Communications in Contemporary Mathematics
Vol. 6, No. 6 (2004) 833–866


# SEIBERG–WITTEN INVARIANTS OF RATIONAL HOMOLOGY 3-SPHERES

LIVIU I. NICOLAESCU

*Department of Mathematics,*
*University of Notre Dame,*
*Notre Dame, IN 46556, USA*
*nicolaescu.1@nd.edu*

Received 23 May 2002

We prove that the Seiberg–Witten invariants of a rational homology sphere are determined by the Casson–Walker invariant and the Reidemeister–Turaev torsion.

*Keywords*: Rational homology 3-spheres; Seiberg–Witten invariants; Reidemeister torsion; Casson–Walker invariant; eta invariants; linking forms; surgery; Fourier transform.

Mathematics Subject Classification 2000: 57M27, 57R57, 58J28, 20C05, 11E81

## 0. Introduction

In 1996 Meng and Taubes [16] have established a relationship between the Seiberg–Witten invariants of a (closed) 3-manifold with $b_1 > 0$ and the Milnor torsion. A bit later Turaev, [29, 30], enhanced Meng–Taubes' result, essentially identifying the Seiberg–Witten invariant with the refined torsion he introduced earlier in [27]. In [29] Turaev raised the question of establishing a connection between these two invariants in the remaining case, that of rational homology spheres.

Around the same time, Lim [12] succeeded in providing a combinatorial description of the Seiberg–Witten invariants of integral homology spheres. Namely, in this case they coincide with the Casson invariant. In [19] we investigated a special class of rational homology spheres, the lens spaces, and we proved that the Seiberg–Witten invariants of such spaces are determined by the Casson–Walker invariant and the Reidemeister–Turaev torsion in a very explicit fashion. In that paper we also raised the question whether this is the case in general. Recently Marcolli and Wang [15] (see also the related work of Ozsváth–Szabó [23]) have shown that the Seiberg–Witten invariants of a $\mathbb{Q}HS$ determine the Casson–Walker invariant. Additionally, they have proved a very general surgery formula involving the Seiberg–Witten invariants.

The main result of the present paper is Theorem 2.4 where we prove that for rational homology spheres we have

$$\text{SW} \Longleftrightarrow \text{CW} + \text{RT} \stackrel{\text{def}}{:=} \text{Casson–Walker invariant} + \text{Reidemeister–Turaev torsion}\,.$$

Our strategy is based on analytic surgery formulae for Seiberg–Witten invariants developed in [13, 15, 23], and topological surgery formulae for the Casson–Walker invariant and the torsion, described in [30–32].

Although this approach is similar in spirit to the one in [16], there is, however, a serious obstacle one has to overcome, which was not present in [16]. More precisely we cannot rely as Meng–Taubes did, on the Seiberg–Witten invariants of 3-manifolds with boundary because the required analytical set-up is incompatible with surgeries leading to rational homology spheres. This has the effect of *severely* limiting the amount of information carried by the various surgery formulae we can employ. We outline below some the difficulties of this approach and the method we propose to get out of trouble.

Both the Seiberg–Witten invariant and the CW+RT-invariant can be thought of as $\mathbb{Q}$-valued functions on the first homology group $H$ of a given rational homology sphere $M$. We denote by $D_M$ the difference of these two functions. Proving the equality of these two invariants is equivalent to showing that $D_M \equiv 0$.

We found it extremely convenient to work not with $D_M$ but with its Fourier transform $\hat{D}_M : H^\sharp \to \mathbb{C}$, where $H^\sharp$ is the Pontryagin dual of $H$. For example, Marcolli–Wang result [15] translates into $\hat{D}_M(1) = 0$, for all rational homology spheres. This is not just an artificial trick. The *true nature* of the surgery formulae is *best displayed* in the Fourier picture. To explain the gist of these formulae consider a 3-manifold $N$ with $b_1 = 1$ and boundary $T^2$. $N$ can be thought of as the complement of a knot in a $\mathbb{Q}HS$. Pick two simple closed curves $c_1$, $c_2$ on $\partial N$ with nontrivial intersection numbers with the longitude $\lambda \in H_1(\partial N, \mathbb{Z})$.

By Dehn surgery with $c_i$ as attaching curves we obtain two rational homology spheres $M_1$, $M_2$ and two knots $K_i \hookrightarrow M_i$, $i = 0, 1$. Let $H_i := H_1(M_i, \mathbb{Z})$, $G := H_1(N, \partial N; \mathbb{Z})$. The knot $K_i$ determines a subgroup $K_i^\perp \subset H_i^\sharp$, consisting of the characters vanishing on $K_i$. These subgroups are naturally isomorphic to $G$ and thus we have a natural isomorphism

$$f : K_1^\perp \to K_2^\perp\,.$$

The surgery formulae have the form[a]

$$\langle \lambda, c_2 \rangle \hat{D}_{M_1}(\chi) = \langle \lambda, c_1 \rangle \hat{D}_{M_2}(f(\chi)) + |G|\mathcal{K}\,, \quad \forall\, \chi \in K_1^\perp\,,$$

where $\langle \bullet, \bullet \rangle$ denotes the intersection pairing on $H_1(\partial N, \mathbb{Z})$, and $\mathcal{K}$ is a universal correction term which depends only on the divisibility $m_0$ of the longitude and the $SL_2(\mathbb{Z})$-orbit of the pair $(c_1, c_2)$ with respect to the obvious action of this group on the space of pairs of primitive vectors in a 2-dimensional lattice. We will thus write

[a]The reader should compare this description of the surgery formula with the ones in [15, 23] (see also Sec. 3 of this paper) to truly appreciate the amazing simplifying power of the Fourier transform.

$\mathcal{K}_{m_0;[c_1,c_2]}$, and call the triplet $(m_0; [c_1, c_2])$ the arithmetic type of the surgery. The results of [23] prove that

$$\mathcal{K}_{1;[c_1,c_2]} \equiv 0\,, \quad \forall\, [c_1, c_2]\,.$$

We call surgeries with $m_0 = 1$ *primitive*, and the surgeries with trivial correction term, *admissible*. We denote by $\mathfrak{X}$ the class of rational homology spheres $M$ such that $\hat{D}_M \equiv 0$. Both the family of admissible surgeries and the family $\mathfrak{X}$ are "time dependent" families, and during our proof we will gradually produce larger and larger classes of surgeries/manifolds inside these families.

The class $\mathfrak{X}$ is closed under connected sums and certain primitive surgeries (see Sec. 4.1). Using this preliminary information we are able to show that all homology lens spaces belong to $\mathfrak{X}$. This follows from the general results in [4] concerning 3-manifolds related by Dehn surgeries of special types. We also present an alternate proof, based on Kirby calculus, which we learned from Saveliev. As a bonus, we can include many more arithmetic types of Dehn surgeries in the class of admissible surgeries.

Loosely speaking, the homology lens spaces have the simplest linking forms. We take this idea seriously, and we define an appropriate notion of complexity of a linking form. The proof then proceeds by induction, including in $\mathfrak{X}$ manifolds of larger and larger complexity. This process also increases the class of admissible surgeries, which can be used at the various inductive steps. Such a proof is feasible if we can produce a large supply of complexity reducing Dehn surgeries. Fortunately, this can be done using elementary arithmetic.

Our proof also shows that the invariant introduced by Ozsváth and Szabó in [23] also coincides with CW+RT-invariant, thus answering a question raised in that paper. Moreover, the main theorem leads to a novel description (in the 3-dimensional case) of the Brumfiel–Morgan [1] correspondence between *spin* structures and quadratic refinements of the linking form. This new description of the Brumfiel–Morgan correspondence plays an important role in our recent investigation [17].

**Basic Notations and Terminology.** We will denote by $M$ a closed, compact, oriented 3-manifold. We will set $H = H_1(M, \mathbb{Z}) \cong H^2(M, \mathbb{Z})$, and we will denote the group operation multiplicatively. Set $\nu_M := |H|$. A homology orientation on $M$ is an orientation on $H \otimes \mathbb{R}$. Define

$$\Theta = \Theta_M := \sum_{h \in \mathrm{Tors}(H)} h \in \mathbb{Z}[H]\,.$$

For any $P = \sum_{h\in H} P_h h \in \mathbb{Z}[H]$ we set $\bar{P} := \sum_{h\in H} P_h h^{-1}$. The letter $N$ will be reserved for compact, oriented three-manifolds with boundary $\partial N \cong T^2$ such that $b_1(N) = 1$. Equivalently, $N$ can be viewed as the complement of a knot in a rational homology sphere. We set $G = H_1(N, \partial N) \cong H^2(N, \mathbb{Z})$.

We will denote by $\mathrm{Spin}^c(M)$ the space of isomorphism classes of spin$^c$ structures on $M$. We will denote a generic spin$^c$ structure by $\sigma$. $\mathrm{Spin}^c(M)$ is an $H$-torsor, and we will denote by

$$\mathrm{Spin}^c(M) \times H \ni (\sigma, h) \mapsto h\sigma$$

the natural action of $H$ on $\mathrm{Spin}^c(M)$. The natural involution on $\mathrm{Spin}^c(M)$ will be denoted by $\sigma \mapsto \bar{\sigma}$. The complex line bundle associated to $\sigma$ will be denoted by $\det(\sigma)$. We can identify $\det(\sigma)$ via the first Chern class and the Poincaré duality with an element in $H$. Note that

$$\det(h\sigma) = h^2 \det(\sigma)\,.$$

We will denote by $\mathrm{Spin}(M)$ the space of isomorphism classes of *spin*-structures on $M$. A generic *spin* structure will be denoted by $\epsilon$. $\mathrm{Spin}(M)$ is naturally a $H^1(M, \mathbb{Z}/2) \cong \mathrm{Hom}(H, \mathbb{Z}/2)$-torsor. We use the same notation to denote the action of $\mathrm{Hom}(H, \mathbb{Z}/2)$ on $\mathrm{Spin}(M)$. Every *spin* structure $\epsilon$ induces a canonical spin$^c$-structure $\sigma(\epsilon)$. Moreover

$$\sigma(\epsilon) = \overline{\sigma(\epsilon)}\,, \quad \sigma(h\epsilon) = \beta(h)\sigma(\epsilon)\,, \quad \forall\, h \in \mathrm{Hom}(H, \mathbb{Z}/2)\,,$$

where $\beta : H^1(M, \mathbb{Z}/2) \to H^2(M, \mathbb{Z})$ denotes the *Bockstein morphism* whose image is the 2-torsion part of $H$.

For any finitely generated Abelian group $A$ we will denote by $A^\sharp$ its Pontryagin dual,

$$A^\sharp = \mathrm{Hom}(A, S^1) \cong \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})\,.$$

Finally for every $\chi \in H^\sharp$ and any $P = \sum_{h \in H} P_h h \in \mathbb{C}[H]$ we set

$$\hat{P}(\chi) := \sum_{h \in H} P_h \chi(h) \in \mathbb{C}\,.$$

The function $H^\sharp \ni \chi \mapsto \hat{P}(\chi)$ is essentially the Fourier transform of $P$. Note that

$$\hat{P}(1) := \sum_{h \in H} P_h\,.$$

Moreover

$$\hat{\Theta}_M(1) = |\mathrm{Tors}(H)|\,, \quad \hat{\Theta}_M(\chi) = 0\,, \quad \text{if } \chi \neq 1\,, \quad \text{and} \quad \exists m > 1 \chi(H) \subset U_m\,.$$

For every positive integer $m$ we denote by $U_m \subset S^1$ the multiplicative group of $m$th roots of 1.

## 1. The Modified Seiberg–Witten Invariants of 3-Manifolds

We want to present, in a form suitable for our goals, some basic structural facts concerning the Seiberg–Witten invariants of 3-manifolds. For more details we refer to [13, 14, 16].

The Poincaré duality defines a natural orientation on $H_*(M, \mathbb{R})$ which by default will be the orientation we use in defining the Seiberg–Witten invariants of $M$. To describe them in more detail we need to differentiate three cases.

### 1.1. *The case $b_1 > 1$*

The Seiberg–Witten invariant of $M$ is a function

$$\mathbf{sw}_M : \mathrm{Spin}^c(M) \to \mathbb{Z}\,.$$

$\mathbf{sw}_M(\sigma)$ is a signed count of $\sigma$-monopoles, objects determined by additional geometric data on $M$, Riemannian metric $g$, and a closed 2-form $\eta$. The canonical orientation on $H_*(M, \mathbb{R})$ associates a sign to each monopole, and the signed count is independent of $g$ and $\eta$. The Seiberg–Witten invariant has the following properties.

- $\mathbf{sw}_M(\sigma) = 0$ for all but finitely many $\sigma$'s.
- $\mathbf{sw}_M(\sigma) = \mathbf{sw}_M(\bar{\sigma})$, $\forall\,\sigma$.

For every $\sigma$ we can form the element

$$\mathbf{SW}_{M,\sigma} \in \mathbb{Z}[H]\,, \quad \mathbf{SW}_{M,\sigma} = \sum_{h\in H} \mathbf{sw}_M(h^{-1}\sigma)h\,.$$

Note that for every $h_0 \in H$ we have

$$\mathbf{SW}_{M,h_0\sigma} = h_0 \mathbf{SW}_{M,\sigma}\,.$$

Moreover

$$\mathbf{SW}_{M,\sigma} = \det(\sigma)\mathbf{SW}_{M,\bar{\sigma}} = \det(\sigma)\overline{\mathbf{SW}}_{M,\sigma}\,.$$

In particular, for any *spin* structure $\epsilon$ we have $\det(\sigma(\epsilon)) = 1$ so that

$$\mathbf{SW}_{M,\sigma(\epsilon)} = \overline{\mathbf{SW}}_{M,\sigma(\epsilon)}\,.$$

For simplicity we set $\mathbf{SW}_{M,\epsilon} := \mathbf{SW}_{M,\sigma(\epsilon)}$, $\forall\,\epsilon$.

