = \vec{n}(H) \subset X.$$

This shows that (2.3a) is satisfied.

In this case $p^{-1}(x,x')$= the set of planes containing $x$ and $x'$. We deduce

$$p^{-1}(x,x') \cong \begin{cases} \mathbb{RP}^{n-2} & \text{if} \quad x \neq x' \\ \mathbb{RP}^{n-1} & \text{if} \quad x = x' \end{cases},$$

so that in this case we have

$$\mu = \mu(n) = \chi(\mathbb{RP}^{n-2}) = \frac{1}{2}(1+(-1)^n), \;\; \nu = \frac{1}{2}(1-(-1)^n)$$

Observe that

$$\nu - \mu = (-1)^{n+1}.$$

In this case we have

$$T_{I^*}T_I(\varphi) = (-1)^{n+1}\varphi + \frac{1}{2}(1+(-1)^n)\Big(\int_X \varphi\Big)\mathbb{1}_X.$$

In particular, if $\varphi = \mathbb{1}_K$, $K$ compact, subanalytic then

$$T_{I^*}T_I(\mathbb{1}_K) = (-1)^{n+1}\mathbb{1}_K + \frac{1}{2}(1+(-1)^n)\chi(K)\mathbb{1}_X = \begin{cases} \mathbb{1}_K & \text{if } n \text{ is odd} \\ -\mathbb{1}_K + \chi(K)\mathbb{1}_X & \text{if } n \text{ is even} \end{cases} \tag{2.5}$$

Here are two striking applications of the last formula.

**Corollary 2.2.** *(a) Suppose $C \subset \mathbb{R}^2$ is continuous, subanalytic simple closed curve. Then if we know the number of intersection points of $C$ with* any affine line *then we can completely determine the shape and location of $C$!*
*(b) Suppose $K$ is a compact subanalytic set in $\mathbb{R}^{2n+1}$. Then we can completely determine $K$ if we know the Euler characteristics of the intersection of $K$ with* any *affine hyperplane.*

### What's hiding behind the curtains

There is a very deep and rich world hiding behind Theorem 1.10. Describing it fully would require a lot more space, but I would not pass the chance to at least hint at it.

It is a deep theorem of Masaki Kashiwara that the Abelian group of constructible functions on the real analytic manifold can be viewed naturally as the $K$-theoretic group of "something". This "something" is a category $D^b_c(X)$ naturally associated to $X$. In this category one can add morphisms, one speak of the direct sum of two objects and one can formulate a notion of short exact sequence of objects. We define its $K$-group as the Abelian group $K(D^b_c(X))$ spanned by the objects of $D^b_c(X)$ modulo the relations

$$B = A + C$$

for every "short exact sequence"

$$0 \to A \to B \to C \to 0$$

The objects in $D^b_c(X)$ are *constructible bounded complexes of sheaves* on $X$. More precisely, a bounded complex of sheaves of real vector spaces

$$0 \to \mathcal{F}^a \to \mathcal{F}^{a+1} \to \cdots \to \mathcal{F}^b \to 0, \;\; a, b \in \mathbb{Z}$$

is called constructible if there exists a locally finite covering $X = \bigsqcup_i X_i$ by subanalytic subsets such that for every $a \leq j \leq b$ the cohomology sheaf $\mathcal{H}^j(\mathcal{F}^\bullet)$ is locally constant along each stratum $X_i$ and for every $x \in X$ the stalk $\mathcal{H}^j(\mathcal{F}^\bullet_x)$ is a finite dimensional vector space.

$D^b_c(X)$ is a full subcategory of the derived category of sheaves on $X$ and as such one can define the direct sum and "short exact sequences"[3] in $D^b_c(X)$.

To a constructible complex $\mathcal{F}^\bullet$ one can associate a constructible function

$$\chi_{\mathcal{F}^\bullet} : X \to \mathbb{Z},\ \ \chi_{\mathcal{F}^\bullet}(x) = \chi(\mathcal{H}^\bullet(\mathcal{F}^\bullet_x)) = \sum_j (-1)^j \dim \mathcal{H}^j(\mathcal{F}^\bullet_x)$$

and this induces a map

$$\chi_- : K(D^b_c(X)) \to CF(X).$$

Kashiwara proved that this map is an isomorphism (see [1]) and that the operations $f^*$, $\int_f$ are then induced by operations between the categories $D^b_c(-)$. This is a highly nontrivial feat requiring a very good microlocal understanding of the behavior of constructible complexes. The integration operation can then be interpreted as an index, more precisely the Euler characteristic of the hypercohomology of a bounded complex.

The story does not end here. The every constructible complex $\mathcal{F}^\bullet$ on $X$ one can associate a (conic) lagrangian cycle $Ch(\mathcal{F}^\bullet)$ on $T^*X$, the so called *characteristic cycle*. It depends only on the class of $\mathcal{F}^\bullet$ in $K(D^b_c(X))$ and thus it is an invariant of the Euler characteristic function $\chi_{\mathcal{F}^\bullet}$. One can think of this characteristic cycle as "the graph of the differential of $\chi_{\mathcal{F}^\bullet}$". The Kashiwara index theorem states that

$$\int_X \chi_{\mathcal{F}^\bullet} = \pm Ch(\mathcal{F}^\bullet) \cap [X]$$

where we regard $X$ as a lagrangian cycle of $T^*X$ via the zero section embedding $X \hookrightarrow T^*X$.

[3] These are technically distinguished triangles in a triangulated category.

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