### 1.2. *The case $b_1 = 1$*

In this case we need to fix an orientation $\mathfrak{o}$ on $H\otimes\mathbb{R}$, i.e. an isomorphism $H\otimes\mathbb{R} \to \mathbb{R}$. To describe the Seiberg–Witten invariant of $M$ we need to recall the rudiments of its construction.

Fix a metric $g$. The chosen orientation on $H \otimes \mathbb{R}$ defines a harmonic 1-form $\omega_g$ such that $\omega_g$ induces the chosen orientation on $H \otimes \mathbb{R}$, and $\|\omega_g\|_{L^2(g)} = 1$. Note that this orientation also produces a surjection

$$\deg_{\mathfrak{o}} : H \to H/\mathrm{Tors}(H) = \mathbb{Z}\,.$$

For $\sigma \in \mathrm{Spin}^c(M)$ denote by $\mathcal{P}_\sigma(g)$ the space of closed 2-forms such that

$$w_{\mathfrak{o}}(\sigma, \eta) := \int_M \omega_g \wedge \eta - 2\pi c_1(\det\sigma) \neq 0\,.$$

It is decomposed into two chambers

$$\mathcal{P}^{\pm}_\sigma(g, \mathfrak{o}) = \{\eta \in \mathcal{P}_\sigma(g)\,; \quad \pm w_{\mathfrak{o}}(\sigma, \eta) > 0\}\,.$$

For $\eta \in \mathcal{P}_\sigma(g)$ we denote by $\mathbf{sw}_M(\sigma,\eta)$ the signed count of $(\sigma, g, \eta)$-monopoles. It is known that

$$\mathbf{sw}_M(\sigma,\eta) = \mathbf{sw}_M(\bar{\sigma}, -\eta)\,,$$

$\mathbf{sw}_M(\sigma,\eta) = 0$ for all but finitely many $\sigma$'s, and

$$\mathbf{sw}_M(\sigma,\eta_1) = \mathbf{sw}_M(\sigma,\eta_2)\,, \quad \text{if } w_{\mathfrak{o}}(\sigma,\eta_1)\cdot w_{\mathfrak{o}}(\sigma,\eta_2) > 0\,.$$

We set

$$\mathbf{sw}^{\pm}_M(\sigma,\mathfrak{o}) := \mathbf{sw}_M(\sigma,\eta,\mathfrak{o})\,, \quad \text{where } \pm w_{\mathfrak{o}}(\sigma,\eta) > 0\,.$$

The wall crossing formula (see [13]) states that

$$\mathbf{sw}^+_M(\sigma,\mathfrak{o}) - \mathbf{sw}^-_M(\sigma,\mathfrak{o}) = \frac{1}{2}\deg_{\mathfrak{o}}(\det\sigma)\,.$$

Set

$$\mathbf{SW}_{M,\sigma,\eta} = \mathbf{sw}_M(h^{-1}\sigma,\eta)h \in \mathbb{Z}[H]\,,$$

$$\mathbf{SW}^+_{M,\sigma} = \sum_{h\in H}\mathbf{sw}^+_M(h^{-1}\sigma)h \in \mathbb{Z}[[H]]\,.$$

Suppose we pick $\sigma = \sigma(\epsilon)$ and $\eta = \eta_0$ such that

$$0 < \left|\int_M \omega\wedge\eta_0\right| \ll 1\,.$$

Fix $T \in H$ such that $\deg_{\mathfrak{o}}(T) = 1$ and set $\deg^{\pm}\mathfrak{o} = \max(\pm\deg_{\mathfrak{o}}, 0)$. We can rephrase the wall crossing formula in the more compact form

$$\mathbf{SW}^+_{M,\sigma(\epsilon)} = \mathbf{SW}_{M,\sigma(\epsilon),\eta_0} + \sum_{h\in H}\deg^+_{\mathfrak{o}}(h^{-1})h = \mathbf{SW}_{M,\sigma(\epsilon),\eta_0} + \frac{\Theta_M T}{(1-T)^2}\,.$$

We set $W_M := \frac{\Theta_M T}{(1-T)^2} \in \mathbb{Z}[[H]]$, and we will refer it as *wall-crossing correction* of $M$. We set

$$\mathbf{SW}^0_{M,\sigma(\epsilon)} := \mathbf{SW}^+_{M,\sigma(\varepsilon)} - \sum_{h\in H}\deg^+_{\mathfrak{o}}(h^{-1})h = \mathbf{SW}_{M,\sigma(\epsilon),\eta_0} \in \mathbb{Z}[H]\,. \tag{1.1}$$

The wall-crossing formula implies the equality

$$\mathbf{SW}^0_{M,\sigma(\epsilon)} = \mathbf{SW}^-_{M,\sigma(\varepsilon)} - \sum_{h\in H}\deg^-_{\mathfrak{o}}(h^{-1})h\,.$$

$\mathbf{SW}^0_{M,\sigma(\epsilon)}$ is a *topological invariant* which satisfies the symmetry condition

$$\mathbf{SW}^0_{M,\sigma(\epsilon)} = \overline{\mathbf{SW}}^0_{M,\sigma(\epsilon)}\,,$$

and the equivariance property

$$\mathbf{SW}^0_{M,\sigma(h_0\epsilon)} = \beta(h_0)\mathbf{SW}^0_{M,\sigma(\epsilon)}\,, \quad \forall\, h_0 \in \mathrm{Hom}(H,\mathbb{Z}/2)\,.$$

We will refer to $\mathbf{SW}^0_{M,\sigma(\epsilon)}$ as the *modified Seiberg–Witten invariant of* $M$.

### 1.3. *The case* $b_1 = 0$

Suppose now that $b_1(M) = 0$, i.e. $M$ is a rational homology sphere. Fix $\sigma \in \mathrm{Spin}^c(M)$. In this case the signed count of $(\sigma, g, \eta)$-monopoles depends on $(g, \eta)$ in a more complicated way. To produce a topological invariant we need to add a correction to this count. For simplicity, we describe this correction only when $\eta = 0$.

Denote by $\mathbb{S}_\sigma$ the bundle of complex spinors determined by $\sigma$. The line bundle $\det \sigma = \det \mathbb{S}_\sigma$ admits a unique equivalence class of flat connections. Pick one such flat connection $A_\sigma$ and denote by $\mathfrak{D}_{A_\sigma}$ the Dirac operator on $\mathbb{S}_\sigma$ determined by the twisting connection $\sigma$. We denote its eta invariant by $\eta_{\mathrm{dir}}(g, \sigma)$. Also, denote by $\eta_{\mathrm{sign}}(g)$ the eta invariant of the odd signature operator determined by $g$. Finally define the Kreck–Stolz invariant of $(g, \sigma)$ by

$$KS(g, \sigma) = 4\eta_{\mathrm{dir}}(g, \sigma) + \eta_{\mathrm{sign}}(g)\,.$$

Define the *modified Seiberg–Witten invariant* of $(M, \sigma)$ by

$$\mathbf{sw}^0_M(\sigma) = \frac{1}{8}KS(g, \sigma) + \mathbf{sw}_M(\sigma) \in \mathbb{Q}\,.$$

As shown in [13], the above quantity is independent of the metric, and it is a topological invariant. Set

$$\mathbf{SW}^0_{M,\sigma} := \sum_{h \in H} \mathbf{sw}_M(h^{-1}\sigma)h \in \mathbb{Q}[H]\,.$$

If $\sigma = \sigma(\epsilon)$ we have

$$\mathbf{SW}^0_{M,\sigma(\epsilon)} = \overline{\mathbf{SW}}^0_{M,\sigma(\epsilon)}\,.$$

### 1.4. *Summary*

Let us coherently organize the facts explained so far. The modified Seiberg–Witten invariant associates to each closed, compact, homologically oriented 3-manifold $M$, and each $\epsilon \in \mathrm{Spin}(M)$ a "Laurent polynomial" $\mathbf{SW}^0_{M,\epsilon} \in \mathbb{Q}[H]$ with the following properties.

$$\mathbf{SW}^0_{M,\epsilon} \in \mathbb{Z}[H]\,, \quad \text{if } b_1(M) > 0\,, \tag{1.2}$$

$$\mathbf{SW}^0_{M,\epsilon} = \overline{\mathbf{SW}}^0_{M,\epsilon}\,, \tag{1.3}$$

and

$$\mathbf{SW}^0_{M,h_0\epsilon} = \beta(h_0)\mathbf{SW}^0_{M,\epsilon}\,, \quad \forall\, h_0 \in \mathrm{Hom}(H, \mathbb{Z}/2)\,. \tag{1.4}$$

## 2. The Modified Reidemeister–Turaev Torsion of 3-Manifolds

In this section we survey the results of Turaev [26–31] in a language appropriate to our goals.

### 2.1. *Turaev's refined torsion*

The Reidemeister–Turaev torsion associates to each homologically oriented 3-manifold $M$, and each spin$^c$ structure $\sigma$ on $M$ a "formal power series" $\mathcal{T}_{M,\sigma} \in \mathbb{Q}[[H]]$ with the following properties.

$$\mathcal{T}_{M,\sigma} \in \mathbb{Z}[[H]]\,, \quad b_1(M) > 1$$

$$(1-T)^2\mathcal{T}_{M,\sigma} \in \mathbb{Z}[H]\,, \quad b_1(M) = 1\,, \quad \deg T = 1$$

$$\mathcal{T}_{M,\sigma} \in \mathbb{Q}[H]\,, \quad \hat{\mathcal{T}}_{M,\sigma}(1) = 0\,, \quad b_1(M) = 0\,.$$

Moreover

$$\mathcal{T}_{M,h_0\sigma} = h_0\mathcal{T}_{M,\sigma}\,, \quad \forall\, h_0 \in H\,,$$

and

$$\mathcal{T}_{M,\sigma} = \det(\sigma)\bar{\mathcal{T}}_{M,\sigma}\,.$$

For $\epsilon \in \mathrm{Spin}(M)$, set $\mathcal{T}_{M,\epsilon} = \mathcal{T}_{M,\sigma(\epsilon)}$. It follows that

$$\mathcal{T}_{M,\epsilon} = \bar{\mathcal{T}}_{M,\epsilon}\,.$$

Using [28, Sec. 4.2] and [30, Appendix 3] we deduce that when $b_1(M) = 1$ we have

$$\mathcal{T}^0_{M,\epsilon} := \mathcal{T}_{M,\epsilon} - W_M \in \mathbb{Z}[H]\,, \tag{2.1}$$

and moreover

$$\mathcal{T}^0_{M,\epsilon} = \bar{\mathcal{T}}^0_{M,\epsilon}\,.$$

When $b_1(M) = 0$ we denote by $CW_M$ the Casson–Walker invariant of $M$ and define

$$\mathcal{T}^0_{M,\epsilon} = \mathcal{T}_{M,\epsilon} - \frac{1}{2}CW_M\Theta_M\,. \tag{2.2}$$

Observe that $-\mathcal{T}^0_{M,\epsilon}(1) = \frac{1}{2}|H|CW_M =$ the Casson–Walker–Lescop invariant of $M$ (see [11, p. 80]).

We we will refer to the quantities $\mathcal{T}^0_{M,\epsilon}$ for $b_1(M) = 0$, 1 the *modified Reidemeister–Turaev torsion of* $M$. For uniformity, we set $\mathcal{T}^0_M = \mathcal{T}_M$ when $b_1(M) > 1$.

Summarizing, we conclude that the modified Reidemeister–Turaev torsion associates to each homologically oriented 3-manifold $M$, and to each *spin* structure $\epsilon$ on $M$ a "Laurent polynomial" $\mathcal{T}^0_{M,\epsilon} \in \mathbb{Q}[H]$ with the following properties.

$$\mathcal{T}^0_{M,\epsilon} \in \mathbb{Z}[H]\,, \quad \text{if } b_1(M) > 0\,, \tag{2.3}$$

$$\mathcal{T}^0_{M,\epsilon} = \bar{\mathcal{T}}^0_{M,\epsilon}\,, \tag{2.4}$$

and

$$\mathcal{T}^0_{M,h_0\epsilon} = \beta(h_0)\mathcal{T}^0_{M,\epsilon}\,, \quad \forall\, h_0 \in \mathrm{Hom}(H,\mathbb{Z}/2)\,. \tag{2.5}$$

### 2.2. *Relations between the torsion and the Seiberg–Witten invariant*

The Seiberg–Witten invariant and the modified Reidemeister torsion are related. More precisely we have the following result.

**Theorem 2.1.** (a) $\mathbf{SW}^0_{M,\epsilon} = \mathfrak{T}^0_{M,\epsilon}$ *if* $b_1(M) = 1$.
(b) $\widehat{\mathbf{SW}}^0_M(1) = \hat{\mathfrak{T}}^0_M(1)$ *if* $b_1(M) = 0$.
(c) $\mathbf{SW}^0_M = \mathfrak{T}^0_M$ *if* $M$ *is a lens space.*

**Proof.** Part (b) follows from [12, 15] while Part (c) follows from [19].[b] Only Part (a) requires a bit of work. According to the results in [16, 29, 30] we have an equality

$$\mathbf{SW}^+_{M,\epsilon} = \pm\mathfrak{T}_{M,\epsilon} \Longleftrightarrow (\mathbf{SW}^0_{M,\epsilon} + W_M) = \pm(\mathfrak{T}^0_{M,\epsilon} + W_M)$$

To prove that the correct choice of signs is "+" we argue by contradiction. Suppose

$$\mathbf{SW}^+_{M,\epsilon} = -\mathfrak{T}_{M,\epsilon}\,.$$

Then this implies that

$$\mathfrak{T}_{M,\epsilon} - W_M = -\mathbf{SW}^0_{M_\epsilon} - 2W_M \notin \mathbb{Z}[H]$$

which contradicts (2.1). □

Part (c) of the above theorem can be slightly strengthened to

$$\mathbf{SW}^0_M = \mathfrak{T}^0_M\,, \quad \text{if } M \text{ is a connected sum of lens spaces}\,. \tag{2.6}$$

This equality follows from the additivity of the torsion and of Casson–Walker invariant under connected sums [31, Sec. XII.1] and the additivity of the Kreck–Stolz invariant which follows from the very general surgery results for eta invariants in [9]. In this case the formulae in [9] simplify considerably since the gluing occurs along a 2-sphere which admits metrics of positive scalar curvature. For a more general result of this type we refer to the discussion at the beginning of Sec. 4.1

Later on we will need the following consequence of Theorem 2.1(a).

**Proposition 2.2.** *If* $M$ *is a homologically oriented* 3*-manifold such that* $b_1(M) = 1$ *then*

$$\hat{\mathfrak{T}}^0_M(1) = \widehat{\mathbf{SW}}^0_M(1) = \frac{1}{2}\Delta''_M(1)\,,$$

[b] There is a sign discrepancy between the definition (2.2), and the definition of the modified torsion in [19]. This is due to the definition in [32, Proposition 6.2] of the lens space $L(p,q)$ as the $p/q$-surgery on the unknot. The traditional definition is as the $-p/q$-surgery on the unknot as in [6, p. 158], or [7, pp. 66–67]. This is the convention we adhere to in this paper.

*where $\Delta_M \in \mathbb{Z}[[T^{1/2}, T^{-1/2}]]$ denotes the symmetrized Alexander polynomial of $M$ normalized such that $\Delta_M(1) = |\mathrm{Tors}(H)|$.*

**Proof.** The projection $\deg : H \twoheadrightarrow \mathbb{Z}$ induces a morphism

$$\mathfrak{aug} : \mathbb{Z}[[H]] \to \mathbb{Z}[[t, t^{-1}]]$$

called *augmentation.* Fix $T \in H$ such that $\deg T = 1$. The symmetrized Alexander polynomial $\Delta_M$ is uniquely determined by the condition

$$\mathfrak{aug}\,\mathcal{T}_{M,\epsilon} = T^{k/2} \frac{\Delta_M(T)}{(1-T)^2}\,,$$

for some $k \in \mathbb{Z}$. Using Theorem 2.1(a) we deduce

$$\begin{aligned} T^{k/2}\frac{\Delta_M(T)}{(1-T)^2} &= \mathfrak{aug}\,\mathbf{SW}_M = \mathfrak{aug}\,\mathbf{SW}^0_M + \mathfrak{aug}(\Theta_M)\frac{T}{(1-T)^2} \\ &= \mathfrak{aug}\,\mathbf{SW}^0_M + |\mathrm{Tors}(H)|\frac{T}{(1-T)^2}\,. \end{aligned}$$

We conclude that

$$T^{k/2-1}\Delta_M(T) = (T - 2 + T^{-1})\mathfrak{aug}\,\mathbf{SW}^0_M(T) + |\mathrm{Tors}(H)|\,.$$

The symmetry of $\mathbf{SW}^0$ implies $\mathbf{SW}^0_M(T) = \mathbf{SW}^0_M(T^{-1})$, and since $\Delta_M$ satisfies a similar symmetry we conclude $k/2 - 1 = 0$. Hence

$$\Delta_M(T) = (T - 2 + T^{-1})\mathfrak{aug}\,\mathbf{SW}^0_M(T) + |\mathrm{Tors}(H)|\,.$$

Differentiating the above equality twice at $T = 1$ we deduce

$$\Delta''_M(1) = 2\mathfrak{aug}\,\mathbf{SW}_M(1) = 2\widehat{\mathbf{SW}}^0(1)\,. \qquad \square$$

**Remark 2.3.** Observe a nice "accident". Suppose $M$ is as in Proposition 2.2. Then

$$W_M = \Theta_M \sum_{n\geq 1} nT^n\,.$$

Formally

$$\begin{aligned} \hat{W}_M(1) &= \hat{\Theta}_M(1)\sum_{n\geq 1} n = |\mathrm{Tors}(H)|\sum_{n\geq 1} n \\ &= |\mathrm{Tors}(H)|\zeta(-1) = -\frac{1}{12}|\mathrm{Tors}(H)|\,, \end{aligned}$$

where $\zeta(s)$ denotes Riemann's zeta function. In particular

$$\widehat{\mathbf{SW}}_M(1) = \widehat{\mathbf{SW}}^0_M(1) + \hat{W}_M(1) = \frac{1}{2}\Delta''_M(1) - \frac{1}{12}|\mathrm{Tors}(H)|\,.$$

The expression in the right-hand side is precisely the Lescop invariant of $M$.

We can now state the main result of this paper.

**Theorem 2.4.**

$$\mathbf{SW}^0_M = \mathcal{T}^0_M$$

*for any rational homology* 3*-sphere* $M$.

## 3. Surgery Formulae

### 3.1. *Dehn surgery*

We want to survey a few basic facts concerning Dehn surgery. For more details and examples we refer to [22].

Consider a 3-manifold $N$ as in the introduction, i.e. $b_1(N) = 1$, $\partial N \cong T^2$, and set $G := H_1(N, \partial N; \mathbb{Z})$. We orient $\partial N$ as boundary of $N$ using the outer-normal first convention. Denote by $\mathbf{j}$ the inclusion induced morphism

$$\mathbf{j} : H_1(\partial N, \mathbb{Z}) \to H_1(N, \mathbb{Z})\,.$$

The kernel of $\mathbf{j}$ is a rank one Abelian group. We can select a generator $\lambda$ of $\ker \mathbf{j}$ by specifying an orientation on $H^1(N, \mathbb{Z}) \cong H_2(N, \partial N; \mathbb{Z})$. We can write $\lambda = m_0 \lambda_0$ where $m_0 > 0$ and $\lambda_0 \in H_1(\partial N, \mathbb{Z})$ is a primitive class. $\lambda$ is called the longitude of $N$ and $m_0$ is called the *divisibility* of $N$. Fix $\mu_0 \in H_1(\partial N, \mathbb{Z})$ such that $\lambda_0 \cdot \mu_0 = 1$, where the dot denotes the intersection pairing on $H_1(\partial N, \mathbb{Z})$.

Denote by $X$ the solid torus $S^1 \times D^2$, so that $\partial X = T^2$. Set $\mathbf{l}_0 = S^1 \times \{\mathbf{pt}\}$ and $\mathbf{m}_0 = \{\mathbf{pt}\} \times \partial D^2$. We regard $\mathbf{l}_0$ and $\mathbf{m}_0$ as elements in $H_1(\partial X, \mathbb{Z})$. They satisfy $\mathbf{m}_0 \cdot \mathbf{l}_0 = 1$. Fix an orientation reversing diffeomorphism $\Gamma : \partial X \to \partial N$ such that

$$\Gamma_*(\mathbf{m}_0) = \mu_0\,, \quad \Gamma_*(\mathbf{l}_0) = \lambda_0\,.$$

Every $\varphi \in SL_2(\mathbb{Z})$ determines an isotopy class of orientation preserving diffeomorphisms of $T^2$. We can use $\varphi$ to construct a closed 3-manifold

$$M_\varphi := X \coprod_{\Gamma \circ \varphi : \partial X \to \partial N} N\,.$$

We say that $M_\varphi$ is obtained by *Dehn surgery* with gluing map $\varphi$. The integer $m_0$ is called the *divisibility* of the surgery. The manifold $M_\varphi$ is uniquely determined up to a diffeomorphism by the attaching curve $c = \Gamma \circ \varphi(\mathbf{m}_0)$. We can write $c = c_{p/q} := p\mu_0 + q\lambda_0$, $(p, q) = 1$. The diffeomorphism type of $M_\varphi$ is uniquely determined by the ratio $p/q$. Instead of $M_\varphi$ we will write $M_{p/q}$. We set $H_{p/q} := H_1(M_{p/q}, \mathbb{Z})$. The core of the solid torus determines an element $K_{p/q} \in H_{p/q}$.

We want to point out that the integer $q$ depends on the choice of $\mu_0$ while $p$ is invariantly determined by the equality $p := \lambda_0 \cdot c$. We refer to $p$ as the *multiplicity* of the surgery.

The group $H_{p/q}$ is determined from the short exact sequence

$$0 \to \langle \mathbf{j} c_{p/q} \rangle \to H_1(N, \mathbb{Z}) \to H_{p/q} \to 0\,.$$

We also have canonical isomorphisms

$$\Phi_{p/q} : G \to H_{p/q}/\langle K_{p/q}\rangle \, .$$

We obtain a natural projection $\pi_{p/q} : H_{p/q} \to G$. The long exact sequence of the pair $(N, \partial N)$ implies

$$G = H_1(N, \mathbb{Z})/\mathbf{j}H_1(\partial N, \mathbb{Z}) \, .$$

We deduce the following result.

**Lemma 3.1.** *The characters of $G$ are precisely the characters of $H_1(N, \mathbb{Z})$ which vanish on $\mathbf{j}H_1(\partial N, \mathbb{Z})$. Also, we can think of the characters of $G$ as characters $\chi$ of $H_{p/q}$ such that $\chi(K_{p/q}) = 1$.*

When $p \neq 0$, $H_{p/q}$ is a finite Abelian group and

$$|H_{p/q}| = pm_0|G| \, .$$

In this case, we denote by $\mathbf{lk}_{p/q}$ the linking form of $M_{p/q}$.

Observe that $b_1(M_{0/1}) = 1$. $K_{0/1}$ can be written as $m_0 h$ where $h \in H_{0/1}$ generates the free part of $H_{0/1}$. $M_{0/1}$ carries a natural homology orientation, induced from the orientation of $H^1(N, \mathbb{Z})$ and $H^1(X, \mathbb{Z})$ (see [30] for more details on this rather painful issue). Fix $T \in H_{0/1}$ such that $\deg(T) = 1$, and $K_{0/1} = m_0 T$. There exists $\chi_0 \in H^\sharp_{0/1}$ uniquely determined by the requirements

$$\chi_0(T) = \rho \, , \quad \chi_0|_{\mathrm{Tors}(H_{0/1})} = 1 \, ,$$

where $\rho$ is a primitive $m_0$th root of 1. According to Lemma 3.1 we can think of $\chi_0$ as a character of $G$.[c]

We then have a natural isomorphism $H_1(X) \cong \mathbb{Z}$, and via the Poincaré duality, a natural isomorphism $H^2(X, \partial X) \cong \mathbb{Z}$. The solid torus is equipped with a canonical relative spin$^c$ structure $\sigma_X$ uniquely determined by the condition

$$c_1(\det \sigma_X) \in 1 \in H^2(X, \partial X) \, .$$

For any orientation reversing homeomorphism $\varphi : \partial X \to \partial N$ we have a gluing map (see [26, Chap. VI])

$$\#_\varphi : \mathrm{Spin}^c(X, \partial X) \times \mathrm{Spin}^c(N, \partial N) \to \mathrm{Spin}^c(M) \, , \quad (\sigma_1, \sigma_2) \mapsto \sigma_1 \#_\varphi \sigma_2 \, .$$

In particular we obtain a surjection

$$\mathrm{Spin}^c(N, \partial N) \to \mathrm{Spin}^c(M) \, , \quad \sigma \mapsto \sigma_\varphi := \sigma_X \#_\varphi \sigma \, .$$

[c]The Universal Coefficients theorem and the Poincaré duality identifies $G^\sharp = H_1(N, \partial N; \mathbb{Z})^\sharp$ with the torsion subgroup of $H_1(N, \mathbb{Z})$. Via this identification we have $\chi_0 = \mathbf{j}\lambda_0 \in H_1(N, \mathbb{Z})$.

### 3.2. *Surgery formula for the modified Seiberg–Witten invariant*

We can now state the main surgery formula for the modified Seiberg–Witten invariant. Let $N$, $M_{p/q}$ etc. be as above.

Any relative spin$^c$ structure $\sigma$ on $N$ induces spin$^c$-structures $\sigma_{p/q}$ on $M_{p,q}$. We fix $\sigma$ so that $\sigma_0$ on $M_{0/1}$ is induced by a *spin* structure $\epsilon_0$ on $M_{0/1}$. For $h \in H_{p,q}$ we will write $\mathbf{sw}^0_{p/q}(h)$ for $\mathbf{sw}^0_{M_{p/q}}(h^{-1}\sigma_{p/q})$.

**Theorem 3.2 ([15, Marcolli–Wang], [23, Ozsváth–Szabó]).**[d] *For every $p, q$ there exists*

$$f_{p,q,m_0} : U_{m_0} \to \mathbb{Q}$$

*which depends only on $p$, $q$, $m_0$ but not on $N$ such that for every $g \in G$ we have*

$$\begin{aligned}\sum_{\pi_{p/q}(h)=g} & \mathbf{sw}^0_{p/q}(h) \\ &= p \sum_{\pi_{1/0}(h)=g} \mathbf{sw}^0_{1/0}(h) + q \sum_{\pi_{0/1}(h)=g} \mathbf{sw}^0_{0/1}(h) + f_{p,q,m_0}(\chi_0(g))\,. \qquad (\dagger_g)\end{aligned}$$

To get more information out of this formula we will take a partial Fourier transform. Let $\chi \in G^\sharp$. Using Lemma 3.1 we can identify $\chi$ with a character of $H_{p/q}$ with the property that $\chi(h) = \chi(h')$ whenever $\pi_{p/q}(h) = \pi_{p/q}(h')$. If we multiply $(\dagger_g)$ by $\chi(g)$ and we sum over $g \in G$ we deduce

$$\widehat{\mathbf{SW}}^0_{p/q}(\chi) = p\widehat{\mathbf{SW}}^0_{1/0}(\chi) + q\widehat{\mathbf{SW}}^0_{0/1}(\chi) + \sum_{g\in G} f_{p,q,m_0}(\chi_0(g))\chi(g)\,.$$

To gain further insight we need to simplify the sum on the right hand side. We have

$$\begin{aligned}\sum_{g\in G} f_{p,q,m_0}(\chi_0(g))\chi(g) &= \sum_{\rho\in U_{m_0}} \left(\sum_{\chi_0(g)=\rho} f_{p,q,m_0}(\rho)\right)\chi(g) \\ &= \sum_{\rho\in U_{m_0}} \left(\sum_{\chi_0(g)=\rho} \chi(g)\right) f_{p,q,m_0}(\rho)\,.\end{aligned}$$

Observe that if $\chi \not\equiv 1$ on $\ker\chi_0$ then

$$\sum_{\chi_0(g)=\rho} \chi(g) = 0\,, \quad \forall\, \rho \in U_{m_0}\,.$$

If $\chi \equiv 1$ on $\ker\chi_0$ then there exists $j \in \mathbb{Z}$ such that $\chi = \chi_0^j$ and

$$\sum_{\chi_0(g)=\rho} \chi(g) = |\ker\chi_0|\rho^j = \frac{|G|}{m_0}\rho^j\,.$$

[d]The surgery formula for the monopole count is contained in [15], while the surgery formula for the Kreck–Stolz invariant is contained in [23]. These involve metrics displaying very long necks around the splitting tori. See also the discussion at the beginning of Sec. 4.1.

Denote by $F_{p,q,m_0}$ the function

$$F_{p,q,m_0} : \mathbb{Z}/m_0\mathbb{Z} \to \mathbb{C}\,, \quad F_{p,q,m_0}(j \bmod \mathbb{Z}) = \frac{1}{m_0} \sum_{\rho \in U_{m_0}} f_{p,q,m_0}(\rho)\rho^j\,.$$

$F_{p,q,m_0}$ is precisely the Fourier transform of $\frac{1}{m_0} f_{p,q,m_0}$. We deduce

$$\begin{aligned} \widehat{\mathbf{SW}}^0_{p/q}(\chi) &= p\widehat{\mathbf{SW}}^0_{1/0}(\chi) + q\widehat{\mathbf{SW}}^0_{0/1}(\chi) \\ &\quad + |G| \begin{cases} F_{p,q,m_0}(j) & \text{if } \chi = \chi_0^j \\ 0 & \text{otherwise} \end{cases}\,. \end{aligned} \tag{3.1}$$

### 3.3. *Surgery formula for the modified Reidemeister–Turaev torsion*

The modified Reidemeister–Turaev torsion satisfies a surgery formula very similar in spirit to (3.1). We first need to survey a few algebraic facts in the special setting of the surgery formula. For details and proofs we refer to [22, Secs. 1.5 and 1.6], [26].

For any finite Abelian group $G$ we set

$$\mathbb{Q}[G]_0 = \{P \in \mathbb{Q}[G]\,; \quad \hat{P}(1) = 0\}\,.$$

Consider a rank 1 Abelian group $A = \mathbb{Z} \oplus \mathrm{Tors}(A)$, $C$ a finite cyclic group of order $m$, and $\varphi : A \to C$ an epimorphism. Fix a generator $u \in A$ of $\mathbb{Z} \subset A$, and let

$$\mathbb{Z}[[A]]_+ := \mathbb{Z}[A] + \Theta_A \mathbb{Z}[u, u^{-1}, (1-u)^{-1}]\,.$$

We refer to [26, 30] or [22, Proposition 1.27(b)] for an invariant definition of $\mathbb{Z}[[A]]_+$, which does not rely on the non-canonical decomposition $A =$ free part $\oplus$ torsion. The morphism $\varphi$ induces an "integration-along-fibers" morphism (see [22, Proposition 1.30])

$$\varphi_\sharp : \mathbb{Z}[[A]]_+ \to \mathbb{Z}[C]_0\,.$$

Its definition is best expressed in terms of Fourier transforms. Think of an element $f \in \mathbb{Z}[[A]]_+$ as a function $f : A \to \mathbb{Z}$. As such, it has a Fourier transform

$$\hat{f} : A^\sharp \to \mathbb{C}\,, \quad \hat{f}(\chi) = \sum_{a \in h} f(a)\chi(h)\,.$$

The above infinite sum may not be convergent for all $\chi$, but the Fourier transform still makes sense as a distribution on the compact Lie group $A^\sharp$ with singular support contained in $\{1\} \subset A^\sharp$ (see [22, Sec. 1.6]). The epimorphism $\varphi$ induces by duality an inclusion

$$\varphi^\sharp : C^\sharp \hookrightarrow A^\sharp\,.$$

If we regard $\hat{f}$ as a (generalized) function on $A^\sharp$ then the Fourier transform of $\varphi_\sharp f$ is the function on $C^\sharp$ obtained by pulling back $\hat{f}$ to $C^\sharp$ via the inclusion $\varphi^\sharp$. The only possible problem arises at $\chi = 1$ where $\hat{f}$ could have a singularity. Let $\chi \in C^\sharp$. In [22, Sec. 1.6] we have shown the following.

- If $f \in \mathbb{Z}[A]$, so that $f$ has finite support as a function $A \to \mathbb{Z}$, then $\hat{f}$ is a genuine function $A^\sharp \to \mathbb{C}$ and we have

$$\widehat{(\varphi_\sharp f)}(\chi) = \begin{cases} \hat{f}(\varphi^\sharp \chi)\,, & \text{if } \varphi^\sharp(\chi) \neq 1\,, \\ 0\,, & \text{if } \varphi^\sharp(\chi) = 1\,. \end{cases} \tag{3.2}$$

- If $f = \Theta_A(1-u)^{-1}$ then $\hat{f}$ is a distribution on $\hat{A}$ with singular support $\{1\}$ and we have

$$\widehat{(\varphi_\sharp f)}(\chi) = \begin{cases} \hat{\Theta}_A(\varphi^\sharp \chi)(1 - \varphi^\sharp\chi(T))^{-1}\,, & \text{if } \varphi^\sharp\chi \neq 1\,, \\ 0\,, & \text{if } \varphi^\sharp\chi = 1\,. \end{cases} \tag{3.3}$$

Suppose now that $\chi : A \to U_m$ is a surjective character, and $f \in \mathbb{Z}[[A]]_+$. The identity function $\iota_m : U_m \to U_m$ is a character of $U_m$ and $\chi^\sharp(\iota_m) = \chi$. We get an element $\chi_\sharp f \in \mathbb{Z}[U_m]_0$. If $f \in \mathbb{Z}[A]$ then

$$(\chi_\sharp f)(\iota_m) = \begin{cases} \hat{f}(\chi)\,, & \text{if } m > 1\,, \\ 0\,, & \text{if } m = 1\,. \end{cases} \tag{3.4}$$

If $f = \frac{\Theta_A}{1-u}$ then

$$\widehat{(\varphi_\sharp f)}(\iota_m) = \begin{cases} \hat{\Theta}_A(\chi)(1 - \chi(T))^{-1}\,, & \text{if } \chi(T) \neq 1\,, \\ 0\,, & \text{if } \chi(T) = 1\,. \end{cases} \tag{3.5}$$

If $\varphi : A \to B$ is a surjective morphism of finite Abelian groups then we get morphisms [22, Sec. 1.6]

$$\varphi_\sharp : \mathbb{Q}[A]_0 \to \mathbb{Q}[B]_0\,, \quad \varphi^\sharp : B^\sharp \to A^\sharp\,.$$

Then for $f \in \mathbb{Q}[A]_0$ and $\chi \in B^\sharp$ we have

$$\widehat{(\varphi_\sharp f)}(\chi) = \begin{cases} \hat{f}(\varphi^\sharp \chi)\,, & \text{if } \chi \neq 1\,, \\ 0\,, & \text{if } \chi = 1\,. \end{cases}$$

We can now return to topology. We will continue to use the notations in the previous section. Applying [30, Lemma 6.2] (or [31, Lemma VIII.1.4]) iteratively we deduce the following result.

**Theorem 3.3.** *Suppose $\chi$ is a nontrivial character of $G = H_1(N, \partial N; \mathbb{Z})$, so that $\chi(G) = U_m$, for some $m > 1$. Then*

$$\chi_\sharp \mathcal{T}_{p/q} = p\chi_\sharp \mathcal{T}_{1/0} + q\chi_\sharp \mathcal{T}_{0/1}\,.$$

Above and in the sequel we use the convention

$$\text{Object}_{p/q} := \text{Object}(M_{p/q})\,.$$

To proceed further we take the Fourier transform of the above formula and we get

$$(\widehat{\chi_\sharp \mathcal{T}_{p/q}})(\iota_m) = (\widehat{p\chi_\sharp \mathcal{T}_{1/0}})(\iota_m) + (\widehat{q\chi_\sharp \mathcal{T}_{0/1}})(\iota_m)\,,$$

where $m = \operatorname{ord}(\chi)$. Recall that

$$\mathcal{T}_{p/q} = \mathcal{T}^0_{p,q} - \frac{1}{2}CW_{p/q}\Theta_{p/q}\,, \quad \mathcal{T}_{1/0} = \mathcal{T}^0_{1/0} - \frac{1}{2}CW_{1/0}\Theta_{1/0}\,,$$

$$\mathcal{T}_{0/1} = \mathcal{T}^0_{0/1} + \frac{\Theta_{0/1}T}{(1-T)^2}\,.$$

Using (3.4) and (3.5) and the identities $\Theta_{p/q}(\chi) = 0 = \Theta_{1/0}(\chi)$ for $\chi \neq 1$ we deduce

$$\hat{\mathcal{T}}^0_{p/q}(\chi) = p\hat{\mathcal{T}}^0_{1/0}(\chi) + q\hat{\mathcal{T}}^0_{0/1}(\chi) + \begin{cases} \hat{\Theta}_{0/1}(\chi)\chi(T)(1-\chi(T))^{-2}\,, & \text{if } \chi(T) \neq 1\,, \\ 0\,, & \text{if } \chi(T) = 1\,. \end{cases}$$

The last term is nontrivial if and only if $\chi(T) \neq 1$ and $\chi|_{\text{Tors}(H_{0/1})} = 1$. This is possible if and only if $\chi = \chi_0^j$, for some $j = 1, 2, \ldots, m_0 - 1$. Additionally, $\Theta_{0/1}(\chi_0^j) = |\text{Tors}(H_{0/1})| = |G|/m_0$. We conclude that if $\chi$ is a nontrivial character of $G$ we have

$$\hat{\mathcal{T}}^0_{p/q}(\chi) = p\hat{\mathcal{T}}^0_{1/0}(\chi) + q\hat{\mathcal{T}}^0_{0/1}(\chi) + \frac{|G|}{m_0}\begin{cases} \dfrac{\chi_0^j}{(1-\chi_0^j)^2}\,, & \text{if } \chi = \chi_0^j\,, \quad j = 1, \ldots, m_0 - 1\,, \\ 0\,, & \text{if } \chi \neq \chi_0^k\,, \quad k = 1, \ldots, m_0\,. \end{cases}$$

We need to relate

$$\hat{\mathcal{T}}^0_{p/q}(1) = \frac{|H_{p/q}|}{2}CW_{p/q} = \frac{pm_0|G|}{2}CW_{p/q}\,,$$

$$\hat{\mathcal{T}}^0_{1/0}(1) = \frac{|H_{1/0}|}{2}CW_{1/0} = \frac{m_0|G|}{2}CW_{1/0}\,,$$

and

$$\hat{\mathcal{T}}^0_{0/1}(1) = \frac{1}{2}\Delta''_{M_{0/1}}(1)\,.$$

This follows from the surgery formula for the Casson–Walker invariant [11, Sec. 4.6], [32, Chap. 4]. More precisely, the arguments in [23, pp. 38–39] yield

$$CW_{p/q} = pCW_{1/0} + \frac{q}{2}\Delta''_{M_{0/1}}(1) + |G|\left(\frac{q(m_0^2-1)}{12m_0} - \frac{pm_0 s(q,p)}{2}\right)\,,$$

where $s(q,p)$ denotes the Dedekind sum of the pair of coprime integers $q$, $p$. Putting all of the above together we deduce that for every pair of coprime integers $(p,q)$, and every positive integer $m_0$ there exists a function

$$G_{p,q,m_0} : \mathbb{Z}/m_0\mathbb{Z} \to \mathbb{C}$$

such that

$$\hat{\mathfrak{T}}^0_{p/q}(\chi) = p\hat{\mathfrak{T}}^0_{1/0}(\chi) + q\hat{\mathfrak{T}}^0_{0/1}(\chi) + |G| \begin{cases} G_{p,q,m_0}(j)\,, & \text{if } \chi = \chi_0^j\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{3.6}$$

The similarity with (3.1) is striking. The results in [15, 23] show that

$$F_{p,q,m_0}(1) = G_{p,q,m_0}(1)\,, \quad \forall\, p,q,m_0\,.$$

In particular

$$F_{p,q,1} = G_{p,q,1}\,, \quad \forall\, p,q\,.$$

Let us briefly comment on the "flavor" of the surgery formulae (3.1) and (3.6). Note first that the first homology group of a rational homology 3-sphere can be naturally identified with its dual using the linking form. We can think of the invariants $\mathfrak{T}^0_M$ and $\mathbf{SW}^0_M$ as functions on $H^1(M,\mathbb{Z})$, as well as functions on the dual.

Suppose we perform Dehn surgery on a knot $K \hookrightarrow M$ to obtain a new rational homology sphere $M(K)$ and a knot $K' \hookrightarrow M(K)$. The surgery formula essentially states that if we know the values of these invariants on homology classes $c \in H_1(M,\mathbb{Z})$ which do not link with $K$ then we can also compute the values of these invariants on homology classes $c \in H_1(M(K),\mathbb{Z})$ which do not link with $K'$.

More rigorously, consider a pair $M_0$, $M_1$ related by a Dehn surgery on a knot $K$. Denote by $N$ the common knot complement, and set

$$H := H_1(N,\mathbb{Z})\,, \quad H_i := H_1(M_i,\mathbb{Z})\,, \quad i = 0,1\,.$$

We have a diagram of surjective morphisms

$$\begin{tikzcd} & H \arrow[dl, two heads, "\pi_0"'] \arrow[dr, two heads, "\pi_1"] & \\ H_0 & & H_1\,. \end{tikzcd}$$

Dualizing we get the diagram

$$\begin{tikzcd} & H^\sharp & \\ H_0^\sharp \arrow[ur, hook, "\pi_0^\sharp"] & & H_1^\sharp\,. \arrow[ul, hook', "\pi_1^\sharp"'] \end{tikzcd}$$

The knot $K$ defines two subgroups

$$K_i^\perp := \{c \in H_i^\sharp\,; \quad c(K_i) = 1\} \cong \{c \in H_i\,; \quad \mathbf{lk}_{M_i}(c,K_i) = 0\}\,,$$

$i = 0$, 1, and we have isomorphisms

$$\pi_i^\sharp : K_i^\perp \to G := \{\chi \in H^\sharp \ \chi(\mathbf{j}c)\,, \quad \forall\, c \in H_1(\partial N)\} \cong H_1(N, \partial N)\,.$$

We can think of $G$ as the graph of a correspondence $T_K : H_0^\sharp \to H_1^\sharp$ induced by the Dehn surgery. The domain of this correspondence is $K_0^\perp$, the range is $K_1^\perp$, and viewed as a correspondence $T_K \subset K_0^\perp \times K_1^\perp$ it is a group isomorphism.

For a surgery along a knot $K \hookrightarrow M$, whose meridian satisfies $\lambda_0 \cdot \mu = 1$, and attaching curve $c = p\mu + q\lambda_0$, we will denote by $\xi := \xi_{K,c}$ the induced p.i. We denote by $G_K$ the group $H_1(M\backslash K, \partial(M\backslash K); \mathbb{Z})$, by $m_0(K, c)$ respectively $p = p(K, c)$ the divisibility, and respectively multiplicity of the surgery. Finally, set $D_M := \mathbf{SW}_M^0 - \mathcal{T}_M^0$. Since $\mathcal{T}_M^0 = \mathbf{SW}_M^0$ if $b_1(M) = 1$ we can now rephrase the surgery formulae (3.1) and (3.6)

$$\hat{D}_{M_{K,c}}(\xi_{K,c}\chi) = p\hat{D}_M(\chi) + |G_K|\mathcal{K}_{m_0;p,q}(\chi)\,, \quad \forall\, \chi \in K^\perp = \mathrm{Dom}\,(\xi_{K,c})\,,$$

where the correction term is a function on $G_K$, which is nontrivial only on the cyclic group of order $m_0$ generated by

$$\mathbf{j}\lambda_0 \in \mathrm{Tors}(H_1(N\backslash K, \mathbb{Z})) = G_K^\sharp\,,$$

and depends only on the arithmetic of the surgery, $p$, $q$, $m_0$. Moreover

$$\mathcal{K}_{1;p,q} = 0\,.$$

More invariantly, consider a 3-manifold $N$ such that $\partial N \cong T^2$, fix a longitude $\lambda_0$, and two primitive classes $c_0$, $c_1$ represented by two simple closed curves. By Dehn surgery with attaching curves $c_0$, $c_1$ we get two manifolds $M_{c_0}$, $M_{c_1}$, with first homology groups $H_{c_0}$, $H_{c_1}$, and distinguished classes $K_{c_i} \in H_{c_i}$, $i = 0$, 1, defined by the core of the attached solid torus. Set $G := H_1(N, \partial N; \mathbb{Z})$, and denote by $\xi_{c_1,c_0}$ the isomorphism $K_{c_0}^\perp \to K_{c_1}^\perp$ described above. We denote by $[c_0, c_1]$ the orbit of $(c_0, c_1)$ relative to the action of $SL_2(\mathbb{Z})$ on the space of pairs of primitive classes $c_0$, $c_1 \in H_1(\partial N, \mathbb{Z})$. Then we have

$$(\lambda_0 \cdot c_0)\hat{D}_{M_{c_1}}(\xi_{c_0,c_1}\chi) = (\lambda_0 \cdot c_1)\hat{D}_{M_{c_0}}(\chi) + |G|\mathcal{K}_{m_0,[c_0,c_1]}(\chi)\,, \tag{3.7}$$

$\forall\, \chi \in K_{c_0}^\perp$. The **arithmetic type** $\alpha$ of a surgery is the pair $(m_0, [c_1, c_2])$. We denote by $\mathcal{A}$ the set of all arithmetic types for which the correction term $\mathcal{K}$ is trivial. We know that

$$(1, [c_1, c_2]) \in \mathcal{A}\,, \quad \forall\, c_1, c_2\,.$$

We will call the surgeries of arithmetic type $\alpha \in \mathcal{A}$ as **admissible**.

**Remark 3.4.** As explained in [22, Remark B.5], the orbit $[c_0, c_0]$ is completely characterized by the extension

$$0 \to \mathbb{Z}\langle c_0\rangle \oplus \mathbb{Z}\langle c_1\rangle \hookrightarrow H^1(\partial N, \mathbb{Z}) \to H^1(\partial N, \mathbb{Z})/(\mathbb{Z}\langle c_0\rangle \oplus \mathbb{Z}\langle c_1\rangle) \to 0\,.$$

More precisely, the quotient group $H^1(\partial N, \mathbb{Z})/(\mathbb{Z}\langle c_0\rangle \oplus \mathbb{Z}\langle c_1\rangle)$ is isomorphic to the cyclic groups of order $|c_0 \cdot c_1|$, and the extension is characterized by a character of this group. Thus, the orbit $[c_0, c_1]$ is described by the integer $c_0 \cdot c_1$, and a character of $\mathbb{Z}_{|c_0 \cdot c_1|}$.

## 4. Seiberg–Witten $\Longleftrightarrow$ Casson–Walker + Reidemeister–Turaev Torsion

### 4.1. *Topological preliminaries*

Denote by $\mathfrak{X}$ the family of all closed, compact oriented 3-manifolds $M$ with $b_1(M) = 0$ such that

$$\mathbf{SW}^0_M = \mathcal{T}^0_M .$$

We want to prove that $\mathfrak{X}$ consists of all 3-manifolds with $b_1(M) = 0$.

We already know that $M \in \mathfrak{X}$ if $M$ is an *integral* homology sphere, or if $M$ is a lens space. Also, we have

$$M_1, M_2 \in \mathfrak{X}\,, \quad b_1(M_1) = b_1(M_2) = 0 \Longrightarrow M_1 \# M_2 \in \mathfrak{X}\,. \tag{4.1}$$

We should perhaps dwell upon this statement which is a consequence of the fact that the modified Seiberg–Witten invariant behaves exactly as the modified Reidemeister–Turaev torsion (see [31, XII.1]) with respect to connected sums.

More precisely we should think of the manifold $M_1 \# M_2$ equipped with a metric $g_L$ with a very long neck $[-L, L] \times \Sigma$, $\Sigma =$ (round sphere), $L \gg 0$, in the region where we perform the connected sum (see Fig. 1).

The Kreck–Stolz invariant can only change by integral jumps due to the possible integral jumps in $\eta_{\rm dir}$. These are due to jumps in the dimension of the kernel of the spin$^c$-Dirac operator on $(M, g_L)$ coupled with a flat connection. Since the cutting

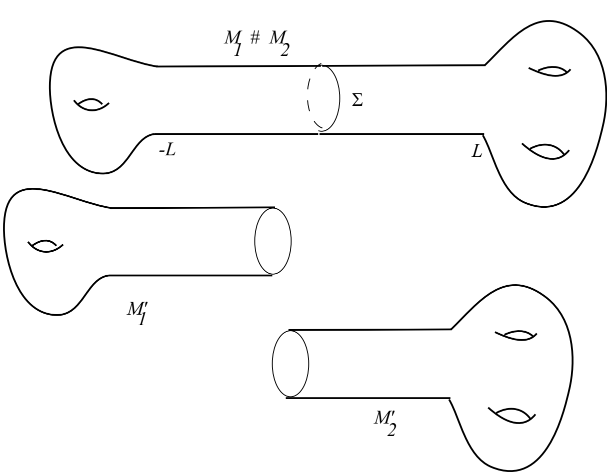


Fig. 1. Stretching a connected sum.

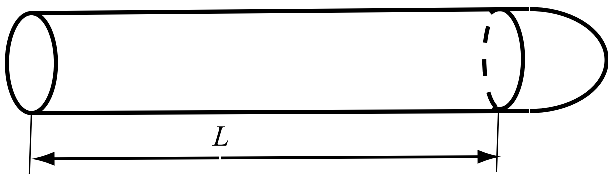


Fig. 2. Elongated thimble.

hypersurface $\Sigma$ has positive scalar curvature, the main result in [21] shows that the dimension of this kernel[e] is independent of $L \gg 0$. In particular this shows that the $g_L$-monopole count must also be independent of $L \gg 0$.

Now cut $(M_1 \# M_2, g_L)$ along the middle hypersurface $\{0\} \times \Sigma$ to obtain two manifolds with boundary $M_1'$ and $M_2'$ which have long collars of the form $[0, L] \times \Sigma$ (see Fig. 1). Now cap $M_k'$ with a long "thimble" of the type depicted in Fig. 2.

We obtain the manifolds $M_k$ equipped with metrics $g_{k,L}$ displaying long thimbles. Denote by $S_L$ the 3-sphere equipped with the Riemann metric $g_{0,L}$ obtained by gluing two thimbles along $\Sigma$. Since $\Sigma$ is equipped with a metric of positive scalar curvature and $b_1(M_k) = 0$ we deduce that the gluing results of [9, Example 8.25] are applicable and we conclude that $\forall\, L \gg 0$, $\forall\, \sigma_k \in \mathrm{Spin}^c(M_k)$

$$KS(M_1 \# M_2, \sigma_1 \# \sigma_2, g_L) = KS(M_1, \sigma_1, g_{1,L}) + KS(M_2, \sigma_2, g_{2,L}) - KS(S_L, g_{0,L})\,,$$

where $\sigma_1 \# \sigma_2$ is the connected sum of Euler structures defined in [31, XII.1]. The positive scalar curvature metric $g_{0,L}$ can be deformed to the round metric through metrics with positive scalar curvature and we deduce that $KS(S_L, g_{0,L}) = 0$.

The gluing techniques in [18, Sec. 4.5] imply that the monopole count is additive as well

$$\begin{aligned} &\mathbf{sw}_{M_1 \# M_2}(\sigma_1 \# \sigma_2, g_L) \\ &\quad = \mathbf{sw}_{M_1}(\sigma_1, g_{1,L}) + \mathbf{sw}_{M_2}(\sigma_2, g_{2,L}) - \mathbf{sw}_{S_L}\,, \quad \forall\, \sigma_k \in \mathrm{Spin}^c(M_k)\,. \end{aligned}$$

Since $g_{0,L}$ has positive scalar curvature we deduce that there are no irreducible monopoles on $S_L$ so that $\mathbf{sw}_{S_L} = 0$. These identities together with [31, Remark XII.1.3] imply immediately (4.1).

**Definition 4.1.** A *deflating primitive surgery* is a Dehn surgery on a knot $K$ in a rational homology sphere $M$ with the following properties.

(a) The longitude $\lambda \in H_1(\partial M \setminus K, \mathbb{Z})$ is a primitive class.
(b) The attaching curve $c$ of the surgery satisfies $c \cdot \lambda = \pm 1$.

An *excellent* surgery is a deflating primitive surgery which does not change the order of the first homology group. Two rational homology 3-spheres will be called *e-related* if one can be obtained from the other by a sequence of excellent surgeries.

[e]In fact, this kernel can be arranged to be trivial for large $L \gg 0$ by suitably and generically choosing the parameters $g$, $\eta$ in the Seiberg–Witten problem.

The attribute deflating is justified by the inequality

$$|H_1(M',\mathbb{Z})| \leq |H_1(M,\mathbb{Z})|$$

when $M'$ is obtained from $M$ by a deflating primitive surgery. The surgery is excellent iff we have equality. Note that if $M_0$ and $M_1$ are e-related then $H_1(M_0,\mathbb{Z}) \cong H_1(M_1,\mathbb{Z})$ and they have isomorphic linking forms. The following result is immediate.

**Lemma 4.2.** *Suppose $M \in \mathfrak{X}$, $b_1(M) = 0$. If $M'$ is obtained from $M$ by a deflating primitive surgery then $M' \in \mathfrak{X}$. In particular, if $M \in \mathfrak{X}$ and $M'$ is e-related to $M$ then $M' \in \mathfrak{X}$.*

**Proof.** Indeed, we have $G := H_1(M',\mathbb{Z}) \cong H_1(M\backslash K, \partial(M\backslash K);\mathbb{Z})$ and $F_{p,q,1} = G_{p,q,1}$. The surgery formulae establish the equality of $\mathcal{T}^0_{M'}$ and $\mathbf{SW}^0_{M'}$ as functions on $G^\sharp$, and $G^\sharp$ turns out to be their maximal domain. □

**Corollary 4.3.** $\mathfrak{X}$ *contains lens spaces, integral homology spheres, and is closed under connected sums and deflating primitive surgeries.*

Before we proceed further we want to briefly recall some basic topological facts. For more details we refer to [6, 24]. Any rational homology sphere can be obtained from $S^3$ by performing Dehn surgery on a link $L = K_1 \cup \cdots \cup K_n$ with surgery coefficients $p_1/q_1, \ldots, p_n/q_n$. Set $\vec{p} = (p_1, \ldots, p_n)$, $\vec{q} = (q_1, \ldots, q_n)$. We denote by $M = M(L, \vec{p}, \vec{q})$ the 3-manifold obtained by this surgery. We say that a surgery diagram belongs to $\mathfrak{X}$ if the corresponding 3-manifold belongs to $\mathfrak{X}$.

All the homological data of $M(L, \vec{p}, \vec{q})$ is contained in the $n \times n$ linking matrix $A = A(L; \vec{p}, \vec{q})$ defined by

$$a_{ij} = \begin{cases} p_i/q_i\,, & \text{if } i = j\,,\\ \ell_{ij}\,, & \text{if } i \neq j\,, \end{cases}$$

where $\ell_{ij} = \mathbf{Lk}(K_i, K_j)$. The surgery diagram is called integral if the linking matrix is integral.

Denote by $\mu_i$ the meridian of $K_i$, set $\Omega := A^{-1}$, and denote by $Q$ the diagonal $n \times n$ matrix $Q := \operatorname{diag}(q_1, \ldots, q_n)$. The manifold $M$ is a rational homology sphere if and only if the linking matrix $A$ is nonsingular. The first homology group admits the presentation

$$0 \to \mathbb{Z}^n \xrightarrow{QA} \mathbb{Z}^n \to H_1(M) \to 0\,,$$

so that its order is $\det(QA)$.

We have a natural isomorphism $H_1(S^3\backslash L) \to \mathbb{Z}^n$ defined by

$$c \mapsto (\mathbf{Lk}(c, K_1), \ldots, \mathbf{Lk}(c, K_n))\,,$$

for any closed curve $c$ disjoint from $L$, where $\mathbf{Lk}$ denotes the linking number of two disjoint knots in $S^3$. More geometrically

$$c = \sum_{i=1}^{m} \mathbf{Lk}(K_i, c)\mu_i \,.$$

Such a closed curve defines a homology class $[c] \in H_1(M)$. We have $[\mu_i] = -q_i[K_i]$. The images of the knots $[K_i]$ generate $H_1(M)$ and we have

$$\mathbf{lk}_M([\mu_i],[\mu_j]) = -\Omega_{ij} \operatorname{mod} \mathbb{Z} \,.$$

Suppose $[c]$ is a homology class in $M$ of order $m$. (We set $m = 1$ if $[c] = 0$.) A surgery on a knot representing $[c]$ has divisibility $m_0$ determined by

$$m_0 := (k, m)\,, \quad \mathbf{lk}_M([c],[c]) = \frac{k}{m} \operatorname{mod} \mathbb{Z} \,.$$

A class $c \in H_1(M, \mathbb{Z})$ is called *primitive* if it has divisibility one. Note that if $K$ is a knot in $M$ representing a primitive class of order $m$, and $M'$ is a manifold obtained from $M$ by deflating surgery then

$$|H_1(M', \mathbb{Z})| = \frac{1}{m}|H_1(M, \mathbb{Z})| \,.$$

The above observations show that the excellent surgeries are precisely the $1/q$-surgeries on a homologically trivial knots.

The **pruning** of a surgery diagram is the operation of removing the components with surgery coefficients $\pm 1$ which are algebraically split from the rest of the diagram. The pruning is equivalent to performing a sequence of excellent surgeries. We say that two surgery diagrams are **p-related** if one can go form one to another by a sequence of Kirby moves and prunings.

**Corollary 4.4.** *If $\mathcal{D}$ is a surgery diagram p-related to a diagram in $\mathfrak{X}$ then $\mathcal{D}$ is also in $\mathfrak{X}$.*

For every $\vec{a} \in \mathbb{Z}^n$ we denote by $[\vec{a}]$ the rational number

$$[\vec{a}] = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cdots}} \,.$$

The following result is a very special case of the general results in [4]. For the reader's convenience we present an elementary proof due to Saveliev [25]

**Lemma 4.5 [4, Corollary 3.9].** *Any homology lens space is e-related to a lens space.*

**Proof.** Any homology lens space $M$ is obtained by Dehn surgery on a knot $K_0$ in an integral homology sphere $M'$ [2]. Denote by $r \in \mathbb{Q}$ the surgery coefficient of $K_0$. We can represent the homology sphere $M'$ as surgery on an algebraically split link

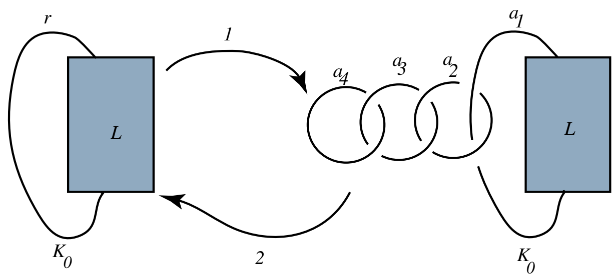


Fig. 3. Slam-dunking $K_0$.

$L = K_1 \cup \cdots \cup K_n$ in $S^3$ with surgery coefficients $\varepsilon_j = \pm 1$, [24]. We can think of $M$ as obtained from $S^3$ by surgery on the link $L_0 = K_0 \cup L$. Suppose

$$r = [\vec{a}]\,, \quad \vec{a} = (a_1, a_2, \ldots, a_m)\,.$$

Performing a sequence of slam-dunks as in [6, Sec. 5.3] we can replace $L_0$ with the link $L \cup K$ as in Fig. 3. We have thus succeeded in presenting $M$ as an integral surgery on a link in $S^3$ with surgery presentation

$$\begin{bmatrix} \pm 1 & 0 & 0 & \cdots & 0 & \ell_1 & 0 & \cdots & \cdots & 0 \\ 0 & \pm 1 & 0 & \cdots & 0 & \ell_2 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \pm 1 & \ell_n & 0 & \cdots & \cdots & 0 \\ \ell_1 & \ell_2 & \cdots & \cdots & \ell_n & a_1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & \cdots & \cdots & 0 & 1 & a_2 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 1 & a_3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 1 & a_m \end{bmatrix}.$$

The first part of this matrix is described by the link $L$, and $\ell_j := \mathbf{Lk}(K_0, K_j)$. By sliding $K_0$ over the components of $L$ we can kill the off-diagonal terms $\ell_i$. This changes the topological type of $K_0$. It becomes a knot $K_0'$, and the surgery coefficient $a_1$ changes to some integer $a_1'$. Now undo the slam-dunks. We get a new link algebraically split link $L_2 = K_0' \cup L$ where the surgery coefficient of $K_0'$ is $r' = [a_1', a_2, \ldots, a_m]$, and the surgery coefficient of $K_j$ is $\pm 1$. By inserting $\infty$-unknots and performing a sequence of Rolfsen twists we can replace $K_0'$ with an unknot $K_0''$. We can thus describe $M$ as surgery on the algebraically split link

$$L_2 = K_0'' \cup K_1 \cdots \cup K_n$$

with surgery coefficients $\varepsilon_0 = r'$, $\varepsilon_1 = \pm 1, \ldots, \varepsilon_n = \pm 1$. The $r'$ surgery on $K_0''$ is a lens space while the surgeries on $K_j$ are excellent surgeries. This shows $M$ is e-related to a lens space. $\square$

### 4.2. *Proof of the main result*

We will present a proof by induction over the "complexity" of a rational homology sphere. To define the notion of complexity we need to present a few algebraic facts about the linking forms of such manifolds. We follow the notations in [8].

For each prime $p > 1$, each $q \in \mathbb{Z}$, and each $k \geq 1$ such that $(p, q) = 1$ denote by $A_p^k(q)$ the linking form on the cyclic group $\mathbb{Z}/p^k$ defined by $\mathbf{g} \cdot \mathbf{g} = \frac{q}{p^k}$, where $\mathbf{g}$ denotes the natural generator of this group. Also denote by $E_0^k$ , $k \geq 1$ and $E_1^k$, $k \geq 2$, the linking forms on $\mathbb{Z}/2^k \oplus \mathbb{Z}/2^k$ defined by the matrices

$$E_0^k = \begin{bmatrix} 0 & 2^{-k} \\ 2^{-k} & 0 \end{bmatrix}, \quad E_1^k = \begin{bmatrix} 2^{1-k} & 2^{-k} \\ 2^{-k} & 2^{1-k} \end{bmatrix}.$$

When referring to $A_p^k(q)$, $E_0^k$, $E_1^k$ we mean the corresponding groups equipped with these linking forms. Define the complexity of $A_p^n$ to be $\kappa(A_p^n) = p^{k+1}$. Define the complexity of $E_i^n$, $i = 0$, $1$ to be $\kappa(E_i^n) = 2^{2n+2}$.

A classical result of Wall [33] shows that every linking form $(G, \mathfrak{q})$ decomposes non-uniquely into an orthogonal sum of $A$'s and $E$'s. If $\mathfrak{q}$ is a linking form on a $p$-group, then we define its complexity to be the product of the complexities of its elementary constituents $A$ and/or $E$ in some orthogonal decomposition. The results in [8] show that this number is independent of the chosen orthogonal decomposition of $\mathfrak{q}$ in elementary parts $A$ and $E$. We denote by $\kappa(\mathfrak{q})$ the complexity of $\mathfrak{q}$. For an arbitrary linking form, we define its complexity to be the product of the complexities of its $p$-group summands. For every $\mathbb{Q}HS$ $M$ we denote by $\nu_M$ the order of $H_1(M, \mathbb{Z})$, by $\mathfrak{q}_M$ the linking form of $M$, and by $\kappa_M$ the complexity of $\mathfrak{q}_M$. We have the following elementary result whose proof is left to the reader.

**Lemma 4.6.** *If $M_1$ and $M_2$ are two rational homology spheres such that $\nu_{M_1} | \nu_{M_2}$ and $\nu_{M_1} < \nu_{M_2}$ then $\kappa_{M_1} < \kappa_{M_2}$.*

We would like to present a few methods of reducing the complexity of a manifold. The primitive deflating surgeries provide one first example.

**Definition 4.7.** Let $K$ be a knot in a rational homology sphere $M$ supporting a **nontrivial** homology class. The knot $K$ is called **good** if $\mathfrak{q}_M(K, K) \neq 0$. Otherwise, it is called **bad**.

Suppose $K$ is a good knot in a rational homology sphere $M$. If $r$ is order of $K$ then

$$\mathfrak{q}(K, K) = \frac{m}{r}, \quad 0 < m < r\,,$$

and the divisibility of any surgery on this knot is $m_0(K) := (m, r)$. Consider any surgery with attaching curve $c$ satisfying $|c \cdot \lambda| = m_0$. This is a surgery of divisibility $m_0$ and of type $(p, q) = (m_0, *)$. We obtain a new rational homology sphere $M'$ such that $\nu_M = \frac{r}{m_0} \nu_{M'}$. Lemma 4.6 shows that the complexity of $M'$ is smaller than the complexity of $M$. We have thus proved the following result.

**Corollary 4.8.** *The complexity of a rational homology sphere can be reduced by performing surgeries on good knots. Moreover if the original manifold has no* 2*-torsion we can arrange that the resulting manifold also has no* 2*-torsion.*

Certain surgeries on certain bad knots also do reduce the complexity. We have the following technical result whose proof is deferred to the Appendix.

**Lemma 4.9.** *Suppose $M$ is a rational homology sphere with linking form $A_p^s(q_1) \oplus A_p^r(q_2)$ $s \geq r$, and $K$ is a bad knot in $M$ of the form*

$$K = c_1 \oplus c_2$$

*where the homology class $c_2$ generates $A_{p^r}(q_2)$. Then one can perform a surgery on $K$ such that the resulting manifold is a homology lens space of the same order as $M$.*

We will call the surgery in this lemma $A_p$-surgery. A knot with the properties in the lemma will be called a **mildly bad** knot. Set

$$\mathcal{Q} := \{\mathfrak{q}\,; \quad \mathfrak{q}_M \cong \mathfrak{q} \Longrightarrow M \in \mathfrak{X}\}\,.$$

We already know that all the linking forms $A_p^k(q)$ belong to $\mathcal{Q}$.

We need to talk a little bit about admissible surgeries, i.e. surgeries for which the correction term in the surgery formula (3.7) is trivial. Observe that if two rational homology spheres in $\mathfrak{X}$ are related by a Dehn surgery then this surgery is admissible.

**Corollary 4.10.** *The $A_p$ surgeries described in Lemma* 4.9 *are admissible.*

**Proof.** Consider a direct sum of two lens spaces with the above linking forms. This is a manifold in $\mathfrak{X}$. The result of this surgery produces a homology lens space which is also a manifold in $\mathfrak{X}$ so the surgery is admissible. □

We also want to mention the following topological result. For a proof we refer to [22].

**Lemma 4.11.** *Suppose $M_1$, $M_2$ are two rational homology spheres and $\phi$ is an isomorphism*

$$\phi : (H_1(M_1, \mathbb{Z}), \mathfrak{q}_{M_1}) \to (H_1(M_2, \mathbb{Z}), \mathfrak{q}_{M_2})\,.$$

*Suppose $K_i$ is a knot in $M_i$, $i = 1, 2$ such that $\phi([K_1]) = [K_2]$. If $M_i'$ $i = 1, 2$, are obtained perform surgeries* of the same arithmetic type $\alpha$ *on $K_1$ and $K_2$, then there exists an isomorphism*

$$\phi_\alpha : (H_1(M_1', \mathbb{Z}), \mathfrak{q}_{M_1'}) \to (H_1(M_2', \mathbb{Z}), \mathfrak{q}_{M_2'})\,.$$

The basic trick used in the proof of Theorem 2.4 is the following immediate consequence of the surgery formula (3.7).

**Lemma 4.12.** *Suppose $M$ is a rational homology sphere, and $\chi$ a character of $H = H_1(M, \mathbb{Z})$. We identify $H$ with its dual using the linking form. Suppose that there exists an admissible surgery on a knot $K$ such that*

$$\mathfrak{q}_M(\chi, K) = 0 \Longleftrightarrow \chi \in K^{\perp}$$

*and the result of the surgery is a manifold in $\mathfrak{X}$. Then $\hat{D}_M(\chi) = 0$. In particular, if for every $\chi$ there exists a knot with the above properties then $\hat{D}_M \equiv 0$.*

The proof of Theorem 2.4 will be carried out in several steps.

• **Step 1.** Fix a prime number $p > 2$, and denote by $\mathcal{R}_p$ the family of rational homology spheres such that $\nu_M = p^r$, $r > 0$. We will show that $\mathcal{R}_p \subset \mathfrak{X}$. The proof will be an induction on the complexity. For $\kappa \geq 0$ denote by $\mathcal{R}_p^{\kappa}$ the manifolds in $\mathcal{R}_p$ of complexity $\leq \kappa$.

The homology lens spaces of order $p$ have minimal complexity $p+1$, and belong to $\mathfrak{X}$ so that $\mathcal{R}_p^{p+1} \subset \mathfrak{X}$. Observe that if $M \in \mathcal{R}_p$ then we have a decomposition

$$\mathfrak{q}_M = \bigoplus_{j=1}^{n} A_p^{s_j}(q_j)\,, \quad 0 < s_1 \leq s_2 \leq \cdots \leq s_k\,,$$

and

$$\kappa_M = p^{s_1 + \cdots + s_k + k}\,.$$

The integer $k$ is called the *rank*, and we denote it by $\rho_M$. Define the **standard model** of $M$ to be the connected sum of lens spaces with the same linking form as $M$. We denote the standard model by $\tilde{M}$. Note that for every $M \in \mathcal{R}_p$ we have $\tilde{M} \in \mathfrak{X}$.

Suppose $\mathcal{R}_p^{\kappa} \subset \mathfrak{X}$. We want to prove that $\mathcal{R}_p^{\kappa+1} \subset \mathfrak{X}$. Let $M \in \mathcal{R}_p^{\kappa+1}$. Set $H := H_1(M, \mathbb{Z})$, and fix a nontrivial character $\chi$ of $H$. We distinguish two cases.

**Case 1.** There exists a good knot $K \in \chi^{\perp}$. Then there exists a good knot $\tilde{K}$ in the model $\tilde{M}$. We can perform a complexity reducing surgery on $\tilde{K}$ to obtain a manifold of smaller complexity which by induction we know is in $\mathfrak{X}$. This show that the arithmetic type of this surgery is admissible. We perform this admissible surgery on the knot $K$ on $M$ and we obtain a manifold of smaller complexity. Lemma 4.12 then implies $\hat{D}_M(\chi) = 0$.

**Case 2.** $\chi^{\perp}$ consists only of bad knots. If the rank of $H$ is 1 then $M$ is a rational homology space so it is in $\mathfrak{X}$. Suppose the rank is $> 1$. $(H, \mathfrak{q}_M)$ decomposes into a nontrivial sum of cyclic $p$ groups

$$H = \mathbb{Z}/p^{s_1} \oplus \cdots \oplus \mathbb{Z}/p^{s_k}\,, \quad 0 < s_1 \leq \cdots \leq s_k\,, \quad k > 1\,.$$

We get a corresponding decomposition $\chi = \chi_1 \oplus \cdots \oplus \chi_k$. Observe that all components must be nonzero. Indeed, if $\chi_j = 0$ then the generator of the $j$th component belongs to $\chi^{\perp}$ and is a good knot. Thus $\chi_1$, $\chi_2 \neq 0$. It is easy to see that

$\chi^\perp \cap (\mathbb{Z}/p^{s_1} \oplus \mathbb{Z}/p^{s_2}) \neq 0$. Pick a mildly bad knot $K$ in this group. Thus all but the first two components of $K$ are zero, and one of the components generates the corresponding summand. Perform an $A_p$ surgery on this knot. Using models as in the first case we conclude that this surgery is admissible. This reduces the complexity of $M$. By induction, the resulting manifold is in $\mathfrak{X}$ that $\hat{D}_M(\chi) = 0$. Thus $\mathcal{R}_p \subset \mathfrak{X}$.

• **Step 2.** If $\nu_M$ is odd then $M \in \mathfrak{X}$. For each vector $\vec{p} = (p_1, \ldots, p_n)$ whose components consist of pairwise distinct of odd primes. Denote by $\mathcal{R}_{\vec{p}}$ the family of rational homology spheres $M$ such that the prime divisors of $\nu_M$ are amongst the primes $p_j$. Again we perform induction on complexity. The first homology group $H$ of $M \in \mathcal{R}_{\vec{p}}$ decomposes as an orthogonal direct sum of $p$-groups

$$H = \bigoplus_{j=1}^{n} G_{p_j}\,, \quad |G_{p_j}| = p_j^{s_j}\,.$$

Each component $G_{p_j}$ decomposes as an orthogonal sums of $A^*_{p_j}(*)$. Denote by $r_j$ the number of such components. We have

$$\kappa(M) = |H| p_1^{r_1} \cdots p_n^{r_n}\,.$$

Suppose $\chi$ is a nontrivial character of $H$. We distinguish again two cases.

**Case 1.** $\chi^\perp$ contains good knots. In this case we perform a surgery as in Lemma 4.8 which produces a manifold of smaller complexity. Using models as in *Step 1* we can prove that such a surgery is admissible. Thus in this case $\hat{D}_M(\chi) = 0$.

**Case 2.** If all $r_j$'s are $= 1$ then $H$ is a cyclic group, $M$ is a homology lens space, so that $M \in \mathfrak{X}$.

Suppose $r_1 > 1$. We set $p := p_1$ and

$$G_p = \bigoplus_{j=1}^{n} A_p^{s_j}(q_j)\,, \quad 0 < s_1 \leq s_2 \leq \cdots \leq s_k\,.$$

We conclude as in Step 1 that all the components of $\chi$ determined by the above decomposition of $G_p$ are nontrivial. By performing an $A_p$ surgery on a mildly bad knot we obtain a manifold $M'$ satisfying

$$\nu_{M'} = \nu_M\,, \quad r_1' = r_1 - 1\,, \quad r_j' = r_j\,, \quad \forall\, j = 2, \ldots, n\,.$$

Thus $\kappa(M') < \kappa(M)$ and we conclude by induction. We can now conclude that any Dehn surgery which transforms an odd order $\mathbb{Q}HS$ to an odd order $\mathbb{Q}HS$ is admissible.

• **Step 3.** If $\mathfrak{q}_M = A_2^n(q) \oplus \mathfrak{q}_1$, where $\mathfrak{q}_1$ is a linking form on a group of odd order, then $M \in \mathfrak{X}$. Denote by $\mathcal{R}_2'$ the family of such rational homology spheres. The complexity of such a manifold is

$$\kappa_M = 2^{n+1}\kappa(\mathfrak{q}_1)\,.$$

The considerations in Step 2 lead to the following complexity reduction trick.

**Lemma 4.13.** *If $K$ is a knot in an odd order $\mathbb{Q}HS$ such that $K^\perp$ is a nontrivial subgroup, then there exists $K' \in K^\perp$ and a surgery on $K'$ producing an odd order $\mathbb{Q}HS$ of smaller complexity. Moreover, such a surgery is admissible.*

Suppose $\mathfrak{q}_M = A_2^n(q) \oplus \mathfrak{q}_1$ and $\chi$ is a nontrivial character of $\mathfrak{q}_M$. Then $\chi^\perp \neq 0$. Decompose

$$\chi = \chi_0 \oplus \chi_1\,, \quad \chi_0 \in A_2^n(q)\,, \quad \chi_1 \in \mathfrak{q}_1\,.$$

It follows that $\chi_1^\perp$ is a nontrivial subgroup in $\mathfrak{q}_1$. Perform a complexity reduction surgery on a knot $K \in \chi_1^\perp \subset \mathfrak{q}_1$ as in Lemma 4.13 to conclude, as we have done before, that $\hat{D}_M(\chi) = 0$.

• **Step 4.** If $\mathfrak{q}_M = \bigoplus_{k=1}^m A_2^{n_k}(q_k) \oplus \mathfrak{q}_1$, $n_1 \geq n_2 \geq \cdots \geq n_m > 0$, where $\mathfrak{q}_1$ is a linking form on a group of odd order, then $M \in \mathfrak{X}$. Denote by $\mathcal{R}_2$ the family of such rational homology spheres. *Redefine* the complexity of such a manifold to be

$$\hat{\kappa}_M := \kappa\left(\bigoplus_{k=1}^m A_2^{n_k}(q_k)\right) = 2^{n_1+\cdots+n_m+m}\,.$$

For every $M \in \mathcal{R}_2$ we define its model $\tilde{M}$ to be a connected sum of lens spaces with the same linking form as $M$. Again we will carry out the proof by induction on the new complexity. The basic complexity reduction technique is contained in the following lemma whose proof is deferred to the Appendix.

**Lemma 4.14.** (a) *Suppose $c \in A_2^s(q_1) \oplus A_2^r(q_2)$, $s \geq r > 0$. Then there exists $K \in c^\perp$ of the form*

$$K = K_1 \oplus K_2 \tag{4.2}$$

*where $K_2$ is a generator of $A_2^r(q_2)$.*

(b) *Suppose $M$ is a rational homology sphere such that $\mathfrak{q}_M = A_2^s(q_1) \oplus A_2^r(q_2)$ and $K$ is a knot in $M$ whose homology class satisfies* (4.2)*. Then there exists a Dehn surgery on $M$ such that the resulting manifold $M'$ is in $\mathcal{R}_2'$ and has smaller complexity. More precisely, we can arrange that*

$$\mathfrak{q}_{M'} = A_2^t(q) \oplus \mathfrak{q}_1\,,$$

*where $t \leq r+s$ and $\mathfrak{q}_1$ is the linking form of some odd order lens space.*

Let $M \in \mathcal{R}_2$. Then we can write

$$\mathfrak{q}_M = \mathfrak{q}_0 \oplus \mathfrak{q}_1 := \left(\bigoplus_{k=1}^m A_2^{n_k}(q_k)\right) \oplus \mathfrak{q}_1\,.$$

If $m = 1$ then $M \in \mathfrak{X}$ according to Step 3. We can assume $m > 1$.

Any nontrivial character $\chi \in \mathfrak{q}_M$ decomposes as

$$\chi = \chi_0 + \chi_1\,, \quad \chi_i \in \mathfrak{q}_i\,, \quad i = 0, 1\,.$$

Pick $K \in A_2^{n_1}(q_1) \oplus A_2^{n_2}(q_2)$ orthogonal to $\chi_0$, and satisfying (4.2). We want to perform a complexity reduction surgery as in Lemma 4.14 but we first must show that any such surgery is admissible. This can be seen by performing this surgery on the model $\tilde{M} \in \mathfrak{X}$. It produces a manifold of smaller complexity which by induction we know it is in $\mathfrak{X}$, and thus proving that the surgery is admissible. We can now conclude, as we have many times before, that $\hat{D}_M(\chi) = 0$. This shows that $\mathfrak{R}_2 \subset \mathfrak{X}$.

• **Step 5 Conclusion.** Suppose $M$ is an arbitrary $\mathbb{Q}HS$. Then $\mathfrak{q}_M = \mathfrak{q}_0 \oplus \mathfrak{q}_1$, where $\mathfrak{q}_0$ is a linking form on a 2 group, and $\mathfrak{q}_1$ is a linking form on an odd order group. The results in [8, Theorem 0.1] show that if we add sufficiently many $A$'s to $\mathfrak{q}_0$ we obtain a linking form isomorphic to a direct sum of $A$'s. Topologically this means that we can find a connected sum $X$ of lens spaces of order $2^s$ such that $M\#X \in \mathcal{R}_2 \subset \mathfrak{X}$. Thus $\hat{D}_{M\#X} = 0$. Since $\hat{D}$ is additive with respect to connected sums we deduce $\hat{D}_M = 0$. This concludes the proof of Theorem 2.4.

## 5. Final Comments

The invariant introduced by Ozsváth and Szabó in [23] satisfies the same surgery formula as the modified Seiberg–Witten invariant, and detects in the same fashion the Casson–Walker invariant. This shows that the strategy presented in this paper also answers a question in [23]. More precisely, their invariant is equivalent to the modified Reidemeister–Turaev torsion.

If we consider the mod $\mathbb{Z}$ reduction of the modified Seiberg–Witten invariant we deduce that

$$\mathbf{sw}_M^0(\sigma) = \frac{1}{8} KS_M(\sigma) \mathrm{mod}\, \mathbb{Z}\,,$$

where $KS_M(\sigma)$ denotes the Kreck–Stolz invariant. In general it depends on the metric but its mod $8\mathbb{Z}$ reduction is metric independent. Fix a *spin* structure $\epsilon$. This choice allows us to think of $\mathcal{T}$ and $\mathbf{SW}$ as functions $H \to \mathbb{Q}$, $H := H_1(M, \mathbb{Z})$.

Denote by $\mathcal{F}_M$ the space of functions $f : H \to \mathbb{Q}/\mathbb{Z}$. For each $h \in H$ define the finite difference operator

$$\Delta_h : \mathcal{F} \to \mathcal{F}\,, \quad (\Delta_h f)(\sigma) := f(h \cdot \sigma) - f(\sigma)\,.$$

In [28] it is shown that for every $h_1$, $h_2 \in H$ we have

$$\mathbf{lk}_M(h_1, h_2) = \Delta_{h_1} \Delta_{h_2} \mathcal{T} \,\mathrm{mod}\, \mathbb{Z}\,.$$

Since the constant functions are killed by $\Delta_\bullet$ we deduce

$$\Delta_\bullet \mathcal{T} = \Delta_\bullet \mathcal{T}^0\,.$$

Our main result now implies the following equality.

$$\mathbf{lk}_M(h_1, h_2) = -\frac{1}{8}\Delta_{h_1}\Delta_{h_2} KS_M \bmod \mathbb{Z}\,, \quad \forall h_1, h_2 \in \mathbb{Z}\,.$$

This shows that the function

$$H \ni h \mapsto \frac{1}{8}KS_M(\sigma(\epsilon)) - \frac{1}{8}KS_M(h \cdot \sigma(\epsilon)) \mod \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$$

is a quadratic refinement of the linking form. A simple application of the Atiyah–Patodi–Singer index theorem shows that this quadratic refinement coincides with the canonical refinement of the linking form associated to the *spin* structure $\epsilon$ described in the work of Brumfiel–Morgan, [1]. This simple observation together with the main result of this paper has important applications in the study of isolated singularities of complex surfaces. We refer to [17, 22] for more details.

Recently, Deloup and Massuyeau [5] gave a purely topological proof of this relationship between the mod $\mathbb{Z}$ reduction of the Reidemeister–Turaev torsion and the quadratic refinements of the linking form.

In [20] we associated to each *spin* structure $\epsilon$ on a rational homology sphere an invariant $c(\epsilon) \in \mathbb{Q}/\mathbb{Z}$ which was powerful enough to distinguish many lens spaces. We can now identify it. We have

$$c(\epsilon) = \frac{1}{8}KS_M(\epsilon) + \frac{1}{2}CW_M \bmod \mathbb{Z}\,.$$

**Acknowledgments**

I want express my gratitude to Frank Connolly from whom I learned about the Reidemeister torsion. I am grateful to Nikolai Saveliev for his patience and expertise in answering the questions I had while working on this project. I want to thank my colleagues Stephan Stolz and Larry Taylor for many illuminating discussions, and to V. Turaev for his interest in this work. I am indebted to Arthur Greenspoon who volunteered to read the paper and smoothed the flow of my hastily written words. Finally, I want to thank the anonymous referee for the many useful comments and suggestions. This work was partially supported by NSF grant DMS-0303601.

## Appendix A. Proofs of Some Technical Results

**Proof of Lemma 4.9.** A simple model of $A_p$ surgery is the manifold given by the surgery diagram

$$\mathcal{D}_n := \{(K_1, p^s/q_1), (K_2, p^r/q_2), (K, n)\}\,, \quad s \geq r > 0\,,$$

where $K_1$ and $K_2$ are unlinked unknots, and $K$ is a knot such that $\mathbf{Lk}(K, K_i) = \ell_i$, $-p^s/2 < \ell_1 < p^s/2$, $-p^r/2 < \ell_2 < p^r/2$. The linking matrix of this diagram is

$$\begin{bmatrix} \dfrac{p^s}{q_1} & 0 & \ell_1 \\ 0 & \dfrac{p^r}{q_2} & \ell_2 \\ \ell_1 & \ell_2 & n \end{bmatrix}.$$

We can view $K$ as a knot in the connected sum of lens spaces $L(p^s, -q_1)\#L(p^r, -q_2)$. $K$ is a bad knot if and only if

$$\frac{q_1\ell_1^2}{p^s} + \frac{q_2\ell_2^2}{p^r} = k \in \mathbb{Z}\,. \tag{$*$}$$

The knot is mildly bad if and only if $(p, \ell_2) = 1$. Denote by $H$ the first homology group of the 3-manifold obtained by performing the surgery indicated by $\mathcal{D}_n$. The matrix

$$B_n := \begin{bmatrix} p^s & 0 & q_1\ell_1 \\ 0 & p^r & q_2\ell_2 \\ \ell_1 & \ell_2 & n \end{bmatrix}$$

is a presentation matrix for $H$ and has determinant

$$\begin{aligned} \det(B_n) &= p^{s+r}n - p^s q_2\ell^2 - p^r q_1\ell_1^2 \\ &= p^{s+r}\left(n - \left(\frac{q_1\ell_1^2}{p^s} + \frac{q_2\ell_2^2}{p^r}\right)\right) = p^{s+r}(n-k)\,. \end{aligned}$$

Observe that $|\det B_n| = p^{s+r}$ when $n = k \pm 1$. Let $n = k + 1$.

Rewrite the condition $(*)$ as

$$q_1\ell_1^2 + p^{s-r}q_2\ell_2^2 = p^s k\,.$$

Since $(q_1\ell_1, p) = 1$ we deduce that $(q_2\ell_2, p) = 1$. To find $H$ we need to find the elementary divisors $d_1|d_2|d_3$ of $B_{k+1}$. Clearly $d_1 = 1$. By looking at the $2\times 2$ minor in the top left hand corner we deduce that $d_2|p^{s+r}$. On the other hand, if we look the $2 \times 2$ minor

$$\begin{vmatrix} 0 & q_2\ell_2 \\ \ell_1 & k+1 \end{vmatrix},$$

we deduce that it is not divisible by $p$. Thus $d_2 = 1$ which shows that $H$ is a cyclic $p$-group of order $p^{s+r}$. $\square$

**Proof of Lemma 4.14.** Part (a) is elementary and is left to the reader. For Part (b) it suffices to look at a concrete realization of the given homological data. Any homology class $K \in A_2^s(q_1) \oplus A_2^r(q_2)$ satisfying (4.2) can be realized as a knot in a connected sum of lens spaces $X := L(2^s, a)\#L(2^r, b)$. We describe $X$ as surgery on

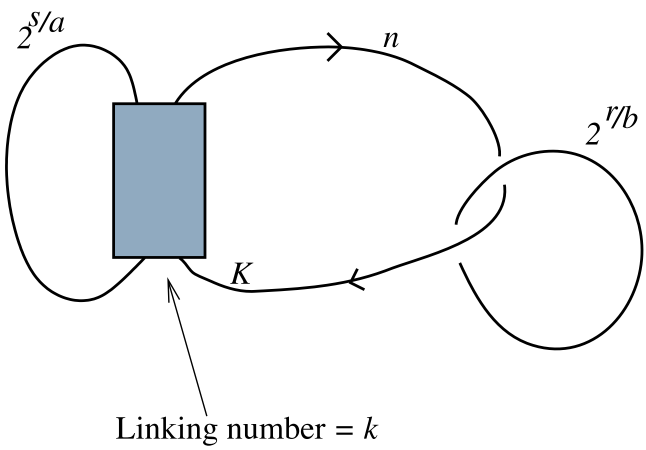


Fig. 4. Modeling a complexity reducing surgery.

two unlinked unknots $K_1$, $K_2$ with surgery coefficients $-2^s/a$, $-2^r/b$. Also we can view $K$ as a knot such that

$$\mathbf{Lk}(K, \mathbf{g}) = 1\,, \quad \mathbf{Lk}(K, K_1) = k\,.$$

Assume $K$ has an integral surgery coefficient $n$ (see Fig. 4). Slam-dunking $K_2$ over $K$ we obtain a surgery presentation with linking matrix

$$A := \begin{bmatrix} -2^s/a & k \\ k & n + b/2^r \end{bmatrix}.$$

The first homology group $H$ of the manifold described by this surgery diagram admits the presentation matrix

$$B := \begin{bmatrix} -2^s & ak \\ 2^r k & 2^r n + b \end{bmatrix}.$$

The order of this group is $|2^{r+s}n + 2^s b + 2^r ak^2|$. Pick $n$ to be any number such that $2^{r+s+1}$ does not divide the order of this group. Then $H$ is a cyclic group of the form $\mathbb{Z}/2^t \oplus \mathbb{Z}/(2m+1)$, $t \leq r + s$. Its $\hat{\kappa}$-complexity is smaller than that of $A_2^s(q_1) \oplus A_2^r(q_2)$. $\square$

## References


[1] G. W. Brumfiel and J. W. Morgan, Quadratic functions, the index modulo 8, and a $\mathbb{Z}/4$-Hirzebruch formula, *Topology* **12** (1973) 105–122.

[2] S. Boyer and D. Lines, Surgery formulae for Casson's invariant and extensions to homology lens spaces, *J. Reine. Angew. Math.* **405** (1990) 181–220.

[3] A. L. Carey, M. Marcolli and B. L. Wang, The geometric triangle for 3-dimensional Seiberg–Witten monopoles, *Commun. Contemp. Math.* **5** (2003) 197–250.

[4] T. Cochran, A. Gerges and K. Orr, Dehn surgery realtions on 3-manifolds, *Math. Proc. Cambridge Philos. Soc.* **131** (2001) 97–127.

[5] F. Deloup and G. Massuyeau, Reidemeister–Turaev torsion modulo one of rational homology 3-spheres, *Geom. Topol.* **7** (2003) 773–787.

[6] R. E. Gompf and A. I. Stipsicz, *An Introduction to 4-Manifolds and Kirby Calculus*, Graduate Studies in Mathematics, Vol. 20 (American Mathematical Society, 1999).

[7] F. Hirzebruch, W. D. Neumann and S. S. Koh, *Differentiable Manifolds and Quadratic Forms*, Lecture Notes in Pure and Applied Mathematical, No. 4 (Marcel Dekker, 1971).

[8] A. Kawauchi and S. Kojima, Algebraic classification of linking pairings of 3-manifolds, *Math. Ann.* **253** (1980) 29–42.

[9] P. Kirk and M. Lesch, The $\eta$-invariant, Maslov index and spectral flow for Dirac-type operators on manifolds with boundary, math.DG/0012123, *Forum Math.* **16** (2004) 553–629.

[10] M. Kreck and S. Stolz, Non connected moduli spaces of positive sectional curvature metrics, *J. Amer. Math. Soc.* **6** (1993) 825–850.

[11] C. Lescop, *Global Surgery Formula for the Casson–Walker Invariant*, Annals of Mathematics Studies, Vol. 140 (Princeton University Press, 1996).

[12] Y. Lim, The equivalence of Seiberg–Witten and Casson invariants for homology 3-spheres, *Math. Res. Letters* **6** (1999) 631–644.

[13] Y. Lim, Seiberg–Witten invariants for 3-manifolds in the case $b_1 = 0$ or 1, *Pacific J. Math.* **195** (2000) 179–204.

[14] Y. Lim, Seiberg–Witten moduli spaces for 3-manifolds with cylindrical-end $T^2 \times \mathbb{R}^+$, *Commun. Contemp. Math.* **2** (2000) 461–509.

[15] M. Marcolli and B. L. Wang, Seiberg–Witten invariant and Casson–Walker invariants for rational homology 3-spheres, math.DG/0101127, *Geom. Dedicata* **91** (2002) 45–58.

[16] G. Meng and C. H. Taubes, $\underline{SW}$ = Milnor torsion, *Math. Res. Letters* **3** (1996) 661–674.

[17] A. Némethi and L. I. Nicolaescu, Seiberg–Witten invariants and complex singularities, math.AG/0111298, *Geom. Topol.* **6** (2002) 268–329.

[18] L. I. Nicolaescu, *Notes on Seiberg–Witten Theory*, Graduate Studies in Mathematics, Vol. 28 (American Mathematics Society, 2000).

[19] L. I. Nicolaescu, Seiberg–Witten invariants of lens spaces, math.DG/9901071, *Canad. J. Math.* **53** (2001) 780–808.

[20] L. I. Nicolaescu, On the Reidemeister torsion of rational homology spheres, math.GT/0006181, *Int. J. Math. Math. Sci.* **25** (2001) 11–17.

[21] L. I. Nicolaescu, On the Cappell–Lee–Miller gluing theorem, *Pacific J. Math.* **206** (2002) 159–185.

[22] L. I. Nicolaescu, *The Reidemeister Torsion of 3-Manifolds*, Studies in Mathematics, Vol. 30 (Walter de Gruyter, 2003).

[23] P. Ozsváth and Z. Szabó, The theta divisor and the Casson–Walker invariant, math.GT/0006194.

[24] N. Saveliev, *Lectures on the Topology of 3-Manifolds. An Introduction to the Casson Invariant*, DeGruyter Textbook (Walter de Gruyter, 1999).

[25] N. Saveliev, private communication.

[26] V. G. Turaev, Reidemeister torsion and the Alexander polynomial, *Math. Sbornik* **101** (1976) 252–270.

[27] V. G. Turaev, Euler structures, nonsingular vector fields and torsions of Reidemeister type, Izvestiya Akad. Nauk **53** (1989); (English version, *Math. USSR Izvestiya* **34** (1990) 627–662.)

[28] V. G. Turaev, Torsion invariants of Spin$^c$-structures on 3-manifolds, *Math. Res. Letters* **4** (1997) 679–695.

[29] V. G. Turaev, A combinatorial formulation for the Seiberg–Witten invariants of 3-manifolds, *Math. Res. Letters* **5** (1998) 583–598.
[30] V. G. Turaev, Surgery formula for torsions and Seiberg–Witten invariants of 3-manifolds, math.GT/0101108.
[31] V. G. Turaev, *Torsions of 3-Dimensional Manifolds*, Progress in Mathematics, Vol. 208 (Birkhäuser, 2002).
[32] K. Walker, *An Extension of Casson's Invariant*, Annals of Mathematical Studies, Vol. 126 (Princeton University Press, 1992).
[33] C. T. C. Wall, Quadratic forms on finite groups, and related topics, *Topology* **2** (1964) 281–298.